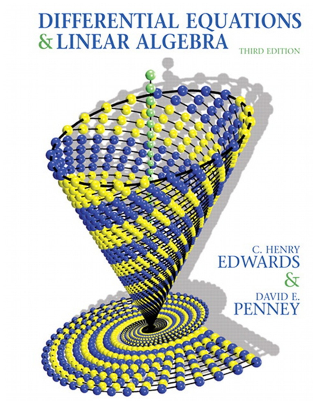
A Solution Manual For

Differential equations and linear algebra, 3rd ed., Edwards and Penney



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May 15, 2024

Contents

1 Section 1.2. Integrals as general and particular solutions. Page 16

1.1 problem 1

Internal problem ID [1]
Internal file name [`OUTPUT/1_Sunday_June_05_2022_01_33_34_AM_9550685/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = 1 + 2x}$$

With initial conditions

$$[y(0) = 3]$$

1.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = 1 + 2x$$

Hence the ode is

$$y' = 1 + 2x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1 + 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.1.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 1 + 2x \,\mathrm{d}x \\ &= x^2 + c_1 + x \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 3$ in the above solution gives an equation to solve for the constant of integration.

$$3 = c_1$$

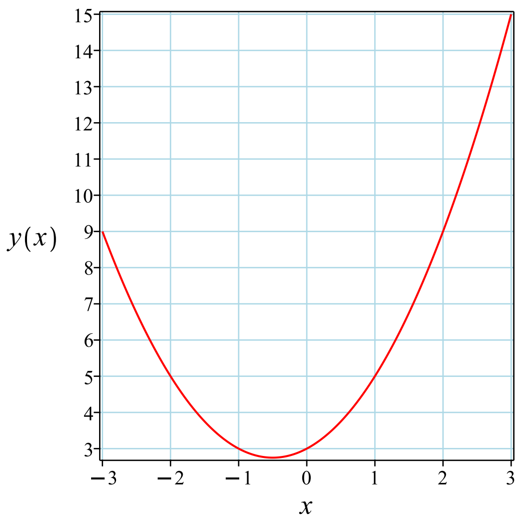
$$c_1 = 3$$

Substituting c_1 found above in the general solution gives

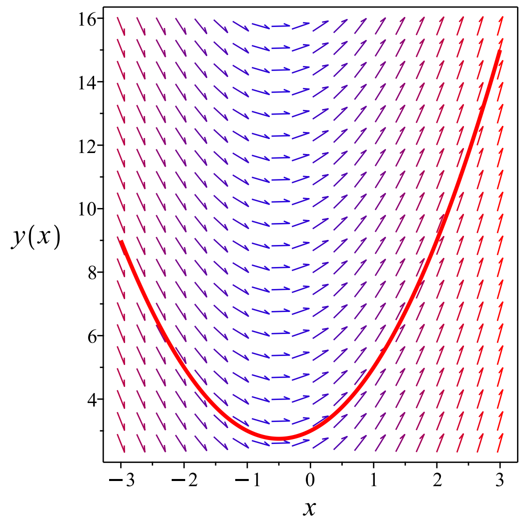
$$y = x^2 + x + 3$$

Summary
The solution(s) found are the following

$$y = x^2 + x + 3 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = x^2 + x + 3$$

Verified OK.

1.1.3 Maple step by step solution

Let's solve

$[y' = 1 + 2x, y(0) = 3]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (1 + 2x) \, dx + c_1$

- Evaluate integral

  $y = x^2 + c_1 + x$

- Solve for y

  $y = x^2 + c_1 + x$

- Use initial condition $y(0) = 3$

  $3 = c_1$

- Solve for c_1

  $c_1 = 3$
- Substitute $c_1 = 3$ into general solution and simplify

  $y = x^2 + x + 3$
- Solution to the IVP

  $y = x^2 + x + 3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([diff(y(x),x) = 1+2*x,y(0) = 3],y(x), singsol=all)
```

$$y(x) = x^2 + x + 3$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 11

```
DSolve[{y'[x]==1+2*x,y[0]==3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 + x + 3$$

1.2 problem 2

Internal problem ID [2]
Internal file name [`OUTPUT/2_Sunday_June_05_2022_01_33_35_AM_53084472/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = (-2 + x)^2}$$

With initial conditions

$$[y(2) = 1]$$

1.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= (-2 + x)^2 \end{aligned}$$

Hence the ode is

$$y' = (-2 + x)^2$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = (-2 + x)^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

1.2.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int (-2 + x)^2 \, \mathrm{d}x \\ &= \frac{(-2 + x)^3}{3} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

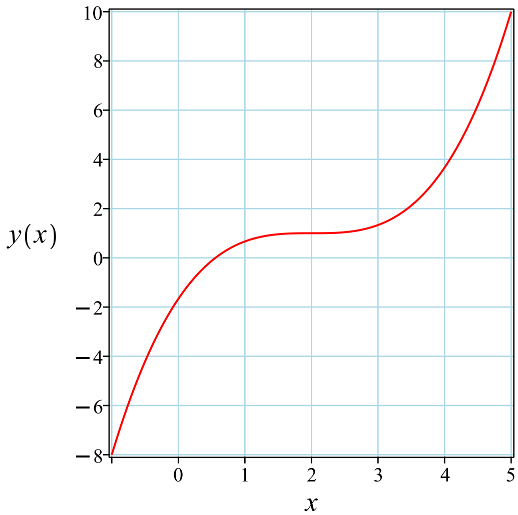
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}$$

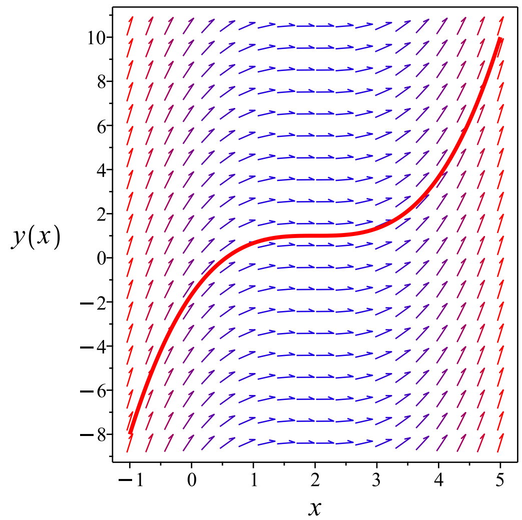
Summary

The solution(s) found are the following

$$y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}$$

Verified OK.

1.2.3 Maple step by step solution

Let's solve

$\left[y' = (-2+x)^2, y(2) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int (-2+x)^2 dx + c_1$

- Evaluate integral

  $y = \frac{(-2+x)^3}{3} + c_1$

- Solve for y

  $y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{8}{3} + c_1$

- Use initial condition $y(2) = 1$

  $1 = c_1$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}$

- Solution to the IVP

  $y = \frac{1}{3}x^3 - 2x^2 + 4x - \frac{5}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(x),x) = (-2+x)^2,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{(-2+x)^3}{3} + 1$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 22

```
DSolve[{y'[x]==(-2+x)^2,y[2]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(x^3 - 6x^2 + 12x - 5)$$

1.3 problem 3

Internal problem ID [3]
Internal file name [`OUTPUT/3_Sunday_June_05_2022_01_33_36_AM_71634107/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \sqrt{x}}$$

With initial conditions

$$[y(4) = 0]$$

1.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \sqrt{x}$$

Hence the ode is

$$y' = \sqrt{x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 4$ is inside this domain. The domain of $q(x) = \sqrt{x}$ is

$$\{0 \leq x\}$$

And the point $x_0 = 4$ is also inside this domain. Hence solution exists and is unique.

1.3.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \sqrt{x}\, \mathrm{d}x \\ &= \frac{2x^{\frac{3}{2}}}{3} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 4$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{16}{3} + c_1$$

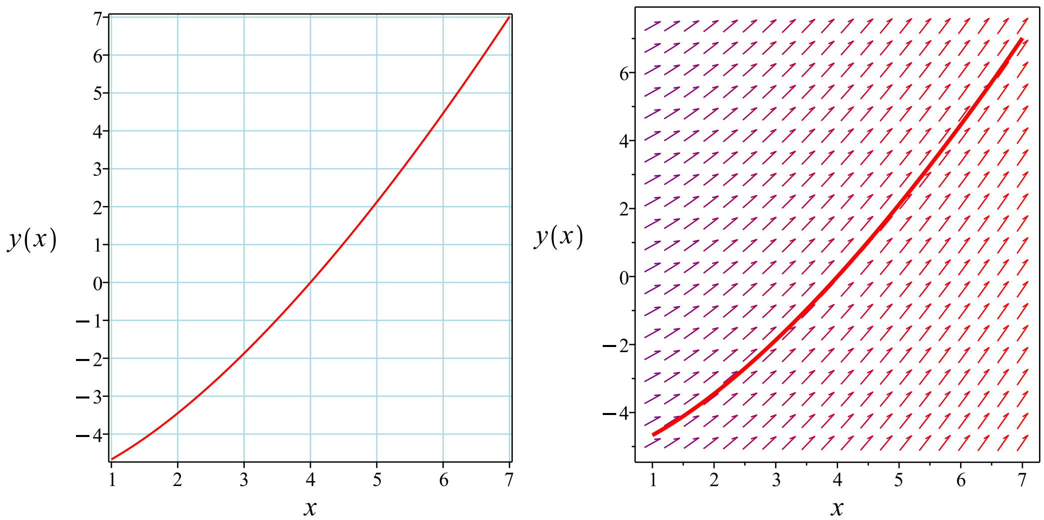
$$c_1 = -\frac{16}{3}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{2x^{\frac{3}{2}}}{3} - \frac{16}{3}$$

Summary
The solution(s) found are the following

$$y = \frac{2x^{\frac{3}{2}}}{3} - \frac{16}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{2x^{\frac{3}{2}}}{3} - \frac{16}{3}$$

Verified OK.

1.3.3 Maple step by step solution

Let's solve

$\left[y' = \sqrt{x}, y(4) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \sqrt{x} dx + c_1$

- Evaluate integral

  $y = \frac{2x^{\frac{3}{2}}}{3} + c_1$

- Solve for y

  $y = \frac{2x^{\frac{3}{2}}}{3} + c_1$

- Use initial condition $y(4) = 0$

$0 = \frac{8\sqrt{4}}{3} + c_1$

- Solve for c_1

  $c_1 = -\frac{8\sqrt{4}}{3}$

- Substitute $c_1 = -\frac{8\sqrt{4}}{3}$ into general solution and simplify

  $y = \frac{2x^{\frac{3}{2}}}{3} - \frac{16}{3}$

- Solution to the IVP

  $y = \frac{2x^{\frac{3}{2}}}{3} - \frac{16}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([diff(y(x),x) = x^(1/2),y(4) = 0],y(x), singsol=all)
```

$$y(x) = \frac{2x^{\frac{3}{2}}}{3} - \frac{16}{3}$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 16

```
DSolve[{y'[x] == x^(1/2),y[4]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}\left(x^{3/2} - 8\right)$$

1.4 problem 4

Internal problem ID [4]
Internal file name [`OUTPUT/4_Sunday_June_05_2022_01_33_37_AM_33243587/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \frac{1}{x^2}}$$

With initial conditions

$$[y(1) = 5]$$

1.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{1}{x^2}$$

Hence the ode is

$$y' = \frac{1}{x^2}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

1.4.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{x^2}\, \mathrm{d}x \\ &= -\frac{1}{x} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = c_1 - 1$$

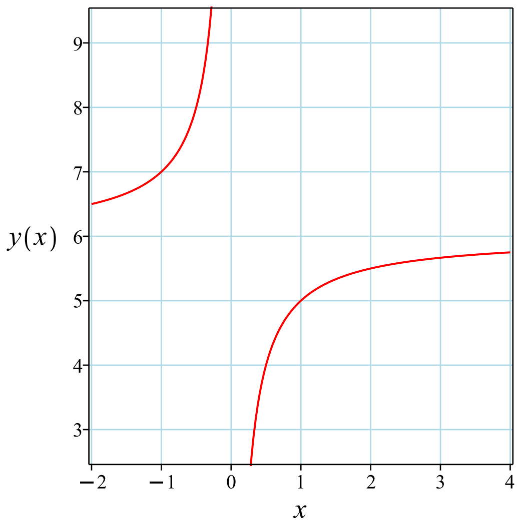
$$c_1 = 6$$

Substituting c_1 found above in the general solution gives

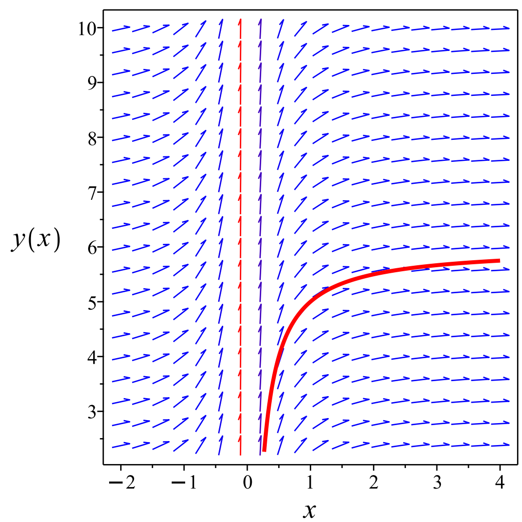
$$y = \frac{-1 + 6x}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{-1 + 6x}{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{-1 + 6x}{x}$$

Verified OK.

1.4.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{x^2}, y(1) = 5\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{x^2} dx + c_1$

- Evaluate integral

  $y = -\frac{1}{x} + c_1$

- Solve for y

  $y = \frac{c_1 x - 1}{x}$

- Use initial condition $y(1) = 5$

  $5 = c_1 - 1$

- Solve for c_1

  $c_1 = 6$

- Substitute $c_1 = 6$ into general solution and simplify

  $y = \frac{-1+6x}{x}$

- Solution to the IVP

  $y = \frac{-1+6x}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x) = 1/x^2,y(1) = 5],y(x), singsol=all)
```

$$y(x) = -\frac{1}{x} + 6$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 12

```
DSolve[{y'[x] == 1/x^2,y[1]==5},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 6 - \frac{1}{x}$$

1.5 problem 5

Internal problem ID [5]
Internal file name [`OUTPUT/5_Sunday_June_05_2022_01_33_38_AM_47063498/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \frac{1}{\sqrt{2+x}}}$$

With initial conditions

$$[y(2) = -1]$$

1.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{1}{\sqrt{2+x}}$$

Hence the ode is

$$y' = \frac{1}{\sqrt{2+x}}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{1}{\sqrt{2+x}}$ is

$$\{-2 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

1.5.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{2+x}}\, \mathrm{d}x \\ &= 2\sqrt{2+x} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = 4 + c_1$$

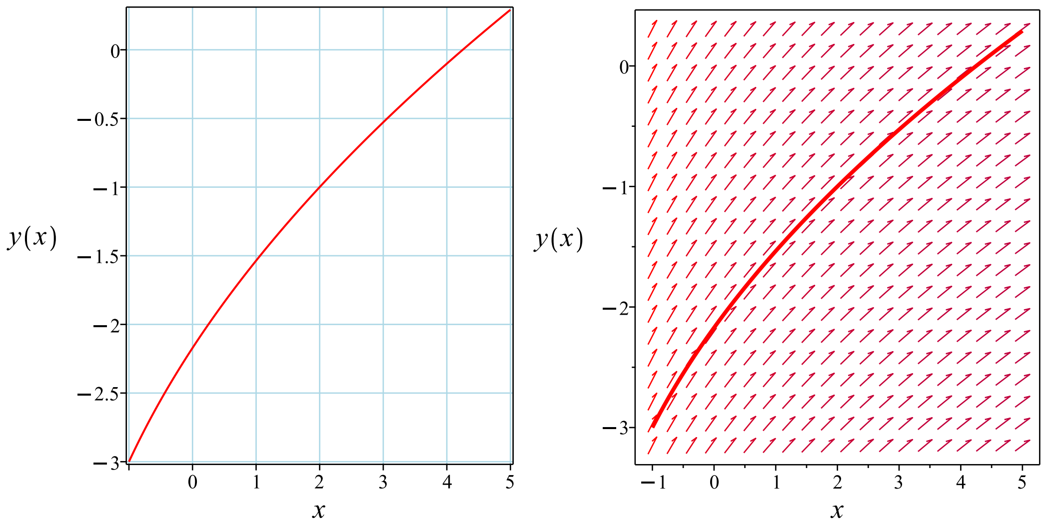
$$c_1 = -5$$

Substituting c_1 found above in the general solution gives

$$y = 2\sqrt{2+x} - 5$$

Summary
The solution(s) found are the following

$$y = 2\sqrt{2+x} - 5 \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\sqrt{2+x} - 5$$

Verified OK.

1.5.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{\sqrt{2+x}}, y(2) = -1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{\sqrt{2+x}} dx + c_1$

- Evaluate integral

  $y = 2\sqrt{2+x} + c_1$

- Solve for y

  $y = 2\sqrt{2+x} + c_1$

- Use initial condition $y(2) = -1$

  $-1 = 2\sqrt{4} + c_1$

- Solve for c_1

  $c_1 = -2\sqrt{4} - 1$

- Substitute $c_1 = -2\sqrt{4} - 1$ into general solution and simplify

  $y = 2\sqrt{2 + x} - 5$

- Solution to the IVP

  $y = 2\sqrt{2 + x} - 5$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([diff(y(x),x) = 1/(2+x)^(1/2),y(2) = -1],y(x), singsol=all)
```

$$y(x) = 2\sqrt{2 + x} - 5$$

✓ Solution by Mathematica

Time used: 0.008 (sec). Leaf size: 16

```
DSolve[{y'[x] == 1/(2+x)^(1/2),y[2]==-1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2\sqrt{x + 2} - 5$$

1.6 problem 6

Internal problem ID [6]
Internal file name [`OUTPUT/6_Sunday_June_05_2022_01_33_39_AM_3851012/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = x\sqrt{x^2+9}}$$

With initial conditions

$$[y(-4) = 0]$$

1.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = x\sqrt{x^2+9}$$

Hence the ode is

$$y' = x\sqrt{x^2+9}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -4$ is inside this domain. The domain of $q(x) = x\sqrt{x^2 + 9}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = -4$ is also inside this domain. Hence solution exists and is unique.

1.6.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x\sqrt{x^2 + 9}\, \mathrm{d}x \\ &= \frac{(x^2 + 9)^{\frac{3}{2}}}{3} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = -4$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{125}{3} + c_1$$

$$c_1 = -\frac{125}{3}$$

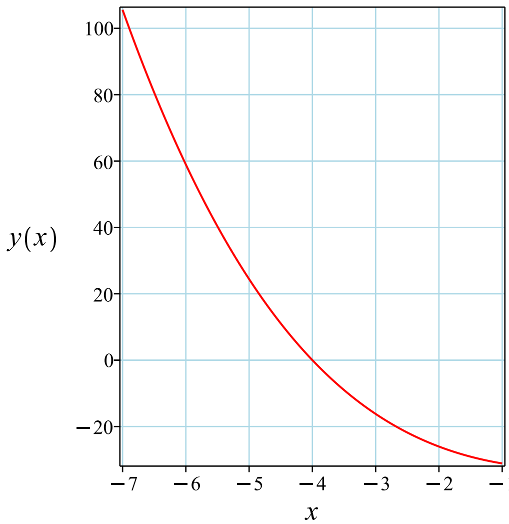
Substituting c_1 found above in the general solution gives

$$y = \frac{(x^2 + 9)^{\frac{3}{2}}}{3} - \frac{125}{3}$$

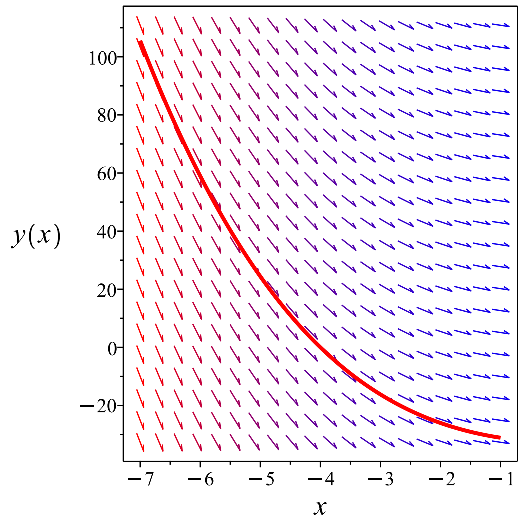
Summary

The solution(s) found are the following

$$y = \frac{(x^2 + 9)^{\frac{3}{2}}}{3} - \frac{125}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(x^2+9)^{\frac{3}{2}}}{3} - \frac{125}{3}$$

Verified OK.

1.6.3 Maple step by step solution

Let's solve

$\left[y' = x\sqrt{x^2+9}, y(-4) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int x\sqrt{x^2+9}dx + c_1$

- Evaluate integral

  $y = \frac{(x^2+9)^{\frac{3}{2}}}{3} + c_1$

- Solve for y

  $y = \frac{(x^2+9)^{\frac{3}{2}}}{3} + c_1$

- Use initial condition $y(-4) = 0$

$0 = \frac{25\sqrt{25}}{3} + c_1$

- Solve for c_1

$c_1 = -\frac{25\sqrt{25}}{3}$

- Substitute $c_1 = -\frac{25\sqrt{25}}{3}$ into general solution and simplify

$y = \frac{x^2\sqrt{x^2+9}}{3} + 3\sqrt{x^2+9} - \frac{125}{3}$

- Solution to the IVP

$y = \frac{x^2\sqrt{x^2+9}}{3} + 3\sqrt{x^2+9} - \frac{125}{3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x) = x*(x^2+9)^(1/2),y(-4) = 0],y(x), singsol=all)
```

$$y(x) = \frac{x^2\sqrt{x^2+9}}{3} + 3\sqrt{x^2+9} - \frac{125}{3}$$

✓ Solution by Mathematica

Time used: 0.005 (sec). Leaf size: 20

```
DSolve[{y'[x] == x*(x^2+9)^(1/2),y[-4]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}\left(\left(x^2+9\right)^{3/2} - 125\right)$$

1.7 problem 7

Internal problem ID [7]
Internal file name [`OUTPUT/7_Sunday_June_05_2022_01_33_40_AM_84918379/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' = \frac{10}{x^2+1}}$$

With initial conditions

$$[y(0) = 0]$$

1.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{10}{x^2+1}$$

Hence the ode is

$$y' = \frac{10}{x^2+1}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{10}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.7.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{10}{x^2+1}\, \mathrm{d}x \\ &= 10 \arctan(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

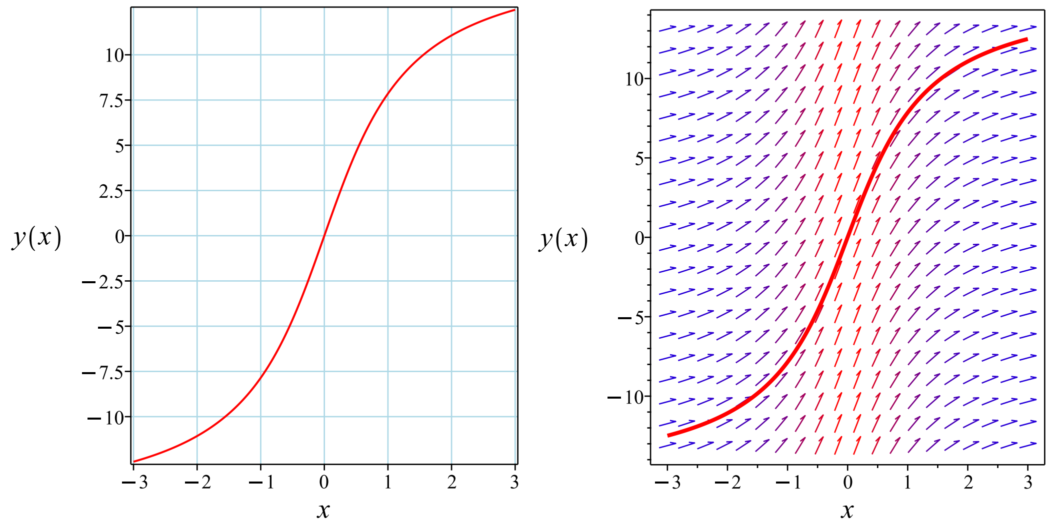
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = 10 \arctan(x)$$

Summary
The solution(s) found are the following

$$y = 10 \arctan(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 10 \arctan(x)$$

Verified OK.

1.7.3 Maple step by step solution

Let's solve

$\left[y' = \frac{10}{x^2+1}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{10}{x^2+1} dx + c_1$

- Evaluate integral

  $y = 10 \arctan(x) + c_1$

- Solve for y

  $y = 10 \arctan(x) + c_1$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = 10 \arctan(x)$
- Solution to the IVP

  $y = 10 \arctan(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple
Time used: 0.047 (sec). Leaf size: 8

```
dsolve([diff(y(x),x) = 10/(x^2+1),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 10 \arctan(x)$$

✓ Solution by Mathematica
Time used: 0.016 (sec). Leaf size: 9

```
DSolve[{y'[x]==10/(x^2+1),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 10 \arctan(x)$$

1.8 problem 8

Internal problem ID [8]
Internal file name [`OUTPUT/8_Sunday_June_05_2022_01_33_41_AM_47998857/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \cos(2x)}$$

With initial conditions

$$[y(0) = 1]$$

1.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \cos(2x)$$

Hence the ode is

$$y' = \cos(2x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \cos(2x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.8.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \cos(2x)\, \mathrm{d}x \\ &= \frac{\sin(2x)}{2} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

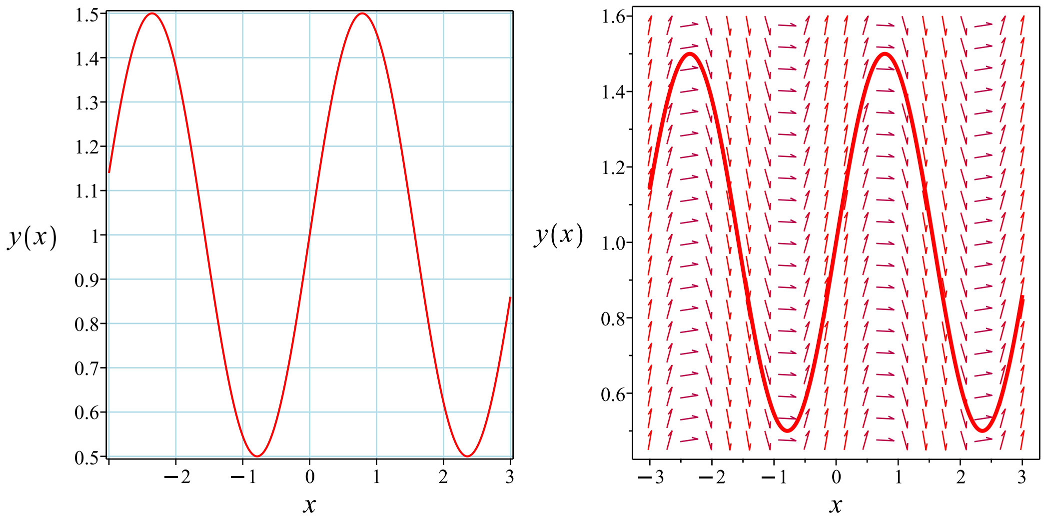
Substituting c_1 found above in the general solution gives

$$y = \frac{\sin(2x)}{2} + 1$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(2x)}{2} + 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(2x)}{2} + 1$$

Verified OK.

1.8.3 Maple step by step solution

Let's solve

$[y' = \cos(2x), y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \cos(2x)\, dx + c_1$

- Evaluate integral

  $y = \frac{\sin(2x)}{2} + c_1$

- Solve for y

  $y = \frac{\sin(2x)}{2} + c_1$

- Use initial condition $y(0) = 1$

  $1 = c_1$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{\sin(2x)}{2} + 1$

- Solution to the IVP

  $y = \frac{\sin(2x)}{2} + 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(y(x),x) = cos(2*x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\sin(2x)}{2} + 1$$

✓ Solution by Mathematica

Time used: 0.055 (sec). Leaf size: 12

```
DSolve[{y'[x] == Cos[2*x],y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sin(x)\cos(x) + 1$$

1.9 problem 9

Internal problem ID [9]
Internal file name [`OUTPUT/9_Sunday_June_05_2022_01_33_42_AM_384554/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = \frac{1}{\sqrt{-x^2+1}}}$$

With initial conditions

$$[y(0) = 0]$$

1.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = \frac{1}{\sqrt{-x^2+1}}$$

Hence the ode is

$$y' = \frac{1}{\sqrt{-x^2+1}}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{1}{\sqrt{-x^2+1}}$ is

$$\{-1 < x < 1\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.9.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{1}{\sqrt{-x^2+1}} \, dx \\ &= \arcsin(x) + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

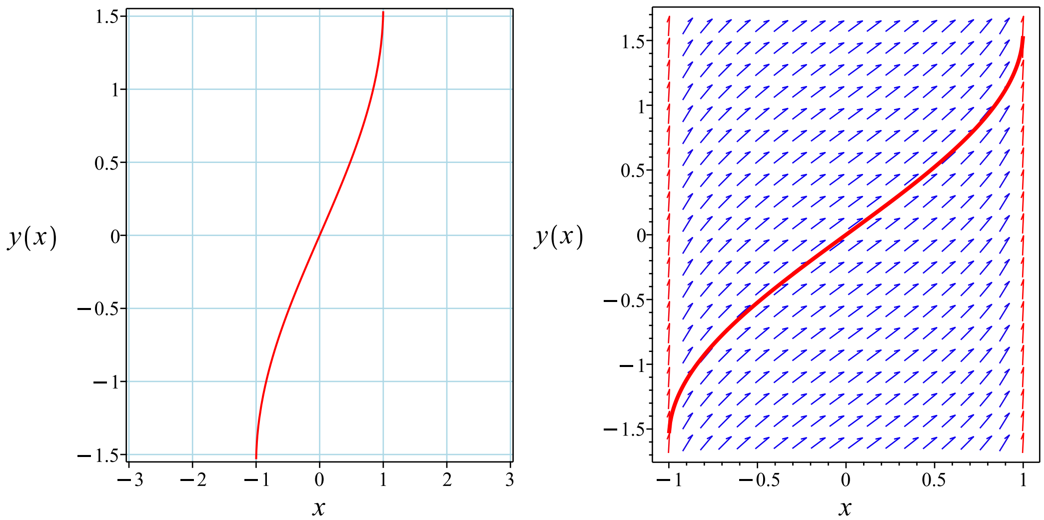
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = \arcsin(x)$$

Summary
The solution(s) found are the following

$$y = \arcsin(x) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \arcsin(x)$$

Verified OK.

1.9.3 Maple step by step solution

Let's solve

$\left[y' = \frac{1}{\sqrt{-x^2+1}}, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{1}{\sqrt{-x^2+1}} dx + c_1$

- Evaluate integral

  $y = \arcsin(x) + c_1$

- Solve for y

  $y = \arcsin(x) + c_1$

- Use initial condition $y(0) = 0$

  $0 = c_1$

- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = \arcsin(x)$
- Solution to the IVP

  $y = \arcsin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 6

```
dsolve([diff(y(x),x) = 1/(-x^2+1)^(1/2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \arcsin(x)$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 31

```
DSolve[{y'[x] == 1/(-x^2+1)^(1/2),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(\pi - 4\arctan\left(\frac{\sqrt{1-x^2}}{x+1}\right)\right)$$

1.10 problem 10

Internal problem ID [10]
Internal file name [`OUTPUT/10_Sunday_June_05_2022_01_33_43_AM_98532069/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.2. Integrals as general and particular solutions. Page 16
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' = x\,\mathrm{e}^{-x}}$$

With initial conditions

$$[y(0) = 1]$$

1.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 0$$
$$q(x) = x\,\mathrm{e}^{-x}$$

Hence the ode is

$$y' = x\,\mathrm{e}^{-x}$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x\,\mathrm{e}^{-x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

1.10.2 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int x\,\mathrm{e}^{-x}\,\mathrm{d}x \\ &= -(x+1)\,\mathrm{e}^{-x} + c_1 \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 1$$

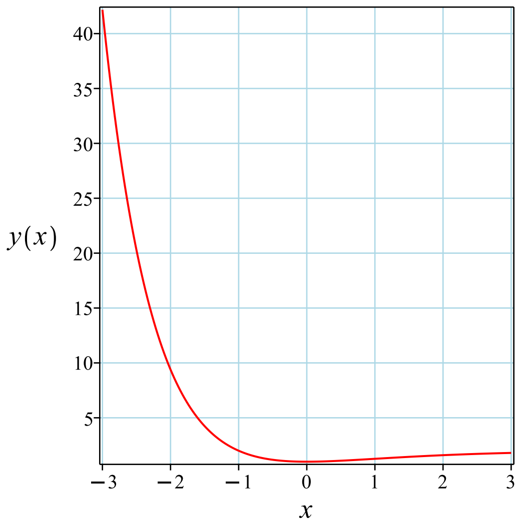
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

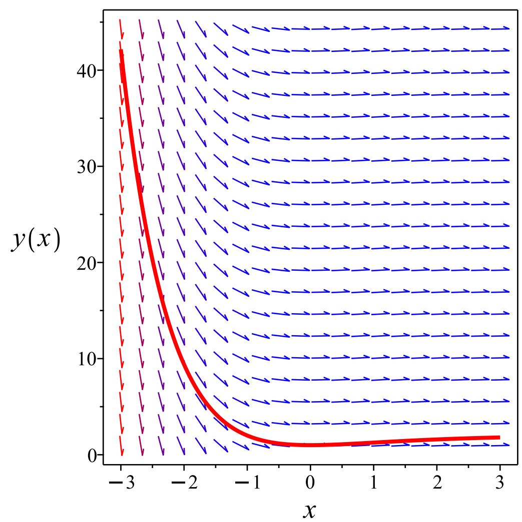
$$y = -x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} + 2$$

Summary
The solution(s) found are the following

$$y = -x\,\mathrm{e}^{-x} - \mathrm{e}^{-x} + 2 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -x\,e^{-x} - e^{-x} + 2$$

Verified OK.

1.10.3 Maple step by step solution

Let's solve

$\left[y' = \frac{x}{e^{x}}, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int y' dx = \int \frac{x}{e^{x}} dx + c_1$

- Evaluate integral

  $y = -\frac{x+1}{e^{x}} + c_1$

- Solve for y

  $y = \frac{c_1 e^{x} - x - 1}{e^{x}}$

- Use initial condition $y(0) = 1$

  $1 = c_1 - 1$

- Solve for c_1

  $c_1 = 2$

- Substitute $c_1 = 2$ into general solution and simplify

  $y = 2 + (-x - 1)\,\mathrm{e}^{-x}$

- Solution to the IVP

  $y = 2 + (-x - 1)\,\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
<- quadrature successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x) = x/exp(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = 2 + (-x - 1)\,\mathrm{e}^{-x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 21

```
DSolve[{y'[x]== x/Exp[x],y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(-x + 2e^x - 1)$$

2 Section 1.3. Slope fields and solution curves. Page 26

2.1 problem 1

Internal problem ID [11]
Internal file name [`OUTPUT/11_Sunday_June_05_2022_01_33_44_AM_42995819/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = -\sin(x)}$$

2.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -\sin(x) \end{aligned}$$

Hence the ode is

$$y' + y = -\sin(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(-\sin(x))$$
$$\frac{d}{dx}(e^x y) = (e^x)(-\sin(x))$$
$$d(e^x y) = (-\sin(x)\, e^x)\, dx$$

Integrating gives

$$e^x y = \int -\sin(x)\, e^x\, dx$$
$$e^x y = \frac{\cos(x)\, e^x}{2} - \frac{\sin(x)\, e^x}{2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(\frac{\cos(x)\, e^x}{2} - \frac{\sin(x)\, e^x}{2}\right) + c_1 e^{-x}$$

which simplifies to

$$y = -\frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x} \tag{1}$$

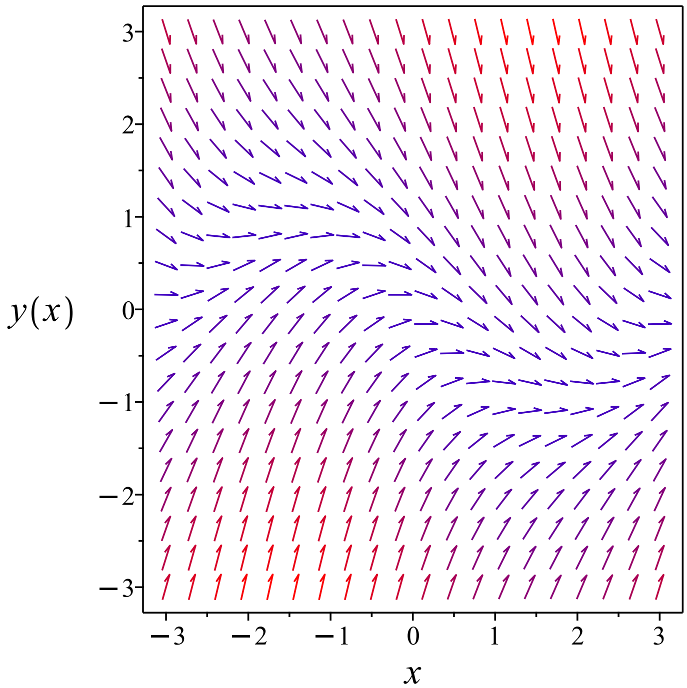


Figure 11: Slope field plot

Verification of solutions

$$y = -\frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x}$$

Verified OK.

2.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\sin(x) - y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 11: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\sin(x) - y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sin(x) e^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sin(R) e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + \frac{e^R(\cos(R) - \sin(R))}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x y = \frac{e^x(\cos(x) - \sin(x))}{2} + c_1$$

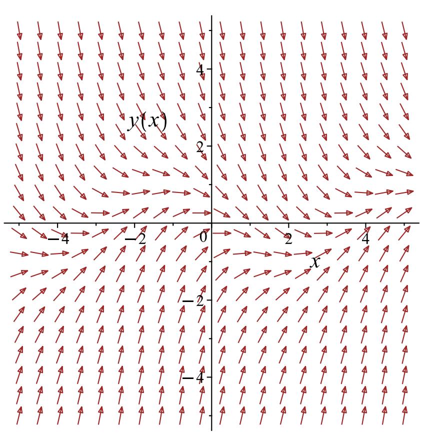
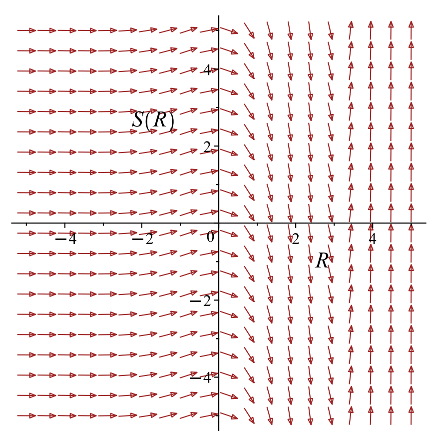
Which simplifies to

$$e^x y = \frac{e^x(\cos(x) - \sin(x))}{2} + c_1$$

Which gives

$$y = -\frac{e^{-x}(\sin(x)\,e^x - \cos(x)\,e^x - 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\sin(x) - y$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = -\sin(R)\,e^R$ 

Summary
The solution(s) found are the following

$$y = -\frac{e^{-x}(\sin(x)\,e^x - \cos(x)\,e^x - 2c_1)}{2} \tag{1}$$

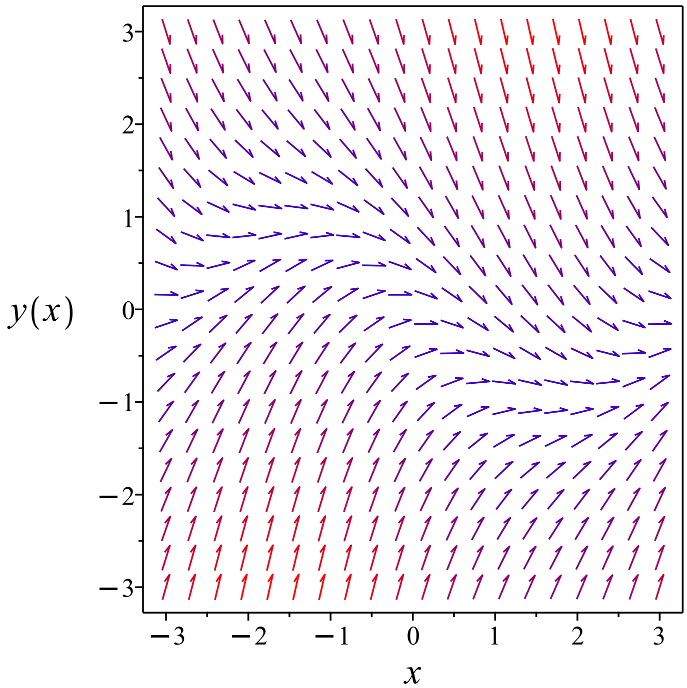


Figure 12: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}(\sin(x)e^{x} - \cos(x)e^{x} - 2c_1)}{2}$$

Verified OK.

2.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-\sin(x) - y)\,\mathrm{d}x$$
$$(\sin(x) + y)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \sin(x) + y$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\sin(x) + y)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}(\sin(x) + y) \\ &= (\sin(x) + y)\,\mathrm{e}^{x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((\sin(x) + y)\,\mathrm{e}^{x}) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int (\sin(x) + y)\, \mathrm{e}^{x}\, \mathrm{d}x$$

$$\phi = \frac{(2y - \cos(x) + \sin(x))\, \mathrm{e}^{x}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(2y - \cos(x) + \sin(x))\, \mathrm{e}^{x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(2y - \cos(x) + \sin(x))\, \mathrm{e}^{x}}{2}$$

The solution becomes

$$y = -\frac{\mathrm{e}^{-x}(\sin(x)\, \mathrm{e}^{x} - \cos(x)\, \mathrm{e}^{x} - 2c_1)}{2}$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-x}(\sin(x)\,e^{x} - \cos(x)\,e^{x} - 2c_1)}{2} \tag{1}$$

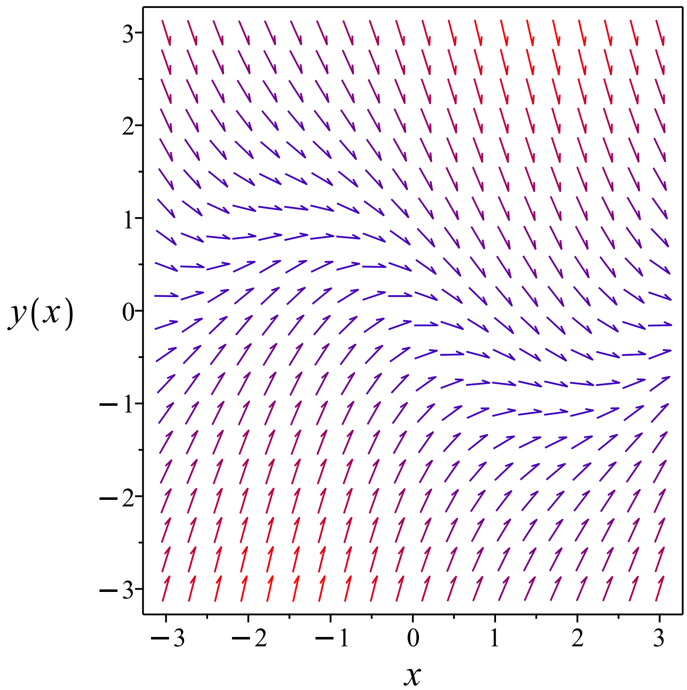


Figure 13: Slope field plot

Verification of solutions

$$y = -\frac{e^{-x}(\sin(x)\,e^{x} - \cos(x)\,e^{x} - 2c_1)}{2}$$

Verified OK.

2.1.4 Maple step by step solution

Let's solve

$y' + y = -\sin(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -\sin(x) - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + y = -\sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)(y' + y) = -\mu(x)\sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)(y' + y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x) = \mu(x)$

- Solve to find the integrating factor

$\mu(x) = e^x$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\sin(x)\,dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y = \int -\mu(x)\sin(x)\,dx + c_1$

- Solve for y

$y = \frac{\int -\mu(x)\sin(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^x$

$y = \frac{\int -\sin(x)e^x dx + c_1}{e^x}$

- Evaluate the integrals on the rhs

$y = \frac{-\frac{\sin(x)e^x}{2} + \frac{\cos(x)e^x}{2} + c_1}{e^x}$

- Simplify

$y = -\frac{\sin(x)}{2} + \frac{\cos(x)}{2} + c_1 e^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x) = -sin(x)-y(x),y(x), singsol=all)
```

$$y(x) = \frac{\cos(x)}{2} - \frac{\sin(x)}{2} + \mathrm{e}^{-x} c_1$$

✓ Solution by Mathematica
Time used: 0.069 (sec). Leaf size: 25

```
DSolve[y'[x]== -Sin[x]-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-\sin(x) + \cos(x) + 2c_1 e^{-x}\right)$$

2.2 problem 2

Internal problem ID [12]
Internal file name [`OUTPUT/12_Sunday_June_05_2022_01_33_45_AM_10607916/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = x}$$

2.2.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = x$$

Hence the ode is

$$y' - y = x$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x)$$
$$\frac{d}{dx}(e^{-x} y) = (e^{-x})(x)$$
$$d(e^{-x} y) = (x e^{-x})\, dx$$

Integrating gives

$$e^{-x} y = \int x e^{-x}\, dx$$
$$e^{-x} y = -(x+1) e^{-x} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = -e^{x}(x+1) e^{-x} + c_1 e^{x}$$

which simplifies to

$$y = c_1 e^{x} - x - 1$$

Summary
The solution(s) found are the following

$$y = c_1 e^{x} - x - 1 \tag{1}$$

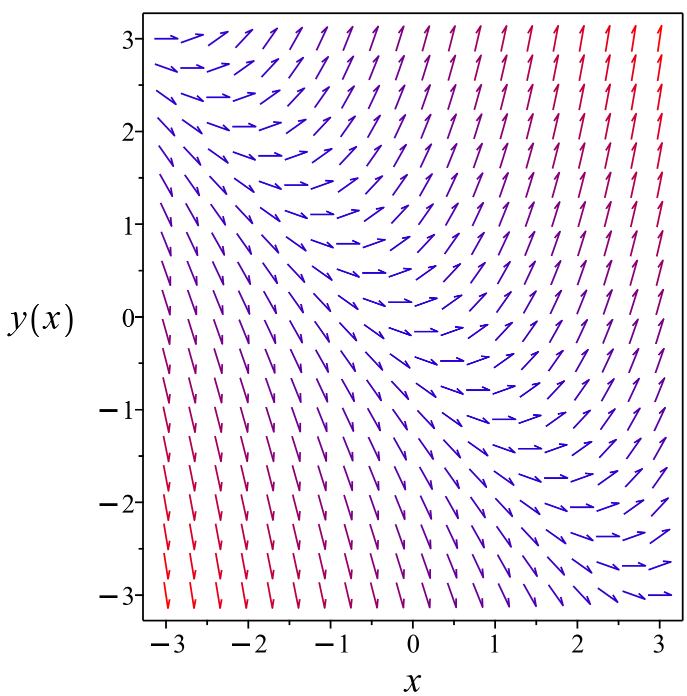


Figure 14: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{x} - x - 1$$

Verified OK.

2.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x + y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 14: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^x
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\, e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\, e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -(R+1)\,\mathrm{e}^{-R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x} y = -(x+1)\,\mathrm{e}^{-x} + c_1$$

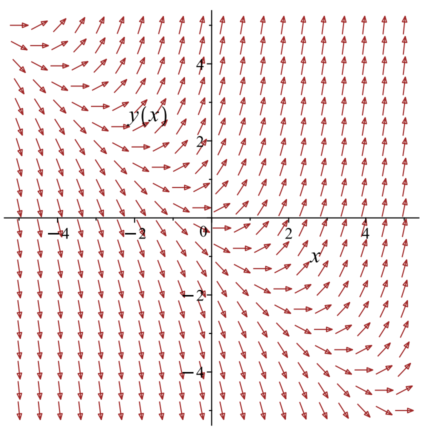
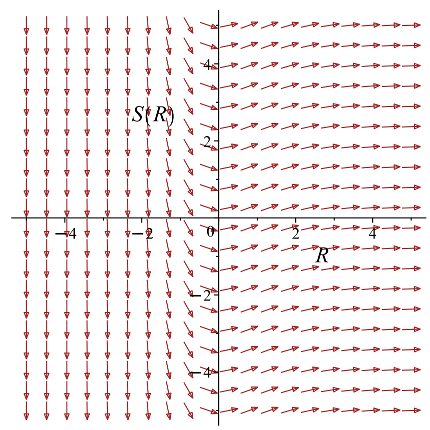
Which simplifies to

$$(x+y+1)\,\mathrm{e}^{-x} - c_1 = 0$$

Which gives

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x + y$ 	$R = x$ $S = \mathrm{e}^{-x} y$	$\frac{dS}{dR} = R\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

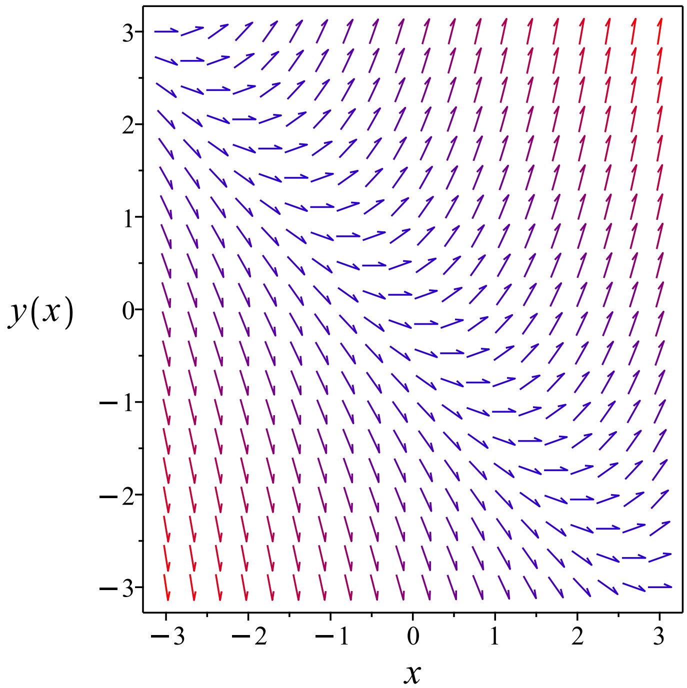


Figure 15: Slope field plot

Verification of solutions

$$y = -\left(x\,e^{-x} + e^{-x} - c_1\right) e^{x}$$

Verified OK.

2.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= (x + y)\, \mathrm{d}x \\ (-x - y)\, \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x - y \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int -1 \,\mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(-x - y) \\ &= -\mathrm{e}^{-x}(x + y) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{-x}(x + y)\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\mathrm{e}^{-x}(x+y) \mathrm{d}x$$

$$\phi = (x+y+1)\mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (x+y+1)\mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (x+y+1)\mathrm{e}^{-x}$$

The solution becomes

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x}$$

Summary
The solution(s) found are the following

$$y = -\left(x\,\mathrm{e}^{-x} + \mathrm{e}^{-x} - c_1\right)\mathrm{e}^{x} \tag{1}$$

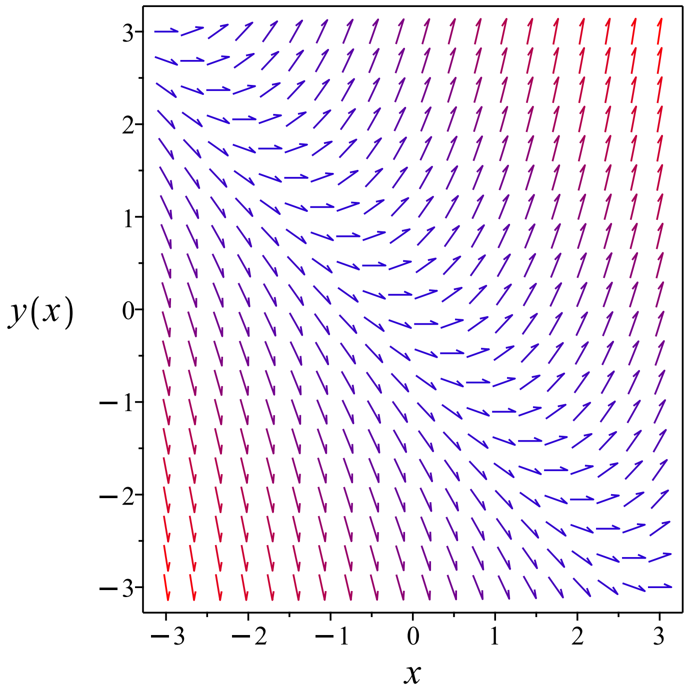


Figure 16: Slope field plot

Verification of solutions

$$y = -\left(x\,e^{-x} + e^{-x} - c_1\right) e^{x}$$

Verified OK.

2.2.4 Maple step by step solution

Let's solve

$y' - y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = x + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' - y) = \mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'-y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = -\mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,x dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{-x}$

  $y = \frac{\int x\,\mathrm{e}^{-x} dx + c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y = \frac{-(x+1)\mathrm{e}^{-x} + c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y = c_1 \mathrm{e}^{x} - x - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x) = x+y(x),y(x), singsol=all)
```

$$y(x) = -x - 1 + e^x c_1$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 16

```
DSolve[y'[x] == x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x + c_1 e^x - 1$$

2.3 problem 3

Internal problem ID [13]
Internal file name [`OUTPUT/13_Sunday_June_05_2022_01_33_46_AM_26965091/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = -\sin(x)}$$

2.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -1$$
$$q(x) = -\sin(x)$$

Hence the ode is

$$y' - y = -\sin(x)$$

The integrating factor μ is

$$\mu = e^{\int(-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(-\sin(x)) \\
\frac{d}{dx}\left(e^{-x} y\right) &= \left(e^{-x}\right)(-\sin(x)) \\
d\left(e^{-x} y\right) &= \left(-\sin(x)\, e^{-x}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x} y &= \int -\sin(x)\, e^{-x}\, dx \\
e^{-x} y &= \frac{\cos(x)\, e^{-x}}{2} + \frac{\sin(x)\, e^{-x}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$
y = e^{x}\left(\frac{\cos(x)\, e^{-x}}{2} + \frac{\sin(x)\, e^{-x}}{2}\right) + c_1 e^{x}
$$

which simplifies to

$$
y = c_1 e^{x} + \frac{\sin(x)}{2} + \frac{\cos(x)}{2}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{x} + \frac{\sin(x)}{2} + \frac{\cos(x)}{2} \tag{1}
$$

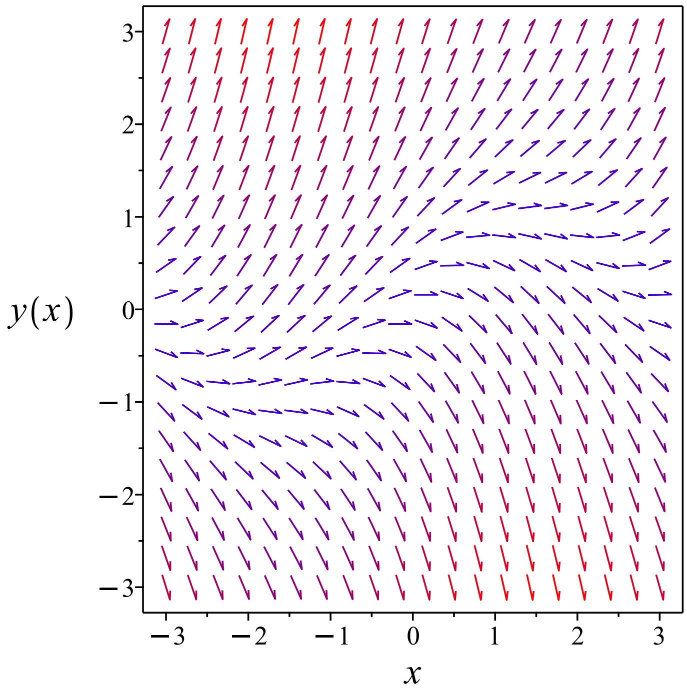


Figure 17: Slope field plot

Verification of solutions

$$y = c_1 e^x + \frac{\sin(x)}{2} + \frac{\cos(x)}{2}$$

Verified OK.

2.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\sin(x) + y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 17: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= e^x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x}} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\sin(x) + y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\sin(x) e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\sin(R) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 + \frac{e^{-R}(\cos(R) + \sin(R))}{2} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x} y = \frac{e^{-x}(\cos(x) + \sin(x))}{2} + c_1$$

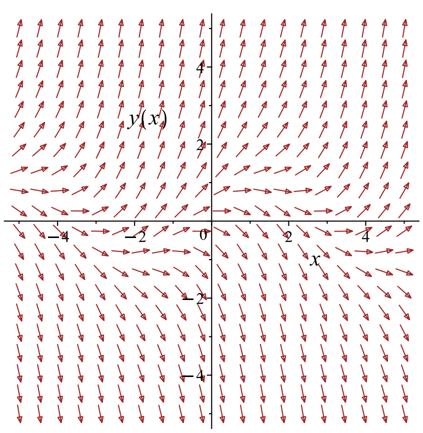
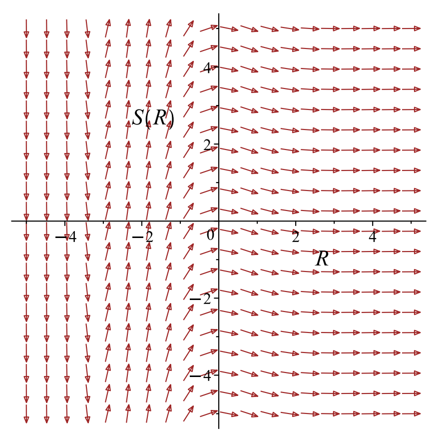
Which simplifies to

$$e^{-x} y = \frac{e^{-x}(\cos(x) + \sin(x))}{2} + c_1$$

Which gives

$$y = \frac{e^{x}(\sin(x) e^{-x} + \cos(x) e^{-x} + 2c_1)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\sin(x) + y$ 	$R = x$ $S = e^{-x} y$	$\frac{dS}{dR} = -\sin(R) e^{-R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{x}(\sin(x) e^{-x} + \cos(x) e^{-x} + 2c_1)}{2} \tag{1}$$

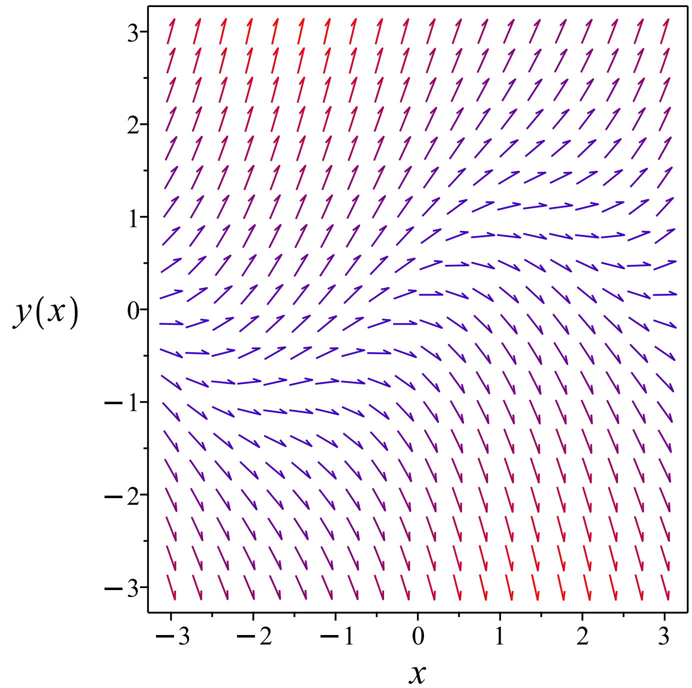


Figure 18: Slope field plot

Verification of solutions

$$y = \frac{e^{x}(\sin(x)\,e^{-x} + \cos(x)\,e^{-x} + 2c_1)}{2}$$

Verified OK.

2.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-\sin(x) + y)\,\mathrm{d}x$$
$$(\sin(x) - y)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \sin(x) - y$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\sin(x) - y) \\ &= -1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(\sin(x) - y) \\ &= (\sin(x) - y)\,\mathrm{e}^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((\sin(x) - y)\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int (\sin(x) - y)\,\mathrm{e}^{-x} \,\mathrm{d}x \\
\phi &= -\frac{\mathrm{e}^{-x}(-2y + \cos(x) + \sin(x))}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$
\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{\mathrm{e}^{-x}(-2y + \cos(x) + \sin(x))}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{\mathrm{e}^{-x}(-2y + \cos(x) + \sin(x))}{2}
$$

The solution becomes

$$
y = \frac{\mathrm{e}^{x}(\sin(x)\,\mathrm{e}^{-x} + \cos(x)\,\mathrm{e}^{-x} + 2c_1)}{2}
$$

Summary
The solution(s) found are the following

$$y = \frac{e^{x}\left(\sin \left(x \right) e^{-x}+\cos \left(x \right) e^{-x}+2 c_{1}\right)}{2} \tag{1}$$

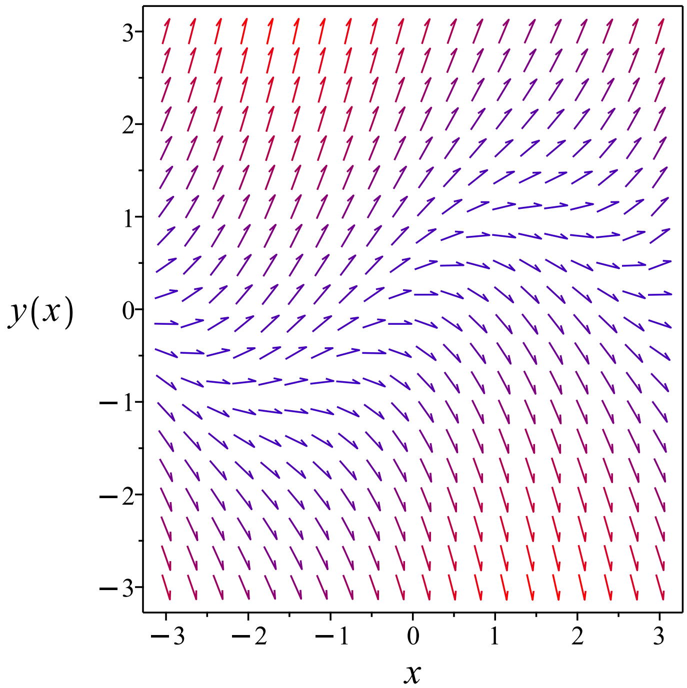


Figure 19: Slope field plot

Verification of solutions

$$y = \frac{e^{x}\left(\sin \left(x \right) e^{-x}+\cos \left(x \right) e^{-x}+2 c_{1}\right)}{2}$$

Verified OK.

2.3.4 Maple step by step solution

Let's solve

$y'-y=-\sin \left(x \right)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -\sin(x) + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' - y = -\sin(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)(y' - y) = -\mu(x)\sin(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)(y' - y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x) = -\mu(x)$

- Solve to find the integrating factor

$\mu(x) = \mathrm{e}^{-x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int -\mu(x)\sin(x)\,dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y = \int -\mu(x)\sin(x)\,dx + c_1$

- Solve for y

$y = \frac{\int -\mu(x)\sin(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-x}$

$y = \frac{\int -\sin(x)\mathrm{e}^{-x}dx + c_1}{\mathrm{e}^{-x}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{\sin(x)\mathrm{e}^{-x}}{2} + \frac{\cos(x)\mathrm{e}^{-x}}{2} + c_1}{\mathrm{e}^{-x}}$

- Simplify

$y = c_1\mathrm{e}^{x} + \frac{\sin(x)}{2} + \frac{\cos(x)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x) = -sin(x)+y(x),y(x), singsol=all)
```

$$y(x) = \frac{\cos(x)}{2} + \frac{\sin(x)}{2} + e^{x} c_1$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 21

```
DSolve[y'[x] == -Sin[x]+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(\sin(x) + \cos(x) + 2c_1 e^x)$$

2.4 problem 4

Internal problem ID [14]
Internal file name [`OUTPUT/14_Sunday_June_05_2022_01_33_47_AM_82648825/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = x}$$

2.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x \end{aligned}$$

Hence the ode is

$$y' + y = x$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^{x} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x)$$
$$\frac{d}{dx}(e^x y) = (e^x)(x)$$
$$d(e^x y) = (x e^x)\ dx$$

Integrating gives

$$e^x y = \int x e^x \, dx$$
$$e^x y = (x-1)e^x + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}(x-1)e^x + c_1 e^{-x}$$

which simplifies to

$$y = x - 1 + c_1 e^{-x}$$

Summary
The solution(s) found are the following

$$y = x - 1 + c_1 e^{-x} \tag{1}$$

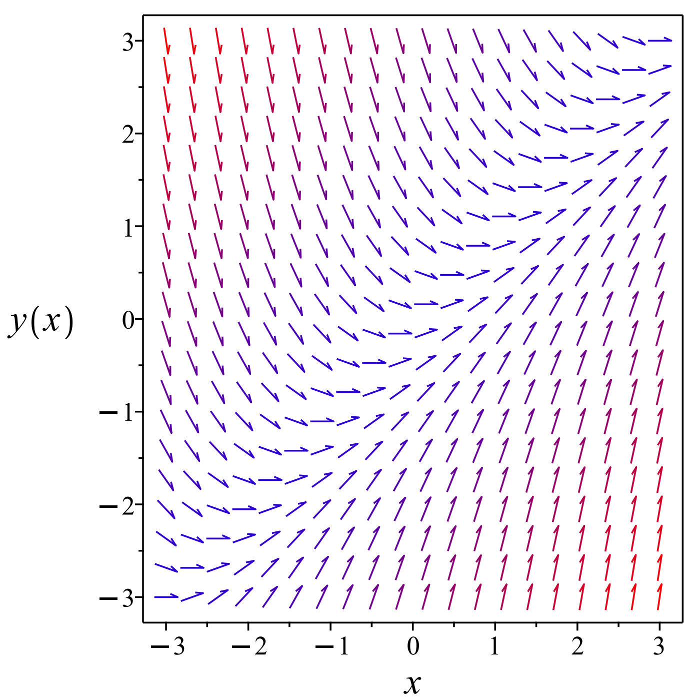


Figure 20: Slope field plot

Verification of solutions

$$y = x - 1 + c_1 e^{-x}$$

Verified OK.

2.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= x - y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 20: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \mathrm{e}^{-x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x - y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y \\ S_y &= \mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x\,\mathrm{e}^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R\,\mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R-1)\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} y = (x-1)\,\mathrm{e}^{x} + c_1$$

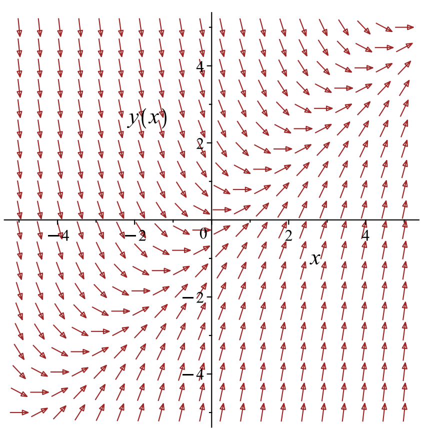
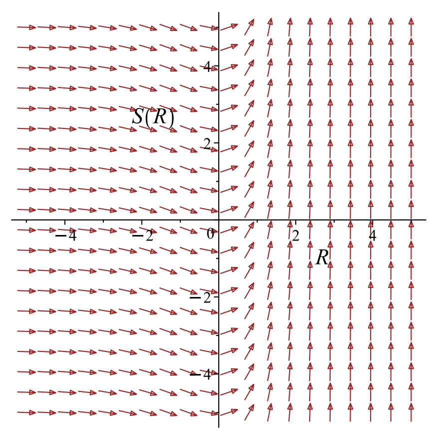
Which simplifies to

$$\mathrm{e}^{x} y = (x-1)\,\mathrm{e}^{x} + c_1$$

Which gives

$$y = \left(x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = x - y$

</td><td>$R = x$
$S = \mathrm{e}^{x} y$</td><td>$\frac{dS}{dR} = R\,\mathrm{e}^{R}$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \left(x\,\mathrm{e}^{x} - \mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x} \tag{1}$$

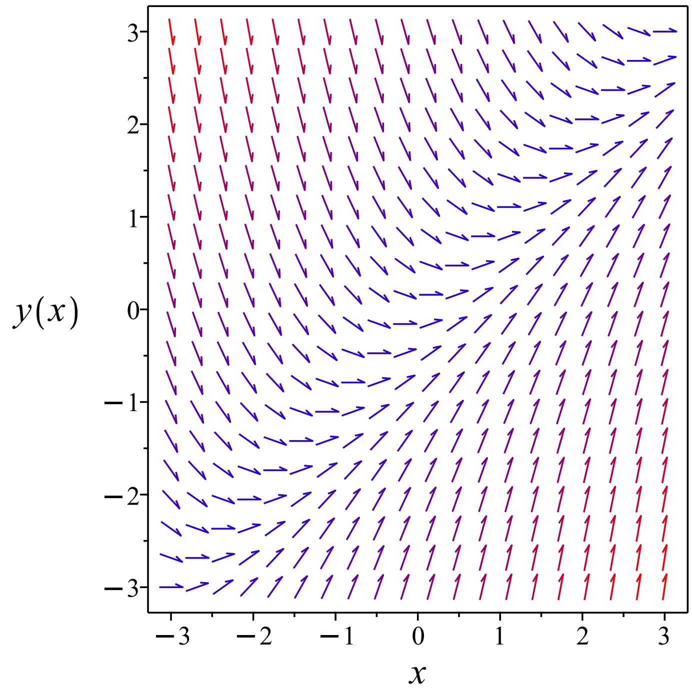


Figure 21: Slope field plot

Verification of solutions

$$y = (x\,e^{x} - e^{x} + c_1)\,e^{-x}$$

Verified OK.

2.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (x-y)\,\mathrm{d}x$$
$$(-x+y)\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x + y$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x+y)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}(-x + y) \\ &= -\mathrm{e}^{x}(x - y) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{x}(x - y)\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, dx = \int \overline{M}\, dx$$
$$\int \frac{\partial \phi}{\partial x}\, dx = \int -e^{x}(x-y)\, dx$$
$$\phi = -(x-y-1)\, e^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{x}$. Therefore equation (4) becomes

$$e^{x} = e^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -(x-y-1)\, e^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -(x-y-1)\, e^{x}$$

The solution becomes

$$y = (x\, e^{x} - e^{x} + c_1)\, e^{-x}$$

Summary
The solution(s) found are the following

$$y = (x\, e^{x} - e^{x} + c_1)\, e^{-x} \tag{1}$$

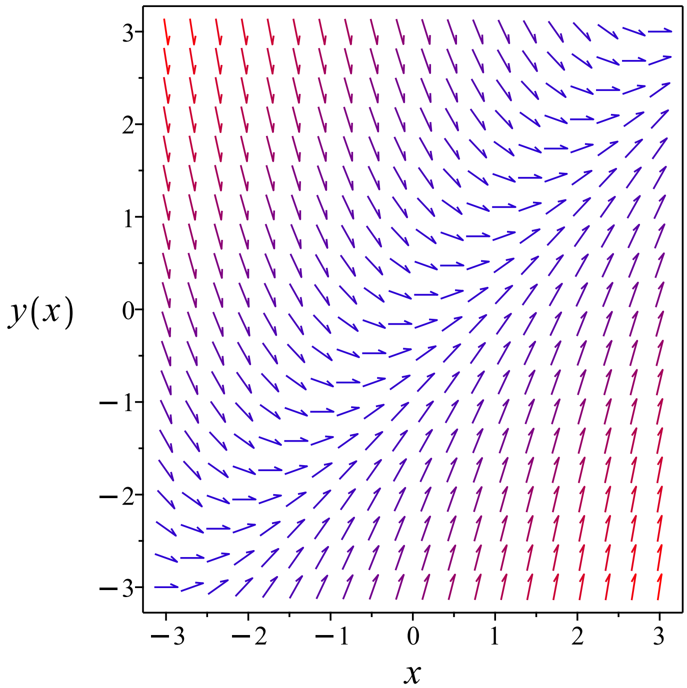


Figure 22: Slope field plot

Verification of solutions

$$y = (x\,e^{x} - e^{x} + c_1)\,e^{-x}$$

Verified OK.

2.4.4 Maple step by step solution

Let's solve

$y' + y = x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = x - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \mu(x)\,x$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\,(y'+y) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x) x dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int x\,\mathrm{e}^{x} dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{(x-1)\mathrm{e}^{x} + c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = x - 1 + c_1 \mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x) = x-y(x),y(x), singsol=all)
```

$$y(x) = x - 1 + e^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 16

```
DSolve[y'[x] == x-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 e^{-x} - 1$$

2.5 problem 5

Internal problem ID [15]
Internal file name [`OUTPUT/15_Sunday_June_05_2022_01_33_48_AM_98169401/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - y = 1 - x}$$

2.5.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -1 \\ q(x) &= 1 - x \end{aligned}$$

Hence the ode is

$$y' - y = 1 - x$$

The integrating factor μ is

$$\mu = e^{\int (-1)dx}$$
$$= e^{-x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(1 - x)$$
$$\frac{d}{dx}(e^{-x}y) = (e^{-x})(1 - x)$$
$$d(e^{-x}y) = \left(-(x - 1)e^{-x}\right) dx$$

Integrating gives

$$e^{-x}y = \int -(x - 1)e^{-x} dx$$
$$e^{-x}y = x e^{-x} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x}$ results in

$$y = e^{x} x e^{-x} + c_1 e^{x}$$

which simplifies to

$$y = x + c_1 e^{x}$$

Summary

The solution(s) found are the following

$$y = x + c_1 e^{x} \tag{1}$$

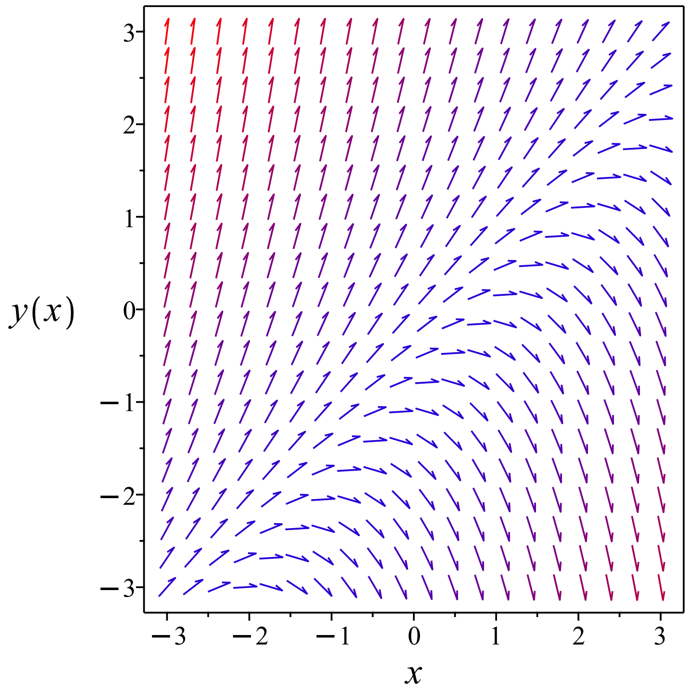


Figure 23: Slope field plot

Verification of solutions

$$y = x + c_1 e^x$$

Verified OK.

2.5.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) - u(x)\, x = 1 - x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(x-1)\,(u-1)}{x} \end{aligned}$$

Where $f(x) = \frac{x-1}{x}$ and $g(u) = u - 1$. Integrating both sides gives

$$\frac{1}{u-1}\, du = \frac{x-1}{x}\, dx$$

$$
\int \frac{1}{u-1}du = \int \frac{x-1}{x}dx
$$
$$
\ln(u-1) = x - \ln(x) + c_2
$$

Raising both side to exponential gives

$$
u - 1 = e^{x-\ln(x)+c_2}
$$

Which simplifies to

$$
u - 1 = c_3 e^{x-\ln(x)}
$$

Which simplifies to

$$
u(x) = \frac{c_3 e^x e^{c_2}}{x} + 1
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= x\left(\frac{c_3 e^x e^{c_2}}{x} + 1\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = x\left(\frac{c_3 e^x e^{c_2}}{x} + 1\right) \tag{1}
$$

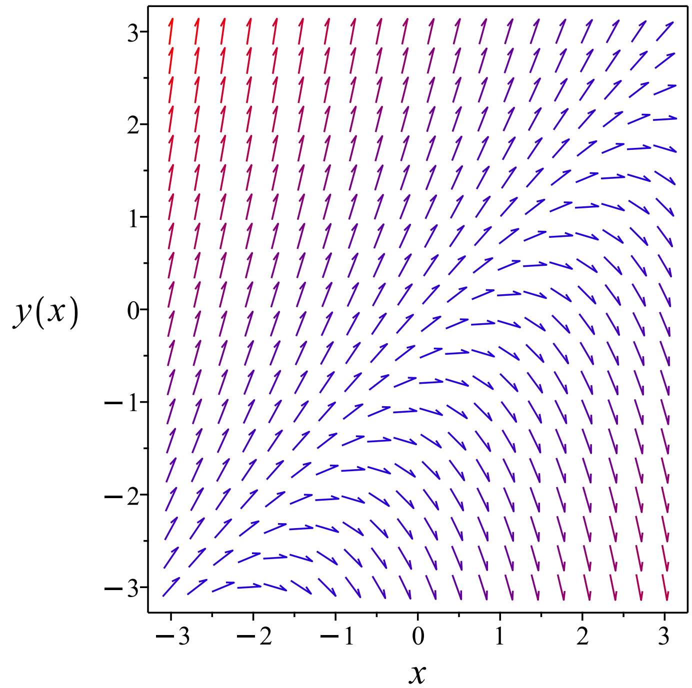


Figure 24: Slope field plot

Verification of solutions

$$y = x\left(\frac{c_3 e^{x} e^{c_2}}{x} + 1\right)$$

Verified OK.

2.5.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -x + y + 1 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 23: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= e^x \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^x} dy \end{aligned}$$

Which results in

$$S = e^{-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -x + y + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -e^{-x} y \\ S_y &= e^{-x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (1 - x) e^{-x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (1 - R) e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{-R}R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x}y = x\,e^{-x} + c_1$$

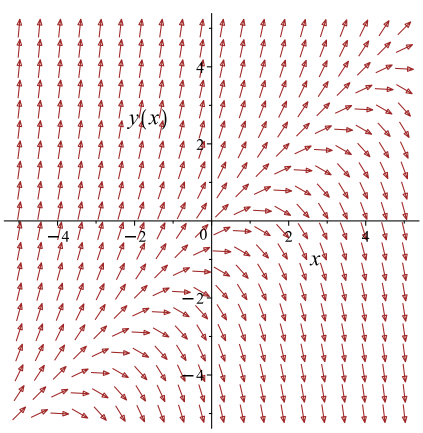
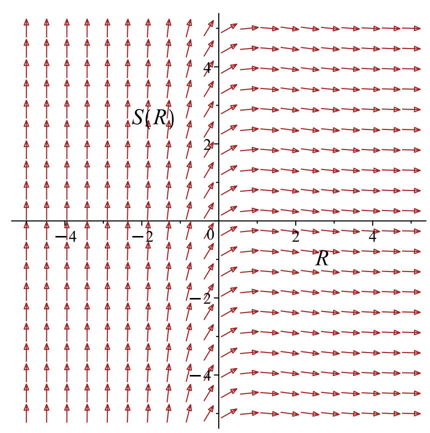
Which simplifies to

$$e^{-x}y = x\,e^{-x} + c_1$$

Which gives

$$y = \left(x\,e^{-x} + c_1\right)e^{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -x + y + 1$ 	$R = x$ $S = e^{-x}y$	$\frac{dS}{dR} = (1 - R)\,e^{-R}$ 

Summary
The solution(s) found are the following

$$y = \left(x\,e^{-x} + c_1\right)e^{x} \tag{1}$$

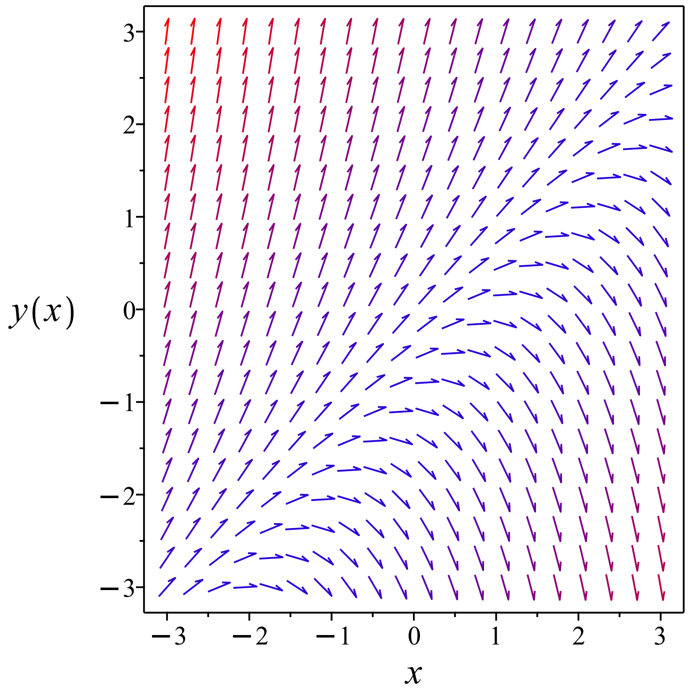


Figure 25: Slope field plot

Verification of solutions

$$y = \left(x\,\mathrm{e}^{-x} + c_1\right)\mathrm{e}^{x}$$

Verified OK.

2.5.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (-x + y + 1)\,\mathrm{d}x \\
(x - y - 1)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= x - y - 1 \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x - y - 1) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((-1) - (0)) \\ &= -1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-x} \\ &= \mathrm{e}^{-x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{-x}(x - y - 1) \\ &= (x - y - 1)\,\mathrm{e}^{-x} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{-x}(1) \\ &= \mathrm{e}^{-x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left((x - y - 1)\,\mathrm{e}^{-x}\right) + \left(\mathrm{e}^{-x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int (x - y - 1) \mathrm{e}^{-x} \mathrm{d}x$$
$$\phi = -(x - y) \mathrm{e}^{-x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x} = \mathrm{e}^{-x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -(x - y) \mathrm{e}^{-x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -(x - y) \mathrm{e}^{-x}$$

The solution becomes

$$y = \left(x \mathrm{e}^{-x} + c_1\right) \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \left(x \mathrm{e}^{-x} + c_1\right) \mathrm{e}^{x} \tag{1}$$

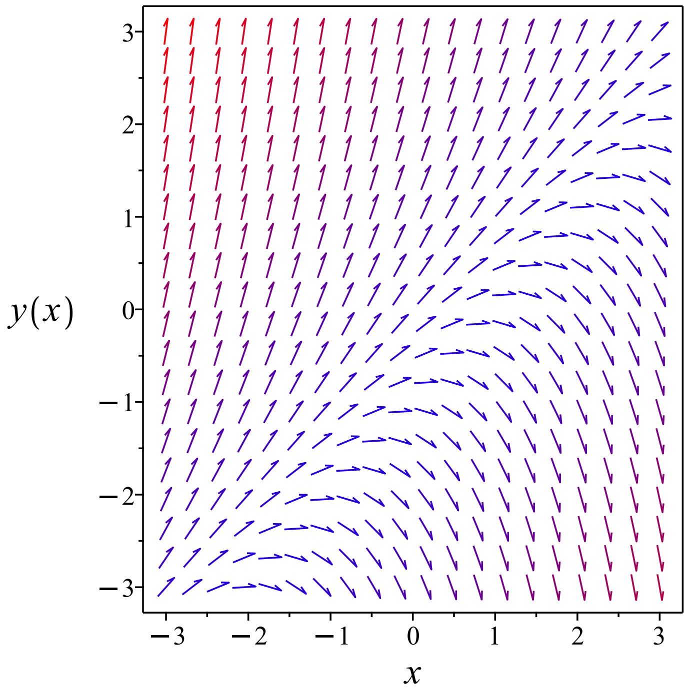


Figure 26: Slope field plot

<u>Verification of solutions</u>

$$y = \left(x\,\mathrm{e}^{-x} + c_1\right)\mathrm{e}^{x}$$

Verified OK.

2.5.5 Maple step by step solution

Let's solve

$y' - y = 1 - x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 - x + y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - y = 1 - x$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - y\right) = \mu(x)\left(1 - x\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'-y)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=-\mu(x)$
- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{-x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\mu(x)\,(1-x)\,dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\mu(x)\,(1-x)\,dx+c_1$
- Solve for y

  $y=\frac{\int\mu(x)(1-x)dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\mathrm{e}^{-x}$

  $y=\frac{\int(1-x)\mathrm{e}^{-x}dx+c_1}{\mathrm{e}^{-x}}$
- Evaluate the integrals on the rhs

  $y=\frac{x\,\mathrm{e}^{-x}+c_1}{\mathrm{e}^{-x}}$
- Simplify

  $y=x+c_1\mathrm{e}^{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(diff(y(x),x) = 1-x+y(x),y(x), singsol=all)
```

$$y(x) = x + e^x c_1$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 13

```
DSolve[y'[x] == 1-x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 e^x$$

2.6 problem 6

Internal problem ID [16]
Internal file name [`OUTPUT/16_Sunday_June_05_2022_01_33_48_AM_22920425/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = x + 1}$$

2.6.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x + 1 \end{aligned}$$

Hence the ode is

$$y' + y = x + 1$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 1dx} \\
&= e^{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(x+1) \\
\frac{d}{dx}(e^{x} y) &= (e^{x})(x+1) \\
d(e^{x} y) &= ((x+1)e^{x})\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{x} y &= \int (x+1)e^{x}\, dx \\
e^{x} y &= x\, e^{x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$
y = e^{x} x\, e^{-x} + c_1 e^{-x}
$$

which simplifies to

$$
y = x + c_1 e^{-x}
$$

Summary

The solution(s) found are the following

$$
y = x + c_1 e^{-x} \tag{1}
$$

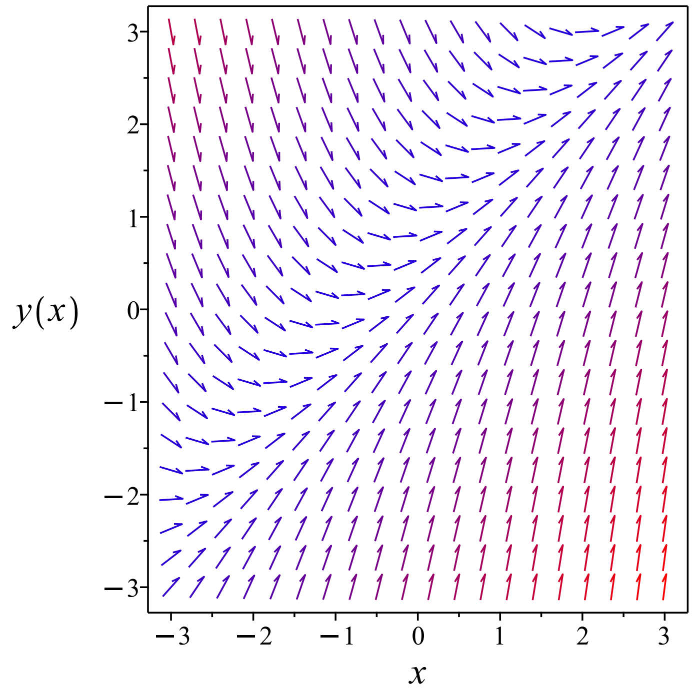


Figure 27: Slope field plot

Verification of solutions

$$y = x + c_1 e^{-x}$$

Verified OK.

2.6.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) + u(x)\, x = x + 1$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{(x+1)(-u+1)}{x} \end{aligned}$$

Where $f(x) = \frac{x+1}{x}$ and $g(u) = -u + 1$. Integrating both sides gives

$$\frac{1}{-u+1}\, du = \frac{x+1}{x}\, dx$$

$$
\int \frac{1}{-u+1} du = \int \frac{x+1}{x} dx
$$

$$
-\ln(u-1) = x + \ln(x) + c_2
$$

Raising both side to exponential gives

$$
\frac{1}{u-1} = e^{x+\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{1}{u-1} = c_3 e^{x+\ln(x)}
$$

Which simplifies to

$$
u(x) = \frac{(c_3 e^x x\, e^{c_2} + 1)\, e^{-x} e^{-c_2}}{c_3 x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{(c_3 e^x x\, e^{c_2} + 1)\, e^{-x} e^{-c_2}}{c_3}
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = \frac{(c_3 e^x x\, e^{c_2} + 1)\, e^{-x} e^{-c_2}}{c_3} \tag{1}
$$

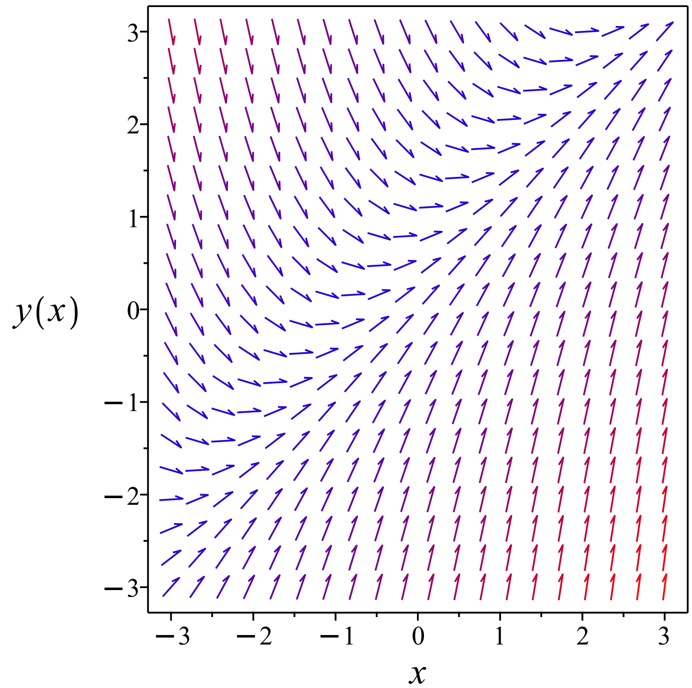


Figure 28: Slope field plot

Verification of solutions

$$y = \frac{(c_3 e^{x} x\, e^{c_2} + 1)\, e^{-x} e^{-c_2}}{c_3}$$

Verified OK.

2.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x - y + 1$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 26: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x - y + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{x} y \\ S_y &= e^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (x+1) e^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R+1) e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^R R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x y = x\,e^x + c_1$$

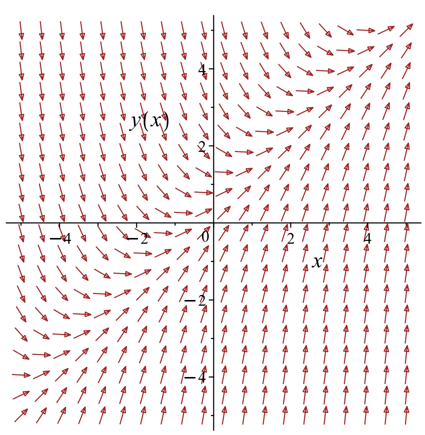
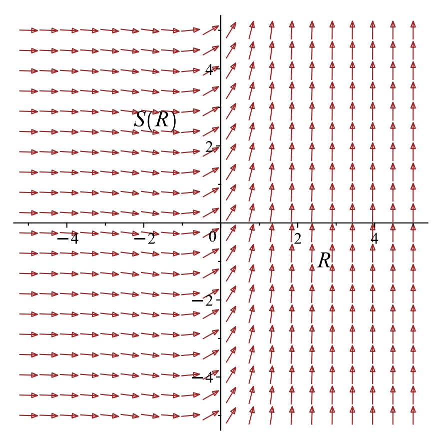
Which simplifies to

$$e^x y = x\,e^x + c_1$$

Which gives

$$y = (x\,e^x + c_1)\,e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x - y + 1$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = (R + 1)\,e^R$ 

Summary
The solution(s) found are the following

$$y = (x\,e^x + c_1)\,e^{-x} \tag{1}$$

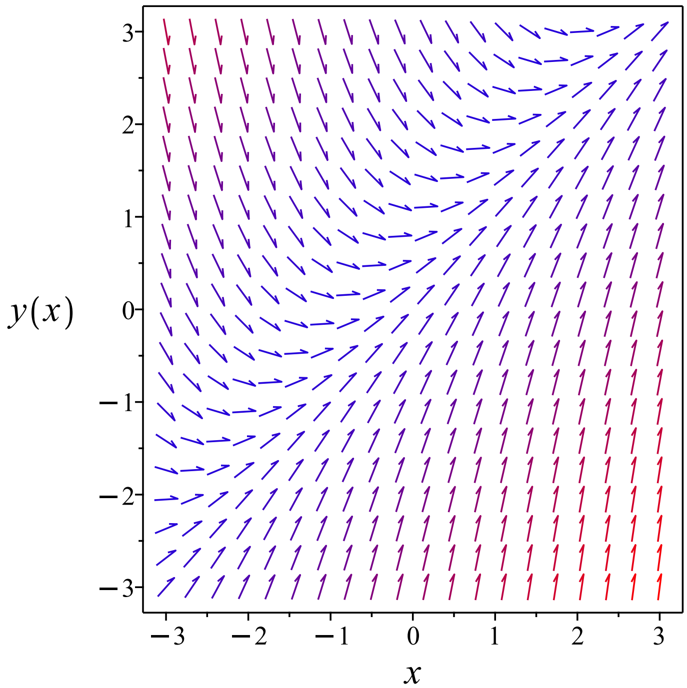


Figure 29: Slope field plot

Verification of solutions

$$y = (x\,e^{x} + c_1)\,e^{-x}$$

Verified OK.

2.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (x - y + 1)\,\mathrm{d}x \\
(-1 - x + y)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -1 - x + y \\
N(x,y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1 - x + y) \\
&= 1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, dx} \\ &= e^{\int 1 \, dx} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}(-1 - x + y) \\ &= -\mathrm{e}^{x}(x - y + 1) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{x}(x - y + 1)\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int -e^{x}(x-y+1) \, dx$$
$$\phi = -e^{x}(x-y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{x}$. Therefore equation (4) becomes

$$e^{x} = e^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -e^{x}(x-y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^{x}(x-y)$$

The solution becomes

$$y = (x e^{x} + c_1) e^{-x}$$

Summary

The solution(s) found are the following

$$y = (x e^{x} + c_1) e^{-x} \tag{1}$$

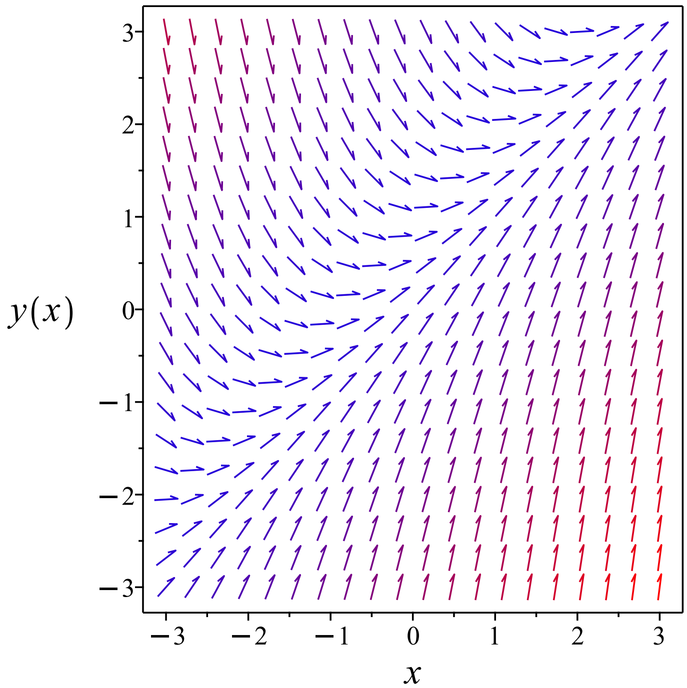


Figure 30: Slope field plot

Verification of solutions

$$y = (x\,e^{x} + c_1)\,e^{-x}$$

Verified OK.

2.6.5 Maple step by step solution

Let's solve

$y' + y = x + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 + x - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = x + 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \mu(x)\,(x + 1)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$
- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(x+1)\,dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,(x+1)\,dx + c_1$
- Solve for y

  $y = \frac{\int \mu(x)(x+1)dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int (x+1)\mathrm{e}^{x}dx + c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y = \frac{x\,\mathrm{e}^{x} + c_1}{\mathrm{e}^{x}}$
- Simplify

  $y = x + c_1\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) = 1+x-y(x),y(x), singsol=all)
```

$$y(x) = x + \mathrm{e}^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 15

```
DSolve[y'[x] == 1+x-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + c_1 e^{-x}$$

2.7 problem 8

Internal problem ID [17]
Internal file name [`OUTPUT/17_Sunday_June_05_2022_01_33_48_AM_76142612/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = x^2}$$

2.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 \end{aligned}$$

Hence the ode is

$$y' + y = x^2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(x^2\right)$$
$$d(e^x y) = \left(x^2 e^x\right) dx$$

Integrating gives

$$e^x y = \int x^2 e^x \, dx$$
$$e^x y = \left(x^2 - 2x + 2\right) e^x + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x}\left(x^2 - 2x + 2\right) e^x + c_1 e^{-x}$$

which simplifies to

$$y = x^2 - 2x + 2 + c_1 e^{-x}$$

Summary
The solution(s) found are the following

$$y = x^2 - 2x + 2 + c_1 e^{-x} \tag{1}$$

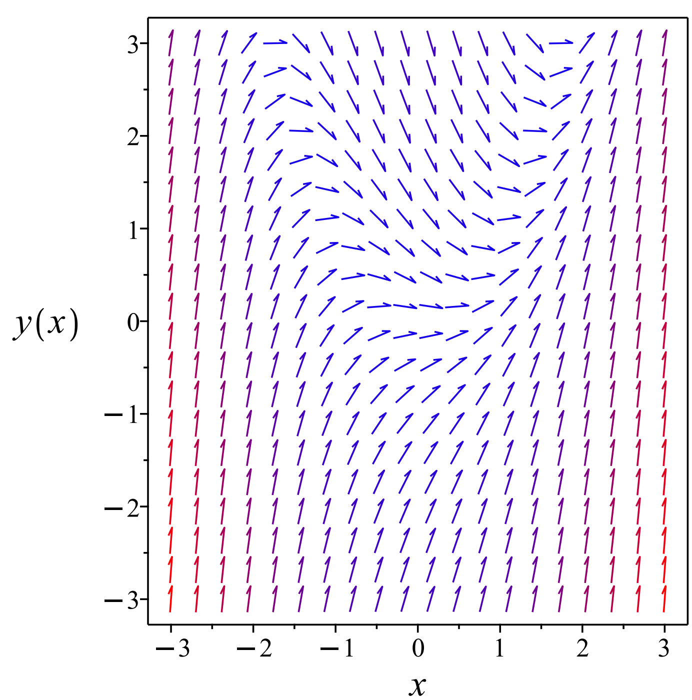


Figure 31: Slope field plot

Verification of solutions

$$y = x^2 - 2x + 2 + c_1 e^{-x}$$

Verified OK.

2.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^2 - y$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 29: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 - y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \mathrm{e}^{x} y \\ S_y &= \mathrm{e}^{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 \mathrm{e}^{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2 \mathrm{e}^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \left(R^2 - 2R + 2\right) e^R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x y = \left(x^2 - 2x + 2\right) e^x + c_1$$

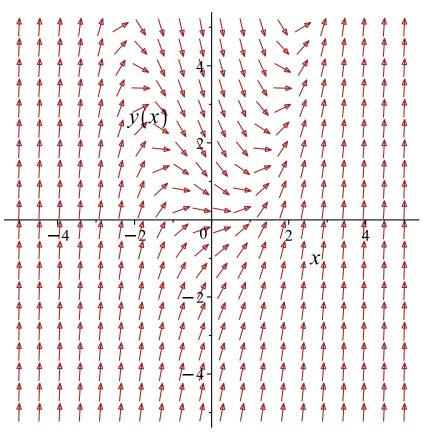
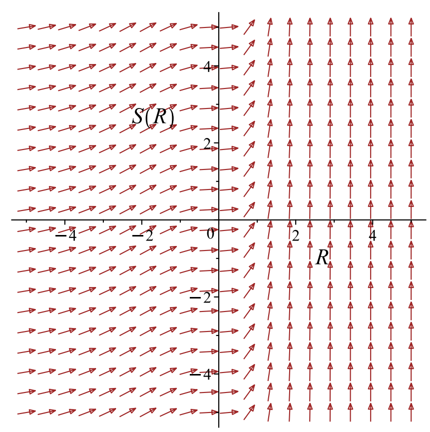
Which simplifies to

$$e^x y = \left(x^2 - 2x + 2\right) e^x + c_1$$

Which gives

$$y = \left(x^2 e^x - 2x\, e^x + 2\, e^x + c_1\right) e^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 - y$ 	$R = x$ $S = e^x y$	$\frac{dS}{dR} = R^2 e^R$ 

Summary
The solution(s) found are the following

$$y = \left(x^2 e^x - 2x\, e^x + 2\, e^x + c_1\right) e^{-x} \tag{1}$$

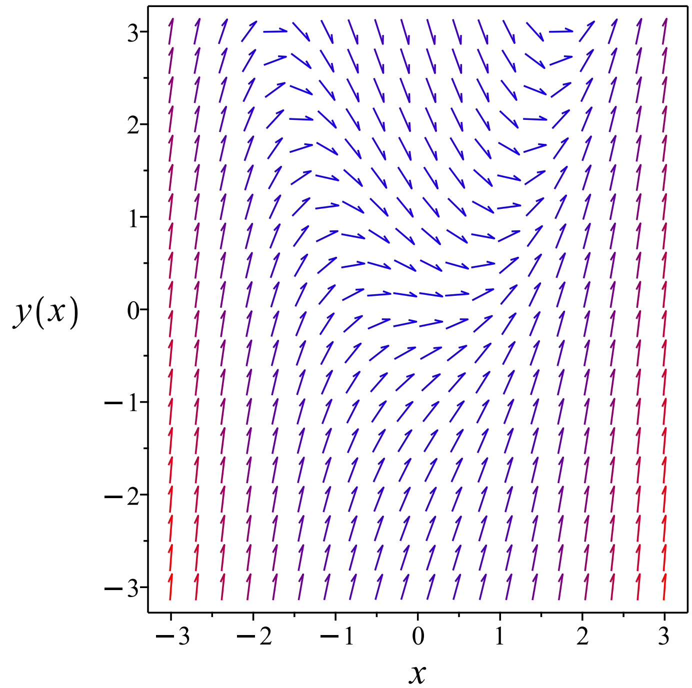


Figure 32: Slope field plot

Verification of solutions

$$y = \left(x^2 e^x - 2x\,e^x + 2\,e^x + c_1\right) e^{-x}$$

Verified OK.

2.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (x^2 - y) \, \mathrm{d}x$$
$$(-x^2 + y) \, \mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2 + y$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2 + y) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(-x^{2} + y\right) \\ &= -\mathrm{e}^{x}\left(x^{2} - y\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{x}\left(x^{2} - y\right)\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\mathrm{e}^{x}\left(x^{2}-y\right)\mathrm{d}x$$

$$\phi = -\left(x^{2}-2x-y+2\right)\mathrm{e}^{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^{2}-2x-y+2\right)\mathrm{e}^{x}+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^{2}-2x-y+2\right)\mathrm{e}^{x}$$

The solution becomes

$$y = \left(x^{2}\mathrm{e}^{x}-2x\,\mathrm{e}^{x}+2\,\mathrm{e}^{x}+c_1\right)\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \left(x^{2}\mathrm{e}^{x}-2x\,\mathrm{e}^{x}+2\,\mathrm{e}^{x}+c_1\right)\mathrm{e}^{-x} \tag{1}$$

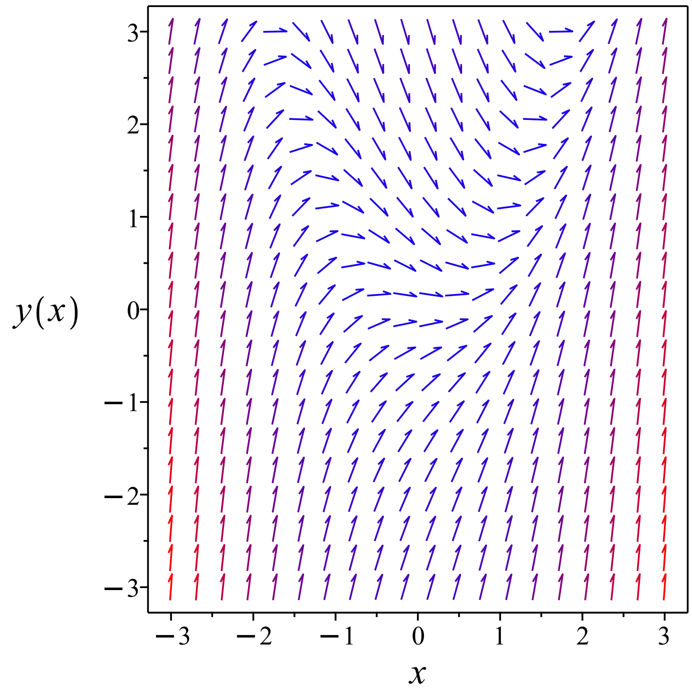


Figure 33: Slope field plot

Verification of solutions

$$y = \left(x^2 e^x - 2x e^x + 2 e^x + c_1\right) e^{-x}$$

Verified OK.

2.7.4 Maple step by step solution

Let's solve

$y' + y = x^2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = x^2 - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + y\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y)=\mu'(x)\,y+\mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x)=\mu(x)$
- Solve to find the integrating factor

  $\mu(x)=\mathrm{e}^{x}$
- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int \mu(x)\,x^{2}dx+c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y=\int \mu(x)\,x^{2}dx+c_1$
- Solve for y

  $y=\frac{\int \mu(x)x^{2}dx+c_1}{\mu(x)}$
- Substitute $\mu(x)=\mathrm{e}^{x}$

  $y=\frac{\int x^{2}\mathrm{e}^{x}dx+c_1}{\mathrm{e}^{x}}$
- Evaluate the integrals on the rhs

  $y=\frac{(x^{2}-2x+2)\,\mathrm{e}^{x}+c_1}{\mathrm{e}^{x}}$
- Simplify

  $y=x^{2}-2x+2+c_1\mathrm{e}^{-x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(diff(y(x),x) = x^2-y(x),y(x), singsol=all)
```

$$y(x) = x^2 - 2x + 2 + e^{-x}c_1$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 21

```
DSolve[y'[x] == x^2-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2 - 2x + c_1 e^{-x} + 2$$

2.8 problem 9

Internal problem ID [18]
Internal file name [`OUTPUT/18_Sunday_June_05_2022_01_33_49_AM_67397934/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = x^2 - 2}$$

2.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= x^2 - 2 \end{aligned}$$

Hence the ode is

$$y' + y = x^2 - 2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 1 dx} \\ &= e^x \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(x^2 - 2\right)$$
$$\frac{d}{dx}(e^x y) = (e^x)\left(x^2 - 2\right)$$
$$d(e^x y) = \left(\left(x^2 - 2\right)e^x\right)\, dx$$

Integrating gives

$$e^x y = \int \left(x^2 - 2\right)e^x\, dx$$
$$e^x y = x(-2 + x)\, e^x + c_1$$

Dividing both sides by the integrating factor $\mu = e^x$ results in

$$y = e^{-x} x(-2 + x)\, e^x + c_1 e^{-x}$$

which simplifies to

$$y = c_1 e^{-x} + x^2 - 2x$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x} + x^2 - 2x \tag{1}$$

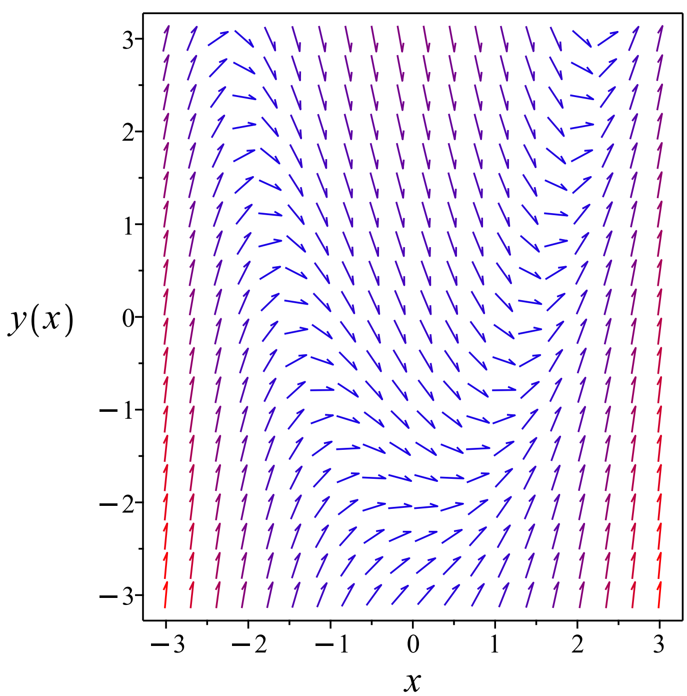


Figure 34: Slope field plot

Verification of solutions

$$y = c_1 e^{-x} + x^2 - 2x$$

Verified OK.

2.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^2 - y - 2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 32: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-x} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x}} dy \end{aligned}$$

Which results in

$$S = e^x y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = x^2 - y - 2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^x y \\ S_y &= e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (x^2 - 2)\, e^x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R^2 - 2)\, e^R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R(R-2)\,\mathrm{e}^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{x} y = x(-2+x)\,\mathrm{e}^{x} + c_1$$

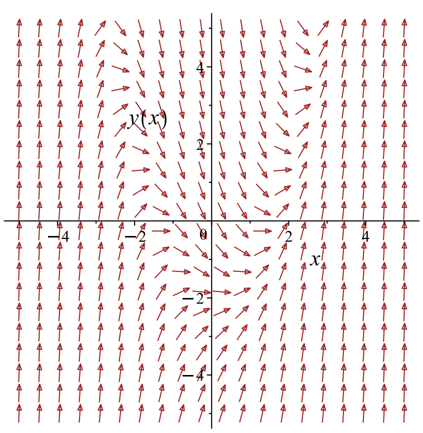
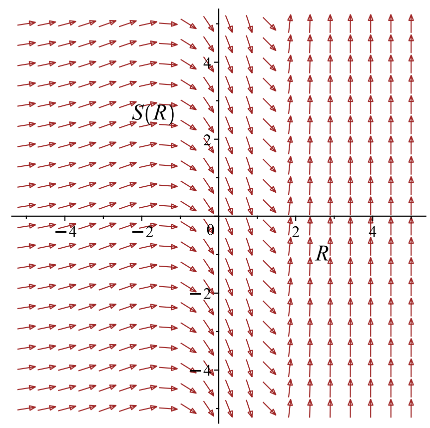
Which simplifies to

$$\mathrm{e}^{x} y = x(-2+x)\,\mathrm{e}^{x} + c_1$$

Which gives

$$y = \left(x^2 \mathrm{e}^{x} - 2x\,\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 - y - 2$ 	$R = x$ $S = \mathrm{e}^{x} y$	$\frac{dS}{dR} = (R^2 - 2)\,\mathrm{e}^{R}$ 

Summary
The solution(s) found are the following

$$y = \left(x^2 \mathrm{e}^{x} - 2x\,\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x} \tag{1}$$

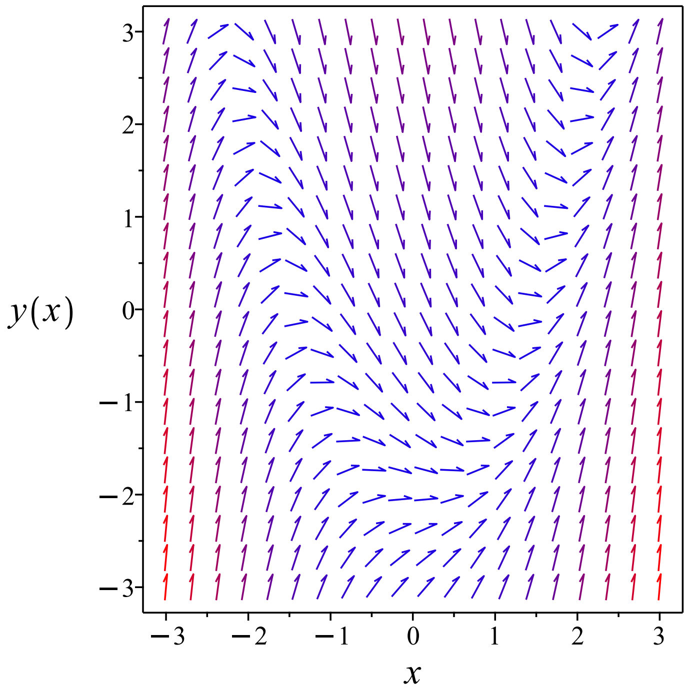


Figure 35: Slope field plot

Verification of solutions

$$y = \left(x^2 e^x - 2x e^x + c_1\right) e^{-x}$$

Verified OK.

2.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(x^2 - y - 2\right) \mathrm{d}x \\ \left(-x^2 + y + 2\right) \mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 + y + 2 \\ N(x, y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 + y + 2\right) \\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= 1((1) - (0)) \\ &= 1 \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int 1 \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{x} \\ &= \mathrm{e}^{x} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \mathrm{e}^{x}\left(-x^{2} + y + 2\right) \\ &= -\mathrm{e}^{x}\left(x^{2} - y - 2\right) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \mathrm{e}^{x}(1) \\ &= \mathrm{e}^{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-\mathrm{e}^{x}\left(x^{2} - y - 2\right)\right) + \left(\mathrm{e}^{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\mathrm{e}^{x}\left(x^{2}-y-2\right)\mathrm{d}x$$
$$\phi = -\left(x^{2}-2x-y\right)\mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\left(x^{2}-2x-y\right)\mathrm{e}^{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^{2}-2x-y\right)\mathrm{e}^{x}$$

The solution becomes

$$y = \left(x^{2}\mathrm{e}^{x} - 2x\,\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = \left(x^{2}\mathrm{e}^{x} - 2x\,\mathrm{e}^{x} + c_1\right)\mathrm{e}^{-x} \tag{1}$$

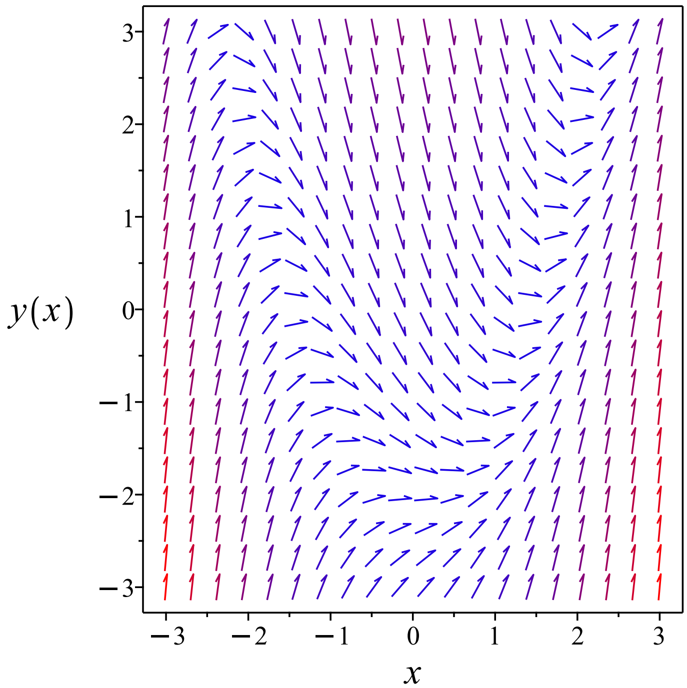


Figure 36: Slope field plot

Verification of solutions

$$y = \left(x^2 e^x - 2x\, e^x + c_1\right) e^{-x}$$

Verified OK.

2.8.4 Maple step by step solution

Let's solve

$y' + y = x^2 - 2$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -2 + x^2 - y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + y = x^2 - 2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(y' + y) = \mu(x)\,(x^2 - 2)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(y'+y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \mu(x)$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,(x^2-2)\,dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,(x^2-2)\,dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)(x^2-2)\,dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{x}$

  $y = \frac{\int (x^2-2)\mathrm{e}^{x}\,dx + c_1}{\mathrm{e}^{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x(-2+x)\mathrm{e}^{x} + c_1}{\mathrm{e}^{x}}$

- Simplify

  $y = c_1\mathrm{e}^{-x} + x^2 - 2x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x) = -2+x^2-y(x),y(x), singsol=all)
```

$$y(x) = x^2 - 2x + e^{-x} c_1$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 19

```
DSolve[y'[x]== -2+x^2-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to (x - 2)x + c_1 e^{-x}$$

2.9 problem 11

Internal problem ID [19]
Internal file name [`OUTPUT/19_Sunday_June_05_2022_01_33_49_AM_64304397/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2x^2y^2 = 0}$$

With initial conditions

$$[y(1) = -1]$$

2.9.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 2x^2y^2 \end{aligned}$$

The x domain of $f(x, y)$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(2x^2y^2) \\ &= 4x^2y\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

2.9.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= 2x^2y^2\end{aligned}$$

Where $f(x) = 2x^2$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2}\, dy &= 2x^2\, dx \\ \int \frac{1}{y^2}\, dy &= \int 2x^2\, dx \\ -\frac{1}{y} &= \frac{2x^3}{3} + c_1\end{aligned}$$

Which results in

$$y = -\frac{3}{2x^3 + 3c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{3}{3c_1 + 2}$$

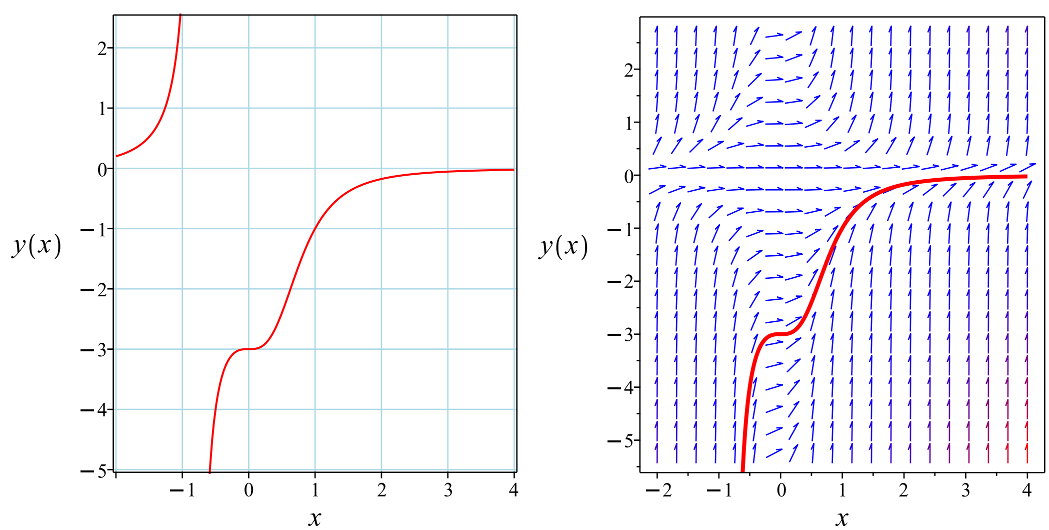
$$c_1 = \frac{1}{3}$$

Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{2x^3 + 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{3}{2x^3 + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -\frac{3}{2x^3 + 1}$$

Verified OK.

2.9.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2x^2y^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 35: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= \frac{1}{2x^2}\\ \eta(x,y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx\\ &= \int \frac{1}{\frac{1}{2x^2}} dx\end{aligned}$$

Which results in

$$S = \frac{2x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2x^2y^2$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 2x^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{2x^3}{3} = -\frac{1}{y} + c_1
$$

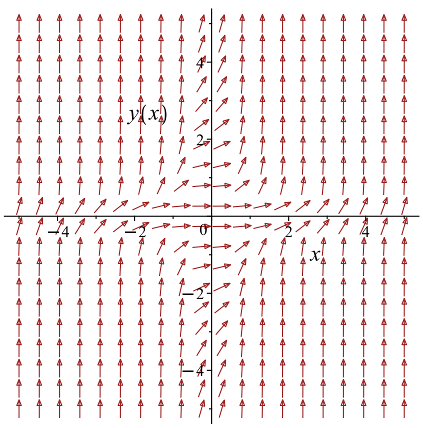
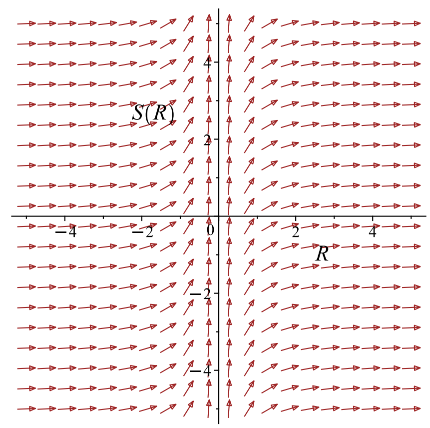
Which simplifies to

$$
\frac{2x^3}{3} = -\frac{1}{y} + c_1
$$

Which gives

$$
y = \frac{3}{-2x^3 + 3c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2x^2y^2$ 	$R = y$ $S = \frac{2x^3}{3}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{3}{3c_1 - 2}$$

$$c_1 = -\frac{1}{3}$$

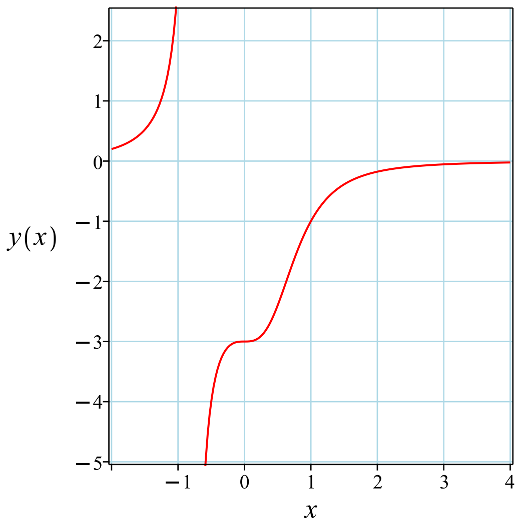
Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{2x^3 + 1}$$

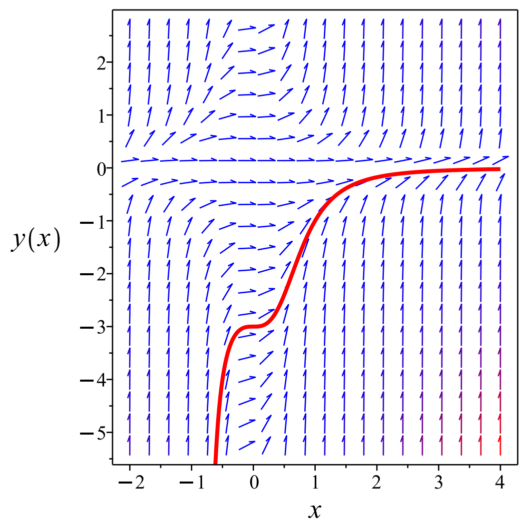
Summary

The solution(s) found are the following

$$y = -\frac{3}{2x^3 + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{2x^3 + 1}$$

Verified OK.

2.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(\frac{1}{2y^2}\right)\mathrm{d}y &= \left(x^2\right)\mathrm{d}x \\ \left(-x^2\right)\mathrm{d}x + \left(\frac{1}{2y^2}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x^2 \\ N(x, y) &= \frac{1}{2y^2} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y^2}\right) \\ &= 0 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x^2 \,\mathrm{d}x
$$

$$
\phi = -\frac{x^3}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y^2}$. Therefore equation (4) becomes

$$
\frac{1}{2y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{2y^2}
$$

Integrating the above w.r.t y gives

$$
\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{2y^2}\right) \mathrm{d}y
$$

$$
f(y) = -\frac{1}{2y} + c_1
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - \frac{1}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - \frac{1}{2y}$$

The solution becomes

$$y = -\frac{3}{2\left(x^3 + 3c_1\right)}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{3}{2\left(3c_1 + 1\right)}$$

$$c_1 = \frac{1}{6}$$

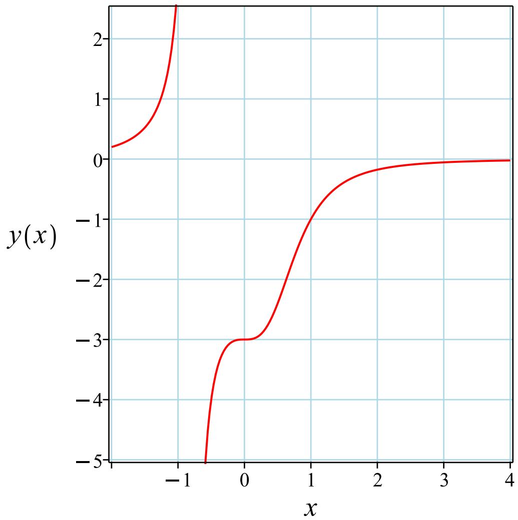
Substituting c_1 found above in the general solution gives

$$y = -\frac{3}{2x^3 + 1}$$

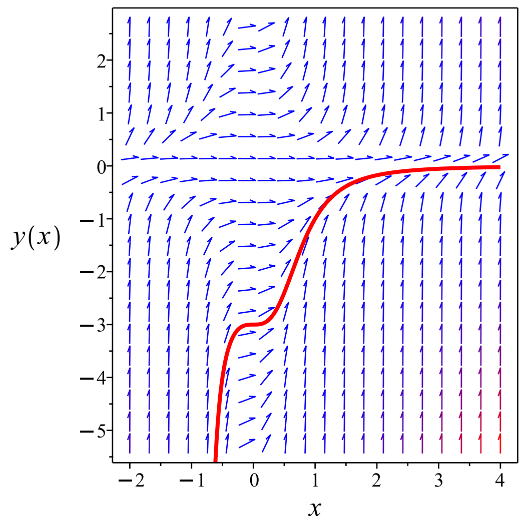
Summary

The solution(s) found are the following

$$y = -\frac{3}{2x^3 + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{3}{2x^3+1}$$

Verified OK.

2.9.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= 2x^2y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 2x^2y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = 2x^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{2x^2u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 4x \\ f_1f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$2x^2u''(x) - 4xu'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2x^3 + c_1$$

The above shows that

$$u'(x) = 3c_2x^2$$

Using the above in (1) gives the solution

$$y = -\frac{3c_2}{2\,(c_2x^3 + c_1)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{3}{2x^3 + 2c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{3}{2\,(c_3 + 1)}$$

$$c_3 = \frac{1}{2}$$

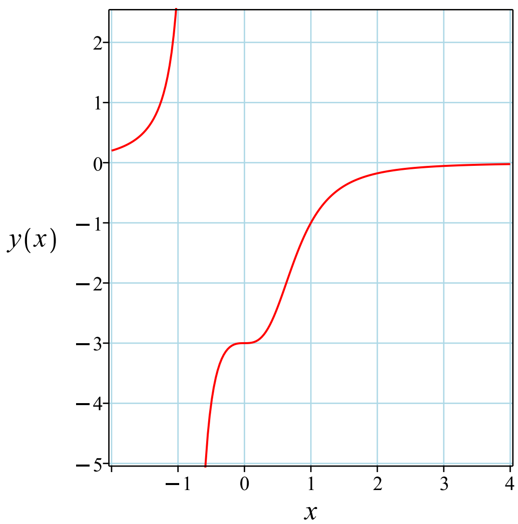
Substituting c_3 found above in the general solution gives

$$y = -\frac{3}{2x^3 + 1}$$

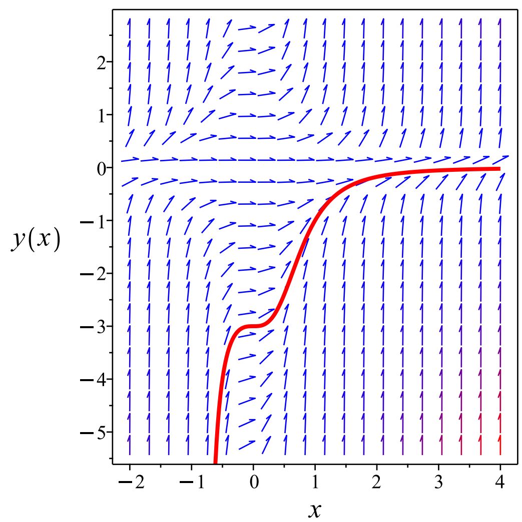
Summary

The solution(s) found are the following

$$y = -\frac{3}{2x^3 + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{3}{2x^3 + 1}$$

Verified OK.

2.9.6 Maple step by step solution

Let's solve

$[y' - 2x^2y^2 = 0, y(1) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = 2x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int 2x^2dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \frac{2x^3}{3} + c_1$

- Solve for y

  $y = -\frac{3}{2x^3+3c_1}$

- Use initial condition $y(1) = -1$

  $-1 = -\frac{3}{3c_1+2}$

- Solve for c_1

  $c_1 = \frac{1}{3}$

- Substitute $c_1 = \frac{1}{3}$ into general solution and simplify

  $y = -\frac{3}{2x^3+1}$

- Solution to the IVP

  $y = -\frac{3}{2x^3+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 15

```
dsolve([diff(y(x),x) = 2*x^2*y(x)^2,y(1) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{3}{2x^3+1}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 16

```
DSolve[{y'[x] == 2*x^2*y[x]^2,y[1]==-1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3}{2x^3 + 1}$$

2.10 problem 12

Internal problem ID [20]
Internal file name [`OUTPUT/20_Sunday_June_05_2022_01_33_49_AM_94980606/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - x\ln(y) = 0}$$

2.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\ln(y) \end{aligned}$$

Where $f(x) = x$ and $g(y) = \ln(y)$. Integrating both sides gives

$$\frac{1}{\ln(y)}\,dy = x\,dx$$

$$\int \frac{1}{\ln(y)}\,dy = \int x\,dx$$

$$-\operatorname{expIntegral}_1(-\ln(y)) = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$e^{-\operatorname{expIntegral}_1(-\ln(y))} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$e^{-\operatorname{expIntegral}_1(-\ln(y))} = c_2 e^{\frac{x^2}{2}}$$

<u>Summary</u>
The solution(s) found are the following

$$y = e^{\operatorname{RootOf}\left(c_2 e^{\frac{x^2}{2}+c_1} e^{\operatorname{expIntegral}_1(-\_Z)}-1\right)} \tag{1}$$

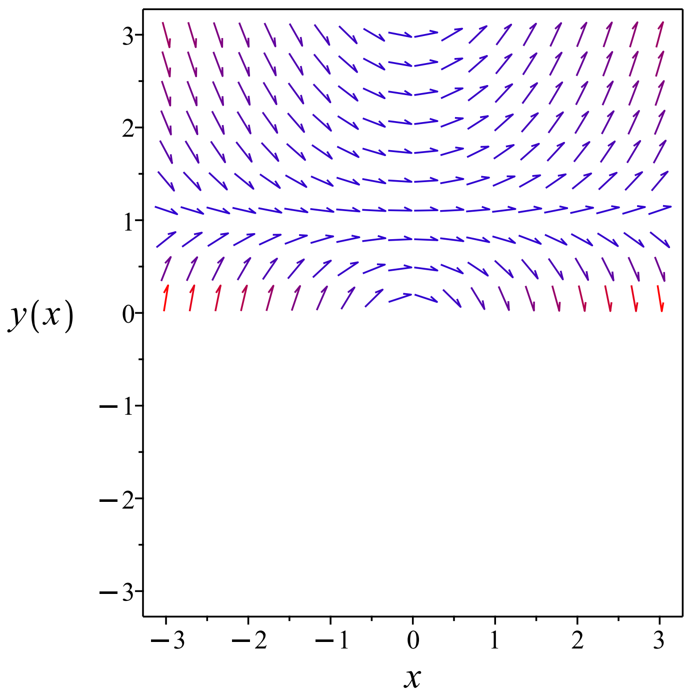


Figure 41: Slope field plot

<u>Verification of solutions</u>

$$y = e^{\operatorname{RootOf}\left(c_2 e^{\frac{x^2}{2}+c_1} e^{\operatorname{expIntegral}_1(-\_Z)}-1\right)}$$

Verified OK.

2.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x\ln(y) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 38: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = x \ln(y)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\ln(y)} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\ln(R)}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\operatorname{expIntegral}_1(-\ln(R)) + c_1 \tag{4}
$$

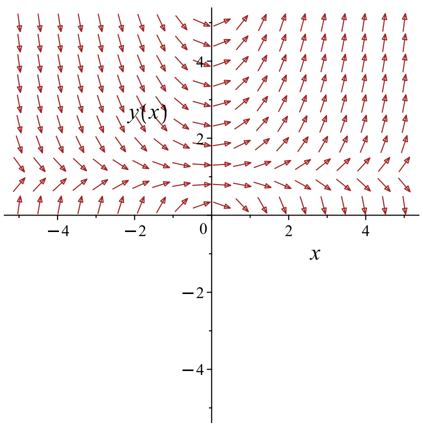
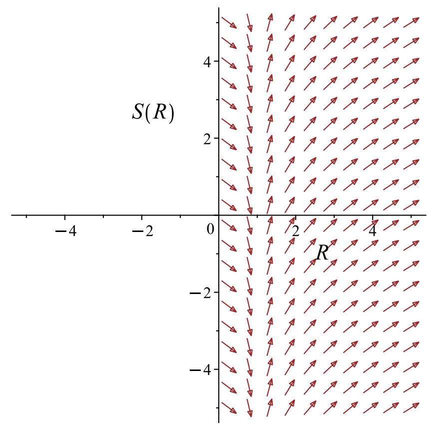
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x^2}{2} = -\operatorname{expIntegral}_1(-\ln(y)) + c_1
$$

Which simplifies to

$$
\frac{x^2}{2} = -\operatorname{expIntegral}_1(-\ln(y)) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = x \ln(y)$

</td><td>$R = y$
$S = \frac{x^2}{2}$</td><td>$\frac{dS}{dR} = \frac{1}{\ln(R)}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = -\operatorname{expIntegral}_1(-\ln(y)) + c_1 \tag{1}$$

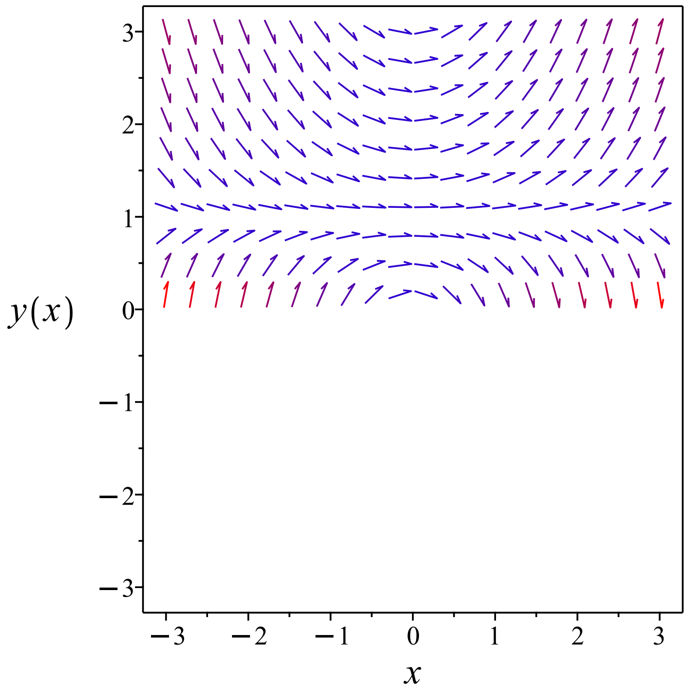


Figure 42: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = -\operatorname{expIntegral}_1\left(-\ln\left(y\right)\right) + c_1$$

Verified OK.

2.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{\ln(y)}\right) \mathrm{d}y &= (x)\, \mathrm{d}x \\
(-x)\, \mathrm{d}x + \left(\frac{1}{\ln(y)}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{\ln(y)}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\ln(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x\,\mathrm{d}x$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\ln(y)}$. Therefore equation (4) becomes

$$\frac{1}{\ln(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\ln(y)}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\ln(y)}\right)\mathrm{d}y$$

$$f(y) = -\operatorname{expIntegral}_1(-\ln(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \text{expIntegral}_1(-\ln(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \text{expIntegral}_1(-\ln(y))$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \text{expIntegral}_1(-\ln(y)) = c_1 \tag{1}$$

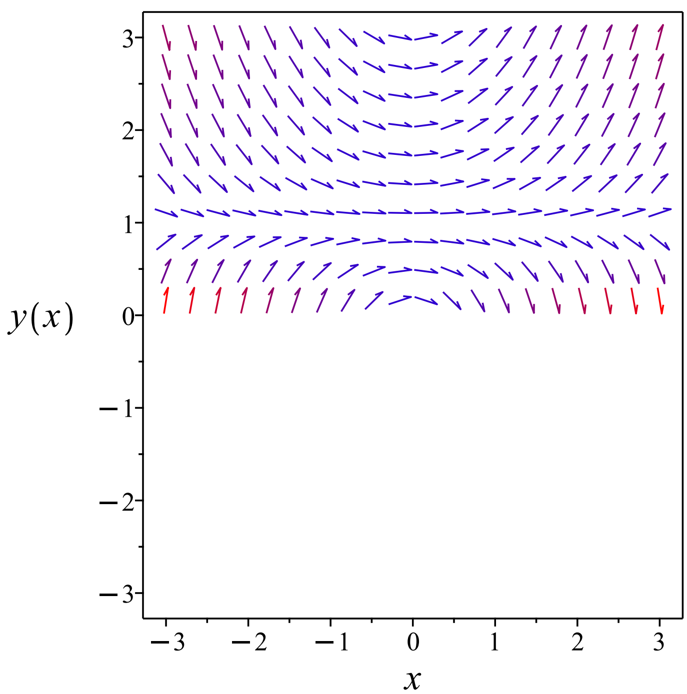


Figure 43: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \text{expIntegral}_1(-\ln(y)) = c_1$$

Verified OK.

2.10.4 Maple step by step solution

Let's solve

$y' - x\ln(y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\ln(y)} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\ln(y)}dx = \int x dx + c_1$

- Evaluate integral

  $-\mathrm{Ei}_1(-\ln(y)) = \frac{x^2}{2} + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(diff(y(x),x) = x*ln(y(x)),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{\mathrm{RootOf}\left(x^2+2\,\mathrm{expIntegral}_1(-\_Z)+2c_1\right)}$$

✓ Solution by Mathematica

Time used: 0.266 (sec). Leaf size: 22

```
DSolve[y'[x] == x*Log[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \text{LogIntegral}^{(-1)}\left(\frac{x^2}{2} + c_1\right)$$
$$y(x) \to 1$$

2.11 problem 13

Internal problem ID [21]
Internal file name [`OUTPUT/21_Sunday_June_05_2022_01_33_50_AM_13432084/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(0) = 1]$$

2.11.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^{\frac{1}{3}} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^{\frac{1}{3}}\right) \\ &= \frac{1}{3y^{\frac{2}{3}}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.11.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{1}{3}}} dy = \int dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = c_1$$

$$c_1 = \frac{3}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{3y^{\frac{2}{3}}}{2} = x + \frac{3}{2}$$

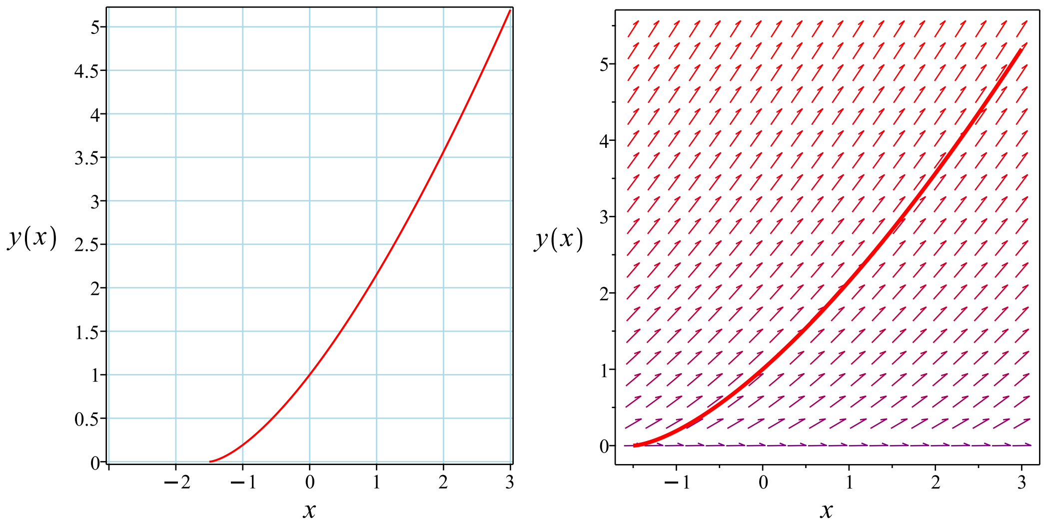
Solving for y from the above gives

$$y = \frac{(2x + 3)\sqrt{6x + 9}}{9}$$

Summary

The solution(s) found are the following

$$y = \frac{(2x + 3)\sqrt{6x + 9}}{9} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{(2x+3)\sqrt{6x+9}}{9}$$

Verified OK.

2.11.3 Maple step by step solution

Let's solve

$\left[y' - y^{\frac{1}{3}} = 0, y(0) = 1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = x + c_1$

- Solve for y

$y = \frac{(6x+6c_1)^{\frac{3}{2}}}{27}$

- Use initial condition $y(0) = 1$

$1 = \frac{2\sqrt{6}\, c_1^{\frac{3}{2}}}{9}$

- Solve for c_1

$c_1 = \frac{3}{2}$

- Substitute $c_1 = \frac{3}{2}$ into general solution and simplify

$y = \frac{(2x+3)\sqrt{6x+9}}{9}$

- Solution to the IVP

$y = \frac{(2x+3)\sqrt{6x+9}}{9}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 13

```
dsolve([diff(y(x),x) = y(x)^(1/3),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{(2x+3)\sqrt{6x+9}}{9}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 23

```
DSolve[{y'[x] == y[x]^(1/3),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{(2x+3)^{3/2}}{3\sqrt{3}}$$

2.12 problem 14

Internal problem ID [22]
Internal file name [`OUTPUT/22_Sunday_June_05_2022_01_33_50_AM_29491422/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^{\frac{1}{3}} = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.12.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= y^{\frac{1}{3}} \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{0 \leq y\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(y^{\frac{1}{3}}\right) \\ &= \frac{1}{3y^{\frac{2}{3}}} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{0 < y\}$$

But the point $y_0 = 0$ is not inside this domain. Hence existence and uniqueness theorem does not apply. Solution exists but no guarantee that unique solution exists.

2.12.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^{\frac{1}{3}}} dy = \int dx$$

$$\frac{3y^{\frac{2}{3}}}{2} = x + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

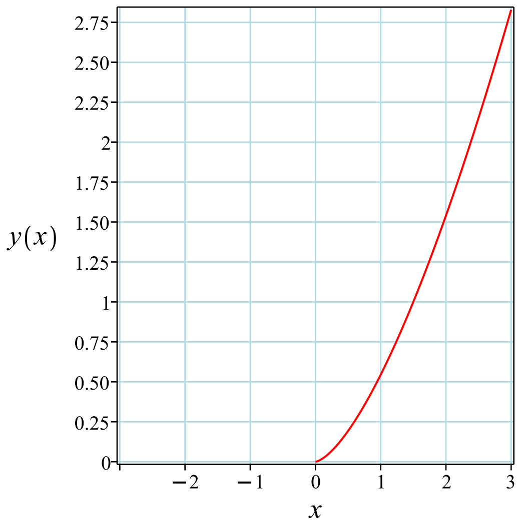
$$\frac{3y^{\frac{2}{3}}}{2} = x$$

Solving for y from the above gives

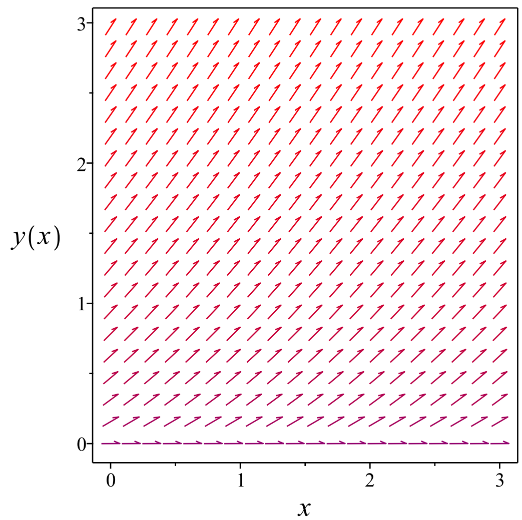
$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Summary
The solution(s) found are the following

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Verified OK.

2.12.3 Maple step by step solution

Let's solve

$\left[y' - y^{\frac{1}{3}} = 0, y(0) = 0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^{\frac{1}{3}}} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^{\frac{1}{3}}}dx = \int 1dx + c_1$

- Evaluate integral

  $\frac{3y^{\frac{2}{3}}}{2} = x + c_1$

- Solve for y

$$y = \frac{(6x+6c_1)^{\frac{3}{2}}}{27}$$

- Use initial condition $y(0) = 0$

$$0 = \frac{2\sqrt{6}\,c_1^{\frac{3}{2}}}{9}$$

- Solve for c_1

$$c_1 = 0$$

- Substitute $c_1 = 0$ into general solution and simplify

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

- Solution to the IVP

$$y = \frac{2x^{\frac{3}{2}}\sqrt{6}}{9}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x) = y(x)^(1/3),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.003 (sec). Leaf size: 21

```
DSolve[{y'[x] == y[x]^(1/3),y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}\sqrt{\frac{2}{3}}x^{3/2}$$

2.13 problem 17

Internal problem ID [23]
Internal file name [`OUTPUT/23_Sunday_June_05_2022_01_33_50_AM_51008168/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{yy' = x - 1}$$

With initial conditions

$$[y(0) = 1]$$

2.13.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x-1}{y} \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{y < 0 \vee 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x-1}{y}\right) \\ &= -\frac{x-1}{y^2}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{y < 0 \vee 0 < y\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

2.13.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x-1}{y}\end{aligned}$$

Where $f(x) = x - 1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}}\, dy &= x - 1\, dx \\ \int \frac{1}{\frac{1}{y}}\, dy &= \int x - 1\, dx \\ \frac{y^2}{2} &= \frac{1}{2}x^2 - x + c_1\end{aligned}$$

Which results in

$$y = \sqrt{x^2 + 2c_1 - 2x}$$
$$y = -\sqrt{x^2 + 2c_1 - 2x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sqrt{c_1}\sqrt{2}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{c_1}\sqrt{2}$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = x - 1$$

But this does not satisfy the initial conditions. Hence no solution can be found.

Verification of solutions N/A

2.13.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x - 1}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (x - 1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(x - 1)\, dx = d\left(\frac{1}{2}x^2 - x\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(\frac{1}{2}x^2 - x\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{x^2 + 2c_1 - 2x} + c_1$$
$$y = -\sqrt{x^2 + 2c_1 - 2x} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\sqrt{c_1}\,\sqrt{2} + c_1$$

$$c_1 = \sqrt{2}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\right) + 1$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{x^2 + 4 + 2\sqrt{3} - 2x} + 2 + \sqrt{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \sqrt{c_1}\,\sqrt{2} + c_1$$

$$c_1 = -\sqrt{2}\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}\right) + 1$$

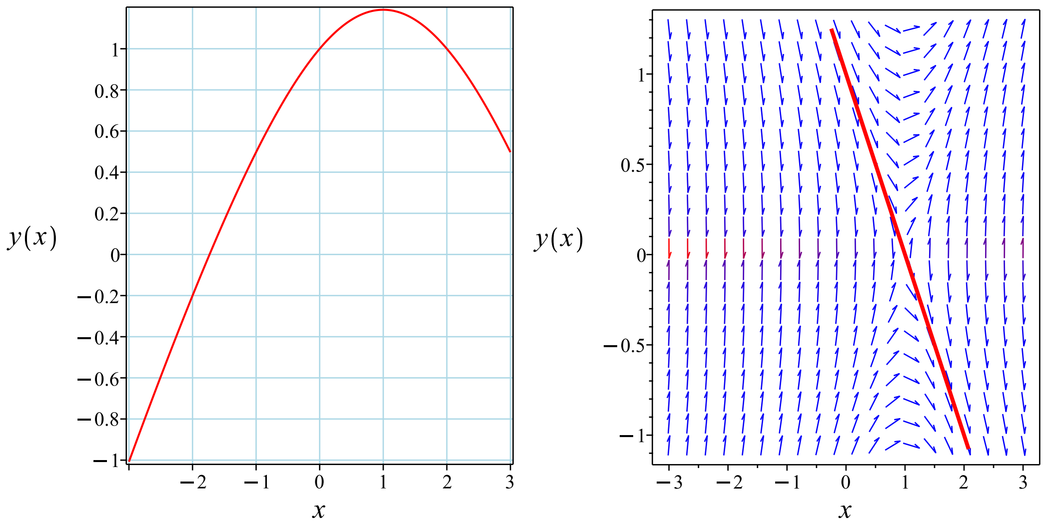
Substituting c_1 found above in the general solution gives

$$y = \sqrt{x^2 + 4 - 2\sqrt{3} - 2x} + 2 - \sqrt{3}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x^2 + 4 - 2\sqrt{3} - 2x} + 2 - \sqrt{3} \tag{1}$$
$$y = -\sqrt{x^2 + 4 + 2\sqrt{3} - 2x} + 2 + \sqrt{3} \tag{2}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \sqrt{x^2 + 4 - 2\sqrt{3} - 2x} + 2 - \sqrt{3}$$

Verified OK. `{positive}`

$$y = -\sqrt{x^2 + 4 + 2\sqrt{3} - 2x} + 2 + \sqrt{3}$$

Verified OK. `{positive}`

2.13.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{X + x_0 - 1}{Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X, Y)$$

In this case, it can be seen that both $M = X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X} X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X} X + u &= \frac{1}{u} \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{1}{u(X)} - u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX} u(X) - \frac{\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX} u(X)\right) u(X)\, X + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X, u) \\ &= f(X) g(u) \\ &= -\frac{u^2 - 1}{uX} \end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = -\frac{1}{X}\,dX$$
$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int -\frac{1}{X}\,dX$$
$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\ln(X) + c_2$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(u-1) + \ln(u+1)) = -\ln(X) + 2c_2$$
$$\ln(u-1) + \ln(u+1) = (2)(-\ln(X) + 2c_2)$$
$$= -2\ln(X) + 4c_2$$

Raising both side to exponential gives

$$e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(X)+2c_2}$$

Which simplifies to

$$u^2 - 1 = \frac{2c_2}{X^2}$$
$$= \frac{c_3}{X^2}$$

The solution is

$$u(X)^2 - 1 = \frac{c_3}{X^2}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$\frac{Y(X)^2}{X^2} - 1 = \frac{c_3}{X^2}$$

Which simplifies to

$$-(X - Y(X))(X + Y(X)) = c_3$$

Using the solution for $Y(X)$

$$-(X - Y(X))(X + Y(X)) = c_3$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x + 1
\end{aligned}
$$

Then the solution in y becomes

$$
-(x - 1 - y)(x - 1 + y) = c_3
$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = c_3
$$

$$
c_3 = 0
$$

Substituting c_3 found above in the general solution gives

$$
-(x - y - 1)(x - 1 + y) = 0
$$

Summary
The solution(s) found are the following

$$
-(x - 1 - y)(x - 1 + y) = 0 \tag{1}
$$

Verification of solutions

$$
-(x - 1 - y)(x - 1 + y) = 0
$$

Verified OK.

2.13.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x - 1}{y} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 43: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= \frac{1}{x - 1} \\ \eta(x, y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x-1}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{2}x^2 - x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x - 1}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x - 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

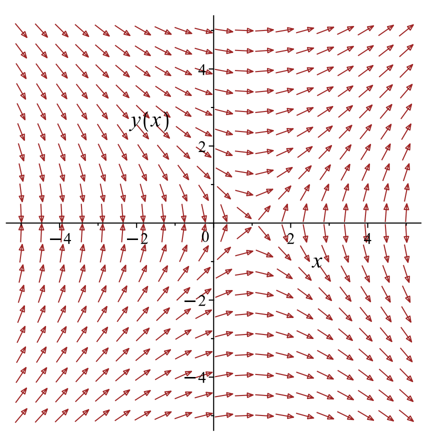
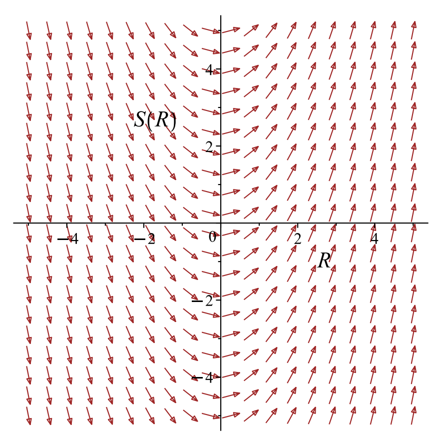
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2}x^2 - x = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{1}{2}x^2 - x = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x-1}{y}$ 	$R = y$ $S = \frac{1}{2}x^2 - x$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{1}{2} + c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{2}x^2 - x = \frac{y^2}{2} - \frac{1}{2}$$

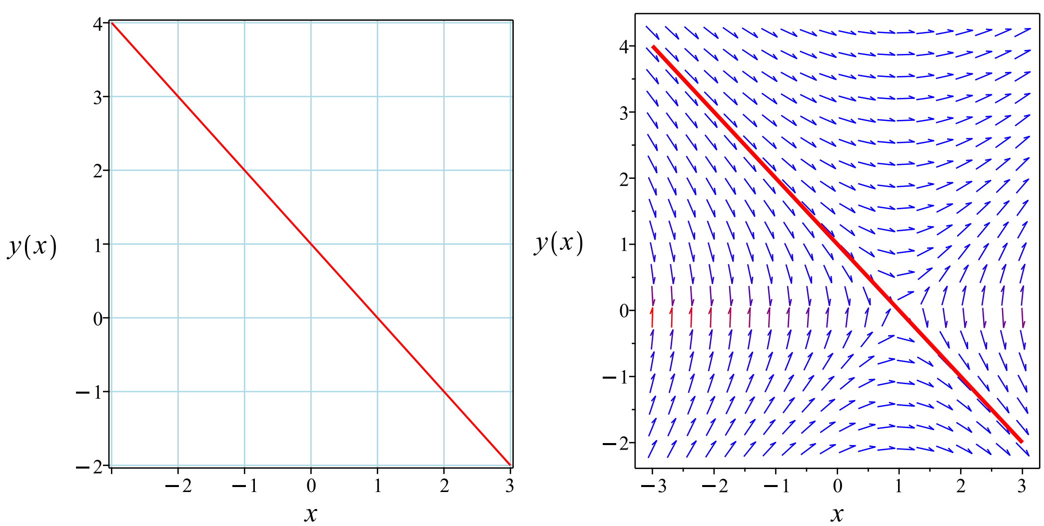
Solving for y from the above gives

$$y = 1 - x$$

Summary

The solution(s) found are the following

$$y = 1 - x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 - x$$

Verified OK. `{positive}`

2.13.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y&=(x-1)\,\mathrm{d}x\\ (1-x)\,\mathrm{d}x+(y)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=1-x\\ N(x,y)&=y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1 - x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 1 - x \, \mathrm{d}x\end{aligned}$$

$$\phi = x - \frac{1}{2}x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + x$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

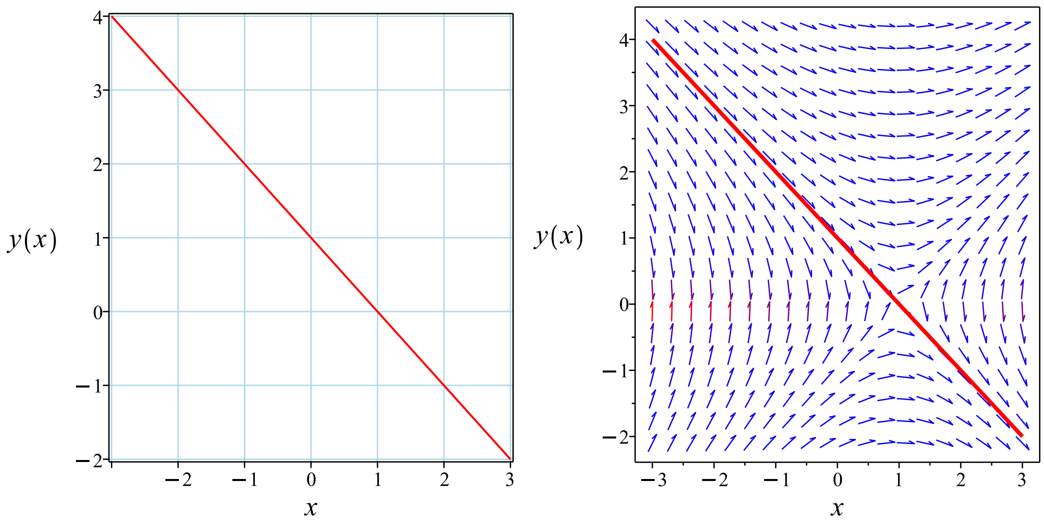
$$-\frac{1}{2}x^2 + \frac{1}{2}y^2 + x = \frac{1}{2}$$

Solving for y from the above gives

$$y = 1 - x$$

<u>Summary</u>
The solution(s) found are the following

$$y = 1 - x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 - x$$

Verified OK. {positive}

2.13.7 Maple step by step solution

Let's solve

$[yy' = x - 1, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int yy'dx = \int (x - 1)\, dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{1}{2}x^2 - x + c_1$

- Solve for y

  $\left\{y = \sqrt{x^2 + 2c_1 - 2x}, y = -\sqrt{x^2 + 2c_1 - 2x}\right\}$

- Use initial condition $y(0) = 1$

  $1 = \sqrt{c_1}\,\sqrt{2}$

- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \text{csgn}(x-1)(x-1)$
- Use initial condition $y(0) = 1$

  $1 = -\sqrt{c_1}\sqrt{2}$
- Solution does not satisfy initial condition
- Solution to the IVP

  $y = \text{csgn}(x-1)(x-1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 9

```
dsolve([y(x)*diff(y(x),x) = -1+x,y(0) = 1],y(x), singsol=all)
```

$$y(x) = 1 - x$$

✓ Solution by Mathematica

Time used: 0.087 (sec). Leaf size: 14

```
DSolve[{y[x]*y'[x] == -1+x,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{(x-1)^2}$$

2.14 problem 18

Internal problem ID [24]
Internal file name [`OUTPUT/24_Sunday_June_05_2022_01_33_51_AM_85751881/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{yy' = x - 1}$$

With initial conditions

$$[y(1) = 0]$$

2.14.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \frac{x-1}{y} \end{aligned}$$

$f(x, y)$ is not defined at $y = 0$ therefore existence and uniqueness theorem do not apply.

2.14.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\&= f(x)g(y)\\&= \frac{x-1}{y}\end{aligned}$$

Where $f(x) = x - 1$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\frac{1}{y}}\,dy &= x - 1\,dx\\\int \frac{1}{\frac{1}{y}}\,dy &= \int x - 1\,dx\\\frac{y^2}{2} &= \frac{1}{2}x^2 - x + c_1\end{aligned}$$

Which results in

$$\begin{aligned}y &= \sqrt{x^2 + 2c_1 - 2x}\\y &= -\sqrt{x^2 + 2c_1 - 2x}\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\sqrt{-1 + 2c_1}$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$y = 1 - x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \sqrt{-1 + 2c_1}$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

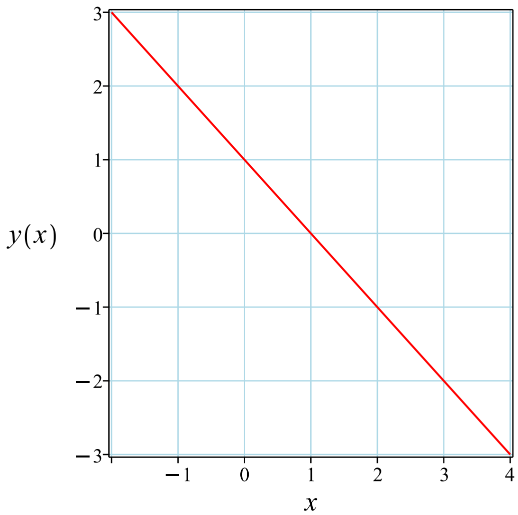
$$y = x - 1$$

Summary

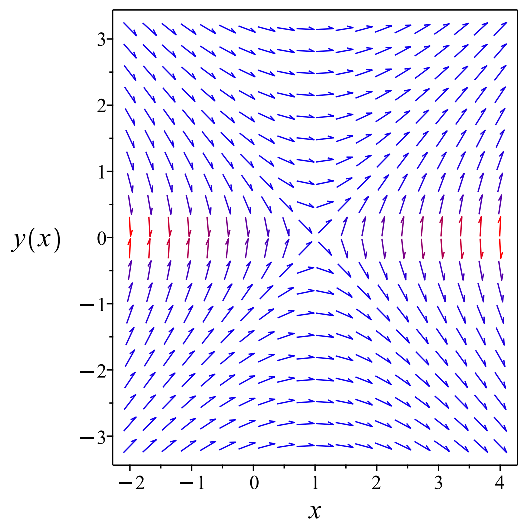
The solution(s) found are the following

$$y = x - 1 \tag{1}$$

$$y = 1 - x \tag{2}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x - 1$$

Verified OK. `{positive}`

$$y = 1 - x$$

Verified OK. `{positive}`

2.14.3 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x-1}{y} \tag{1}$$

Which becomes

$$(y)\, dy = (x-1)\, dx \tag{2}$$

But the RHS is complete differential because

$$(x-1)\, dx = d\left(\frac{1}{2}x^2 - x\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(\frac{1}{2}x^2 - x\right)$$

Integrating both sides gives gives these solutions

$$y = \sqrt{x^2 + 2c_1 - 2x} + c_1$$
$$y = -\sqrt{x^2 + 2c_1 - 2x} + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = -\sqrt{-1 + 2c_1} + c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = -\sqrt{x^2 - 2x + 2} + 1$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

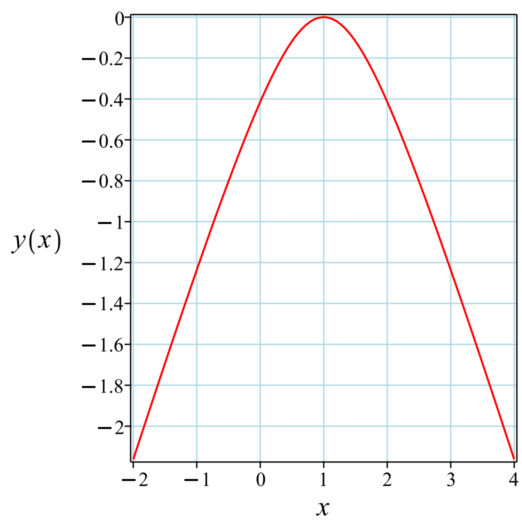
$$0 = \sqrt{-1 + 2c_1} + c_1$$

Warning: Unable to solve for constant of integration.

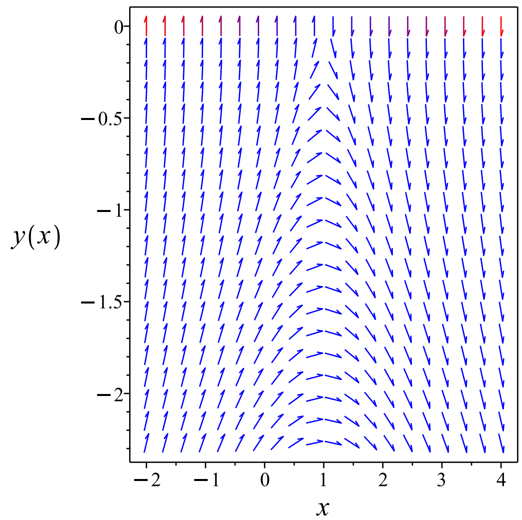
Summary

The solution(s) found are the following

$$y = -\sqrt{x^2 - 2x + 2}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\sqrt{x^2 - 2x + 2} + 1$$

Verified OK. `{positive}`

2.14.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = \frac{X + x_0 - 1}{Y(X) + y_0}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = 1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = \frac{X}{Y(X)}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X, Y) \\ &= \frac{X}{Y} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = X$ and $N = Y$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned}\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= \frac{1}{u}\\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{\frac{1}{u(X)} - u(X)}{X}\end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{\frac{1}{u(X)} - u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)u(X)\,X + u(X)^2 - 1 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(X,u)\\ &= f(X)g(u)\\ &= -\frac{u^2-1}{uX}\end{aligned}$$

Where $f(X) = -\frac{1}{X}$ and $g(u) = \frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du = -\frac{1}{X}\,dX$$

$$\int \frac{1}{\frac{u^2-1}{u}}\,du = \int -\frac{1}{X}\,dX$$

$$\frac{\ln(u-1)}{2} + \frac{\ln(u+1)}{2} = -\ln(X) + c_2$$

The above can be written as

$$
\begin{aligned}
\left(\frac{1}{2}\right)(\ln(u-1)+\ln(u+1)) &= -\ln(X)+2c_2 \\
\ln(u-1)+\ln(u+1) &= (2)(-\ln(X)+2c_2) \\
&= -2\ln(X)+4c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(u-1)+\ln(u+1)} = e^{-2\ln(X)+2c_2}
$$

Which simplifies to

$$
\begin{aligned}
u^2-1 &= \frac{2c_2}{X^2} \\
&= \frac{c_3}{X^2}
\end{aligned}
$$

The solution is

$$
u(X)^2-1 = \frac{c_3}{X^2}
$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$
\frac{Y(X)^2}{X^2}-1 = \frac{c_3}{X^2}
$$

Which simplifies to

$$
-(X-Y(X))(X+Y(X)) = c_3
$$

Using the solution for $Y(X)$

$$
-(X-Y(X))(X+Y(X)) = c_3
$$

And replacing back terms in the above solution using

$$
\begin{aligned}
Y &= y+y_0 \\
X &= x+x_0
\end{aligned}
$$

Or

$$
\begin{aligned}
Y &= y \\
X &= x+1
\end{aligned}
$$

Then the solution in y becomes

$$-(x-1-y)(x-1+y) = c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_3$$

$$c_3 = 0$$

Substituting c_3 found above in the general solution gives

$$-(x-y-1)(x-1+y) = 0$$

Summary
The solution(s) found are the following

$$-(x-1-y)(x-1+y) = 0 \tag{1}$$

Verification of solutions

$$-(x-1-y)(x-1+y) = 0$$

Verified OK.

2.14.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x-1}{y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 46: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{1}{x - 1}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x-1}} dx \end{aligned}$$

Which results in

$$S = \frac{1}{2}x^2 - x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x-1}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x - 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

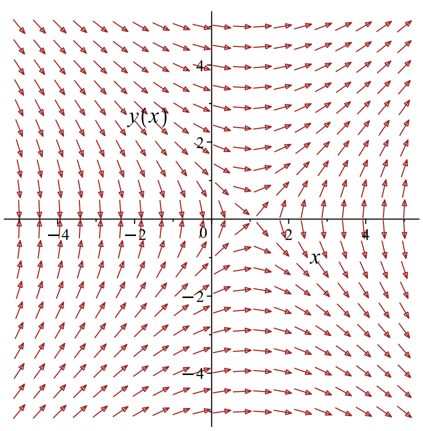
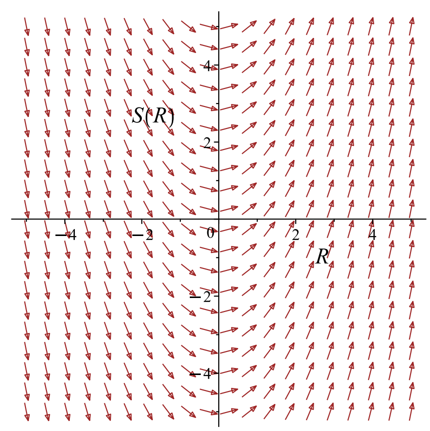
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{2}x^2 - x = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{1}{2}x^2 - x = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x-1}{y}$ 	$R = y$ $S = \frac{1}{2}x^2 - x$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$-\frac{1}{2} = c_1$$

$$c_1 = -\frac{1}{2}$$

Substituting c_1 found above in the general solution gives

$$\frac{1}{2}x^2 - x = \frac{y^2}{2} - \frac{1}{2}$$

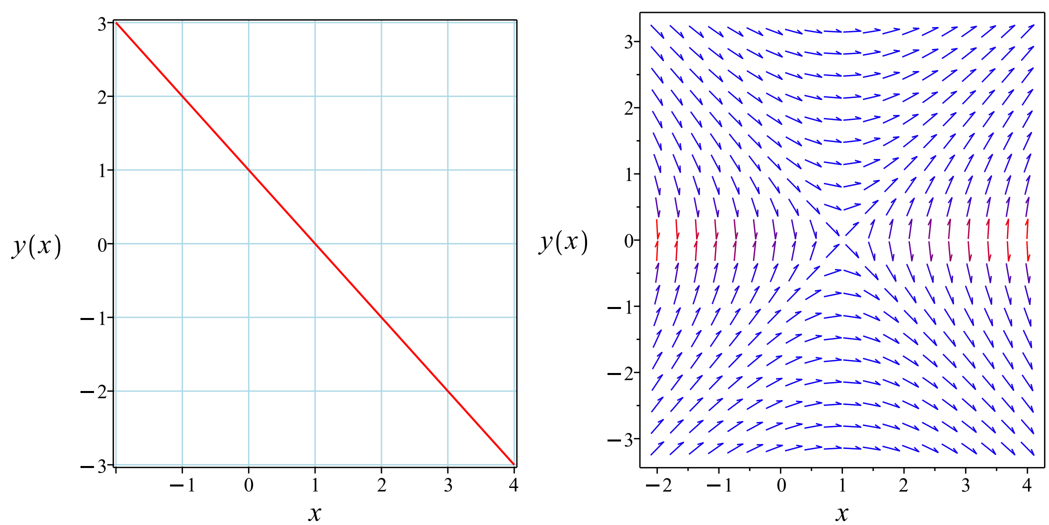
Solving for y from the above gives

$$y = x - 1$$
$$y = 1 - x$$

Summary

The solution(s) found are the following

$$y = x - 1 \tag{1}$$
$$y = 1 - x \tag{2}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x - 1$$

Verified OK. `{positive}`

$$y = 1 - x$$

Verified OK. `{positive}`

2.14.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(y)\,\mathrm{d}y&=(x-1)\,\mathrm{d}x\\ (1-x)\,\mathrm{d}x+(y)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=1-x\\ N(x,y)&=y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(1 - x) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(y) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 1 - x \, \mathrm{d}x \\ \phi &= x - \frac{1}{2}x^2 + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = y$. Therefore equation (4) becomes

$$y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (y)\,\mathrm{d}y$$

$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{2}x^2 + \frac{1}{2}y^2 + x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{1}{2} = c_1$$

$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

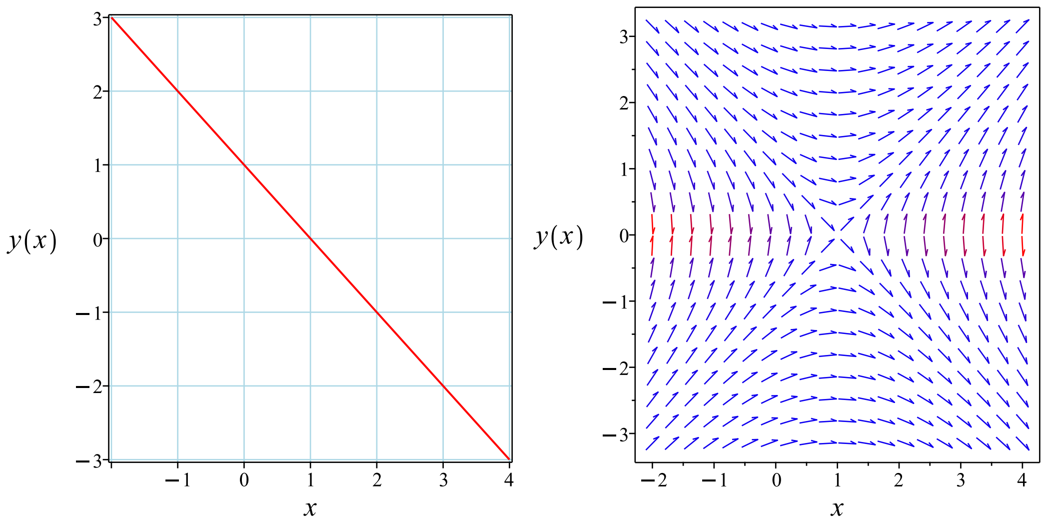
$$-\frac{1}{2}x^2 + \frac{1}{2}y^2 + x = \frac{1}{2}$$

Solving for y from the above gives

$$y = x - 1$$
$$y = 1 - x$$

Summary
The solution(s) found are the following

$$y = x - 1 \tag{1}$$
$$y = 1 - x \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x - 1$$

Verified OK. `{positive}`

$$y = 1 - x$$

Verified OK. `{positive}`

2.14.7 Maple step by step solution

Let's solve

$[yy' = x - 1, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int yy'dx = \int (x - 1)\,dx + c_1$

- Evaluate integral

  $\frac{y^2}{2} = \frac{1}{2}x^2 - x + c_1$
- Solve for y

  $\{y = \sqrt{x^2 + 2c_1 - 2x}, y = -\sqrt{x^2 + 2c_1 - 2x}\}$
- Use initial condition $y(1) = 0$

  $0 = \sqrt{-1 + 2c_1}$
- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = \text{csgn}(x - 1)(x - 1)$
- Use initial condition $y(1) = 0$

  $0 = -\sqrt{-1 + 2c_1}$
- Solve for c_1

  $c_1 = \frac{1}{2}$
- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

  $y = (1 - x)\,\text{csgn}(x - 1)$
- Solutions to the IVP

  $\{y = (1 - x)\,\text{csgn}(x - 1), y = \text{csgn}(x - 1)(x - 1)\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([y(x)*diff(y(x),x) = -1+x,y(1) = 0],y(x), singsol=all)
```

$$y(x) = 1 - x$$
$$y(x) = x - 1$$

✓ Solution by Mathematica

Time used: 0.052 (sec). Leaf size: 29

```
DSolve[{y[x]*y'[x] == -1+x,y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{(x-1)^2}$$
$$y(x) \to \sqrt{(x-1)^2}$$

2.15 problem 19

Internal problem ID [25]
Internal file name [`OUTPUT/25_Sunday_June_05_2022_01_33_51_AM_10109649/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - \ln\left(1 + y^2\right) = 0}$$

With initial conditions

$$[y(0) = 0]$$

2.15.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= \ln\left(y^2 + 1\right) \end{aligned}$$

The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\ln\left(y^2 + 1\right)\right) \\ &= \frac{2y}{y^2 + 1} \end{aligned}$$

The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

2.15.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{\ln(y^2+1)} dy = \int dx$$
$$\int^y \frac{1}{\ln(\_a^2+1)} d\_a = x + c_1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$\int^0 \frac{1}{\ln(\_a^2+1)} d\_a = c_1$$

$$c_1 = \int^0 \frac{1}{\ln(\_a^2+1)} d\_a$$

Substituting c_1 found above in the general solution gives

$$\int^y \frac{1}{\ln(\_a^2+1)} d\_a = x + \int^0 \frac{1}{\ln(\_a^2+1)} d\_a$$

Solving for y from the above gives

$$y = \text{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\ln(\_a^2+1)} d\_a\right) + x + \int^0 \frac{1}{\ln(\_a^2+1)} d\_a\right)$$

Summary
The solution(s) found are the following

$$y = \text{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\ln(\_a^2+1)} d\_a\right) + x + \int^0 \frac{1}{\ln(\_a^2+1)} d\_a\right) \tag{1}$$

Verification of solutions

$$y = \text{RootOf}\left(-\left(\int^{\_Z} \frac{1}{\ln(\_a^2+1)} d\_a\right) + x + \int^0 \frac{1}{\ln(\_a^2+1)} d\_a\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([diff(y(x),x) = ln(1+y(x)^2),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 6

```
DSolve[{y'[x] == Log[1+y[x]^2],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

2.16 problem 20

Internal problem ID [26]
Internal file name [`OUTPUT/26_Sunday_June_05_2022_01_33_51_AM_88102550/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.3. Slope fields and solution curves. Page 26
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati"**

Maple gives the following as the ode type

`[_Riccati]`

$$\boxed{y' + y^2 = x^2}$$

2.16.1 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x^2 - y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = 0$ and $f_2(x) = -1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-u''(x) + x^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \left(\text{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \text{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1\right) \sqrt{x}$$

The above shows that

$$u'(x) = \left(-\text{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \text{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x^{\frac{3}{2}}$$

Using the above in (1) gives the solution

$$y = \frac{\left(-\text{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) c_2 + \text{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_1\right) x}{\text{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_2 + \text{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\left(-\text{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \text{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{\text{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \text{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(-\text{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \text{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{\text{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \text{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_3} \tag{1}$$

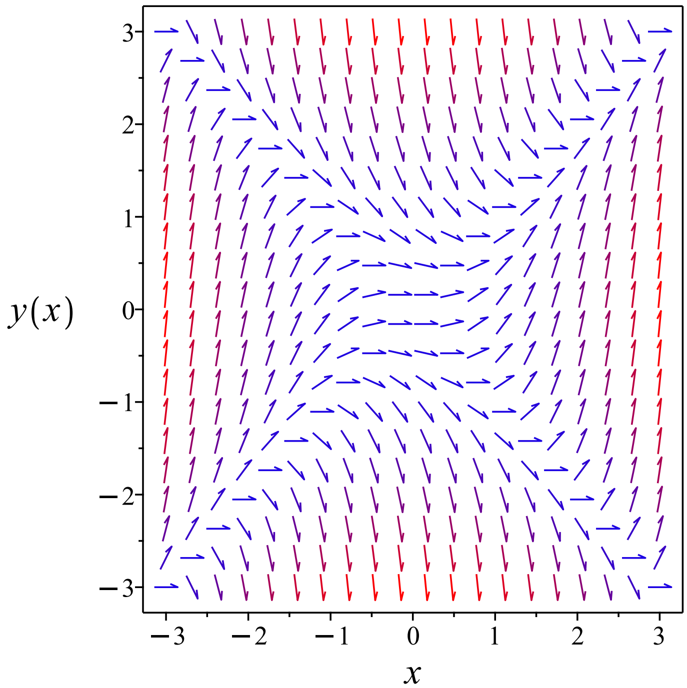


Figure 53: Slope field plot

Verification of solutions

$$y = \frac{\left(-\operatorname{BesselK}\left(\frac{3}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(-\frac{3}{4}, \frac{x^2}{2}\right) c_3\right) x}{\operatorname{BesselK}\left(\frac{1}{4}, \frac{x^2}{2}\right) + \operatorname{BesselI}\left(\frac{1}{4}, \frac{x^2}{2}\right) c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying Riccati
trying Riccati Special
<- Riccati Special successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(diff(y(x),x) = x^2-y(x)^2,y(x), singsol=all)
```

$$y(x)=\frac{x\left(\text{BesselI}\left(-\frac{3}{4},\frac{x^2}{2}\right)c_1-\text{BesselK}\left(\frac{3}{4},\frac{x^2}{2}\right)\right)}{c_1\,\text{BesselI}\left(\frac{1}{4},\frac{x^2}{2}\right)+\text{BesselK}\left(\frac{1}{4},\frac{x^2}{2}\right)}$$

✓ Solution by Mathematica

Time used: 0.109 (sec). Leaf size: 197

```
DSolve[y'[x]== x^2-y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{-ix^2\left(2\,\text{BesselJ}\left(-\frac{3}{4},\frac{ix^2}{2}\right)+c_1\left(\text{BesselJ}\left(-\frac{5}{4},\frac{ix^2}{2}\right)-\text{BesselJ}\left(\frac{3}{4},\frac{ix^2}{2}\right)\right)\right)-c_1\,\text{BesselJ}\left(-\frac{1}{4},\frac{ix^2}{2}\right)}{2x\left(\text{BesselJ}\left(\frac{1}{4},\frac{ix^2}{2}\right)+c_1\,\text{BesselJ}\left(-\frac{1}{4},\frac{ix^2}{2}\right)\right)}$$

$$y(x)\to\frac{ix^2\,\text{BesselJ}\left(-\frac{5}{4},\frac{ix^2}{2}\right)-ix^2\,\text{BesselJ}\left(\frac{3}{4},\frac{ix^2}{2}\right)+\text{BesselJ}\left(-\frac{1}{4},\frac{ix^2}{2}\right)}{2x\,\text{BesselJ}\left(-\frac{1}{4},\frac{ix^2}{2}\right)}$$

3 Section 1.4. Separable equations. Page 43

3.1 problem 1

Internal problem ID [27]
Internal file name [`OUTPUT/27_Sunday_June_05_2022_01_33_52_AM_84822014/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2yx + y' = 0}$$

3.1.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -2yx \end{aligned}$$

Where $f(x) = -2x$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= -2x\,dx \\
\int \frac{1}{y}\,dy &= \int -2x\,dx \\
\ln(y) &= -x^2 + c_1 \\
y &= e^{-x^2+c_1} \\
&= c_1 e^{-x^2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-x^2} \tag{1}$$

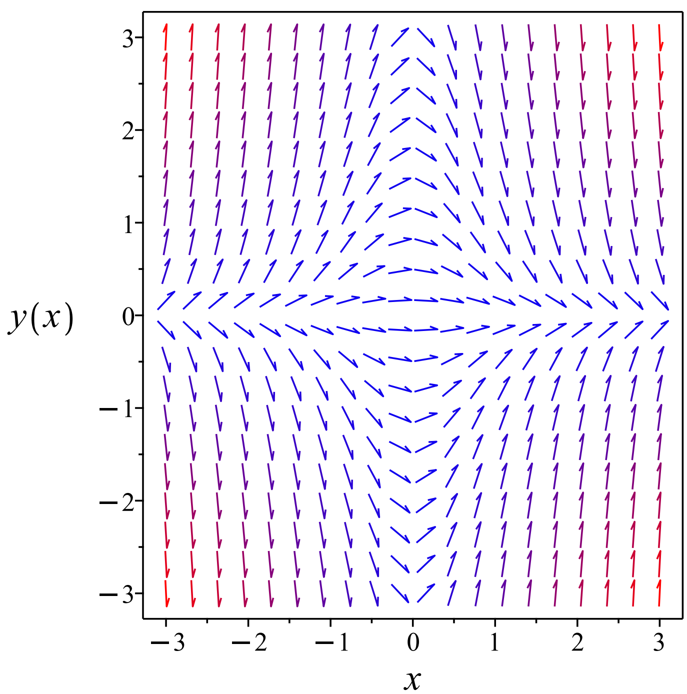


Figure 54: Slope field plot

Verification of solutions

$$y = c_1 e^{-x^2}$$

Verified OK.

3.1.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 2x \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$2yx + y' = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 2x dx} \\ &= \mathrm{e}^{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x^2} y\right) &= 0 \end{aligned}$$

Integrating gives

$$\mathrm{e}^{x^2} y = c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{x^2}$ results in

$$y = c_1 \mathrm{e}^{-x^2}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x^2} \tag{1}$$

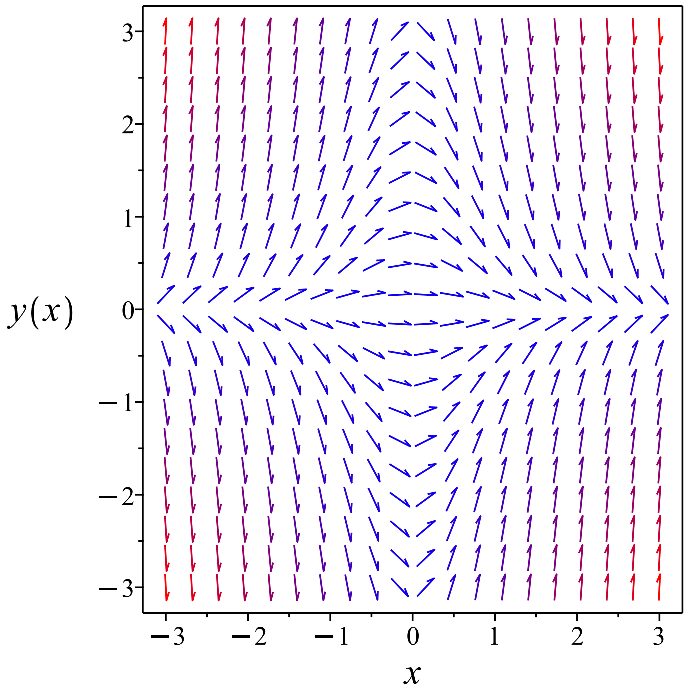


Figure 55: Slope field plot

Verification of solutions

$$y = c_1 e^{-x^2}$$

Verified OK.

3.1.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x^2 + u'(x)\,x + u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(2x^2+1)}{x} \end{aligned}$$

Where $f(x) = -\frac{2x^2+1}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{2x^2+1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2x^2+1}{x}\,dx \\
\ln(u) &= -x^2 - \ln(x) + c_2 \\
u &= e^{-x^2-\ln(x)+c_2} \\
&= c_2 e^{-x^2-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 e^{-x^2}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= c_2 e^{-x^2}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_2 e^{-x^2} \tag{1}
$$

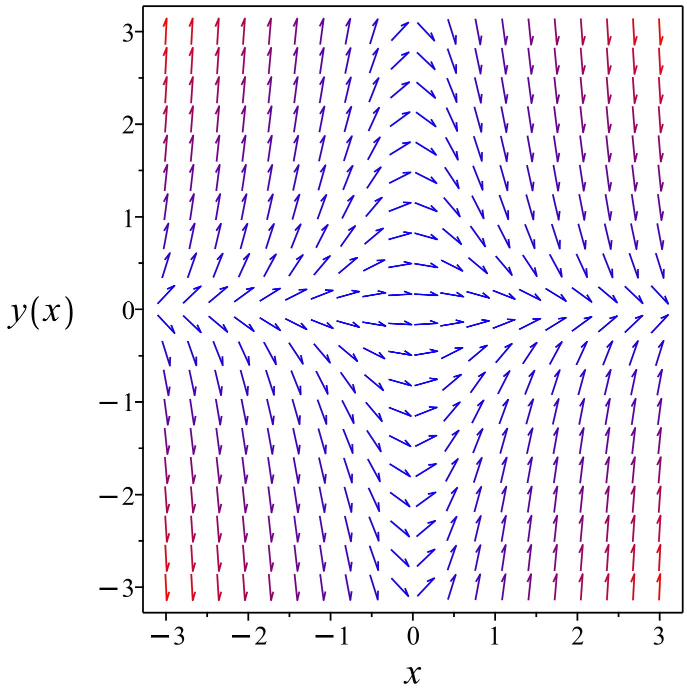


Figure 56: Slope field plot

Verification of solutions

$$y = c_2 \mathrm{e}^{-x^2}$$

Verified OK.

3.1.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2yx$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 49: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -2yx$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\, e^{x^2} y \\ S_y &= e^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = c_1$$

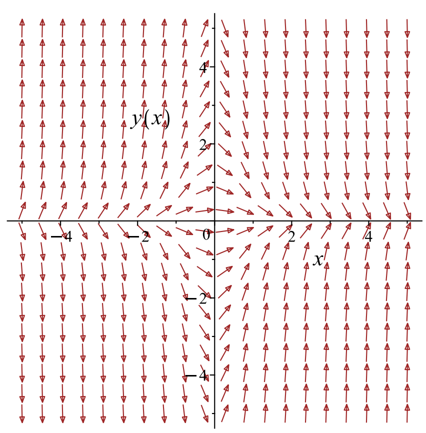
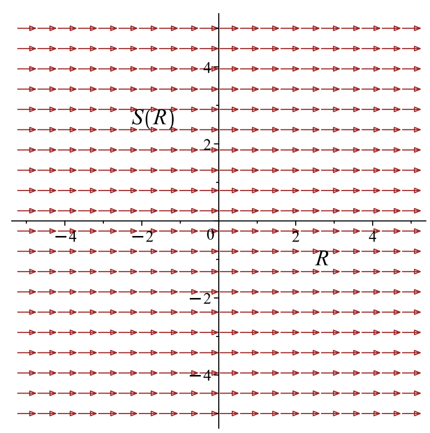
Which simplifies to

$$e^{x^2} y = c_1$$

Which gives

$$y = c_1 e^{-x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2yx$ 	$R = x$ $S = e^{x^2} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{-x^2} \tag{1}$$

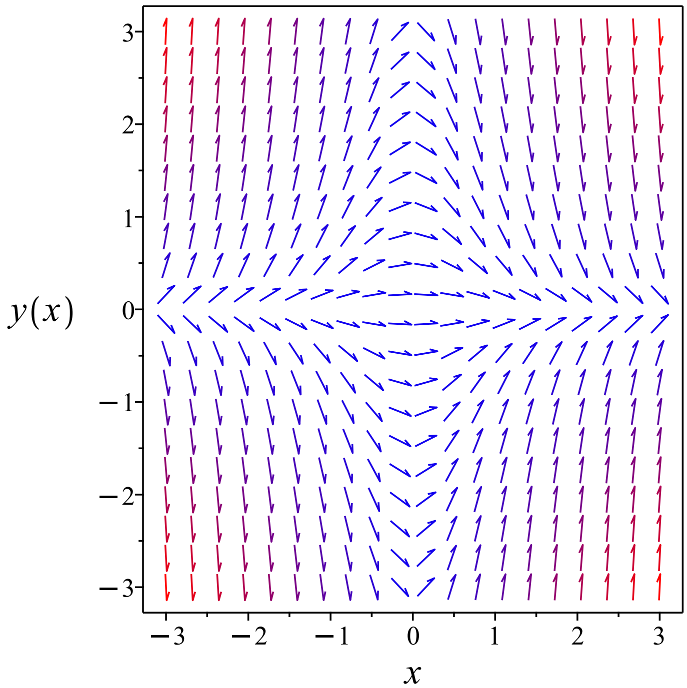


Figure 57: Slope field plot

Verification of solutions

$$y = c_1 e^{-x^2}$$

Verified OK.

3.1.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(-\frac{1}{2y}\right) \mathrm{d}y &= (x)\,\mathrm{d}x \\
(-x)\,\mathrm{d}x + \left(-\frac{1}{2y}\right) \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= -\frac{1}{2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$
-\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(-\frac{1}{2y}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y)}{2}$$

The solution becomes

$$y = \mathrm{e}^{-x^2-2c_1}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-x^2-2c_1} \tag{1}$$

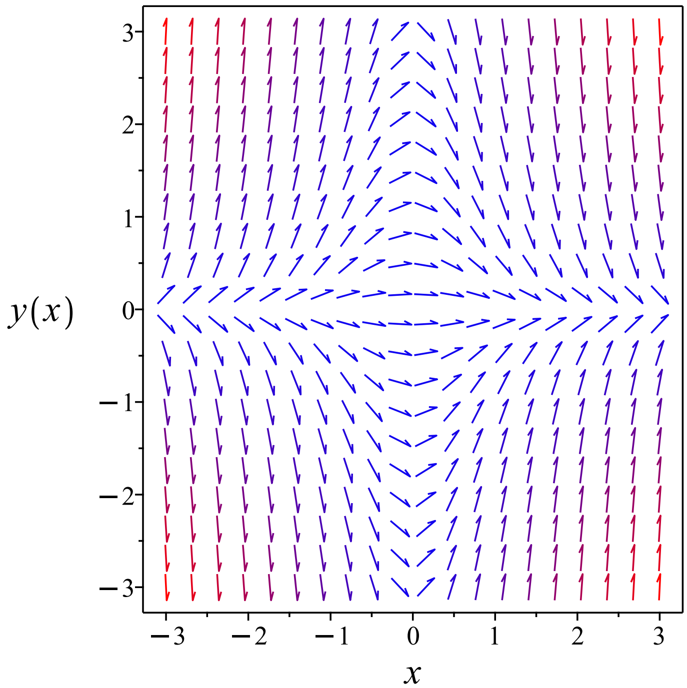


Figure 58: Slope field plot

Verification of solutions

$$y = e^{-x^2-2c_1}$$

Verified OK.

3.1.6 Maple step by step solution

Let's solve

$2yx + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int -2x dx + c_1$

- Evaluate integral

$$\ln(y) = -x^2 + c_1$$

- Solve for y

$$y = e^{-x^2+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(2*x*y(x)+diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = c_1 e^{-x^2}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 20

```
DSolve[2*x*y[x]+y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x^2}$$
$$y(x) \to 0$$

3.2 problem 2

Internal problem ID [28]
Internal file name [`OUTPUT/28_Sunday_June_05_2022_01_33_52_AM_4751393/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2xy^2 + y' = 0}$$

3.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -2x\,y^2 \end{aligned}$$

Where $f(x) = -2x$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = -2x\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -2x\,dx$$

$$-\frac{1}{y} = -x^2 + c_1$$

Which results in

$$y = -\frac{1}{-x^2 + c_1}$$

Summary
The solution(s) found are the following

$$y = -\frac{1}{-x^2 + c_1} \tag{1}$$

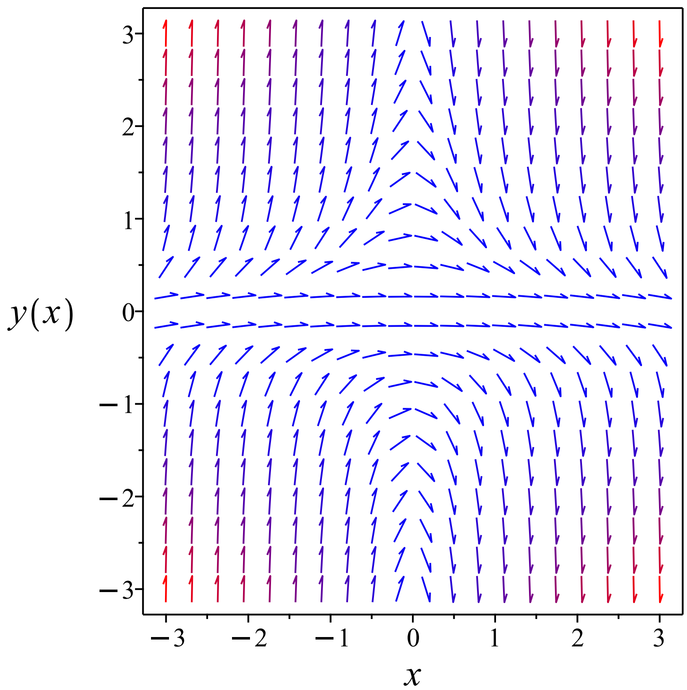


Figure 59: Slope field plot

Verification of solutions

$$y = -\frac{1}{-x^2 + c_1}$$

Verified OK.

3.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -2x\,y^2 \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 52: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = -\frac{1}{2x}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{-\frac{1}{2x}} dx$$

Which results in

$$S = -x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2x\, y^2$$

Evaluating all the partial derivatives gives

$$R_x = 0$$
$$R_y = 1$$
$$S_x = -2x$$
$$S_y = 0$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-x^2 = -\frac{1}{y} + c_1$$

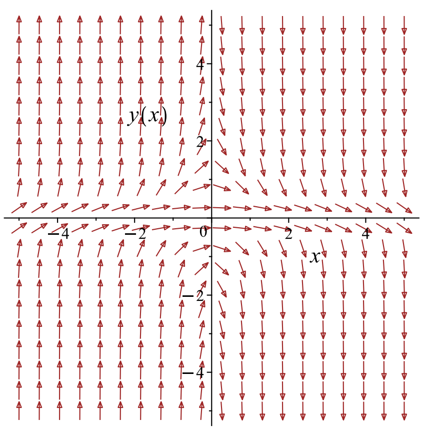
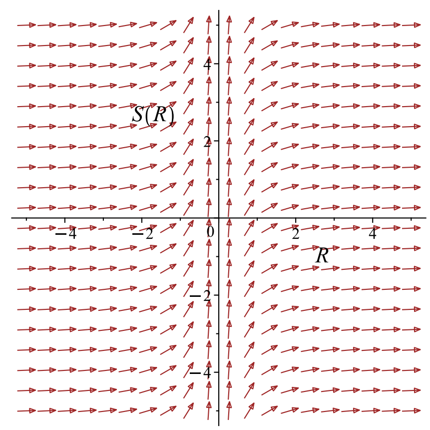
Which simplifies to

$$-x^2 = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{1}{x^2 + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2x\, y^2$ 	$R = y$ $S = -x^2$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 + c_1} \tag{1}$$

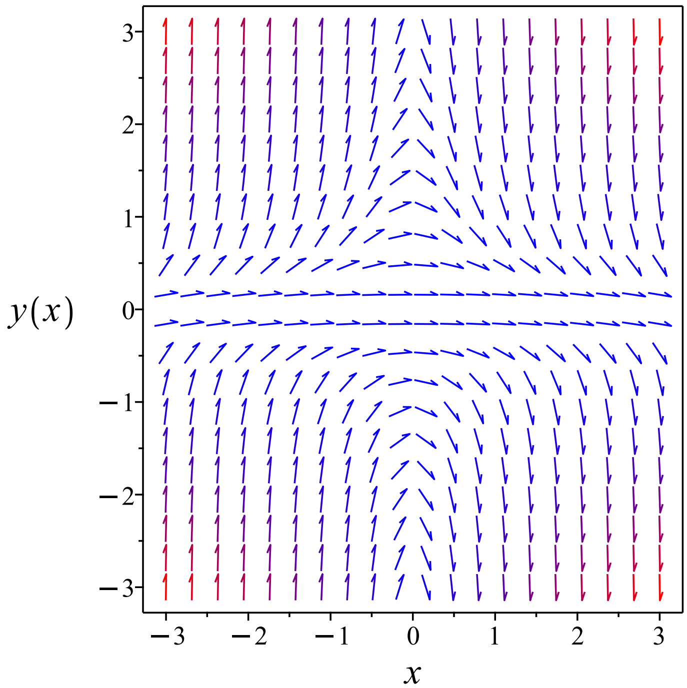


Figure 60: Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 + c_1}$$

Verified OK.

3.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{2y^2}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(-\frac{1}{2y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = -\frac{1}{2y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y^2}$. Therefore equation (4) becomes

$$
-\frac{1}{2y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{2y^2}\right)\mathrm{d}y$$
$$f(y) = \frac{1}{2y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{1}{2y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{1}{2y}$$

The solution becomes

$$y = \frac{1}{x^2 + 2c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 + 2c_1} \tag{1}$$

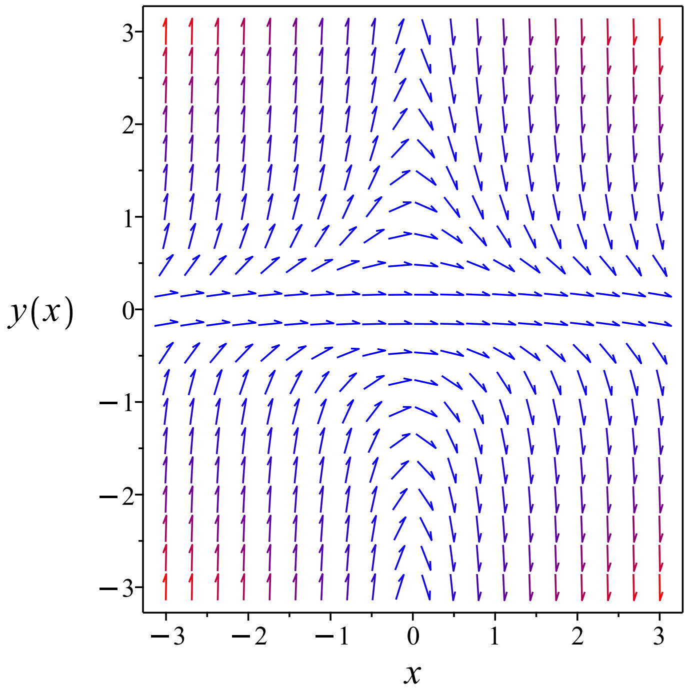


Figure 61: Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 + 2c_1}$$

Verified OK.

3.2.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -2x\, y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -2x\, y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -2x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-2xu} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -2 \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-2xu''(x) + 2u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 x^2 + c_1$$

The above shows that

$$u'(x) = 2c_2 x$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{c_2 x^2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{1}{x^2 + c_3}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{x^2 + c_3} \tag{1}$$

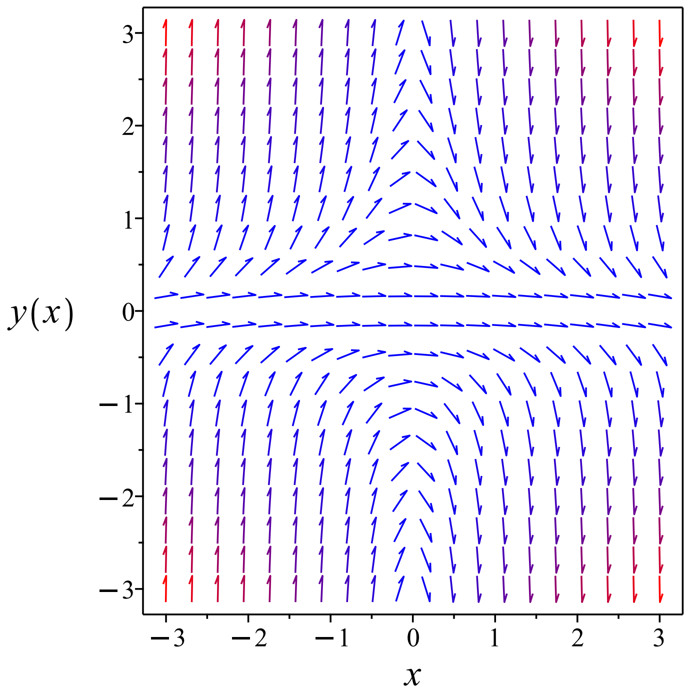


Figure 62: Slope field plot

Verification of solutions

$$y = \frac{1}{x^2 + c_3}$$

Verified OK.

3.2.5 Maple step by step solution

Let's solve

$2xy^2 + y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int -2xdx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -x^2 + c_1$

- Solve for y

  $y = -\frac{1}{-x^2+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(2*x*y(x)^2+diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{1}{x^2 + c_1}$$

✓ Solution by Mathematica

Time used: 0.098 (sec). Leaf size: 20

```
DSolve[2*x*y[x]^2+y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow \frac{1}{x^2 - c_1}$$
$$y(x) \rightarrow 0$$

3.3 problem 3

Internal problem ID [29]
Internal file name [`OUTPUT/29_Sunday_June_05_2022_01_33_52_AM_46698786/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \sin(x)\, y = 0}$$

3.3.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \sin(x)\, y \end{aligned}$$

Where $f(x) = \sin(x)$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= \sin(x)\, dx \\
\int \frac{1}{y}\, dy &= \int \sin(x)\, dx \\
\ln(y) &= -\cos(x) + c_1 \\
y &= e^{-\cos(x)+c_1} \\
&= c_1 e^{-\cos(x)}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-\cos(x)} \tag{1}
$$

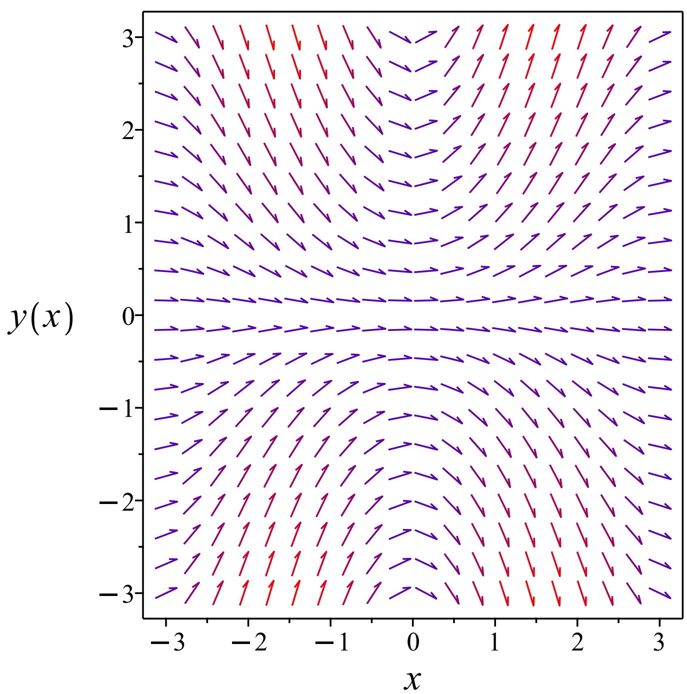


Figure 63: Slope field plot

Verification of solutions

$$
y = c_1 e^{-\cos(x)}
$$

Verified OK.

3.3.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\sin(x) \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \sin(x)\, y = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\sin(x)dx} \\ &= e^{\cos(x)} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{\cos(x)}y\right) &= 0 \end{aligned}$$

Integrating gives

$$e^{\cos(x)}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\cos(x)}$ results in

$$y = c_1 e^{-\cos(x)}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\cos(x)} \tag{1}$$

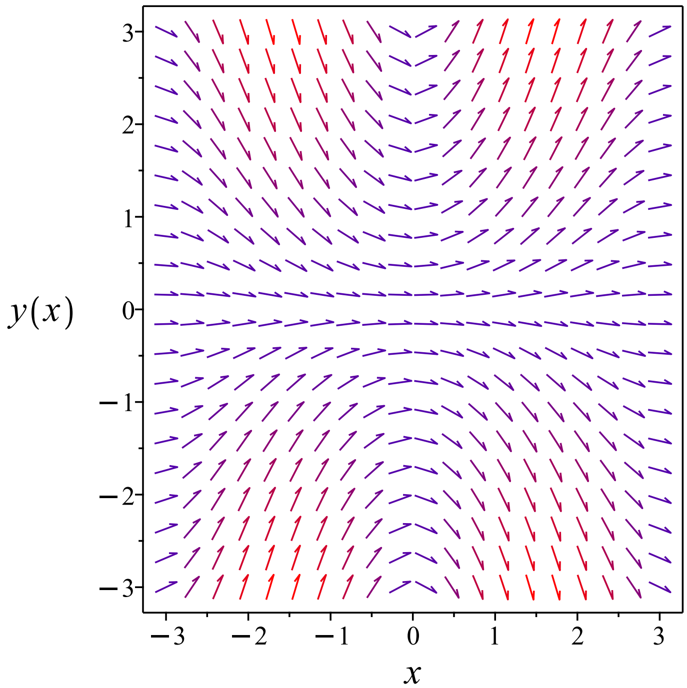


Figure 64: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\cos(x)}$$

Verified OK.

3.3.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \sin(x)\,u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= \frac{u(-1+\sin(x)\,x)}{x}
\end{aligned}$$

Where $f(x) = \frac{-1+\sin(x)x}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{-1+\sin(x)\,x}{x}\,dx\\ \int \frac{1}{u}\,du &= \int \frac{-1+\sin(x)\,x}{x}\,dx\\ \ln(u) &= -\ln(x) - \cos(x) + c_2\\ u &= \mathrm{e}^{-\ln(x)-\cos(x)+c_2}\\ &= c_2\mathrm{e}^{-\ln(x)-\cos(x)}\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2\mathrm{e}^{-\cos(x)}}{x}$$

Therefore the solution y is

$$\begin{aligned}y &= xu\\ &= c_2\mathrm{e}^{-\cos(x)}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_2\mathrm{e}^{-\cos(x)} \tag{1}$$

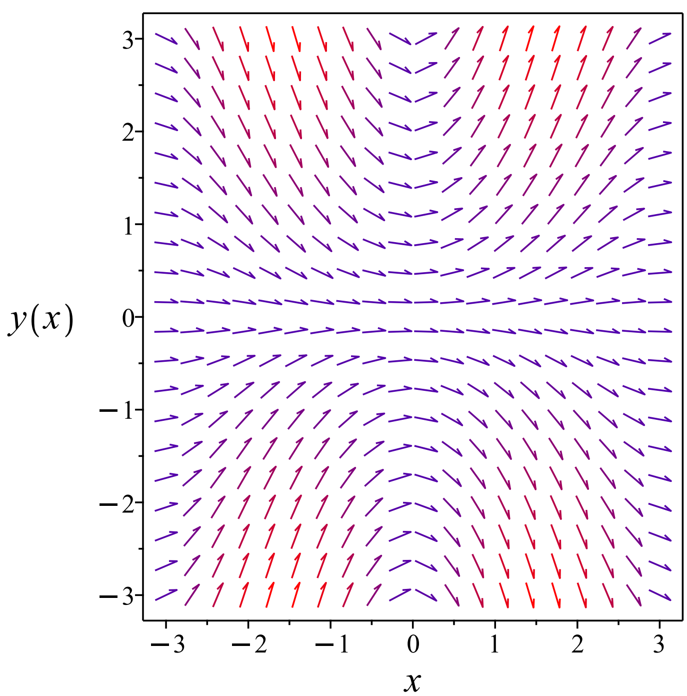


Figure 65: Slope field plot

Verification of solutions

$$y = c_2 \mathrm{e}^{-\cos(x)}$$

Verified OK.

3.3.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \sin(x) y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 55: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-\cos(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\cos(x)}} dy \end{aligned}$$

Which results in

$$S = e^{\cos(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sin(x) y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\sin(x) e^{\cos(x)} y \\ S_y &= e^{\cos(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\cos(x)} y = c_1$$

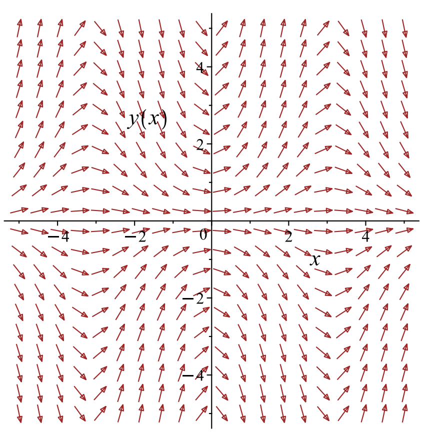
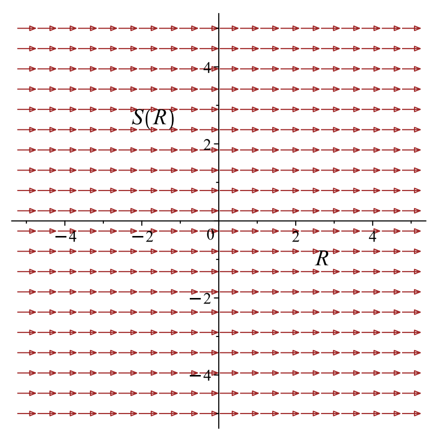
Which simplifies to

$$e^{\cos(x)} y = c_1$$

Which gives

$$y = c_1 e^{-\cos(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sin(x) y$ 	$R = x$ $S = e^{\cos(x)} y$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = c_1 e^{-\cos(x)} \tag{1}$$

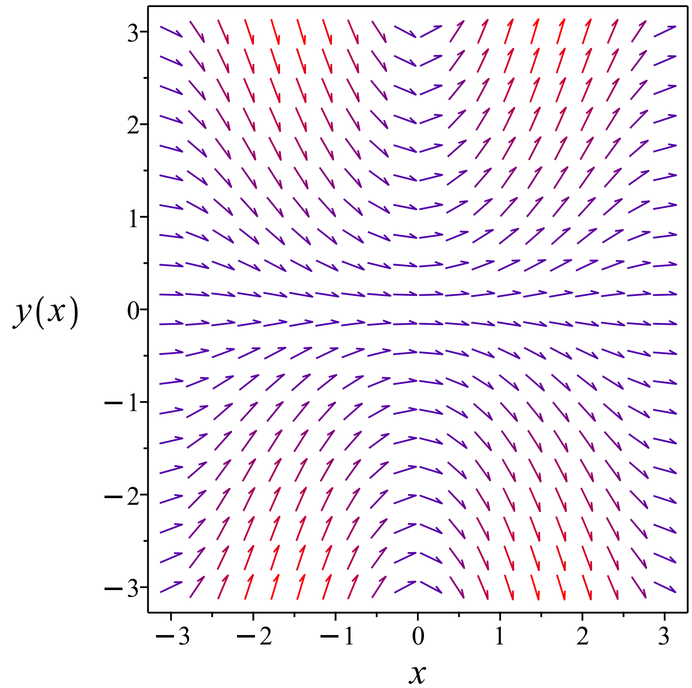


Figure 66: Slope field plot

Verification of solutions

$$y = c_1 e^{-\cos(x)}$$

Verified OK.

3.3.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y &= (\sin(x))\,\mathrm{d}x\\ (-\sin(x))\,\mathrm{d}x + \left(\frac{1}{y}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\sin(x)\\ N(x, y) &= \frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\sin(x))\\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\ = 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\sin(x) \mathrm{d}x$$

$$\phi = \cos(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$

$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \cos(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \cos(x) + \ln(y)$$

The solution becomes

$$y = e^{-\cos(x)+c_1}$$

Summary

The solution(s) found are the following

$$y = e^{-\cos(x)+c_1} \tag{1}$$

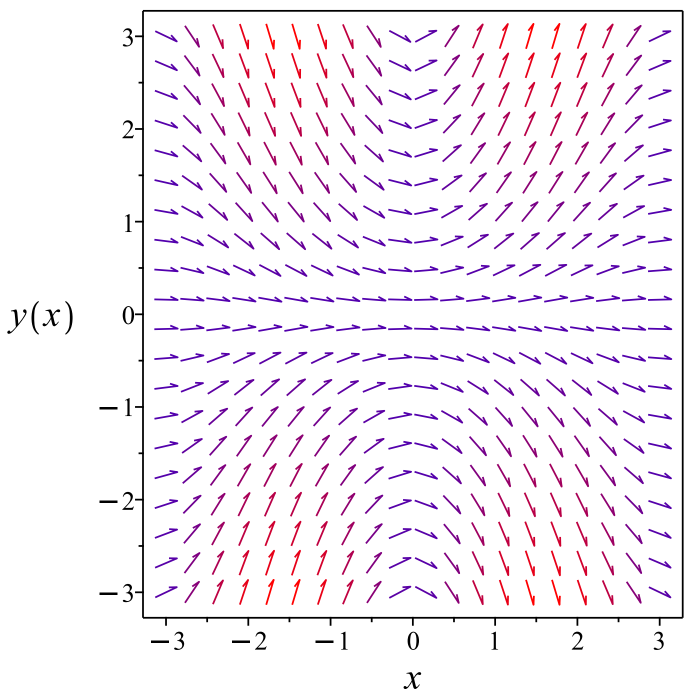


Figure 67: Slope field plot

Verification of solutions

$$y = e^{-\cos(x)+c_1}$$

Verified OK.

3.3.6 Maple step by step solution

Let's solve

$y' - \sin(x)\, y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \sin(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \sin(x)\, dx + c_1$

- Evaluate integral

  $\ln(y) = -\cos(x) + c_1$

- Solve for y

  $y = \mathrm{e}^{-\cos(x)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve(diff(y(x),x) = sin(x)*y(x),y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\cos(x)}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 19

```
DSolve[y'[x] == Sin[x]*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-\cos(x)}$$
$$y(x) \to 0$$

3.4 problem 4

Internal problem ID [30]
Internal file name [`OUTPUT/30_Sunday_June_05_2022_01_33_53_AM_72715024/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "homogeneousTypeMapleC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{(x+1)\,y' - 4y = 0}$$

3.4.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{4y}{x+1} \end{aligned}$$

Where $f(x) = \frac{4}{x+1}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\,dy &= \frac{4}{x+1}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{4}{x+1}\,dx \\
\ln(y) &= 4\ln(x+1) + c_1 \\
y &= e^{4\ln(x+1)+c_1} \\
&= c_1(x+1)^4
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1(x+1)^4 \tag{1}
$$

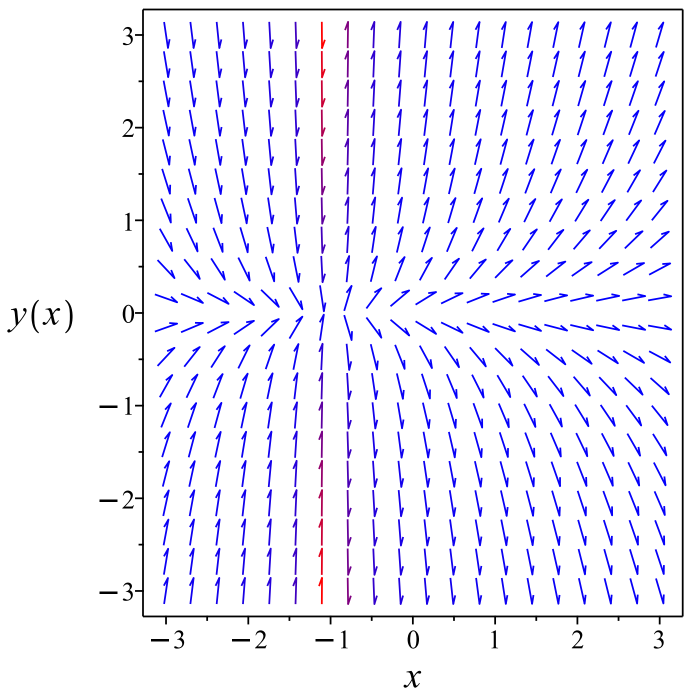


Figure 68: Slope field plot

Verification of solutions

$$
y = c_1(x+1)^4
$$

Verified OK.

3.4.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{4}{x+1} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - \frac{4y}{x+1} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{4}{x+1}dx} \\ &= \frac{1}{(x+1)^4} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{(x+1)^4}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{y}{(x+1)^4} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{(x+1)^4}$ results in

$$y = c_1(x+1)^4$$

Summary
The solution(s) found are the following

$$y = c_1(x+1)^4 \tag{1}$$

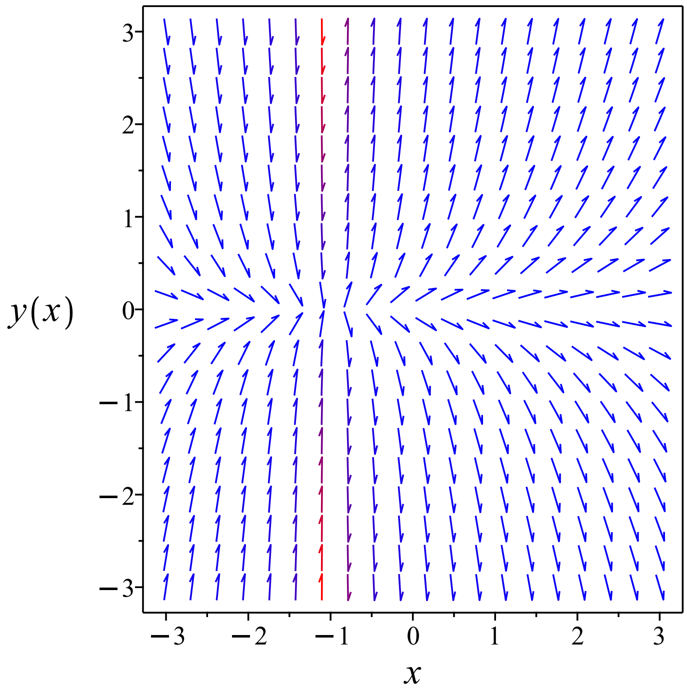


Figure 69: Slope field plot

Verification of solutions

$$y = c_1(x+1)^4$$

Verified OK.

3.4.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x+1)\,(u'(x)\,x + u(x)) - 4u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{u(3x-1)}{(x+1)\,x}
\end{aligned}$$

Where $f(x) = \frac{3x-1}{x(x+1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{3x-1}{x\,(x+1)}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{3x-1}{x\,(x+1)}\,dx \\
\ln (u) &= -\ln (x) + 4\ln (x+1) + c_2 \\
u &= \mathrm{e}^{-\ln(x)+4\ln(x+1)+c_2} \\
&= c_2 \mathrm{e}^{-\ln(x)+4\ln(x+1)}
\end{aligned}$$

Which simplifies to

$$u(x) = c_2 \left(x^3 + 4x^2 + 6x + 4 + \frac{1}{x}\right)$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= x c_2 \left(x^3 + 4x^2 + 6x + 4 + \frac{1}{x}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x c_2 \left(x^3 + 4x^2 + 6x + 4 + \frac{1}{x}\right) \tag{1}$$

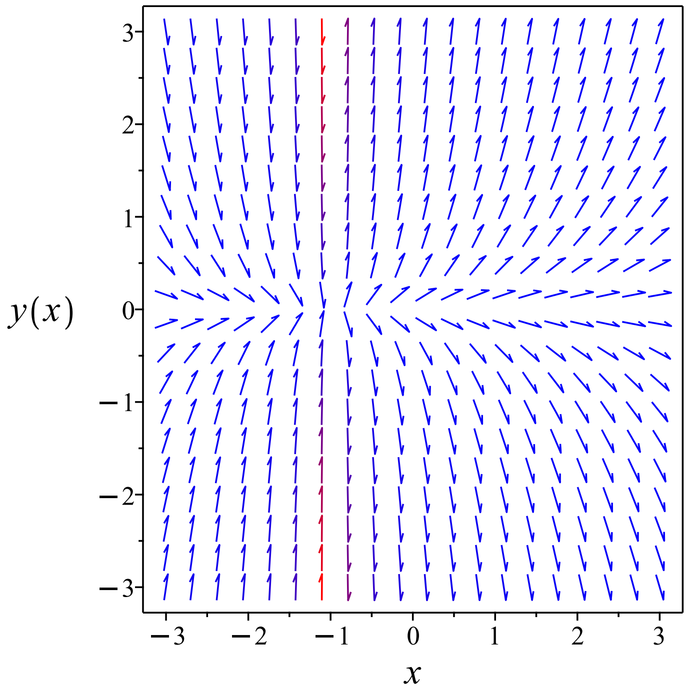


Figure 70: Slope field plot

Verification of solutions

$$y = x c_2 \left(x^3 + 4x^2 + 6x + 4 + \frac{1}{x}\right)$$

Verified OK.

3.4.4 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX} Y(X) = \frac{4Y(X) + 4y_0}{X + x_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX} Y(X) = \frac{4Y(X)}{X}$$

In canonical form, the ODE is

$$
\begin{aligned}
Y' &= F(X, Y) \\
&= \frac{4Y}{X}
\end{aligned}
\tag{1}
$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X, Y)$ and $N(X, Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X, Y)$ is homogeneous of order n if

$$
f(t^n X, t^n Y) = t^n f(X, Y)
$$

In this case, it can be seen that both $M = 4Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$
\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u
$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$
\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}X}X + u &= 4u \\
\frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{3u(X)}{X}
\end{aligned}
$$

Or

$$
\frac{d}{dX}u(X) - \frac{3u(X)}{X} = 0
$$

Or

$$
\left(\frac{d}{dX}u(X)\right) X - 3u(X) = 0
$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(X, u) \\
&= f(X)g(u) \\
&= \frac{3u}{X}
\end{aligned}
$$

Where $f(X) = \frac{3}{X}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{3}{X}\,dX \\
\int \frac{1}{u}\,du &= \int \frac{3}{X}\,dX \\
\ln(u) &= 3\ln(X) + c_2 \\
u &= e^{3\ln(X)+c_2} \\
&= c_2 X^3
\end{aligned}$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = X^4 c_2$$

Using the solution for $Y(X)$

$$Y(X) = X^4 c_2$$

And replacing back terms in the above solution using

$$\begin{aligned}
Y &= y + y_0 \\
X &= x + x_0
\end{aligned}$$

Or

$$\begin{aligned}
Y &= y \\
X &= x - 1
\end{aligned}$$

Then the solution in y becomes

$$y = c_2(x+1)^4$$

Summary

The solution(s) found are the following

$$y = c_2(x+1)^4 \tag{1}$$

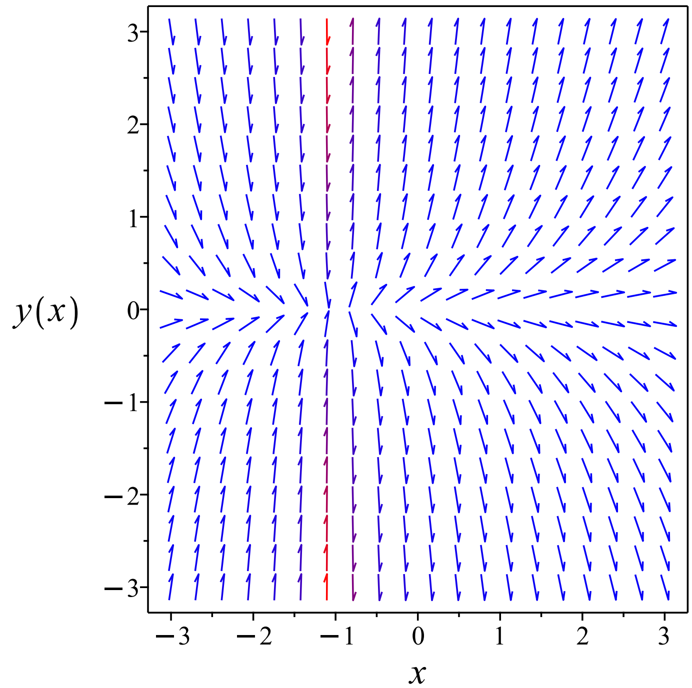


Figure 71: Slope field plot

<u>Verification of solutions</u>

$$y = c_2(x+1)^4$$

Verified OK.

3.4.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{4y}{x+1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 58: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= (x + 1)^4
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{(x+1)^4} dy \end{aligned}$$

Which results in

$$S = \frac{y}{(x+1)^4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{4y}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{4y}{(x+1)^5} \\ S_y &= \frac{1}{(x+1)^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{(x+1)^4} = c_1$$

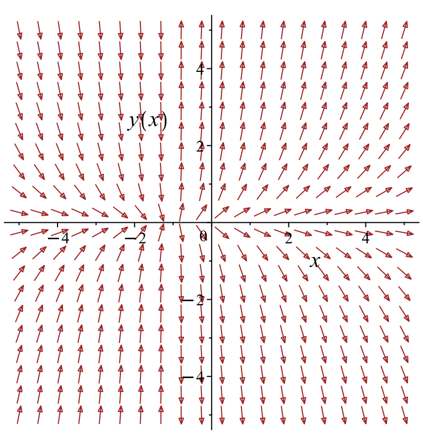
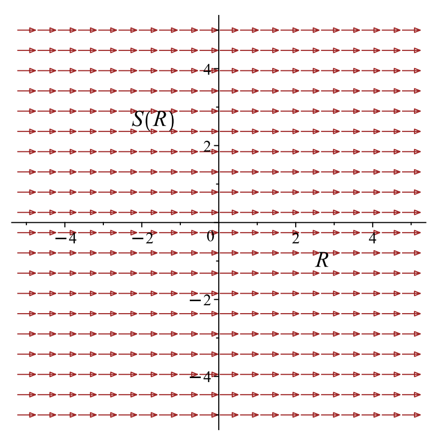
Which simplifies to

$$\frac{y}{(x+1)^4} = c_1$$

Which gives

$$y = c_1 (x+1)^4$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{4y}{x+1}$ 	$R = x$ $S = \frac{y}{(x+1)^4}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = c_1 (x+1)^4 \tag{1}$$

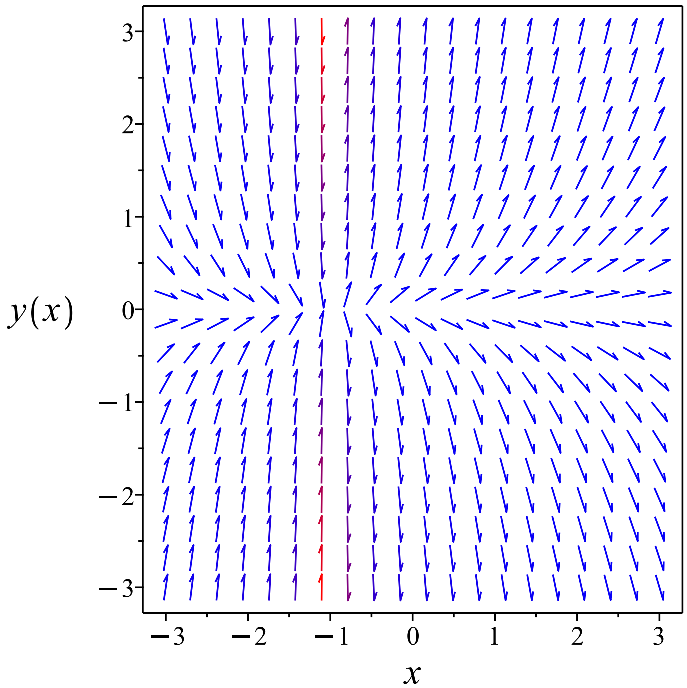


Figure 72: Slope field plot

Verification of solutions

$$y = c_1(x+1)^4$$

Verified OK.

3.4.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{4y}\right) \mathrm{d}y &= \left(\frac{1}{x+1}\right) \mathrm{d}x \\ \left(-\frac{1}{x+1}\right) \mathrm{d}x + \left(\frac{1}{4y}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x+1} \\ N(x, y) &= \frac{1}{4y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x+1}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{4y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{1}{x+1} \, \mathrm{d}x$$
$$\phi = -\ln(x+1) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{4y}$. Therefore equation (4) becomes

$$\frac{1}{4y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{4y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{4y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(y)}{4} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x+1) + \frac{\ln(y)}{4} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x+1) + \frac{\ln(y)}{4}$$

The solution becomes

$$y = e^{4c_1}(x+1)^4$$

Summary

The solution(s) found are the following

$$y = e^{4c_1}(x+1)^4 \tag{1}$$

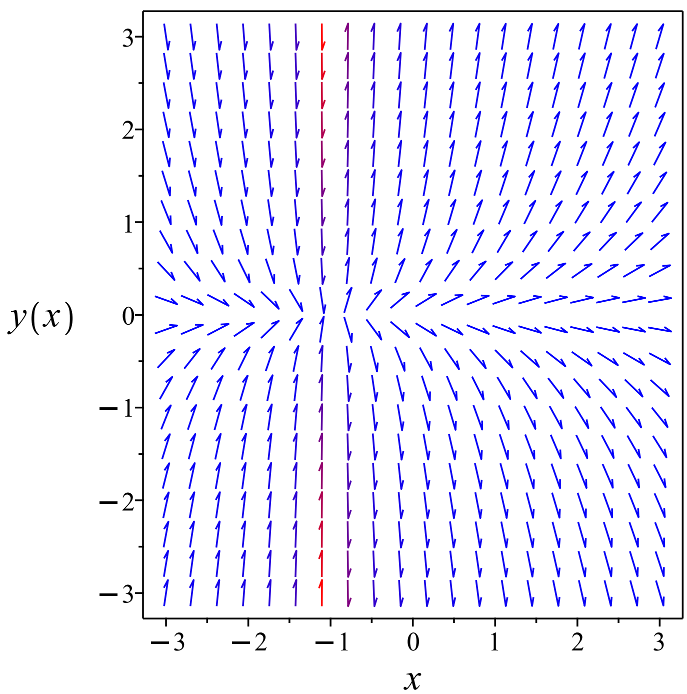


Figure 73: Slope field plot

Verification of solutions

$$y = e^{4c_1}(x+1)^4$$

Verified OK.

3.4.7 Maple step by step solution

Let's solve

$(x+1)\,y' - 4y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{4}{x+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{4}{x+1}dx + c_1$

- Evaluate integral

  $\ln(y) = 4\ln(x+1) + c_1$

- Solve for y

  $y = e^{c_1}(x+1)^4$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((1+x)*diff(y(x),x) = 4*y(x),y(x), singsol=all)
```

$$y(x) = c_1(x+1)^4$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 18

```
DSolve[(1+x)*y'[x] == 4*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1(x+1)^4$$
$$y(x) \to 0$$

3.5 problem 5

Internal problem ID [31]
Internal file name [`OUTPUT/31_Sunday_June_05_2022_01_33_53_AM_21188209/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2\sqrt{x}\, y' - \sqrt{1-y^2} = 0}$$

3.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{\sqrt{-y^2+1}}{2\sqrt{x}} \end{aligned}$$

Where $f(x) = \frac{1}{2\sqrt{x}}$ and $g(y) = \sqrt{-y^2+1}$. Integrating both sides gives

$$\begin{aligned} \frac{1}{\sqrt{-y^2+1}}\, dy &= \frac{1}{2\sqrt{x}}\, dx \\ \int \frac{1}{\sqrt{-y^2+1}}\, dy &= \int \frac{1}{2\sqrt{x}}\, dx \\ \arcsin(y) &= \sqrt{x} + c_1 \end{aligned}$$

Which results in

$$y = \sin\left(\sqrt{x} + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \sin\left(\sqrt{x} + c_1\right) \tag{1}$$

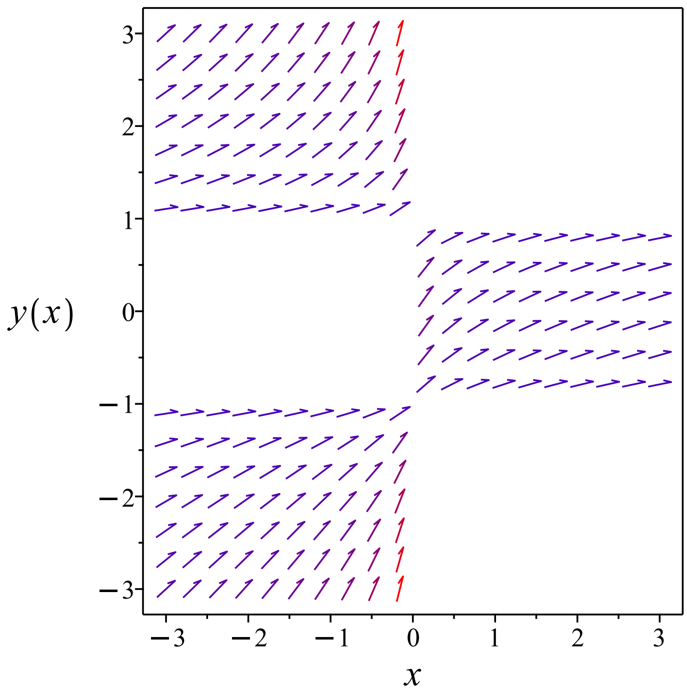


Figure 74: Slope field plot

Verification of solutions

$$y = \sin\left(\sqrt{x} + c_1\right)$$

Verified OK.

3.5.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\sqrt{-y^2+1}}{2\sqrt{x}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 61: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 2\sqrt{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{2\sqrt{x}} dx
\end{aligned}
$$

Which results in

$$
S = \sqrt{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\sqrt{-y^2+1}}{2\sqrt{x}}
$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{2\sqrt{x}} \\
S_y &= 0
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{-y^2+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arcsin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x} = \arcsin(y) + c_1$$

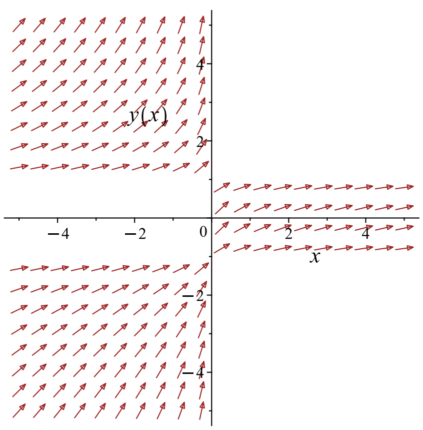
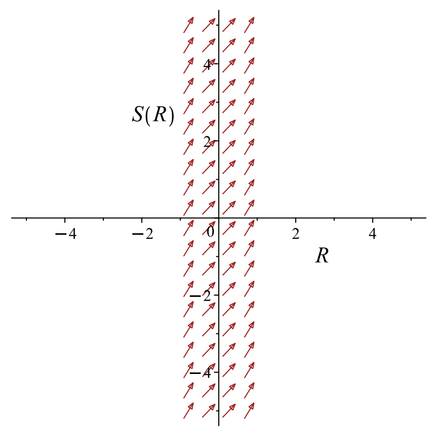
Which simplifies to

$$\sqrt{x} = \arcsin(y) + c_1$$

Which gives

$$y = -\sin\left(-\sqrt{x} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\sqrt{-y^2+1}}{2\sqrt{x}}$ 	$R = y$ $S = \sqrt{x}$	$\frac{dS}{dR} = \frac{1}{\sqrt{-R^2+1}}$ 

Summary
The solution(s) found are the following

$$y = -\sin\left(-\sqrt{x} + c_1\right) \quad (1)$$

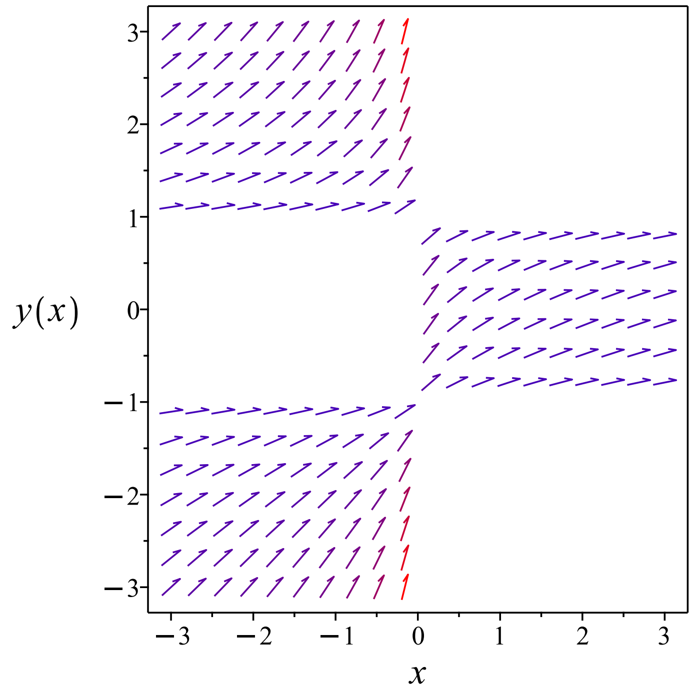


Figure 75: Slope field plot

Verification of solutions

$$y = -\sin\left(-\sqrt{x} + c_1\right)$$

Verified OK.

3.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2}{\sqrt{-y^2+1}}\right)\mathrm{d}y = \left(\frac{1}{\sqrt{x}}\right)\mathrm{d}x$$
$$\left(-\frac{1}{\sqrt{x}}\right)\mathrm{d}x + \left(\frac{2}{\sqrt{-y^2+1}}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{\sqrt{x}}$$
$$N(x, y) = \frac{2}{\sqrt{-y^2+1}}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\sqrt{x}}\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2}{\sqrt{-y^2+1}}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{\sqrt{x}} \, \mathrm{d}x \\
\phi &= -2\sqrt{x} + f(y) && (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{\sqrt{-y^2+1}}$. Therefore equation (4) becomes

$$
\frac{2}{\sqrt{-y^2+1}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{2}{\sqrt{-y^2+1}}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{2}{\sqrt{-y^2+1}}\right) \mathrm{d}y \\
f(y) &= 2\arcsin(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\sqrt{x} + 2\arcsin(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\sqrt{x} + 2\arcsin(y)$$

The solution becomes

$$y = \sin\left(\sqrt{x} + \frac{c_1}{2}\right)$$

Summary

The solution(s) found are the following

$$y = \sin\left(\sqrt{x} + \frac{c_1}{2}\right) \tag{1}$$

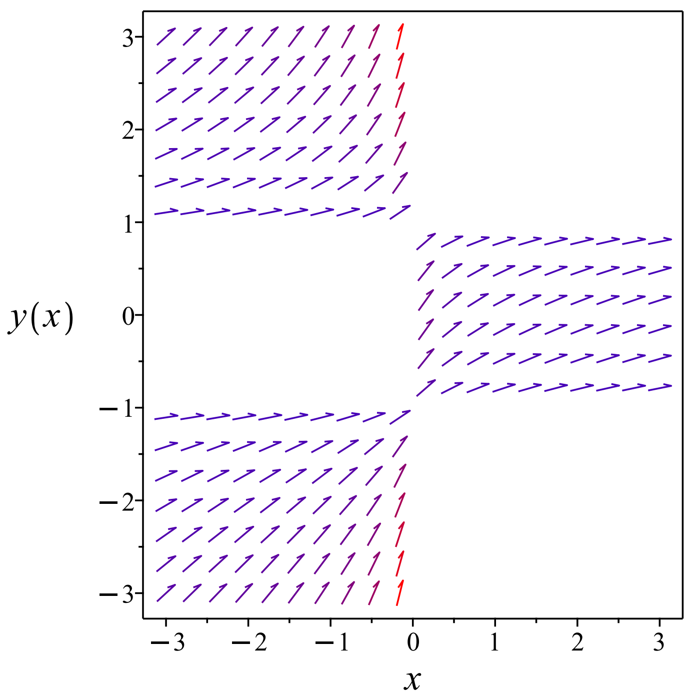


Figure 76: Slope field plot

Verification of solutions

$$y = \sin\left(\sqrt{x} + \frac{c_1}{2}\right)$$

Verified OK.

3.5.4 Maple step by step solution

Let's solve

$2\sqrt{x}\, y' - \sqrt{1 - y^2} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{1-y^2}} = \frac{1}{2\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{1-y^2}} dx = \int \frac{1}{2\sqrt{x}} dx + c_1$

- Evaluate integral

  $\arcsin(y) = \sqrt{x} + c_1$

- Solve for y

  $y = \sin\left(\sqrt{x} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve(2*x^(1/2)*diff(y(x),x) = (1-y(x)^2)^(1/2),y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{x} + \frac{c_1}{2}\right)$$

✓ Solution by Mathematica

Time used: 0.221 (sec). Leaf size: 32

```
DSolve[2*x^(1/2)*y'[x] == (1-y[x]^2)^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos\left(\sqrt{x} + c_1\right)$$
$$y(x) \to -1$$
$$y(x) \to 1$$
$$y(x) \to \text{Interval}[\{-1, 1\}]$$

3.6 problem 6

Internal problem ID [32]
Internal file name [`OUTPUT/32_Sunday_June_05_2022_01_33_53_AM_61324329/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y' - 3\sqrt{yx} = 0}$$

3.6.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = 3\sqrt{yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + 3\sqrt{yx}\,(b_3 - a_2) - 9yxa_3 - \frac{3y(xa_2 + ya_3 + a_1)}{2\sqrt{yx}} - \frac{3x(xb_2 + yb_3 + b_1)}{2\sqrt{yx}} = 0 \quad \text{(5E)}$$

Putting the above in normal form gives

$$-\frac{18yxa_3\sqrt{yx} + 9xya_2 - 3xyb_3 + 3x^2b_2 + 3y^2a_3 - 2b_2\sqrt{yx} + 3xb_1 + 3ya_1}{2\sqrt{yx}} = 0$$

Setting the numerator to zero gives

$$-18yxa_3\sqrt{yx} - 3x^2b_2 - 9xya_2 + 3xyb_3 - 3y^2a_3 + 2b_2\sqrt{yx} - 3xb_1 - 3ya_1 = 0 \quad \text{(6E)}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, \sqrt{yx}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, \sqrt{yx} = v_3\}$$

The above PDE (6E) now becomes

$$-18v_2v_1a_3v_3 - 9v_1v_2a_2 - 3v_2^2a_3 - 3v_1^2b_2 + 3v_1v_2b_3 - 3v_2a_1 - 3v_1b_1 + 2b_2v_3 = 0 \quad \text{(7E)}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$-3v_1^2b_2 - 18v_2v_1a_3v_3 + (-9a_2 + 3b_3)\,v_1v_2 - 3v_1b_1 - 3v_2^2a_3 - 3v_2a_1 + 2b_2v_3 = 0 \quad \text{(8E)}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-3a_1 &= 0 \\
-18a_3 &= 0 \\
-3a_3 &= 0 \\
-3b_1 &= 0 \\
-3b_2 &= 0 \\
2b_2 &= 0 \\
-9a_2 + 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= a_2\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= 3a_2
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= 3y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= 3y - \left(3\sqrt{yx}\right)(x)\\
&= -3x\sqrt{yx} + 3y\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{-3x\sqrt{yx} + 3y} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^3+y)}{3} + \frac{\ln(-x^2+\sqrt{yx})}{3} - \frac{\ln(x^2+\sqrt{yx})}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3\sqrt{yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1\\
R_y &= 0\\
S_x &= \frac{\left(x^{\frac{3}{2}}+\sqrt{y}\right)\sqrt{x}}{x^3-y}\\
S_y &= -\frac{x^{\frac{3}{2}}+\sqrt{y}}{\sqrt{y}\,(3x^3-3y)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\left(x^{\frac{3}{2}}+\sqrt{y}\right)\left(\sqrt{y}\,\sqrt{x}-\sqrt{yx}\right)}{\sqrt{y}\,(x^3-y)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

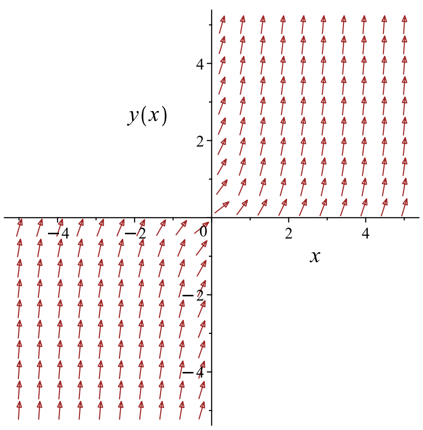
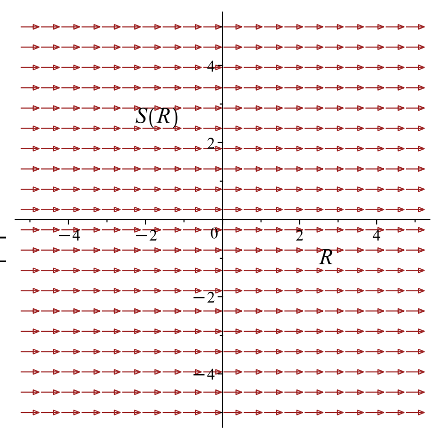
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(-x^3+y)}{3} + \frac{\ln\left(-x^2+\sqrt{y}\,\sqrt{x}\right)}{3} - \frac{\ln\left(x^2+\sqrt{y}\,\sqrt{x}\right)}{3} = c_1$$

Which simplifies to

$$\frac{\ln(-x^3+y)}{3} + \frac{\ln\left(-x^2+\sqrt{y}\,\sqrt{x}\right)}{3} - \frac{\ln\left(x^2+\sqrt{y}\,\sqrt{x}\right)}{3} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3\sqrt{yx}$ 	$R = x$ $S = \frac{\ln(-x^3+y)}{3} + \ln(-$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(-x^3+y)}{3} + \frac{\ln\left(-x^2+\sqrt{y}\,\sqrt{x}\right)}{3} - \frac{\ln\left(x^2+\sqrt{y}\,\sqrt{x}\right)}{3} = c_1 \tag{1}$$

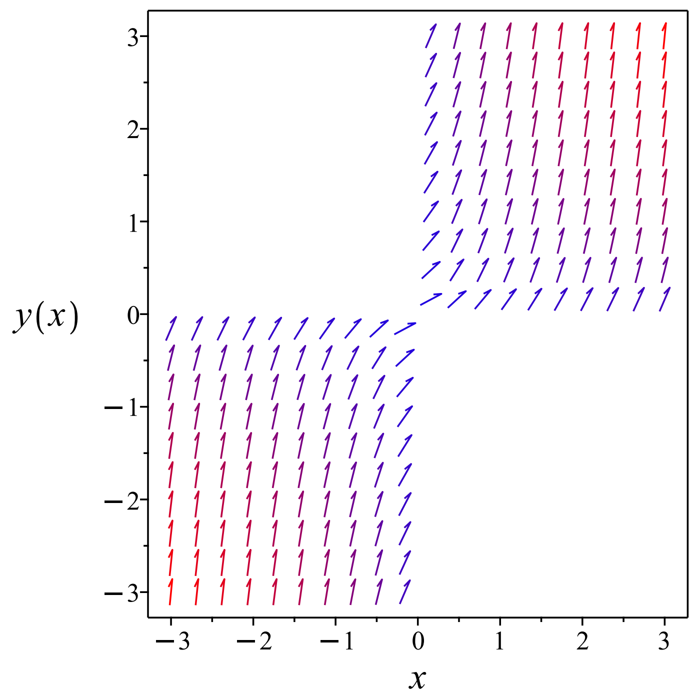


Figure 77: Slope field plot

Verification of solutions

$$\frac{\ln \left(-x^{3}+y\right)}{3}+\frac{\ln \left(-x^{2}+\sqrt{y}\, \sqrt{x}\right)}{3}-\frac{\ln \left(x^{2}+\sqrt{y}\, \sqrt{x}\right)}{3} = c_{1}$$

Verified OK.

3.6.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\mathrm{d}y &= (3\sqrt{yx})\, \mathrm{d}x \\
(-3\sqrt{yx})\, \mathrm{d}x + \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -3\sqrt{yx} \\
N(x, y) &= 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-3\sqrt{yx}) \\
&= -\frac{3x}{2\sqrt{yx}}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{3x}{2\sqrt{yx}}\right) - (0)\right) \\
&= -\frac{3x}{2\sqrt{yx}}
\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{3\sqrt{yx}}\left((0) - \left(-\frac{3x}{2\sqrt{yx}}\right)\right) \\
&= -\frac{1}{2y}
\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{1}{2y} \, \mathrm{d}y}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\frac{\ln(y)}{2}} \\
&= \frac{1}{\sqrt{y}}
\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{y}}(-3\sqrt{yx}) \\
&= -\frac{3\sqrt{yx}}{\sqrt{y}}
\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{y}}(1) \\
&= \frac{1}{\sqrt{y}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{3\sqrt{yx}}{\sqrt{y}}\right) + \left(\frac{1}{\sqrt{y}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{3\sqrt{yx}}{\sqrt{y}} \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{2x\sqrt{yx}}{\sqrt{y}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x\sqrt{yx}}{y^{\frac{3}{2}}} - \frac{x^2}{\sqrt{y}\sqrt{yx}} + f'(y) \tag{4}
$$

$$
= 0 + f'(y)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{\sqrt{y}}$. Therefore equation (4) becomes

$$
\frac{1}{\sqrt{y}} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y$$

$$f(y) = 2\sqrt{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{2x\sqrt{yx}}{\sqrt{y}} + 2\sqrt{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{2x\sqrt{yx}}{\sqrt{y}} + 2\sqrt{y}$$

Summary

The solution(s) found are the following

$$-\frac{2x\sqrt{yx}}{\sqrt{y}} + 2\sqrt{y} = c_1 \tag{1}$$

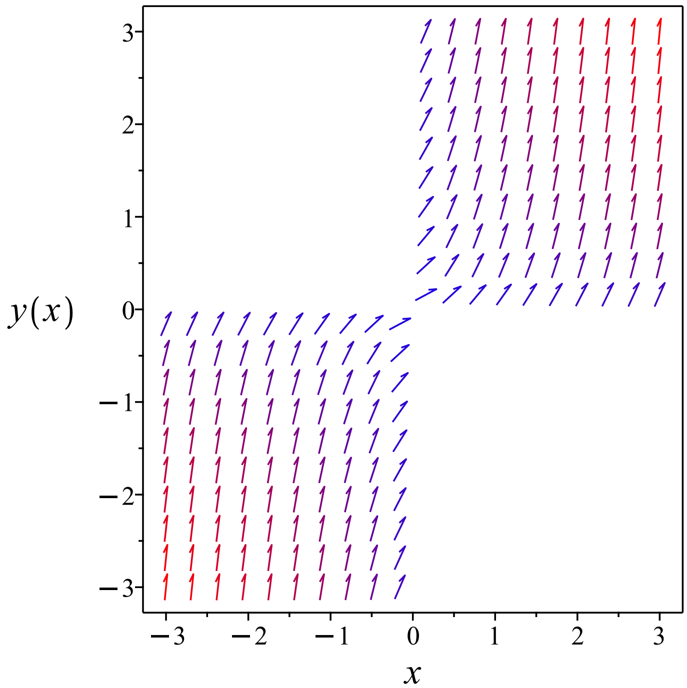


Figure 78: Slope field plot

Verification of solutions

$$-\frac{2x\sqrt{yx}}{\sqrt{y}}+2\sqrt{y}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 65

```
dsolve(diff(y(x),x) = 3*(x*y(x))^(1/2),y(x), singsol=all)
```

$$\frac{(c_1x^3 - y(x)\, c_1 + 1)\, \sqrt{xy\,(x)} - x^2(c_1x^3 - y(x)\, c_1 - 1)}{(x^3 - y\,(x))\left(x^2 - \sqrt{xy\,(x)}\right)} = 0$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 26

```
DSolve[y'[x] == 3*(x*y[x])^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}\left(2x^{3/2} + c_1\right)^2$$
$$y(x) \to 0$$

3.7 problem 7

Internal problem ID [33]
Internal file name [`OUTPUT/33_Sunday_June_05_2022_01_33_54_AM_75111295/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`]]
```

$$\boxed{y' - 4(yx)^{\frac{1}{3}} = 0}$$

3.7.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = 4(yx)^{\frac{1}{3}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+4(yx)^{\frac{1}{3}}(b_3-a_2)-16(yx)^{\frac{2}{3}}a_3-\frac{4y(xa_2+ya_3+a_1)}{3(yx)^{\frac{2}{3}}}-\frac{4x(xb_2+yb_3+b_1)}{3(yx)^{\frac{2}{3}}}=0 \quad (5E)$$

Putting the above in normal form gives

$$-\frac{48(yx)^{\frac{4}{3}}a_3+16xya_2-8xyb_3-3b_2(yx)^{\frac{2}{3}}+4x^2b_2+4y^2a_3+4xb_1+4ya_1}{3(yx)^{\frac{2}{3}}}=0$$

Setting the numerator to zero gives

$$-48(yx)^{\frac{4}{3}}a_3+3b_2(yx)^{\frac{2}{3}}-4x^2b_2-16xya_2+8xyb_3-4y^2a_3-4xb_1-4ya_1=0 \quad (6E)$$

Since the PDE has radicals, simplifying gives

$$-48yx(yx)^{\frac{1}{3}}a_3-4x^2b_2-16xya_2+8xyb_3+3b_2(yx)^{\frac{2}{3}}-4y^2a_3-4xb_1-4ya_1=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,(yx)^{\frac{1}{3}},(yx)^{\frac{2}{3}}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\left\{x=v_1,y=v_2,(yx)^{\frac{1}{3}}=v_3,(yx)^{\frac{2}{3}}=v_4\right\}$$

The above PDE (6E) now becomes

$$-48v_2v_1v_3a_3-16v_1v_2a_2-4v_2^2a_3-4v_1^2b_2+8v_1v_2b_3-4v_2a_1-4v_1b_1+3b_2v_4=0 \quad (7E)$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2,v_3,v_4\}$$

Equation (7E) now becomes

$$-4v_1^2b_2-48v_2v_1v_3a_3+(-16a_2+8b_3)v_1v_2-4v_1b_1-4v_2^2a_3-4v_2a_1+3b_2v_4=0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
-4a_1 &= 0 \\
-48a_3 &= 0 \\
-4a_3 &= 0 \\
-4b_1 &= 0 \\
-4b_2 &= 0 \\
3b_2 &= 0 \\
-16a_2 + 8b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= a_2 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= 2a_2
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= 2y
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$
\begin{aligned}
\frac{dy}{dx} &= \frac{\eta}{\xi} \\
&= \frac{2y}{x} \\
&= \frac{2y}{x}
\end{aligned}
$$

This is easily solved to give

$$y = c_1 x^2$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x^2}$$

And S is found from

$$\begin{aligned} dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x} \end{aligned}$$

Integrating gives

$$\begin{aligned} S &= \int \frac{dx}{T} \\ &= \ln(x) \end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 4(yx)^{\frac{1}{3}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{2y}{x^3} \\ R_y &= \frac{1}{x^2} \\ S_x &= \frac{1}{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^2}{4x\,(yx)^{\frac{1}{3}} - 2y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{4R^{\frac{1}{3}} - 2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R^2 - 8)}{4} - \frac{\ln\left(R^{\frac{2}{3}} - 2\right)}{2} + \frac{\ln\left(2R^{\frac{2}{3}} + R^{\frac{4}{3}} + 4\right)}{4} + c_1 \tag{4}$$

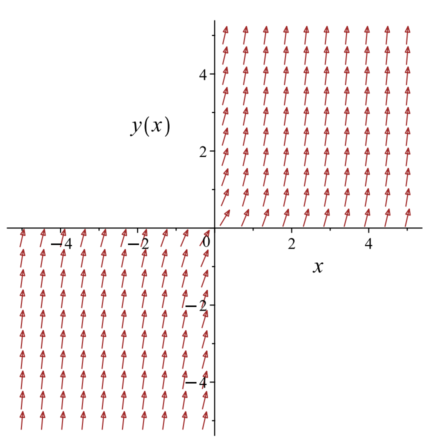
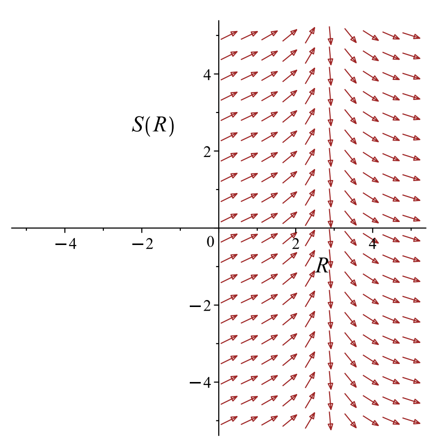
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\ln(x) = -\frac{\ln\left(\frac{y^2}{x^4} - 8\right)}{4} - \frac{\ln\left(\left(\frac{y}{x^2}\right)^{\frac{2}{3}} - 2\right)}{2} + \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}} + \left(\frac{y}{x^2}\right)^{\frac{4}{3}} + 4\right)}{4} + c_1$$

Which simplifies to

$$\ln(x) = -\frac{\ln\left(\frac{y^2}{x^4} - 8\right)}{4} - \frac{\ln\left(\left(\frac{y}{x^2}\right)^{\frac{2}{3}} - 2\right)}{2} + \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}} + \left(\frac{y}{x^2}\right)^{\frac{4}{3}} + 4\right)}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 4(yx)^{\frac{1}{3}}$ 	$R = \frac{y}{x^2}$ $S = \ln(x)$	$\frac{dS}{dR} = \frac{1}{4R^{\frac{1}{3}} - 2R}$ 

Summary
The solution(s) found are the following

$$\ln(x) = -\frac{\ln\left(\frac{y^2}{x^4}-8\right)}{4} - \frac{\ln\left(\left(\frac{y}{x^2}\right)^{\frac{2}{3}}-2\right)}{2} + \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}}+\left(\frac{y}{x^2}\right)^{\frac{4}{3}}+4\right)}{4} + c_1 \tag{1}$$

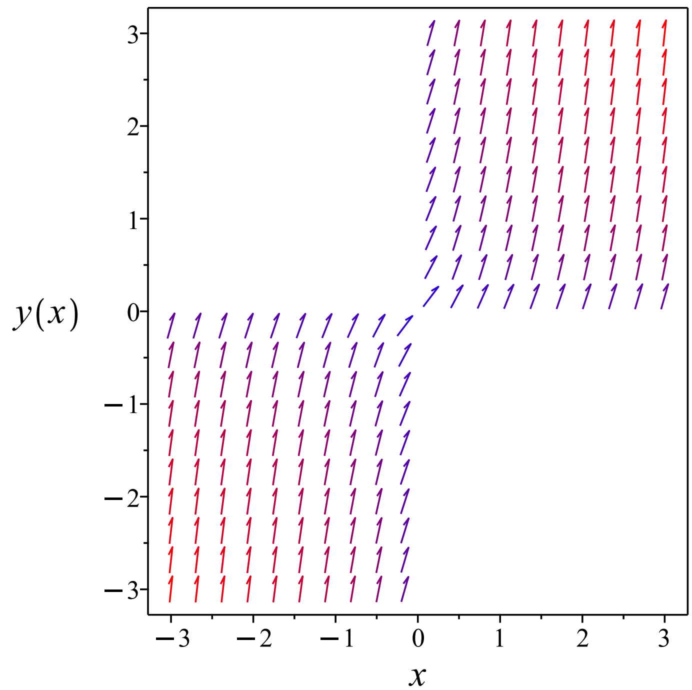


Figure 79: Slope field plot

Verification of solutions

$$\ln(x) = -\frac{\ln\left(\frac{y^2}{x^4}-8\right)}{4} - \frac{\ln\left(\left(\frac{y}{x^2}\right)^{\frac{2}{3}}-2\right)}{2} + \frac{\ln\left(2\left(\frac{y}{x^2}\right)^{\frac{2}{3}}+\left(\frac{y}{x^2}\right)^{\frac{4}{3}}+4\right)}{4} + c_1$$

Verified OK.

3.7.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y &= \left(4(yx)^{\frac{1}{3}}\right)\mathrm{d}x\\ \left(-4(yx)^{\frac{1}{3}}\right)\mathrm{d}x + \mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -4(yx)^{\frac{1}{3}}\\ N(x,y) &= 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-4(yx)^{\frac{1}{3}}\right) \\
&= -\frac{4x}{3\,(yx)^{\frac{2}{3}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1\left(\left(-\frac{4x}{3\,(yx)^{\frac{2}{3}}}\right) - (0)\right) \\
&= -\frac{4x}{3\,(yx)^{\frac{2}{3}}}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\
&= -\frac{1}{4\,(yx)^{\frac{1}{3}}}\left((0) - \left(-\frac{4x}{3\,(yx)^{\frac{2}{3}}}\right)\right) \\
&= -\frac{1}{3y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y} \\
&= e^{\int -\frac{1}{3y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(y)}{3}} \\
&= \frac{1}{y^{\frac{1}{3}}}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{y^{\frac{1}{3}}}\left(-4(yx)^{\frac{1}{3}}\right) \\
&= -\frac{4(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^{\frac{1}{3}}}(1) \\
&= \frac{1}{y^{\frac{1}{3}}}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{4(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}}\right) + \left(\frac{1}{y^{\frac{1}{3}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x
$$

$$
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{4(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}}\,\mathrm{d}x
$$

$$
\phi = -\frac{3x(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y} &= -\frac{x^2}{(yx)^{\frac{2}{3}} y^{\frac{1}{3}}} + \frac{x(yx)^{\frac{1}{3}}}{y^{\frac{4}{3}}} + f'(y) \\ &= 0 + f'(y)\end{aligned} \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^{\frac{1}{3}}}$. Therefore equation (4) becomes

$$\frac{1}{y^{\frac{1}{3}}} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^{\frac{1}{3}}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^{\frac{1}{3}}}\right) \mathrm{d}y$$

$$f(y) = \frac{3y^{\frac{2}{3}}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{3x(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}} + \frac{3y^{\frac{2}{3}}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{3x(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}} + \frac{3y^{\frac{2}{3}}}{2}$$

Summary

The solution(s) found are the following

$$-\frac{3x(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}}+\frac{3y^{\frac{2}{3}}}{2}=c_1 \tag{1}$$

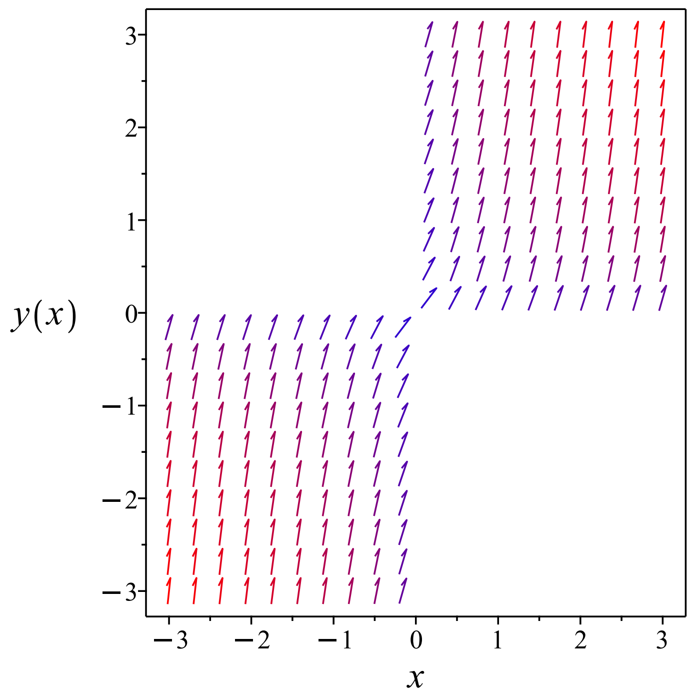


Figure 80: Slope field plot

Verification of solutions

$$-\frac{3x(yx)^{\frac{1}{3}}}{y^{\frac{1}{3}}}+\frac{3y^{\frac{2}{3}}}{2}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 91

```
dsolve(diff(y(x),x) = 4*(x*y(x))^(1/3),y(x), singsol=all)
```

$$-\frac{32x\left(\left(-c_1x^5+\frac{y(x)^2c_1x}{8}+\frac{x}{16}\right)(xy(x))^{\frac{2}{3}}+\left(c_1x^4-\frac{y(x)^2c_1}{8}+\frac{1}{8}\right)\left(x^3+\frac{y(x)(xy(x))^{\frac{1}{3}}}{4}\right)\right)}{\left(8x^4-y\left(x\right)^2\right)\left(-\left(xy\left(x\right)\right)^{\frac{2}{3}}+2x^2\right)^2}=0$$

✓ Solution by Mathematica

Time used: 4.813 (sec). Leaf size: 35

```
DSolve[y'[x] == 4*(x*y[x])^(1/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to\frac{2}{3}\sqrt{\frac{2}{3}}\left(3x^{4/3}+c_1\right)^{3/2}$$
$$y(x)\to 0$$

3.8 problem 8

Internal problem ID [34]
Internal file name [`OUTPUT/34_Sunday_June_05_2022_01_33_55_AM_57455569/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 2x\sec(y) = 0}$$

3.8.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= 2x\sec(y) \end{aligned}$$

Where $f(x) = 2x$ and $g(y) = \sec(y)$. Integrating both sides gives

$$\frac{1}{\sec(y)}\,dy = 2x\,dx$$

$$\int \frac{1}{\sec(y)}\,dy = \int 2x\,dx$$

$$\sin(y) = x^2 + c_1$$

Which results in

$$y = \arcsin\left(x^2 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \arcsin\left(x^2 + c_1\right) \tag{1}$$

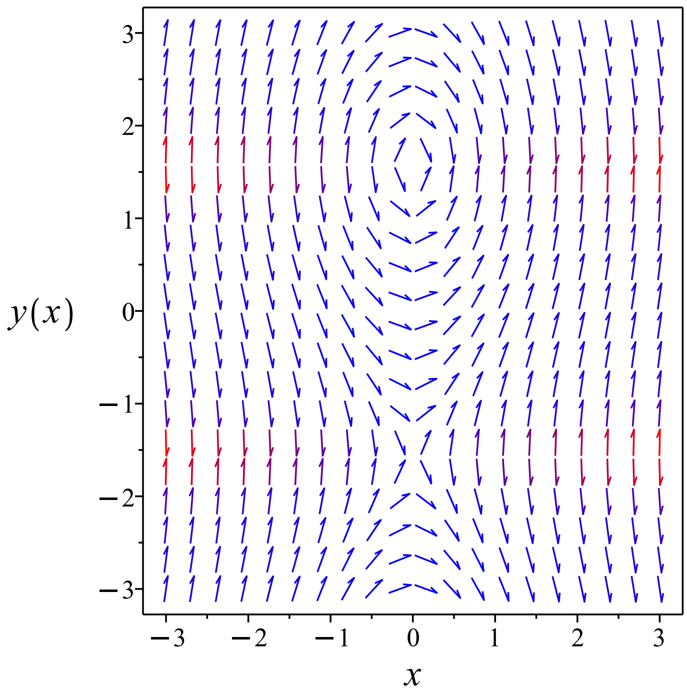


Figure 81: Slope field plot

Verification of solutions

$$y = \arcsin\left(x^2 + c_1\right)$$

Verified OK.

3.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 2x\sec(y) \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 64: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{1}{2x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{2x}} dx \end{aligned}$$

Which results in

$$S = x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2x\sec(y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 2x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 = \sin(y) + c_1$$

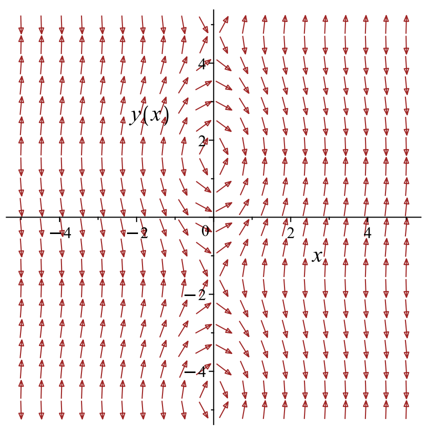
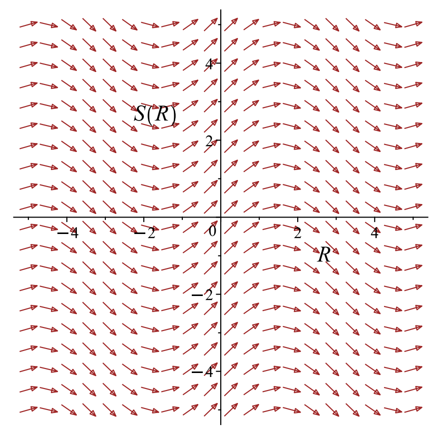
Which simplifies to

$$x^2 = \sin(y) + c_1$$

Which gives

$$y = -\arcsin\left(-x^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2x \sec(y)$ 	$R = y$ $S = x^2$	$\frac{dS}{dR} = \cos(R)$ 

Summary
The solution(s) found are the following

$$y = -\arcsin\left(-x^2 + c_1\right) \quad (1)$$

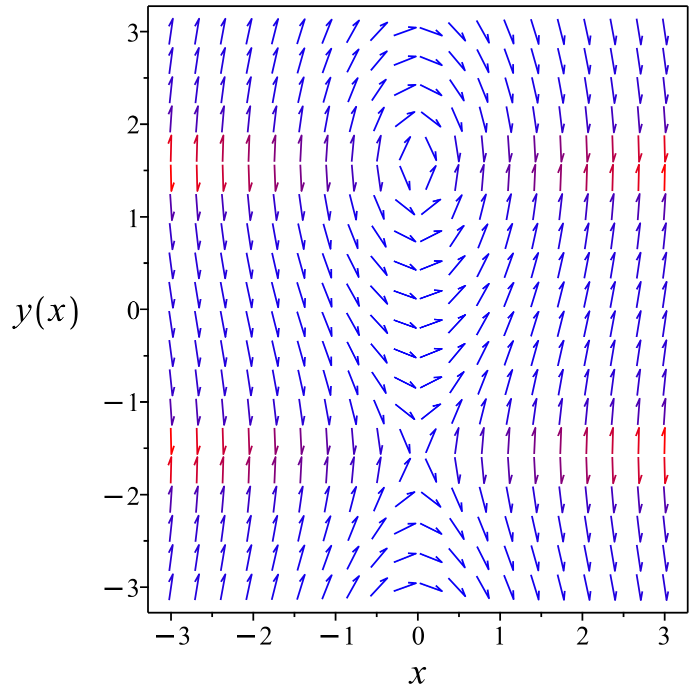


Figure 82: Slope field plot

Verification of solutions

$$y = -\arcsin\left(-x^2 + c_1\right)$$

Verified OK.

3.8.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{2\sec(y)}\right)\mathrm{d}y = (x)\,\mathrm{d}x$$
$$(-x)\,\mathrm{d}x + \left(\frac{1}{2\sec(y)}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x$$
$$N(x, y) = \frac{1}{2\sec(y)}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2\sec(y)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -x\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2\sec(y)}$. Therefore equation (4) becomes

$$\frac{1}{2\sec(y)} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned}f'(y) &= \frac{1}{2\sec(y)}\\ &= \frac{\cos(y)}{2}\end{aligned}$$

Integrating the above w.r.t y results in

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{\cos(y)}{2}\right) \mathrm{d}y$$
$$f(y) = \frac{\sin(y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\sin(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\sin(y)}{2}$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{\sin(y)}{2} = c_1 \tag{1}$$

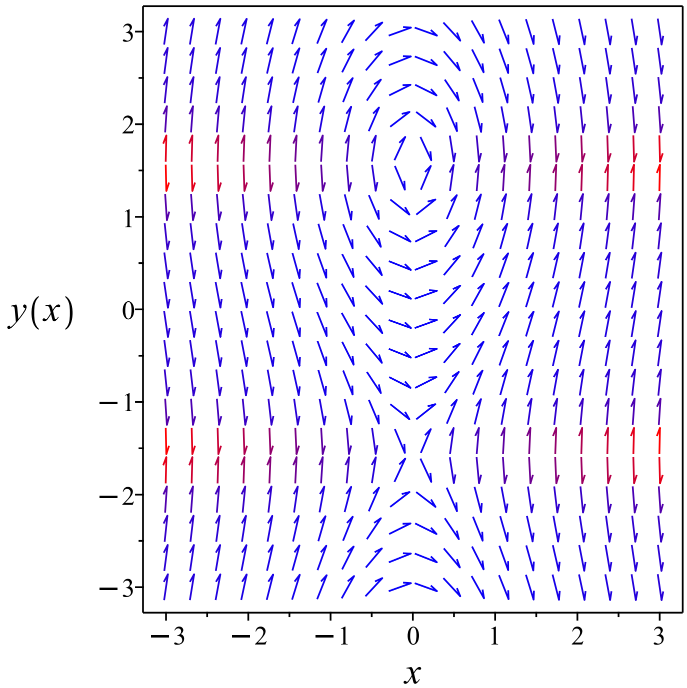


Figure 83: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{\sin(y)}{2} = c_1$$

Verified OK.

3.8.4 Maple step by step solution

Let's solve

$y' - 2x\sec(y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sec(y)} = 2x$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sec(y)}dx = \int 2xdx + c_1$

- Evaluate integral

$$\sin(y) = x^2 + c_1$$

- Solve for y

$$y = \arcsin(x^2 + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) = 2*x*sec(y(x)),y(x), singsol=all)
```

$$y(x) = \arcsin(x^2 + 2c_1)$$

✓ Solution by Mathematica

Time used: 0.841 (sec). Leaf size: 12

```
DSolve[y'[x]==2*x*Sec[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \arcsin(x^2 + c_1)$$

3.9 problem 9

Internal problem ID [35]
Internal file name [`OUTPUT/35_Sunday_June_05_2022_01_33_55_AM_55264386/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{\left(-x^2+1\right)y'-2y=0}$$

3.9.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2y}{x^2-1} \end{aligned}$$

Where $f(x) = -\frac{2}{x^2-1}$ and $g(y) = y$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{y}\, dy &= -\frac{2}{x^2-1}\, dx \\
\int \frac{1}{y}\, dy &= \int -\frac{2}{x^2-1}\, dx \\
\ln(y) &= 2 \operatorname{arctanh}(x) + c_1 \\
y &= \mathrm{e}^{2 \operatorname{arctanh}(x) + c_1} \\
&= \frac{c_1 (x+1)^2}{-x^2+1}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{c_1 (x+1)^2}{-x^2+1} \tag{1}
$$

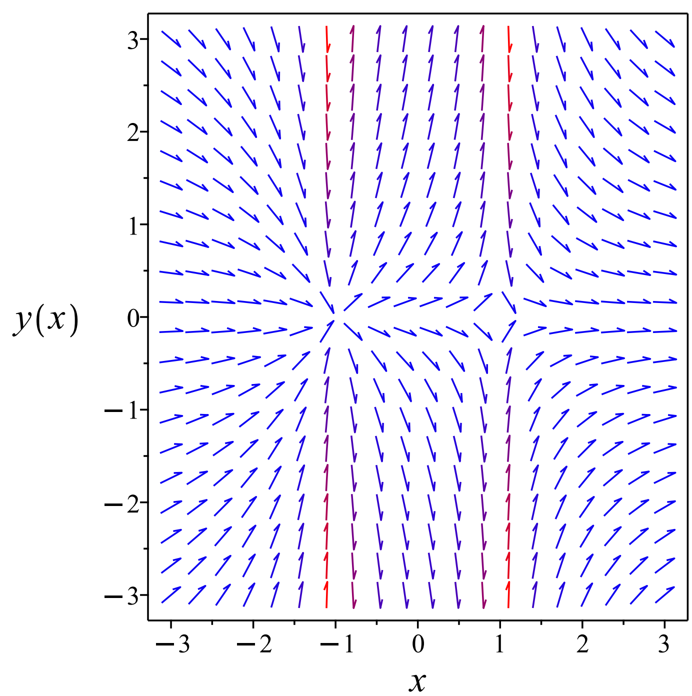


Figure 84: Slope field plot

Verification of solutions

$$
y = \frac{c_1 (x+1)^2}{-x^2+1}
$$

Verified OK.

3.9.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x^2 - 1} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' + \frac{2y}{x^2 - 1} = 0$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{2}{x^2-1} dx} \\ &= \frac{-x^2 + 1}{(x + 1)^2} \end{aligned}$$

Which simplifies to

$$\mu = \frac{1 - x}{x + 1}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(1 - x)\, y}{x + 1}\right) &= 0 \end{aligned}$$

Integrating gives

$$\frac{(1 - x)\, y}{x + 1} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1-x}{x+1}$ results in

$$y = \frac{c_1(-x - 1)}{x - 1}$$

which simplifies to

$$y = -\frac{(x + 1)\, c_1}{x - 1}$$

Summary
The solution(s) found are the following

$$y = -\frac{(x+1) c_1}{x-1} \tag{1}$$

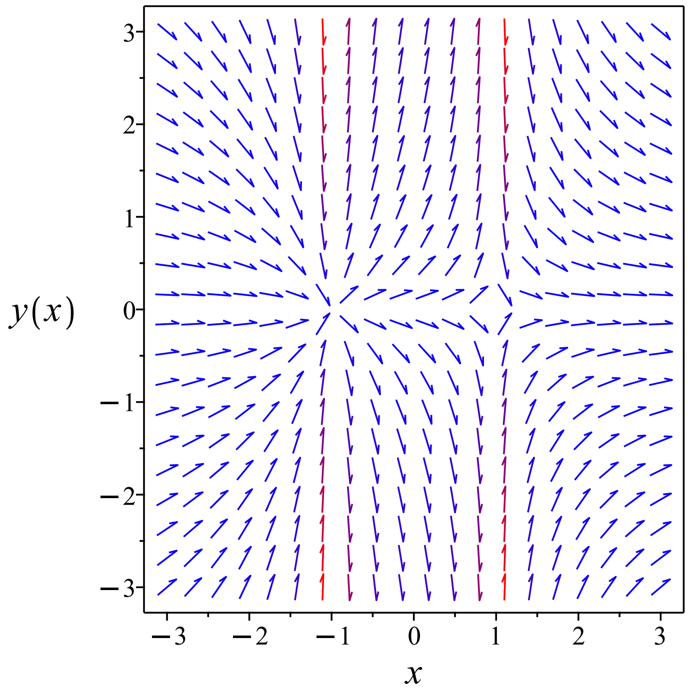


Figure 85: Slope field plot

Verification of solutions

$$y = -\frac{(x+1) c_1}{x-1}$$

Verified OK.

3.9.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$\left(-x^2+1\right)\left(u'(x)\, x + u(x)\right) - 2u(x)\, x = 0$$

In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{u(x^2+2x-1)}{(x^2-1)\,x}
\end{aligned}
$$

Where $f(x) = -\frac{x^2+2x-1}{(x^2-1)x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{x^2+2x-1}{(x^2-1)\,x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x^2+2x-1}{(x^2-1)\,x}\,dx \\
\ln(u) &= -\ln(x) + \ln(x+1) - \ln(x-1) + c_2 \\
u &= \mathrm{e}^{-\ln(x)+\ln(x+1)-\ln(x-1)+c_2} \\
&= c_2\mathrm{e}^{-\ln(x)+\ln(x+1)-\ln(x-1)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_2\left(\frac{1}{x-1} + \frac{1}{x\,(x-1)}\right)
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= xc_2\left(\frac{1}{x-1} + \frac{1}{x\,(x-1)}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = xc_2\left(\frac{1}{x-1} + \frac{1}{x\,(x-1)}\right) \tag{1}
$$

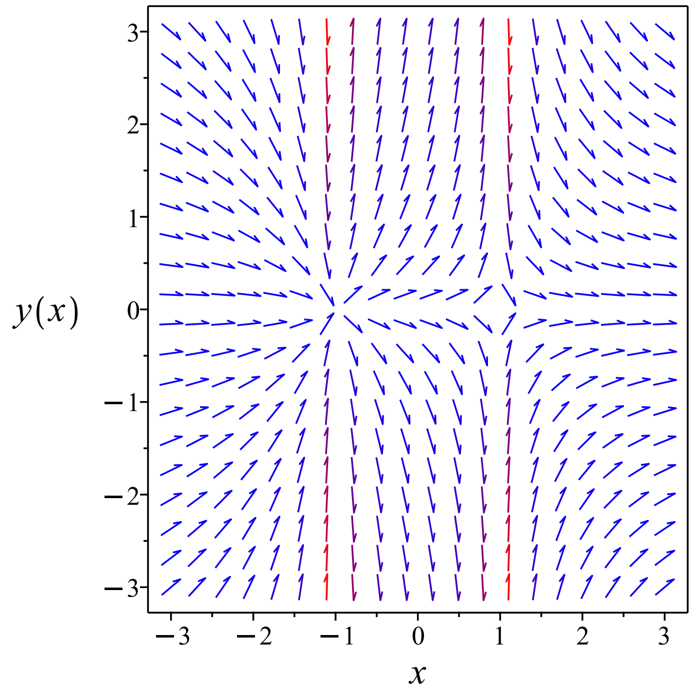


Figure 86: Slope field plot

Verification of solutions

$$y = xc_2\left(\frac{1}{x-1} + \frac{1}{x(x-1)}\right)$$

Verified OK.

3.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y}{x^2-1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 67: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{(x+1)^2}{-x^2+1}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{(x+1)^2}{-x^2+1}} dy \end{aligned}$$

Which results in

$$S = \frac{(-x^2+1)\, y}{(x+1)^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y}{x^2-1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{(x+1)^2} \\ S_y &= \frac{1-x}{x+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{(x-1)\, y}{x+1} = c_1$$

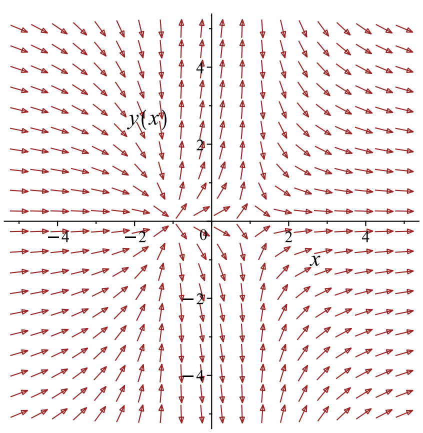
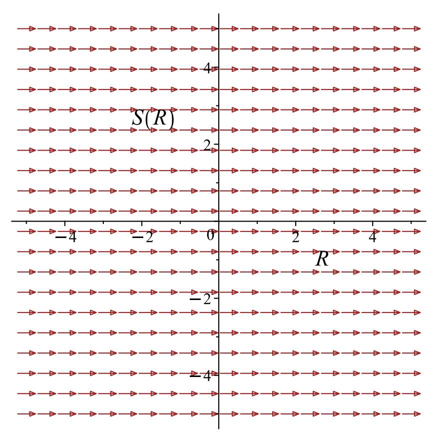
Which simplifies to

$$-\frac{(x-1)\, y}{x+1} = c_1$$

Which gives

$$y = -\frac{(x+1)\, c_1}{x-1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y}{x^2-1}$ 	$R = x$ $S = -\frac{(x-1)\, y}{x+1}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = -\frac{(x+1)\,c_1}{x-1} \tag{1}$$

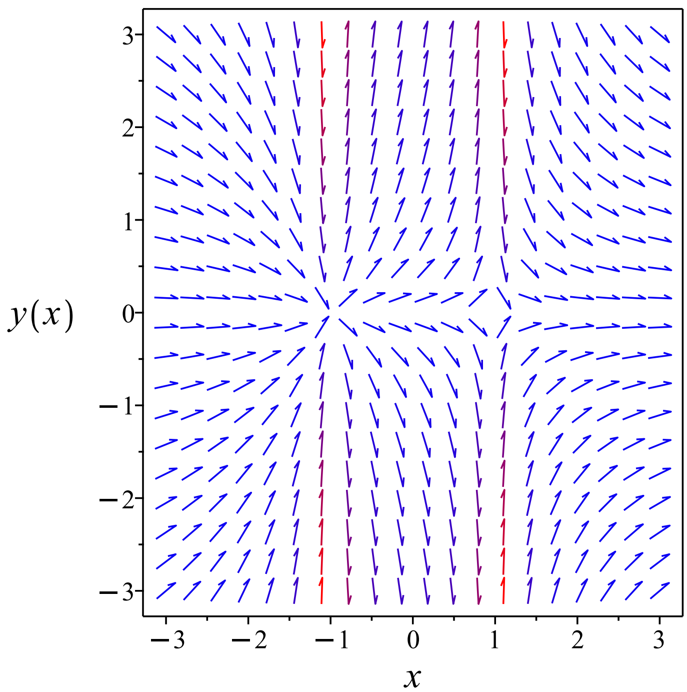


Figure 87: Slope field plot

Verification of solutions

$$y = -\frac{(x+1)\,c_1}{x-1}$$

Verified OK.

3.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{2y}\right)\mathrm{d}y&=\left(\frac{1}{x^2-1}\right)\mathrm{d}x\\ \left(-\frac{1}{x^2-1}\right)\mathrm{d}x+\left(-\frac{1}{2y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{1}{x^2-1}\\ N(x,y)&=-\frac{1}{2y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{1}{x^2-1}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{x^2 - 1} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \operatorname{arctanh}(x) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{2y}$. Therefore equation (4) becomes

$$
-\frac{1}{2y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{2y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(-\frac{1}{2y}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln(y)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \operatorname{arctanh}(x) - \frac{\ln(y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \operatorname{arctanh}(x) - \frac{\ln(y)}{2}$$

The solution becomes

$$y = e^{2\,\operatorname{arctanh}(x) - 2c_1}$$

Summary

The solution(s) found are the following

$$y = e^{2\,\operatorname{arctanh}(x) - 2c_1} \tag{1}$$

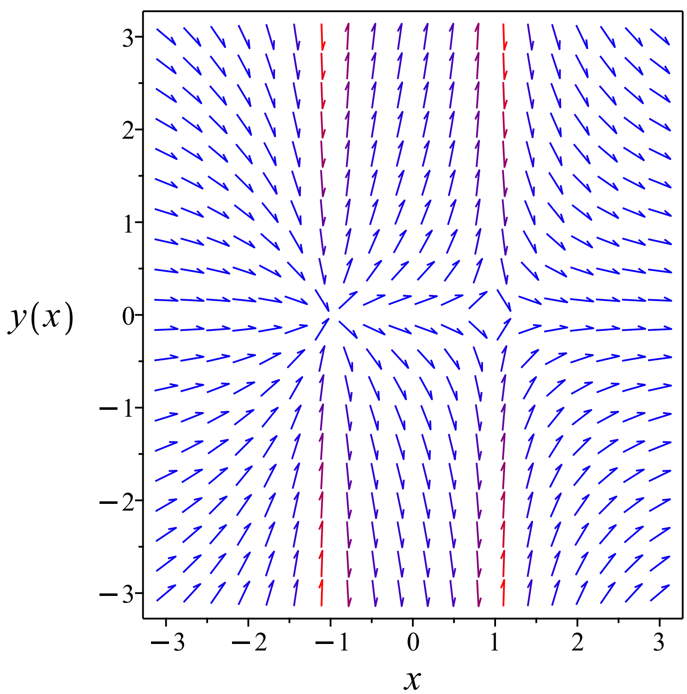


Figure 88: Slope field plot

Verification of solutions

$$y = e^{2\,\mathrm{arctanh}(x)-2c_1}$$

Verified OK.

3.9.6 Maple step by step solution

Let's solve

$(-x^2+1)\,y' - 2y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2}{-x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{2}{-x^2+1}dx + c_1$

- Evaluate integral

  $\ln(y) = 2\,\mathrm{arctanh}(x) + c_1$

- Solve for y

  $y = e^{2\,\mathrm{arctanh}(x)+c_1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve((-x^2+1)*diff(y(x),x) = 2*y(x),y(x), singsol=all)
```

$$y(x) = -\frac{(x+1)\,c_1}{x-1}$$

✓ Solution by Mathematica

Time used: 0.028 (sec). Leaf size: 22

```
DSolve[(-x^2+1)*y'[x] == 2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{c_1(x+1)}{x-1}$$
$$y(x) \to 0$$

3.10 problem 10

Internal problem ID [36]
Internal file name [`OUTPUT/36_Sunday_June_05_2022_01_33_56_AM_69487583/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\,y' - (1+y)^2 = 0}$$

3.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(y+1)^2}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{1}{x^2+1}$ and $g(y) = (y+1)^2$. Integrating both sides gives

$$\begin{aligned} \frac{1}{(y+1)^2}\,dy &= \frac{1}{x^2+1}\,dx \\ \int \frac{1}{(y+1)^2}\,dy &= \int \frac{1}{x^2+1}\,dx \end{aligned}$$

$$-\frac{1}{y+1} = \arctan(x) + c_1$$

Which results in

$$y = -\frac{\arctan(x) + c_1 + 1}{\arctan(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{\arctan(x) + c_1 + 1}{\arctan(x) + c_1} \tag{1}$$

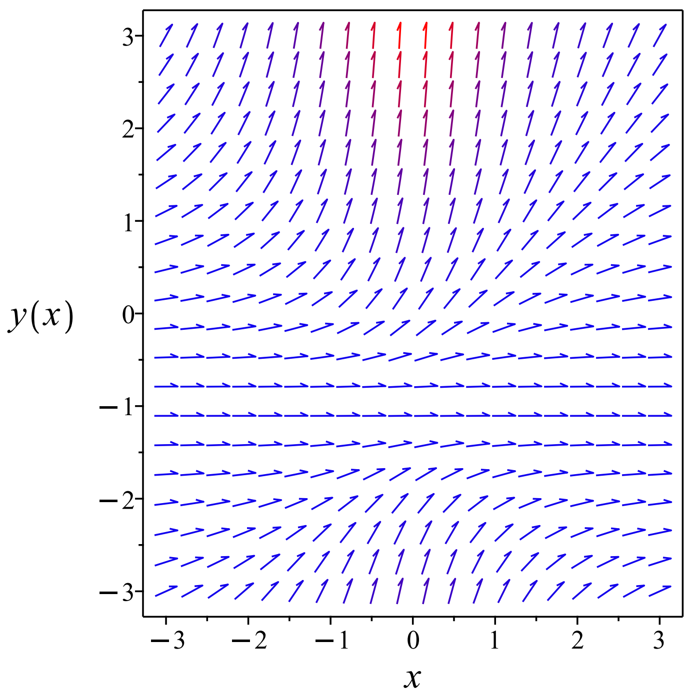


Figure 89: Slope field plot

Verification of solutions

$$y = -\frac{\arctan(x) + c_1 + 1}{\arctan(x) + c_1}$$

Verified OK.

3.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{(y+1)^2}{x^2+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 70: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= x^2 + 1 \\ \eta(x, y) &= 0\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{x^2 + 1} dx\end{aligned}$$

Which results in

$$S = \arctan(x)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{(y + 1)^2}{x^2 + 1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{(y+1)^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{(R+1)^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R+1} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\arctan(x) = -\frac{1}{1+y} + c_1
$$

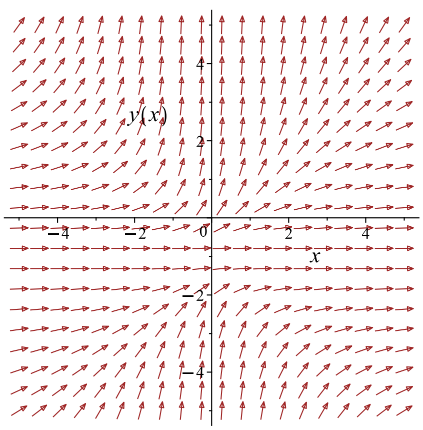
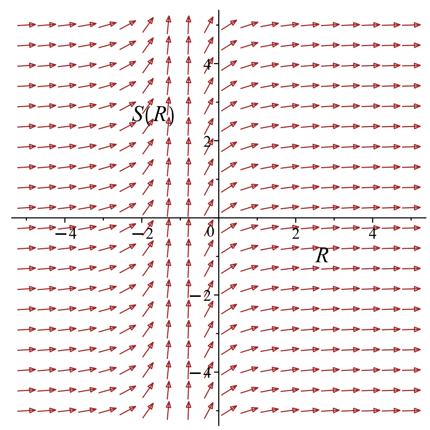
Which simplifies to

$$
\arctan(x) = -\frac{1}{1+y} + c_1
$$

Which gives

$$
y = -\frac{\arctan(x) - c_1 + 1}{\arctan(x) - c_1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{(y+1)^2}{x^2+1}$ 	$R = y$ $S = \arctan(x)$	$\frac{dS}{dR} = \frac{1}{(R+1)^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\arctan(x) - c_1 + 1}{\arctan(x) - c_1} \tag{1}$$

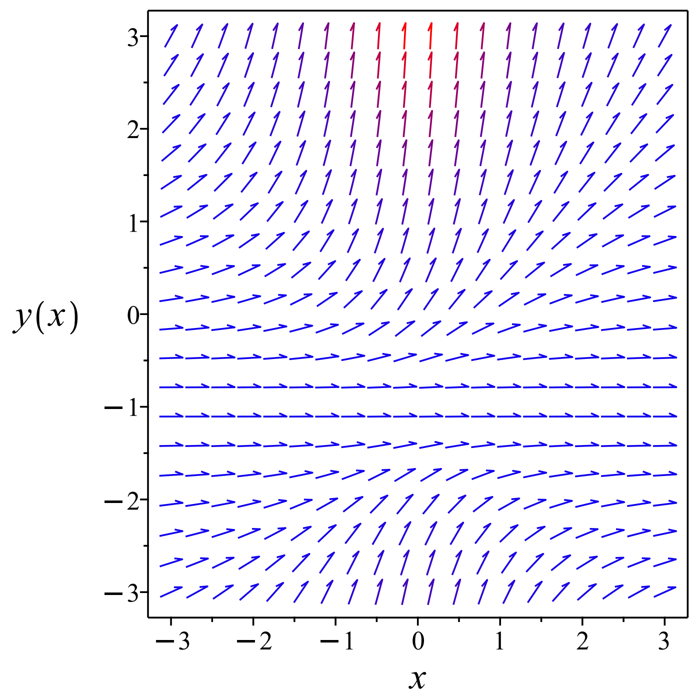


Figure 90: Slope field plot

Verification of solutions

$$y = -\frac{\arctan(x) - c_1 + 1}{\arctan(x) - c_1}$$

Verified OK.

3.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{(y+1)^2}\right) \mathrm{d}y = \left(\frac{1}{x^2+1}\right) \mathrm{d}x$$
$$\left(-\frac{1}{x^2+1}\right) \mathrm{d}x + \left(\frac{1}{(y+1)^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{x^2+1}$$
$$N(x, y) = \frac{1}{(y+1)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{1}{x^2+1}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{(y+1)^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -\frac{1}{x^2+1}\,\mathrm{d}x$$

$$\phi = -\arctan(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{(y+1)^2}$. Therefore equation (4) becomes

$$\frac{1}{(y+1)^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{(y+1)^2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{(y+1)^2}\right)\mathrm{d}y$$

$$f(y) = -\frac{1}{y+1} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\arctan(x) - \frac{1}{y+1} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\arctan(x) - \frac{1}{y+1}$$

Summary

The solution(s) found are the following

$$-\arctan(x) - \frac{1}{1+y} = c_1 \tag{1}$$

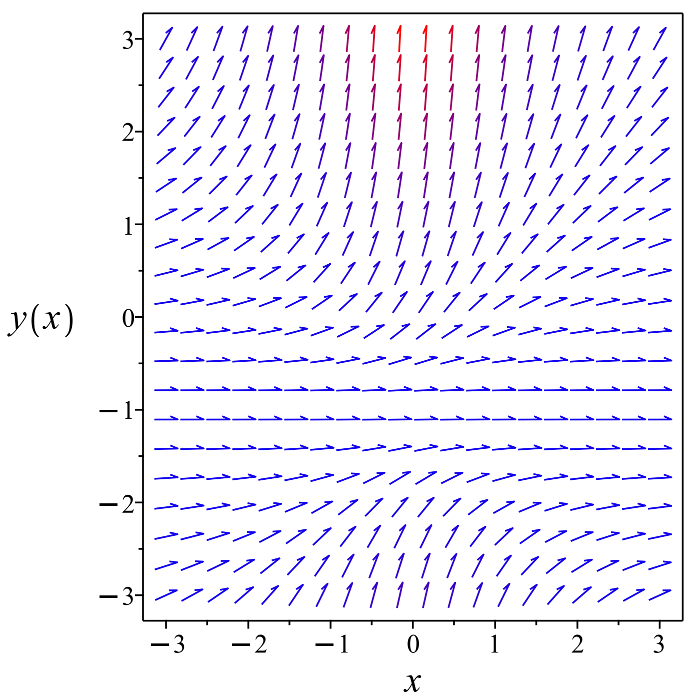


Figure 91: Slope field plot

Verification of solutions

$$-\arctan(x) - \frac{1}{1+y} = c_1$$

Verified OK.

3.10.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{(y+1)^2}{x^2+1} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x^2+1} + \frac{2y}{x^2+1} + \frac{1}{x^2+1}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = \frac{1}{x^2+1}$, $f_1(x) = \frac{2}{x^2+1}$ and $f_2(x) = \frac{1}{x^2+1}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2+1}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2x}{(x^2+1)^2} \\ f_1 f_2 &= \frac{2}{(x^2+1)^2} \\ f_2^2 f_0 &= \frac{1}{(x^2+1)^3} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2+1} - \left(-\frac{2x}{(x^2+1)^2} + \frac{2}{(x^2+1)^2}\right) u'(x) + \frac{u(x)}{(x^2+1)^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = (\arctan (x) c_2 + c_1) \left(\frac{x+i}{x-i}\right)^{\frac{i}{2}}$$

The above shows that

$$u'(x) = \frac{\left(\frac{-x-i}{-x+i}\right)^{\frac{i}{2}} (\arctan (x) c_2 + c_1 + c_2)}{x^2+1}$$

Using the above in (1) gives the solution

$$y = -\frac{\left(\frac{-x-i}{-x+i}\right)^{\frac{i}{2}} (\arctan (x) c_2 + c_1 + c_2) \left(\frac{x+i}{x-i}\right)^{-\frac{i}{2}}}{\arctan (x) c_2 + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\arctan (x) - c_3 - 1}{\arctan (x) + c_3}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{-\arctan (x) - c_3 - 1}{\arctan (x) + c_3} \tag{1}$$

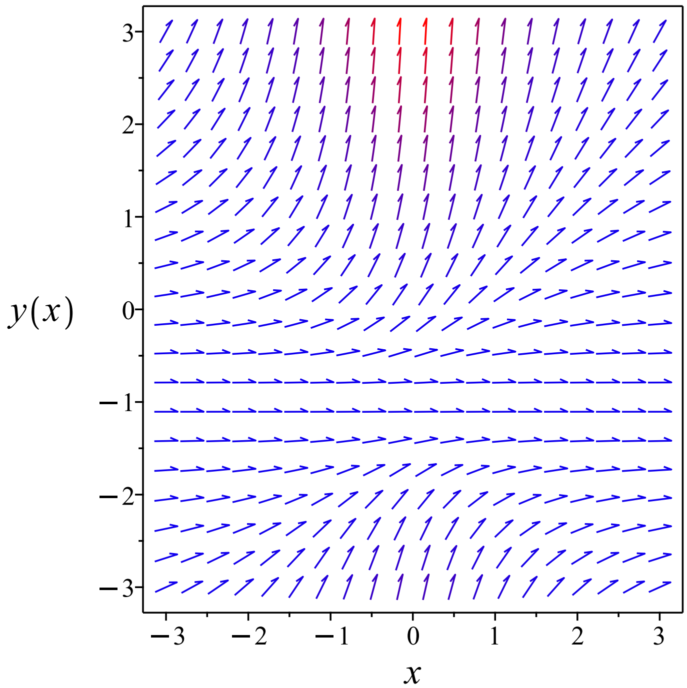


Figure 92: Slope field plot

Verification of solutions

$$y = \frac{-\arctan(x) - c_3 - 1}{\arctan(x) + c_3}$$

Verified OK.

3.10.5 Maple step by step solution

Let's solve

$(x^2+1)\,y' - (1+y)^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{(1+y)^2} = \frac{1}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{(1+y)^2}dx = \int \frac{1}{x^2+1}dx + c_1$

- Evaluate integral

$$-\frac{1}{1+y} = \arctan(x) + c_1$$

- Solve for y

$$y = -\frac{\arctan(x)+c_1+1}{\arctan(x)+c_1}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve((x^2+1)*diff(y(x),x) = (1+y(x))^2,y(x), singsol=all)
```

$$y(x) = \frac{-\arctan(x) - c_1 - 1}{\arctan(x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.19 (sec). Leaf size: 25

```
DSolve[(x^2+1)*y'[x]== (1+y[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\arctan(x) + 1 + c_1}{\arctan(x) + c_1}$$
$$y(x) \to -1$$

3.11 problem 11

Internal problem ID [37]
Internal file name [`OUTPUT/37_Sunday_June_05_2022_01_33_56_AM_89088322/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - xy^3 = 0}$$

3.11.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= x\,y^3 \end{aligned}$$

Where $f(x) = x$ and $g(y) = y^3$. Integrating both sides gives

$$\frac{1}{y^3}\,dy = x\,dx$$

$$\int \frac{1}{y^3}\,dy = \int x\,dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$$

Which results in

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}}$$

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}} \tag{1}$$

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}} \tag{2}$$

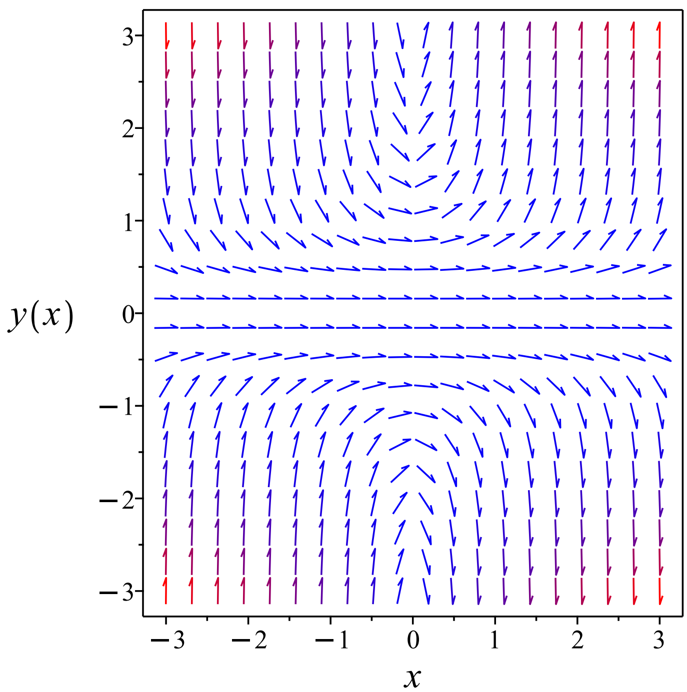


Figure 93: Slope field plot

Verification of solutions

$$y = -\frac{1}{\sqrt{-x^2 - 2c_1}}$$

Verified OK.

$$y = \frac{1}{\sqrt{-x^2 - 2c_1}}$$

Verified OK.

3.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x\,y^3$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 73: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{x^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x\, y^3$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{2R^2} + c_1 \tag{4}$$

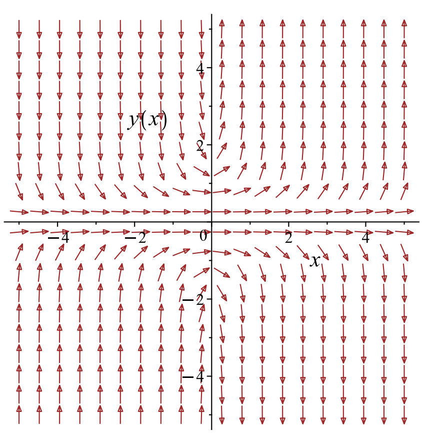
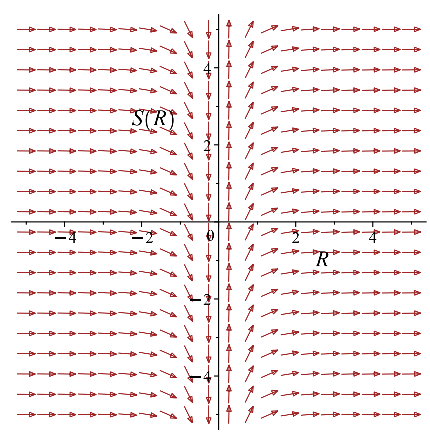
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x y^3$ 	$R = y$ $S = \frac{x^2}{2}$	$\frac{dS}{dR} = \frac{1}{R^3}$ 

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1 \tag{1}$$

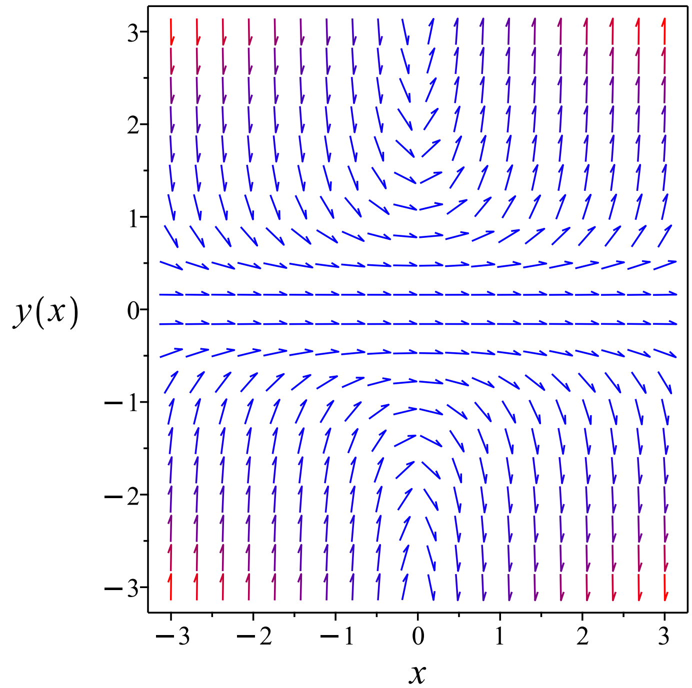


Figure 94: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = -\frac{1}{2y^2} + c_1$$

Verified OK.

3.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^3}\right)\mathrm{d}y &= (x)\,\mathrm{d}x \\ (-x)\,\mathrm{d}x + \left(\frac{1}{y^3}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x \\ N(x, y) &= \frac{1}{y^3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^3}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3}$. Therefore equation (4) becomes

$$
\frac{1}{y^3} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^3}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^3}\right) \mathrm{d}y$$

$$f(y) = -\frac{1}{2y^2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{1}{2y^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{1}{2y^2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{1}{2y^2} = c_1 \tag{1}$$

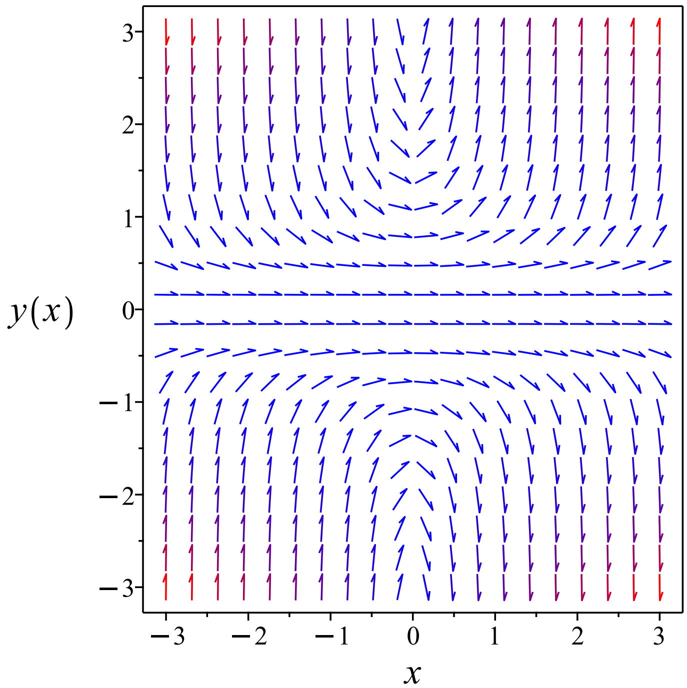


Figure 95: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{1}{2y^2} = c_1$$

Verified OK.

3.11.4 Maple step by step solution

Let's solve

$y' - xy^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3} dx = \int x dx + c_1$

- Evaluate integral

$$-\frac{1}{2y^2} = \frac{x^2}{2} + c_1$$

- Solve for y

$$\left\{y = \frac{1}{\sqrt{-x^2-2c_1}}, y = -\frac{1}{\sqrt{-x^2-2c_1}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x) = x*y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{-x^2+c_1}}$$
$$y(x) = -\frac{1}{\sqrt{-x^2+c_1}}$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 44

```
DSolve[y'[x] == x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{\sqrt{-x^2-2c_1}}$$
$$y(x) \to \frac{1}{\sqrt{-x^2-2c_1}}$$
$$y(x) \to 0$$

3.12 problem 12

Internal problem ID [38]
Internal file name [`OUTPUT/38_Sunday_June_05_2022_01_33_57_AM_44493475/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{yy' - x(1 + y^2) = 0}$$

3.12.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \frac{x(y^2 + 1)}{y} \end{aligned}$$

Where $f(x) = x$ and $g(y) = \frac{y^2+1}{y}$. Integrating both sides gives

$$\frac{1}{\frac{y^2+1}{y}}\, dy = x\, dx$$

$$\int \frac{1}{\frac{y^2+1}{y}}\, dy = \int x\, dx$$

$$\frac{\ln (y^2+1)}{2} = \frac{x^2}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y^2+1} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\sqrt{y^2+1} = c_2 e^{\frac{x^2}{2}}$$

The solution is

$$\sqrt{1+y^2} = c_2 e^{\frac{x^2}{2}+c_1}$$

Summary
The solution(s) found are the following

$$\sqrt{1+y^2} = c_2 e^{\frac{x^2}{2}+c_1} \tag{1}$$

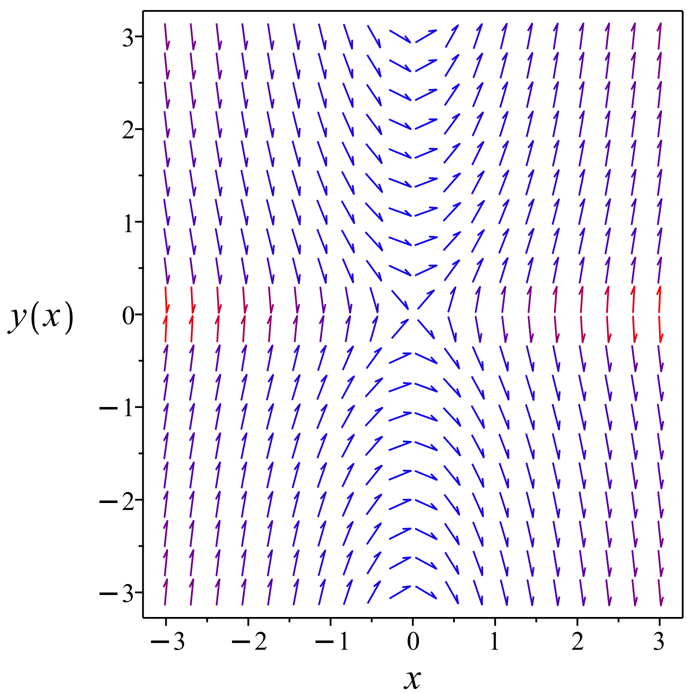


Figure 96: Slope field plot

Verification of solutions

$$\sqrt{1+y^2} = c_2 e^{\frac{x^2}{2}+c_1}$$

Verified OK.

3.12.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x(y^2+1)}{y} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 76: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{x(y^2+1)}{y}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{y}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\ln(R^2+1)}{2} + c_1 \tag{4}$$

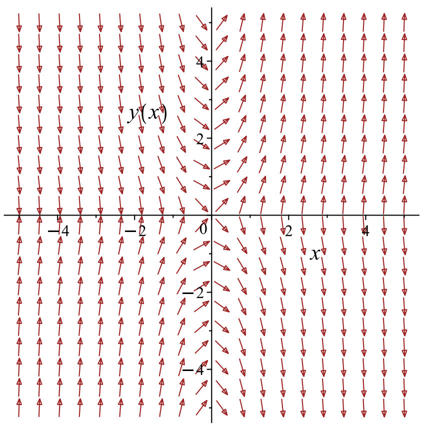
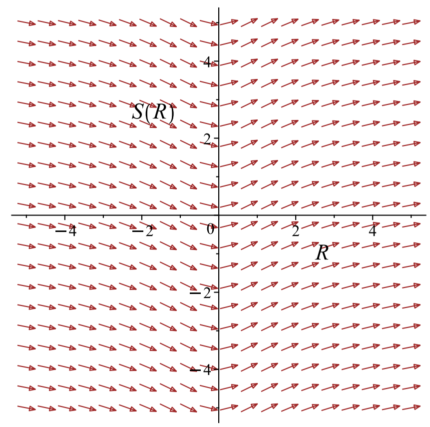
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Which simplifies to

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x(y^2+1)}{y}$
</td><td>$R = y$
$S = \frac{x^2}{2}$</td><td>$\frac{dS}{dR} = \frac{R}{R^2+1}$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1 \tag{1}$$

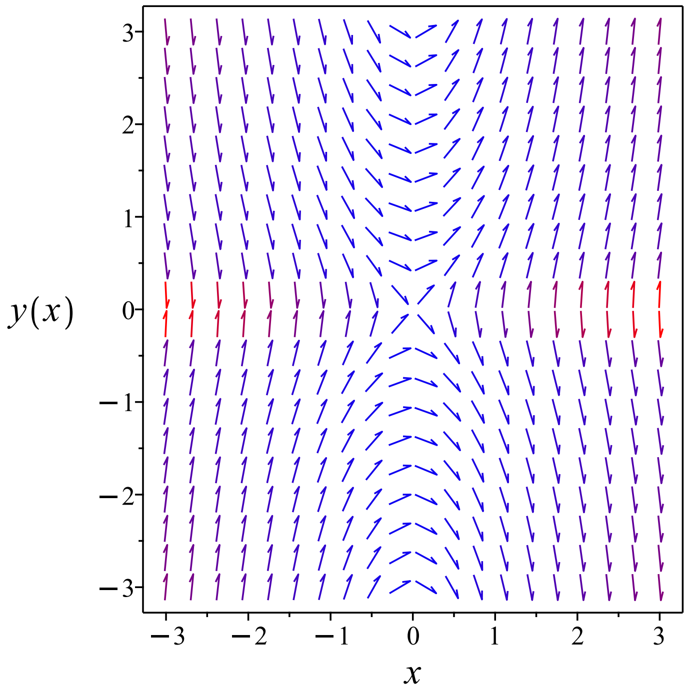


Figure 97: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = \frac{\ln(1+y^2)}{2} + c_1$$

Verified OK.

3.12.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{x(y^2+1)}{y} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = xy + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= x \\ f_1(x) &= x \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = x\,y^2 + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= w(x)\,x + x \\ w' &= 2xw + 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 2x \end{aligned}$$

Hence the ode is

$$w'(x) - 2w(x)\,x = 2x$$

The integrating factor μ is

$$\mu = e^{\int -2x dx}$$
$$= e^{-x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)(2x)$$
$$\frac{d}{dx}\left(e^{-x^2} w\right) = \left(e^{-x^2}\right)(2x)$$
$$d\left(e^{-x^2} w\right) = \left(2x\, e^{-x^2}\right) dx$$

Integrating gives

$$e^{-x^2} w = \int 2x\, e^{-x^2} dx$$
$$e^{-x^2} w = -e^{-x^2} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$w(x) = -e^{x^2} e^{-x^2} + c_1 e^{x^2}$$

which simplifies to

$$w(x) = -1 + c_1 e^{x^2}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -1 + c_1 e^{x^2}$$

Solving for y gives

$$y(x) = \sqrt{-1 + c_1 e^{x^2}}$$
$$y(x) = -\sqrt{-1 + c_1 e^{x^2}}$$

Summary

The solution(s) found are the following

$$y = \sqrt{-1 + c_1 e^{x^2}} \tag{1}$$
$$y = -\sqrt{-1 + c_1 e^{x^2}} \tag{2}$$

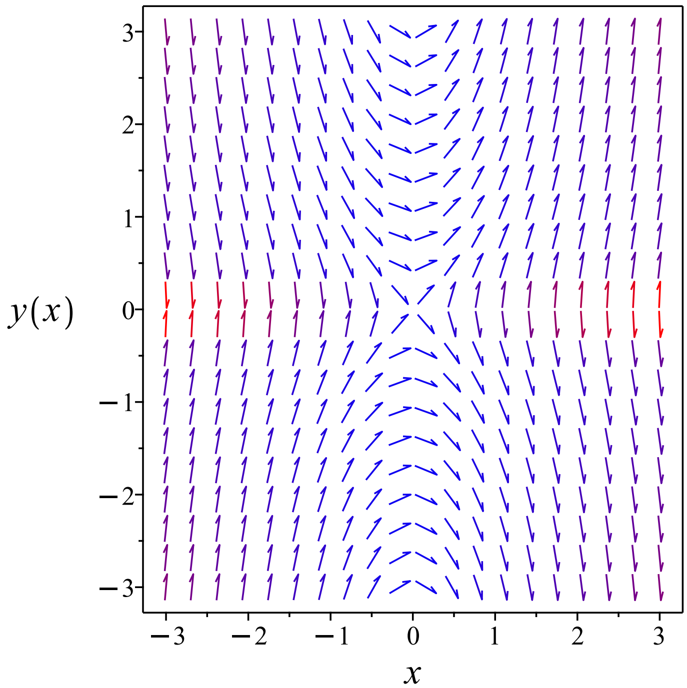


Figure 98: Slope field plot

Verification of solutions

$$y = \sqrt{-1 + c_1 e^{x^2}}$$

Verified OK.

$$y = -\sqrt{-1 + c_1 e^{x^2}}$$

Verified OK.

3.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{y}{y^2+1}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{y}{y^2+1}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=\frac{y}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{y}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{y}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{y}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{y}{y^2+1}\right) \mathrm{d}y \\
f(y) &= \frac{\ln(y^2+1)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} + \frac{\ln(y^2+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} + \frac{\ln(y^2+1)}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} + \frac{\ln(1+y^2)}{2} = c_1 \tag{1}$$

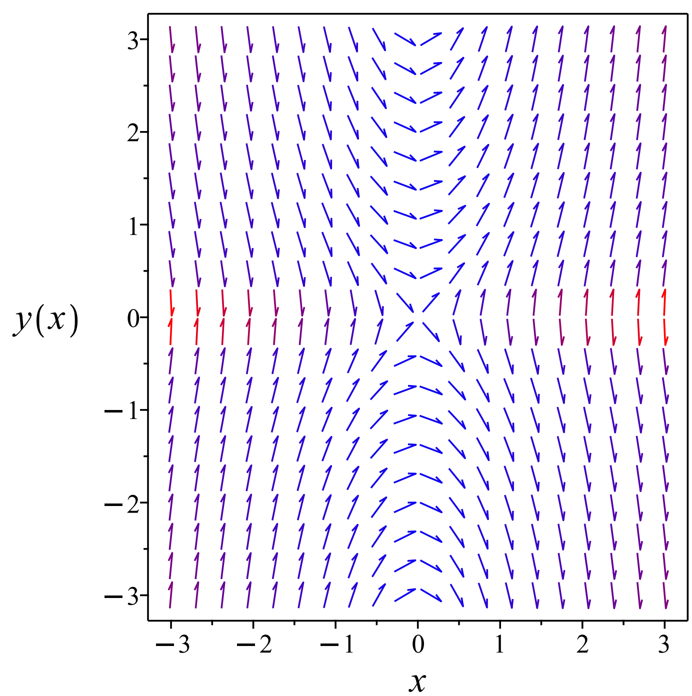


Figure 99: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} + \frac{\ln(1+y^2)}{2} = c_1$$

Verified OK.

3.12.5 Maple step by step solution

Let's solve

$yy' - x(1+y^2) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{yy'}{1+y^2} = x$

- Integrate both sides with respect to x

  $\int \frac{yy'}{1+y^2} dx = \int x dx + c_1$

- Evaluate integral

  $\frac{\ln(1+y^2)}{2} = \frac{x^2}{2} + c_1$

- Solve for y

  $\left\{y = \sqrt{-1 + e^{x^2+2c_1}}, y = -\sqrt{-1 + e^{x^2+2c_1}}\right\}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(y(x)*diff(y(x),x) = x*(1+y(x)^2),y(x), singsol=all)
```

$$y(x) = \sqrt{e^{x^2} c_1 - 1}$$
$$y(x) = -\sqrt{e^{x^2} c_1 - 1}$$

✓ Solution by Mathematica

Time used: 6.961 (sec). Leaf size: 57

```
DSolve[y[x]*y'[x] == x*(1+y[x]^2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\sqrt{-1 + e^{x^2+2c_1}}$$
$$y(x) \to \sqrt{-1 + e^{x^2+2c_1}}$$
$$y(x) \to -i$$
$$y(x) \to i$$

3.13 problem 14

Internal problem ID [39]
Internal file name [`OUTPUT/39_Sunday_June_05_2022_01_33_57_AM_34740163/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "differentialType", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{1+\sqrt{x}}{1+\sqrt{y}} = 0}$$

3.13.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{1+\sqrt{x}}{1+\sqrt{y}} \end{aligned}$$

Where $f(x) = 1+\sqrt{x}$ and $g(y) = \frac{1}{1+\sqrt{y}}$. Integrating both sides gives

$$\frac{1}{\frac{1}{1+\sqrt{y}}}\, dy = 1+\sqrt{x}\, dx$$

$$\int \frac{1}{\frac{1}{1+\sqrt{y}}} dy = \int 1 + \sqrt{x}\, dx$$

$$y + \frac{2y^{\frac{3}{2}}}{3} = \frac{2x^{\frac{3}{2}}}{3} + x + c_1$$

The solution is

$$y + \frac{2y^{\frac{3}{2}}}{3} - \frac{2x^{\frac{3}{2}}}{3} - x - c_1 = 0$$

Summary

The solution(s) found are the following

$$y + \frac{2y^{\frac{3}{2}}}{3} - \frac{2x^{\frac{3}{2}}}{3} - x - c_1 = 0 \tag{1}$$

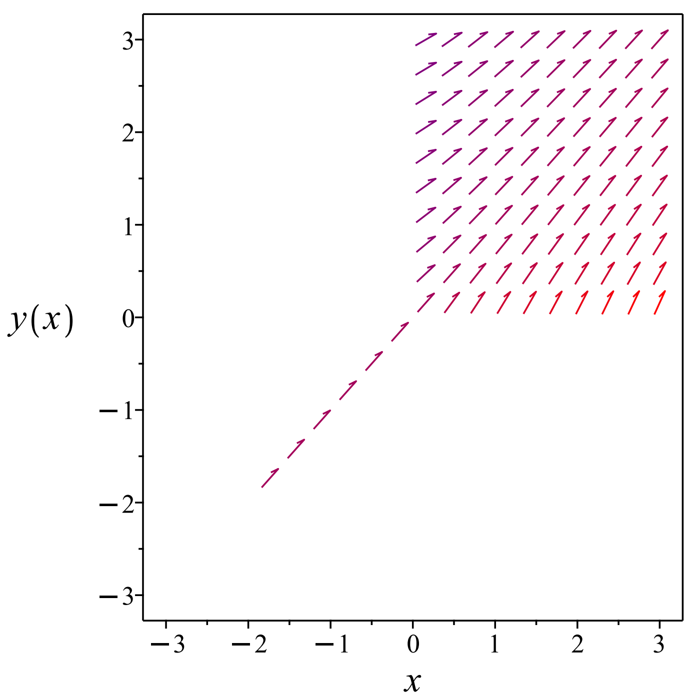


Figure 100: Slope field plot

Verification of solutions

$$y + \frac{2y^{\frac{3}{2}}}{3} - \frac{2x^{\frac{3}{2}}}{3} - x - c_1 = 0$$

Verified OK.

3.13.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{1+\sqrt{x}}{1+\sqrt{y}} \tag{1}$$

Which becomes

$$(1+\sqrt{y})\,dy = (1+\sqrt{x})\,dx \tag{2}$$

But the RHS is complete differential because

$$(1+\sqrt{x})\,dx = d\left(\frac{2x^{\frac{3}{2}}}{3}+x\right)$$

Hence (2) becomes

$$(1+\sqrt{y})\,dy = d\left(\frac{2x^{\frac{3}{2}}}{3}+x\right)$$

Integrating both sides gives gives these solutions

$$y = \left(\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{2} + \right.$$

$$y = \left(-\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{4} - \right.$$

$$y = \left(-\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{4} - \right.$$

Summary

The solution(s) found are the following

$$y = \left(\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{2} + \frac{\quad}{2\left(\right.} \right. + c_1 \tag{1}$$

$$y = \left(-\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{4} - \frac{\quad}{4} \right. + c_1 \tag{2}$$

$$y = \left(-\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{4} - \frac{\quad}{4} \right. + c_1 \tag{3}$$

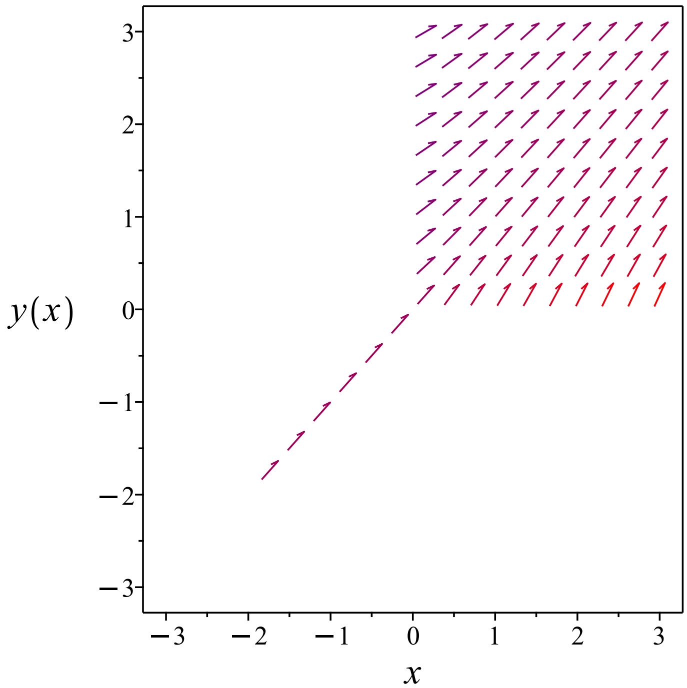


Figure 101: Slope field plot

Verification of solutions

$$y = \left(\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{2} + \frac{}{2\left(\right.} \right. + c_1$$

Verified OK.

$$y = \left(-\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{4} - \frac{}{4} \right. + c_1$$

Verified OK.

$$y = \left(-\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{4} - \frac{}{4} \right. + c_1$$

Verified OK.

3.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{1+\sqrt{x}}{1+\sqrt{y}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 79: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{1+\sqrt{x}} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x,y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta=0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{1+\sqrt{x}}} dx \end{aligned}$$

Which results in

$$S = \frac{2x^{\frac{3}{2}}}{3} + x$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{1+\sqrt{x}}{1+\sqrt{y}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 1+\sqrt{x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1+\sqrt{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1+\sqrt{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{2R^{\frac{3}{2}}}{3} + R + c_1 \tag{4}$$

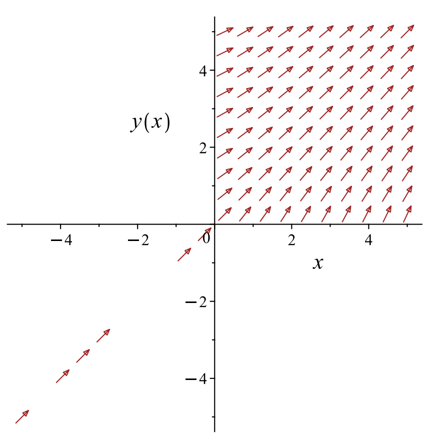
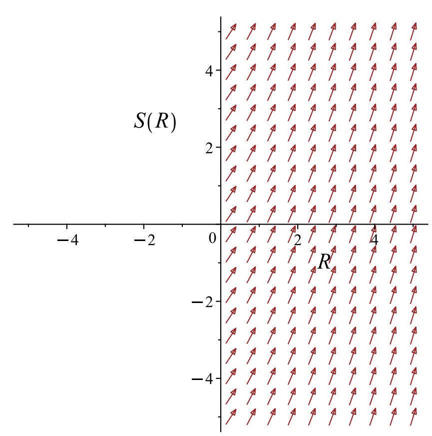
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2x^{\frac{3}{2}}}{3} + x = \frac{2y^{\frac{3}{2}}}{3} + y + c_1$$

Which simplifies to

$$\frac{2x^{\frac{3}{2}}}{3} + x = \frac{2y^{\frac{3}{2}}}{3} + y + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{1+\sqrt{x}}{1+\sqrt{y}}$ 	$R = y$ $S = \frac{2x^{\frac{3}{2}}}{3} + x$	$\frac{dS}{dR} = 1 + \sqrt{R}$ 

Summary
The solution(s) found are the following

$$\frac{2x^{\frac{3}{2}}}{3} + x = \frac{2y^{\frac{3}{2}}}{3} + y + c_1 \tag{1}$$

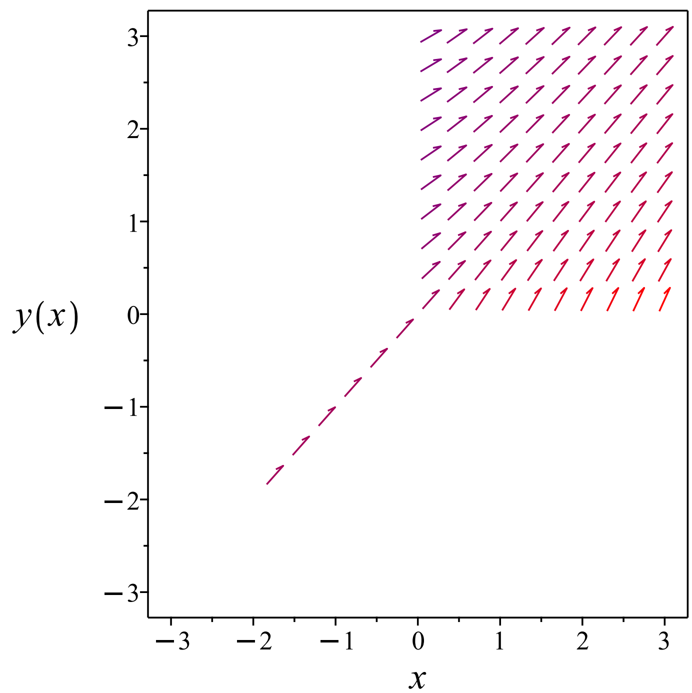


Figure 102: Slope field plot

Verification of solutions

$$\frac{2x^{\frac{3}{2}}}{3}+x=\frac{2y^{\frac{3}{2}}}{3}+y+c_1$$

Verified OK.

3.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(1 + \sqrt{y})\,\mathrm{d}y &= \left(1 + \sqrt{x}\right)\mathrm{d}x\\ \left(-1 - \sqrt{x}\right)\mathrm{d}x + (1 + \sqrt{y})\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -1 - \sqrt{x}\\ N(x, y) &= 1 + \sqrt{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-1 - \sqrt{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1 + \sqrt{y})\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -1 - \sqrt{x} \, \mathrm{d}x$$

$$\phi = -x - \frac{2x^{\frac{3}{2}}}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 1 + \sqrt{y}$. Therefore equation (4) becomes

$$1 + \sqrt{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1 + \sqrt{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (1 + \sqrt{y}) \, \mathrm{d}y$$

$$f(y) = y + \frac{2y^{\frac{3}{2}}}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \frac{2x^{\frac{3}{2}}}{3} + y + \frac{2y^{\frac{3}{2}}}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \frac{2x^{\frac{3}{2}}}{3} + y + \frac{2y^{\frac{3}{2}}}{3}$$

Summary
The solution(s) found are the following

$$-\frac{2x^{\frac{3}{2}}}{3} - x + \frac{2y^{\frac{3}{2}}}{3} + y = c_1 \tag{1}$$

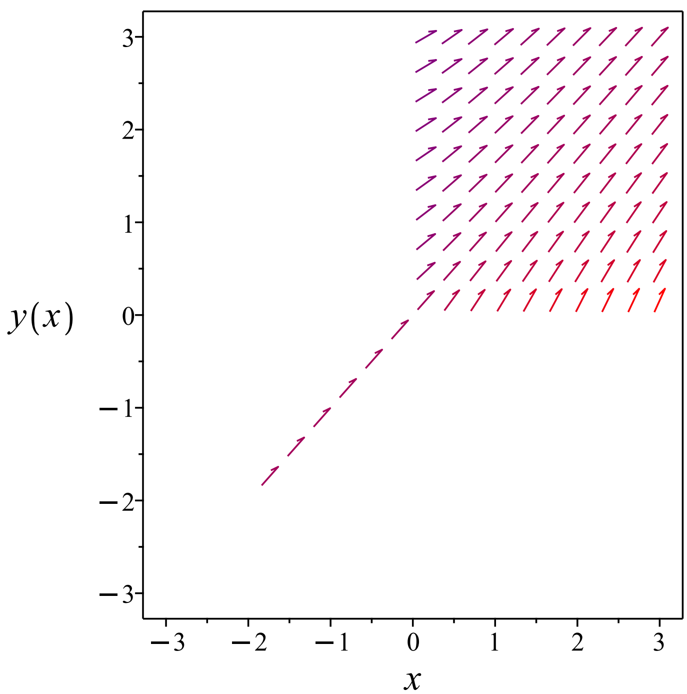


Figure 103: Slope field plot

Verification of solutions

$$-\frac{2x^{\frac{3}{2}}}{3} - x + \frac{2y^{\frac{3}{2}}}{3} + y = c_1$$

Verified OK.

3.13.5 Maple step by step solution

Let's solve

$y' - \frac{1+\sqrt{x}}{1+\sqrt{y}} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\left(1+\sqrt{y}\right) y' = 1+\sqrt{x}$

- Integrate both sides with respect to x

  $\int \left(1+\sqrt{y}\right) y' dx = \int \left(1+\sqrt{x}\right) dx + c_1$

- Evaluate integral

  $y + \frac{2y^{\frac{3}{2}}}{3} = \frac{2x^{\frac{3}{2}}}{3} + x + c_1$

- Solve for y

  $y = \left(\frac{\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c_1-3x+4x^3+12x^{\frac{3}{2}}c_1+12x^{\frac{5}{2}}+9c_1^2+18c_1x+9x^2}\right)^{\frac{1}{3}}}{2} + \frac{}{2\left(-1+4x^{\frac{3}{2}}+6c_1+6x+2\sqrt{-2x^{\frac{3}{2}}-3c\ldots}\right.}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x) = (1+x^(1/2))/(1+y(x)^(1/2)),y(x), singsol=all)
```

$$x + \frac{2x^{\frac{3}{2}}}{3} - y(x) - \frac{2y(x)^{\frac{3}{2}}}{3} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 4.529 (sec). Leaf size: 796

```
DSolve[y'[x]== (1+x^(1/2))/(1+y[x]^(1/2)),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-16x^{3/2} + \left(96x^{5/2} + 24(-3+4c_1)x^{3/2} + 8\sqrt{(2x^{3/2}+3x-1+3c_1)\left(2x^{3/2}+3x+3c_1\right)^3} + 32x^3 + 72x^2 \ldots\right.}{4\sqrt[3]{96x^{5/2} + 24(-3 \ldots}}$$

$$y(x) \to \frac{1}{16}\left(\frac{2(1+i\sqrt{3})\left(16x^{3/2}+24x-9+24c_1\right)}{\sqrt[3]{96x^{5/2} + 24(-3+4c_1)x^{3/2} + 8\sqrt{(2x^{3/2}+3x-1+3c_1)\left(2x^{3/2}+3x+3c_1\right)^3} + 32x^3 + 72x^2 + 3 \ldots}} + 2i\left(\sqrt{3}+i\right)\sqrt[3]{96x^{5/2} + 24(-3+4c_1)x^{3/2} + 8\sqrt{(2x^{3/2}+3x-1+3c_1)\left(2x^{3/2}+3x+3c_1\right)^3} + 32x^3 + 7 \ldots} + 12\right)$$

$$y(x) \to \frac{1}{16}\left(\frac{2(1-i\sqrt{3})\left(16x^{3/2}+24x-9+24c_1\right)}{\sqrt[3]{96x^{5/2} + 24(-3+4c_1)x^{3/2} + 8\sqrt{(2x^{3/2}+3x-1+3c_1)\left(2x^{3/2}+3x+3c_1\right)^3} + 32x^3 + 72x^2 + 3 \ldots}} - 2\left(1+i\sqrt{3}\right)\sqrt[3]{96x^{5/2} + 24(-3+4c_1)x^{3/2} + 8\sqrt{(2x^{3/2}+3x-1+3c_1)\left(2x^{3/2}+3x+3c_1\right)^3} + 32x^3 + 7 \ldots} + 12\right)$$

3.14 problem 15

Internal problem ID [40]
Internal file name [`OUTPUT/40_Sunday_June_05_2022_01_33_58_AM_70028064/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{(x-1)\, y^5}{x^2\,(-y+2y^3)} = 0}$$

3.14.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^4(x-1)}{x^2\,(2y^2-1)} \end{aligned}$$

Where $f(x) = \frac{x-1}{x^2}$ and $g(y) = \frac{y^4}{2y^2-1}$. Integrating both sides gives

$$\frac{1}{\frac{y^4}{2y^2-1}}\, dy = \frac{x-1}{x^2}\, dx$$

$$\int \frac{1}{\frac{y^4}{2y^2-1}}\, dy = \int \frac{x-1}{x^2}\, dx$$

$$-\frac{2}{y}+\frac{1}{3y^3}=\ln(x)+\frac{1}{x}+c_1$$

Which results in

$$y = \frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2 x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+18x\ln(x)\right.\right.}{}$$

$$+\frac{}{3(x\ln(x)+c_1x+1)\left(x\left(9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2\right.\right.}}$$

$$-\frac{2x}{3(x\ln(x)+c_1x+1)}$$

$$y = -\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2 x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+18x\ln(\right.\right.}{}$$

$$-\frac{}{3(x\ln(x)+c_1x+1)\left(x\left(9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2\right.\right.}}$$

$$-\frac{2x}{3(x\ln(x)+c_1x+1)}$$

$$+\frac{i\sqrt{3}\left(\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+18x\ln(x)+18c_1x-32x^2+9}\,x+3\sqrt{9\ln(x)^2x^2+\right.\right.}{}\right.}{}$$

$$y =$$

$$-\frac{4^{\frac{1}{3}}\left(x\left(9\ln\left(x\right)^{2}x^{2}+18\ln\left(x\right)c_{1}x^{2}+9c_{1}^{2}x^{2}+3\ln\left(x\right)\sqrt{9\ln\left(x\right)^{2}x^{2}+18\ln\left(x\right)c_{1}x^{2}+9c_{1}^{2}x^{2}+18x\ln\left(\right.}\right.\right.}{}$$

$$-\frac{}{3\left(x\ln\left(x\right)+c_{1}x+1\right)\left(x\left(9\ln\left(x\right)^{2}x^{2}+18\ln\left(x\right)c_{1}x^{2}+9c_{1}^{2}x^{2}+3\ln\left(x\right)\sqrt{9\ln\left(x\right)^{2}x^{2}+18\ln\left(x\right)c_{1}x^{2}}\right.\right.}$$

$$-\frac{2x}{3\left(x\ln\left(x\right)+c_{1}x+1\right)}$$

$$-\frac{i\sqrt{3}\left(\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^{2}x^{2}+18\ln(x)c_{1}x^{2}+9c_{1}^{2}x^{2}+3\ln(x)\sqrt{9\ln(x)^{2}x^{2}+18\ln(x)c_{1}x^{2}+9c_{1}^{2}x^{2}+18x\ln(x)+18c_{1}x-32x^{2}+9}\,x+3\sqrt{9\ln(x)^{2}x^{2}+}\right.\right.}{}\right.}{}$$

Summary

The solution(s) found are the following

$$y = \frac{4^{\frac{1}{3}} \Big(x \Big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 18 x \ln(x)} \cdots}{\cdots} + \frac{\cdots}{3 (x \ln(x) + c_1 x + 1) \Big(x \Big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2} \cdots} - \frac{2x}{3 (x \ln(x) + c_1 x + 1)} \tag{1}$$

$$y = -\frac{4^{\frac{1}{3}} \Big(x \Big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 18 x \ln(} \cdots}{\cdots} - \frac{\cdots}{3 (x \ln(x) + c_1 x + 1) \Big(x \Big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2} \cdots} - \frac{2x}{3 (x \ln(x) + c_1 x + 1)} + \frac{i\sqrt{3} \Bigg(\frac{4^{\frac{1}{3}} \big(x \big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 18 x \ln(x) + 18 c_1 x - 32 x^2 + 9}\, x + 3 \sqrt{9 \ln(x)^2 x^2 +} \cdots}{\cdots} \cdots}{\cdots} \tag{2}$$

$$y = -\frac{4^{\frac{1}{3}} \Big(x \Big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 18 x \ln(} \cdots}{\cdots} - \frac{\cdots}{3 (x \ln(x) + c_1 x + 1) \Big(x \Big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2} \cdots} - \frac{2x}{3 (x \ln(x) + c_1 x + 1)} - \frac{i\sqrt{3} \Bigg(\frac{4^{\frac{1}{3}} \big(x \big(9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 3 \ln(x) \sqrt{9 \ln(x)^2 x^2 + 18 \ln(x) c_1 x^2 + 9 c_1^2 x^2 + 18 x \ln(x) + 18 c_1 x - 32 x^2 + 9}\, x + 3 \sqrt{9 \ln(x)^2 x^2 +} \cdots}{\cdots} \cdots}{\cdots} \tag{3}$$

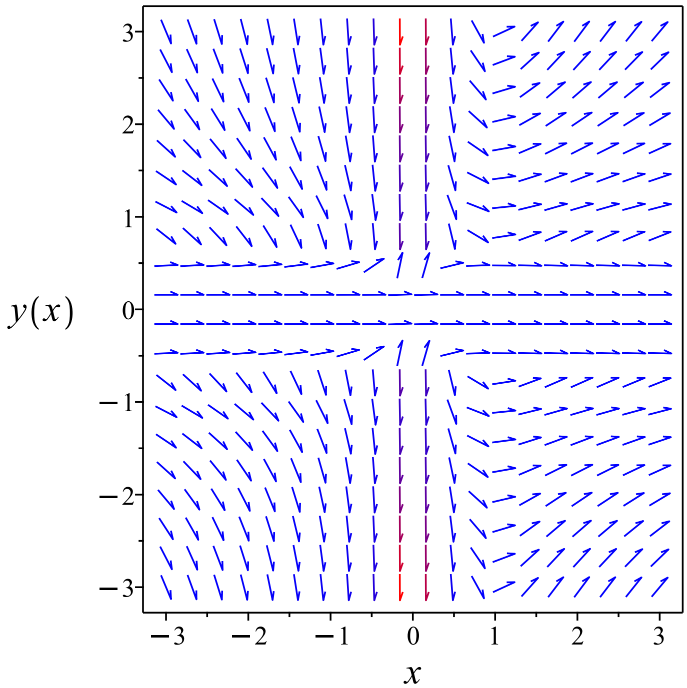


Figure 104: Slope field plot

Verification of solutions

$$y = \frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 3\ln(x)\sqrt{9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 18x\ln(x)\right.\right.}{}$$

$$+ \frac{}{3(x\ln(x) + c_1 x + 1)\left(x\left(9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 3\ln(x)\sqrt{9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2\right.\right.}}$$

$$- \frac{2x}{3(x\ln(x) + c_1 x + 1)}$$

Verified OK.

$$y = -\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 3\ln(x)\sqrt{9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 18x\ln(\right.\right.}{}$$

$$- \frac{}{3(x\ln(x) + c_1 x + 1)\left(x\left(9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 3\ln(x)\sqrt{9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2\right.\right.}}$$

$$- \frac{2x}{3(x\ln(x) + c_1 x + 1)}$$

$$+ \frac{i\sqrt{3}\left(\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+18x\ln(x)+18c_1x-32x^2+9}\,x+3\sqrt{9\ln(x)^2x^2+\right.\right.}{}\right.}{}$$

Verified OK.

$$y = -\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 3\ln(x)\sqrt{9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 18x\ln(\right.\right.}{}$$

$$- \frac{}{3(x\ln(x) + c_1 x + 1)\left(x\left(9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2 + 9c_1^2 x^2 + 3\ln(x)\sqrt{9\ln(x)^2 x^2 + 18\ln(x) c_1 x^2\right.\right.}}$$

$$- \frac{2x}{3(x\ln(x) + c_1 x + 1)}$$

$$- \frac{i\sqrt{3}\left(\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+18x\ln(x)+18c_1x-32x^2+9}\,x+3\sqrt{9\ln(x)^2x^2+\right.\right.}{}\right.}{}$$

Verified OK.

3.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^4(x-1)}{x^2(2y^2-1)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 82: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{x^2}{x-1}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$S = \int \frac{1}{\xi} dx$$
$$= \int \frac{1}{\frac{x^2}{x-1}} dx$$

Which results in

$$S = \ln(x) + \frac{1}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y^4(x-1)}{x^2(2y^2-1)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 0\\ R_y &= 1\\ S_x &= \frac{x-1}{x^2}\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2y^2-1}{y^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R^2-1}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2}{R} + \frac{1}{3R^3} + c_1 \tag{4}$$

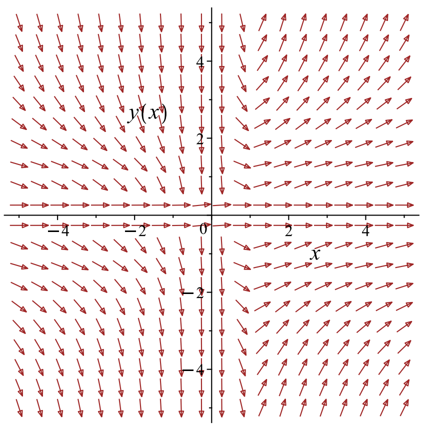
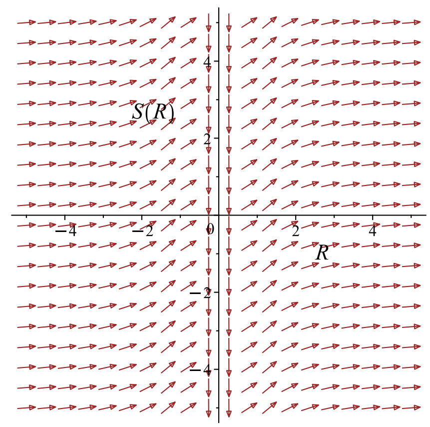
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x\ln(x)+1}{x} = -\frac{2}{y} + \frac{1}{3y^3} + c_1$$

Which simplifies to

$$\frac{x\ln(x)+1}{x} = -\frac{2}{y} + \frac{1}{3y^3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^4(x-1)}{x^2(2y^2-1)}$ 	$R = y$ $S = \frac{x \ln(x) + 1}{x}$	$\frac{dS}{dR} = \frac{2R^2-1}{R^4}$ 

Summary

The solution(s) found are the following

$$\frac{x \ln(x) + 1}{x} = -\frac{2}{y} + \frac{1}{3y^3} + c_1 \tag{1}$$

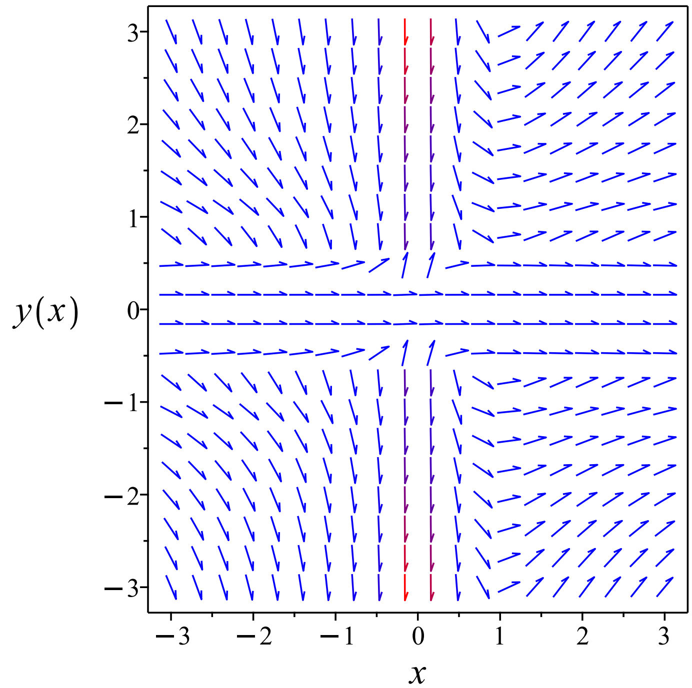


Figure 105: Slope field plot

Verification of solutions

$$\frac{x\ln(x)+1}{x} = -\frac{2}{y} + \frac{1}{3y^3} + c_1$$

Verified OK.

3.14.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2y^2-1}{y^4}\right)\mathrm{d}y = \left(\frac{x-1}{x^2}\right)\mathrm{d}x$$
$$\left(-\frac{x-1}{x^2}\right)\mathrm{d}x + \left(\frac{2y^2-1}{y^4}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x-1}{x^2}$$
$$N(x,y) = \frac{2y^2-1}{y^4}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x-1}{x^2}\right)$$
$$= 0$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2y^2-1}{y^4}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x-1}{x^2}\,\mathrm{d}x\\ \phi &= -\ln(x) - \frac{1}{x} + f(y)\end{aligned} \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y^2-1}{y^4}$. Therefore equation (4) becomes

$$\frac{2y^2-1}{y^4} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2y^2-1}{y^4}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{2y^2-1}{y^4}\right) \mathrm{d}y$$
$$f(y) = -\frac{2}{y} + \frac{1}{3y^3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{1}{x} - \frac{2}{y} + \frac{1}{3y^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{1}{x} - \frac{2}{y} + \frac{1}{3y^3}$$

Summary
The solution(s) found are the following

$$-\ln(x) - \frac{1}{x} - \frac{2}{y} + \frac{1}{3y^3} = c_1 \tag{1}$$

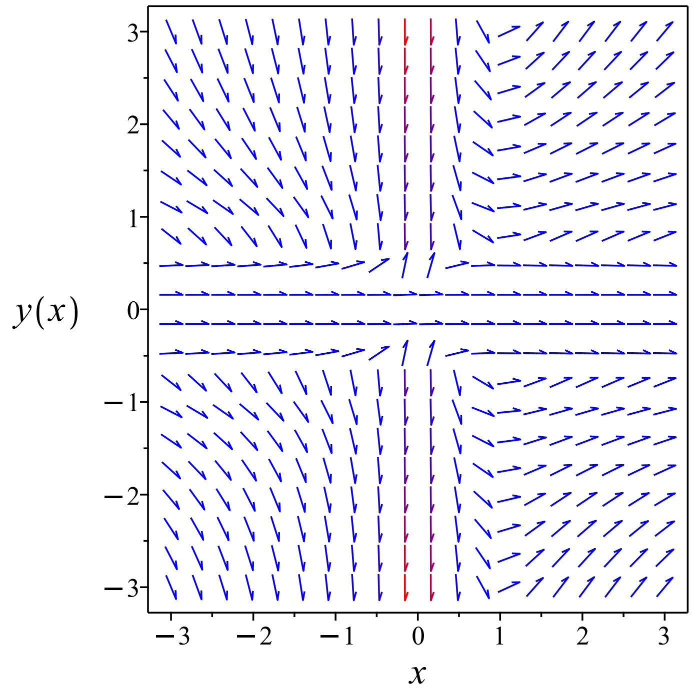


Figure 106: Slope field plot

Verification of solutions

$$-\ln(x) - \frac{1}{x} - \frac{2}{y} + \frac{1}{3y^3} = c_1$$

Verified OK.

3.14.4 Maple step by step solution

Let's solve

$y' - \frac{(x-1)y^5}{x^2(-y+2y^3)} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'(-y+2y^3)}{y^5} = \frac{x-1}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'(-y+2y^3)}{y^5} dx = \int \frac{x-1}{x^2} dx + c_1$

- Evaluate integral

  $-\frac{2}{y}+\frac{1}{3y^3}=\ln(x)+\frac{1}{x}+c_1$

- Solve for y

  $$y=\frac{4^{\frac{1}{3}}\left(x\left(9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+3\ln(x)\sqrt{9\ln(x)^2x^2+18\ln(x)c_1x^2+9c_1^2x^2+18x\ln(x)+18c_1x-32x^2+9}\,x+3\sqrt{9\ln(x)^2x^2-}\right.\right.}{}$$

<u>Maple trace</u>

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 844

```
dsolve(diff(y(x),x) = (-1+x)*y(x)^5/x^2/(-y(x)+2*y(x)^3),y(x), singsol=all)
```

$$y(x) = \frac{8x^2 2^{\frac{1}{3}} - 4x\left(3x(x\ln(x)+c_1x+1)\sqrt{9+9\ln(x)^2x^2+18(c_1x^2+x)\ln(x)+(9c_1^2-32)x^2+18c_1x}+9\right.}{\left(3x(x\ln(x)+c_1x+1)\sqrt{9+9\right.}$$

$$y(x) = -\frac{8x\left(3x(x\ln(x)+c_1x+1)\sqrt{9+9\ln(x)^2x^2+18(c_1x^2+x)\ln(x)+(9c_1^2-32)x^2+18c_1x}+9(x\ln(x\right.}{\left(3x(x\ln(x)+c_1x-\right.}$$

$$y(x) = \frac{-8x\left(3x(x\ln(x)+c_1x+1)\sqrt{9+9\ln(x)^2x^2+18(c_1x^2+x)\ln(x)+(9c_1^2-32)x^2+18c_1x}+9(x\ln(x\right.}{\left(3x(x\ln(x)+c_1x+\right.}$$

✓ Solution by Mathematica

Time used: 19.626 (sec). Leaf size: 842

```
DSolve[y'[x] == (-1+x)*y[x]^5/x^2/(-y[x]+2*y[x]^3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{8\sqrt[3]{2}x^2}{\sqrt[3]{16x^3 - 9x^3\log^2(x) - 9c_1{}^2x^3 - 18c_1x^2 + 3\sqrt{x^2(x\log(x) + c_1x + 1)^2\left(9x^2\log^2(x) + \left(-32 + 9c_1{}^2\right)x^2}}}$$

$$y(x) \to \frac{8\sqrt[3]{2}\left(1+i\sqrt{3}\right)x^2}{\sqrt[3]{16x^3 - 9x^3\log^2(x) - 9c_1{}^2x^3 - 18c_1x^2 + 3\sqrt{x^2(x\log(x) + c_1x + 1)^2\left(9x^2\log^2(x) + \left(-32 + 9c_1{}^2\right)x^2 -}}}$$

$$y(x) \to \frac{8\sqrt[3]{2}\left(1-i\sqrt{3}\right)x^2}{\sqrt[3]{16x^3 - 9x^3\log^2(x) - 9c_1{}^2x^3 - 18c_1x^2 + 3\sqrt{x^2(x\log(x) + c_1x + 1)^2\left(9x^2\log^2(x) + \left(-32 + 9c_1{}^2\right)x^2 -}}}$$

$$y(x) \to 0$$

3.15 problem 16

Internal problem ID [41]
Internal file name [`OUTPUT/41_Sunday_June_05_2022_01_33_59_AM_89324613/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{(x^2+1)\tan(y)\,y' = x}$$

3.15.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x}{(x^2+1)\tan(y)} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+1}$ and $g(y) = \frac{1}{\tan(y)}$. Integrating both sides gives

$$\frac{1}{\frac{1}{\tan(y)}}\,dy = \frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{\frac{1}{\tan(y)}}\,dy = \int \frac{x}{x^2+1}\,dx$$

$$-\ln(\cos(y)) = \frac{\ln(x^2+1)}{2} + c_1$$

Raising both side to exponential gives

$$\frac{1}{\cos(y)} = e^{\frac{\ln(x^2+1)}{2}+c_1}$$

Which simplifies to

$$\sec(y) = c_2\sqrt{x^2+1}$$

Which simplifies to

$$y = \operatorname{arcsec}\left(c_2\sqrt{x^2+1}\,e^{c_1}\right)$$

Summary

The solution(s) found are the following

$$y = \operatorname{arcsec}\left(c_2\sqrt{x^2+1}\,e^{c_1}\right) \tag{1}$$

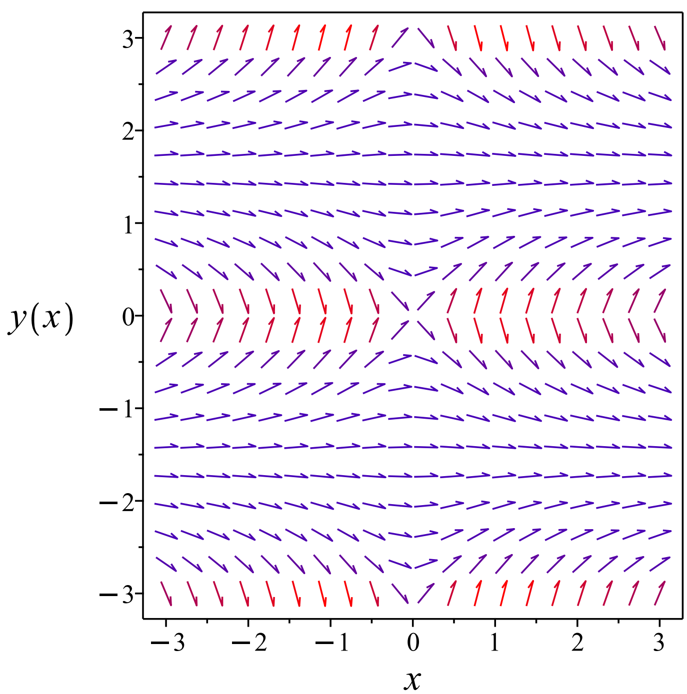


Figure 107: Slope field plot

Verification of solutions

$$y = \operatorname{arcsec}\left(c_2\sqrt{x^2+1}\,e^{c_1}\right)$$

Verified OK.

3.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x}{(x^2+1)\tan(y)} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 85: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = \frac{x^2+1}{x}$$
$$\eta(x,y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{x^2+1}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{\ln(x^2+1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{(x^2+1)\tan(y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x}{x^2+1} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \tan(y) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \tan(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(\cos(R)) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+1)}{2} = -\ln(\cos(y)) + c_1$$

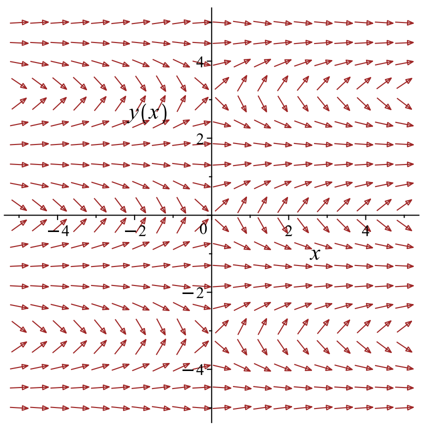
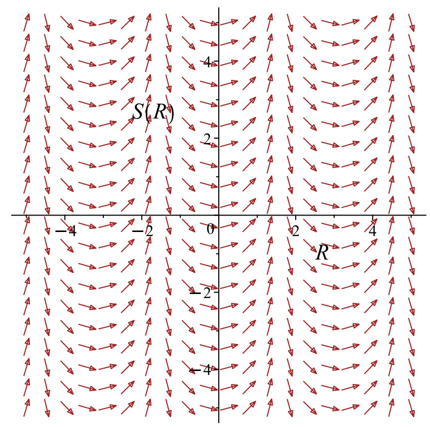
Which simplifies to

$$\frac{\ln(x^2+1)}{2} = -\ln(\cos(y)) + c_1$$

Which gives

$$y = \arccos\left(\mathrm{e}^{-\frac{\ln(x^2+1)}{2}+c_1}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{(x^2+1)\tan(y)}$ 	$R = y$ $S = \frac{\ln(x^2+1)}{2}$	$\frac{dS}{dR} = \tan(R)$ 

Summary

The solution(s) found are the following

$$y = \arccos\left(\mathrm{e}^{-\frac{\ln(x^2+1)}{2}+c_1}\right) \tag{1}$$

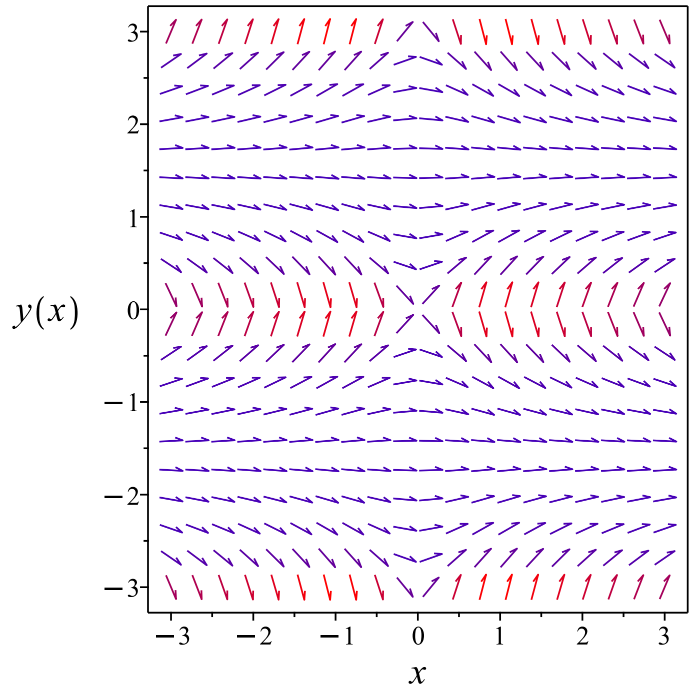


Figure 108: Slope field plot

Verification of solutions

$$y = \arccos\left(e^{-\frac{\ln(x^2+1)}{2}+c_1}\right)$$

Verified OK.

3.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(\tan(y))\,\mathrm{d}y = \left(\frac{x}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + (\tan(y))\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x}{x^2+1}$$
$$N(x,y) = \tan(y)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\tan(y))$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -\frac{x}{x^2+1} \, \mathrm{d}x$$
$$\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \tan(y)$. Therefore equation (4) becomes

$$\tan(y) = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \tan(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\tan(y)) \, \mathrm{d}y$$
$$f(y) = -\ln(\cos(y)) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+1)}{2} - \ln(\cos(y)) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+1)}{2} - \ln(\cos(y))$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+1)}{2} - \ln(\cos(y)) = c_1 \tag{1}$$

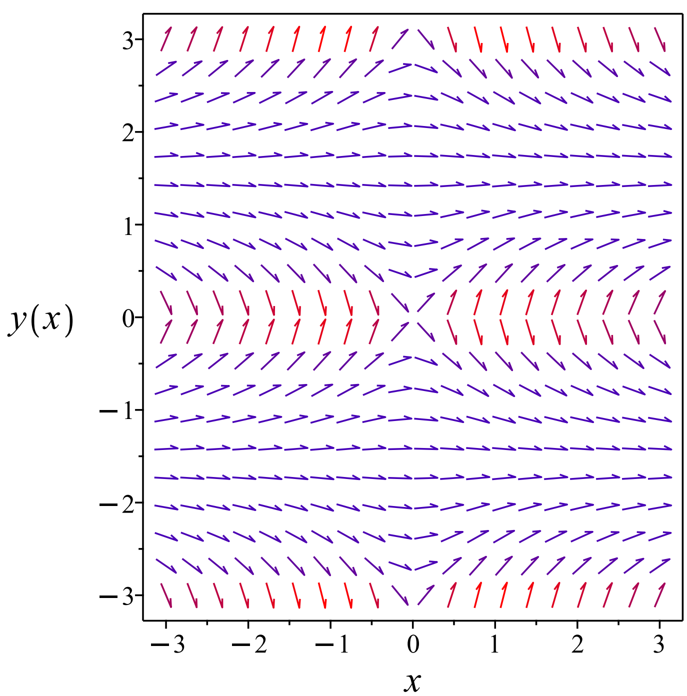


Figure 109: Slope field plot

Verification of solutions

$$-\frac{\ln(x^2+1)}{2} - \ln(\cos(y)) = c_1$$

Verified OK.

3.15.4 Maple step by step solution

Let's solve

$(x^2+1)\tan(y)\,y' = x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\tan(y)\,y' = \frac{x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \tan(y)\,y'dx = \int \frac{x}{x^2+1}dx + c_1$

- Evaluate integral

  $-\ln(\cos(y)) = \frac{\ln(x^2+1)}{2} + c_1$

- Solve for y

  $y = \arccos\left(\mathrm{e}^{-\frac{\ln(x^2+1)}{2}-c_1}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve((x^2+1)*tan(y(x))*diff(y(x),x) = x,y(x), singsol=all)
```

$$y(x) = \arccos\left(\frac{1}{\sqrt{x^2+1}\,c_1}\right)$$

✓ Solution by Mathematica

Time used: 15.547 (sec). Leaf size: 63

```
DSolve[(x^2+1)*Tan[y[x]]*y'[x] == x,y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -\arccos\left(\frac{e^{-c_1}}{\sqrt{x^2+1}}\right) \\
y(x) &\to \arccos\left(\frac{e^{-c_1}}{\sqrt{x^2+1}}\right) \\
y(x) &\to -\frac{\pi}{2} \\
y(x) &\to \frac{\pi}{2}
\end{aligned}
$$

3.16 problem 17

Internal problem ID [42]
Internal file name [`OUTPUT/42_Sunday_June_05_2022_01_34_00_AM_31480899/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - y - yx = x + 1}$$

3.16.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y+1)(x+1) \end{aligned}$$

Where $f(x) = x + 1$ and $g(y) = y + 1$. Integrating both sides gives

$$\frac{1}{y+1}\,dy = x + 1\,dx$$

$$\int \frac{1}{y+1}\,dy = \int x + 1\,dx$$

$$\ln(y+1) = \frac{1}{2}x^2 + x + c_1$$

Raising both side to exponential gives

$$y + 1 = e^{\frac{1}{2}x^2 + x + c_1}$$

Which simplifies to

$$y + 1 = c_2 e^{\frac{1}{2}x^2 + x}$$

Summary

The solution(s) found are the following

$$y = c_2 e^{\frac{1}{2}x^2 + x + c_1} - 1 \tag{1}$$

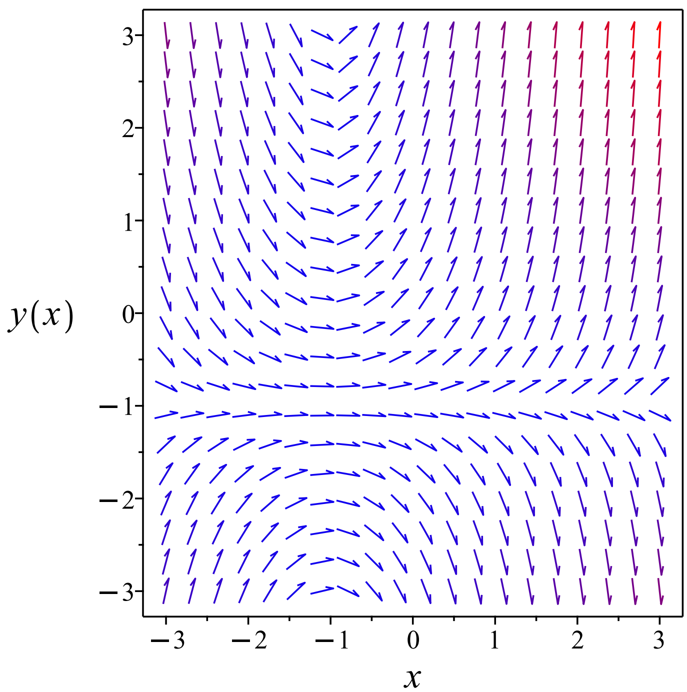


Figure 110: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{1}{2}x^2 + x + c_1} - 1$$

Verified OK.

3.16.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -x - 1\\ q(x) &= x + 1\end{aligned}$$

Hence the ode is

$$y' + (-x - 1)\, y = x + 1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int(-x-1)dx}\\ &= \mathrm{e}^{-\frac{1}{2}x^2-x}\end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{-\frac{x(2+x)}{2}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\,(x+1)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-\frac{x(2+x)}{2}} y\right) &= \left(\mathrm{e}^{-\frac{x(2+x)}{2}}\right)(x+1)\\ \mathrm{d}\left(\mathrm{e}^{-\frac{x(2+x)}{2}} y\right) &= \left((x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-\frac{x(2+x)}{2}} y &= \int (x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}\,\mathrm{d}x\\ \mathrm{e}^{-\frac{x(2+x)}{2}} y &= -\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-\frac{x(2+x)}{2}}$ results in

$$y = -\mathrm{e}^{\frac{x(2+x)}{2}}\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1\mathrm{e}^{\frac{x(2+x)}{2}}$$

which simplifies to

$$y = -1 + c_1\mathrm{e}^{\frac{x(2+x)}{2}}$$

Summary

The solution(s) found are the following

$$y = -1 + c_1 e^{\frac{x(2+x)}{2}} \tag{1}$$

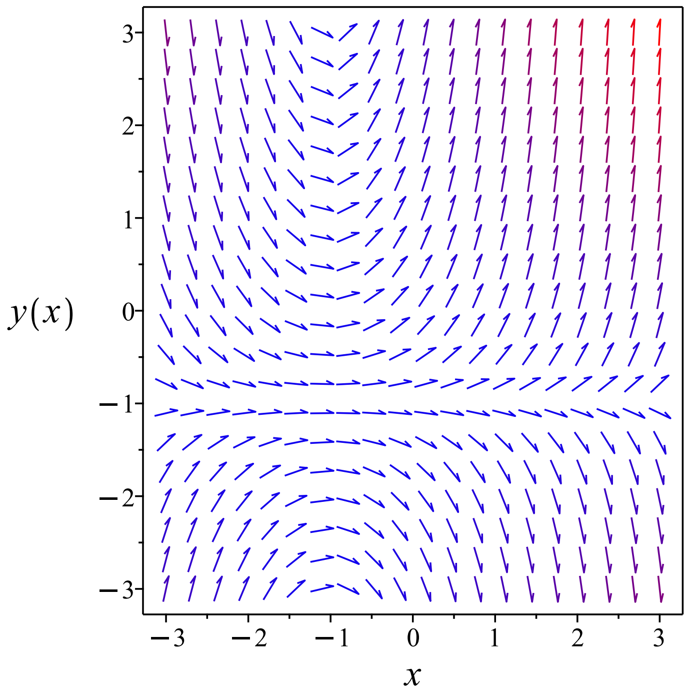


Figure 111: Slope field plot

Verification of solutions

$$y = -1 + c_1 e^{\frac{x(2+x)}{2}}$$

Verified OK.

3.16.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = yx + x + y + 1$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 88: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{\frac{1}{2}x^2+x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x}+\eta \frac{\partial}{\partial y}\right) S(x, y)=1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi=0$ then in this special case

$$R=x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} d y \\ &= \int \frac{1}{\mathrm{e}^{\frac{1}{2} x^{2}+x}} d y \end{aligned}$$

Which results in

$$S=\mathrm{e}^{-\frac{1}{2} x^{2}-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{d S}{d R}=\frac{S_{x}+\omega(x, y) S_{y}}{R_{x}+\omega(x, y) R_{y}} \tag{2}$$

Where in the above $R_{x}, R_{y}, S_{x}, S_{y}$ are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y)=y x+x+y+1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_{x} &= 1 \\ R_{y} &= 0 \\ S_{x} &= -(x+1) \mathrm{e}^{-\frac{x(2+x)}{2}} y \\ S_{y} &= \mathrm{e}^{-\frac{x(2+x)}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{d S}{d R}=(x+1) \mathrm{e}^{-\frac{x(2+x)}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{d S}{d R}=(R+1) \mathrm{e}^{-\frac{R(2+R)}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-\frac{R(2+R)}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x(2+x)}{2}} y = -\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1$$

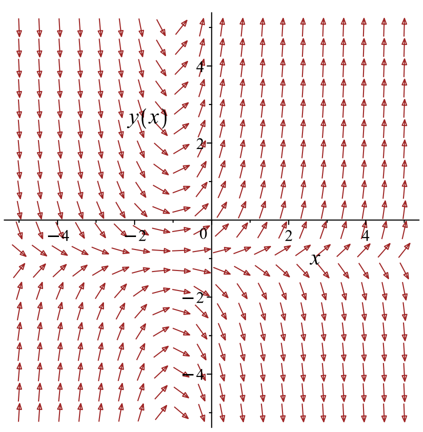
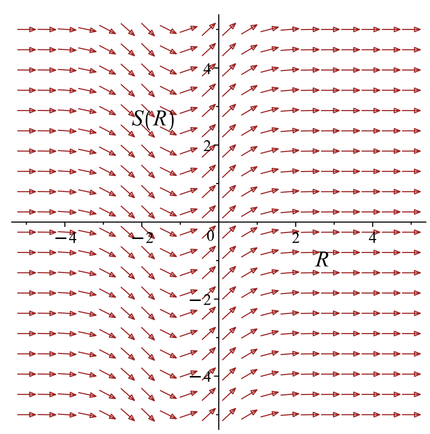
Which simplifies to

$$\mathrm{e}^{-\frac{x(2+x)}{2}} y = -\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1$$

Which gives

$$y = -\left(\mathrm{e}^{-\frac{x(2+x)}{2}} - c_1\right) \mathrm{e}^{\frac{x(2+x)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = yx + x + y + 1$ 	$R = x$ $S = \mathrm{e}^{-\frac{x(2+x)}{2}} y$	$\frac{dS}{dR} = (R + 1)\,\mathrm{e}^{-\frac{R(2+R)}{2}}$ 

Summary
The solution(s) found are the following

$$y = -\left(\mathrm{e}^{-\frac{x(2+x)}{2}} - c_1\right) \mathrm{e}^{\frac{x(2+x)}{2}} \tag{1}$$

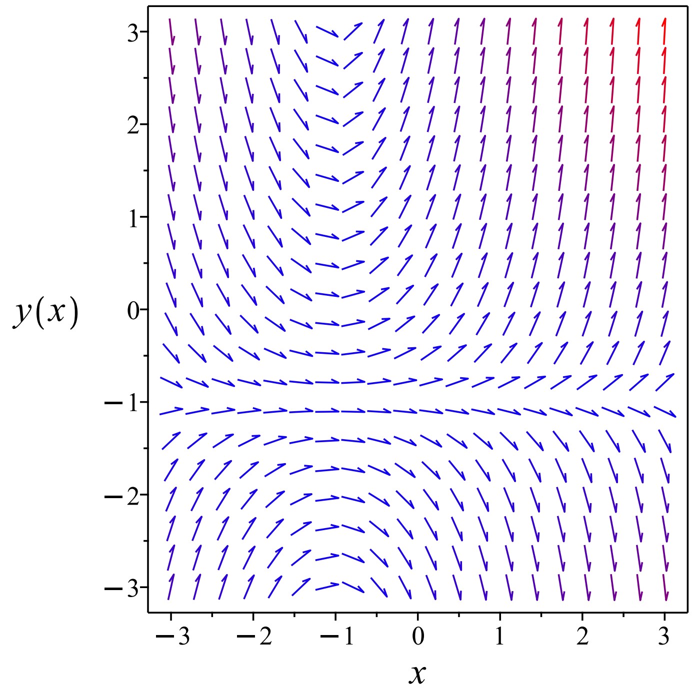


Figure 112: Slope field plot

Verification of solutions

$$y = -\left(\mathrm{e}^{-\frac{x(2+x)}{2}} - c_1\right) \mathrm{e}^{\frac{x(2+x)}{2}}$$

Verified OK.

3.16.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y+1}\right) \mathrm{d}y = (x+1)\, \mathrm{d}x$$
$$(-x-1)\, \mathrm{d}x + \left(\frac{1}{y+1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x - 1$$
$$N(x, y) = \frac{1}{y+1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x-1)$$
$$= 0$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y+1}\right) \\ = 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -x - 1 \,\mathrm{d}x$$

$$\phi = -\frac{1}{2}x^2 - x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$\frac{1}{y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y+1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y+1}\right) \mathrm{d}y$$

$$f(y) = \ln(y+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - x + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - x + \ln(y+1)$$

The solution becomes

$$y = e^{\frac{1}{2}x^2+x+c_1} - 1$$

Summary

The solution(s) found are the following

$$y = e^{\frac{1}{2}x^2+x+c_1} - 1 \tag{1}$$

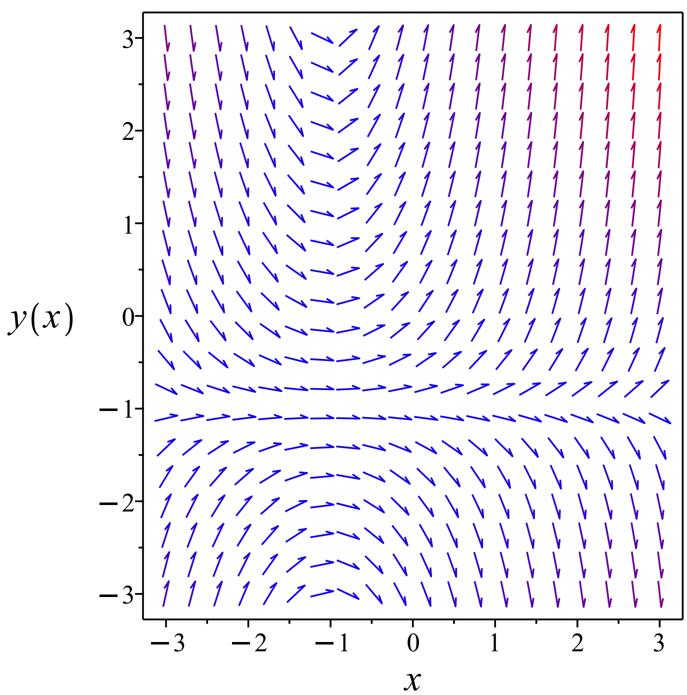


Figure 113: Slope field plot

Verification of solutions

$$y = e^{\frac{1}{2}x^2+x+c_1} - 1$$

Verified OK.

3.16.5 Maple step by step solution

Let's solve

$y' - y - yx = x + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = x + 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y} dx = \int (x + 1) dx + c_1$

- Evaluate integral

  $\ln(1 + y) = \frac{1}{2}x^2 + x + c_1$

- Solve for y

  $y = e^{\frac{1}{2}x^2+x+c_1} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x) = 1+x+y(x)+x*y(x),y(x), singsol=all)
```

$$y(x) = -1 + e^{\frac{x(2+x)}{2}} c_1$$

✓ Solution by Mathematica

Time used: 0.074 (sec). Leaf size: 25

```
DSolve[y'[x] == 1+x+y[x]+x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -1 + c_1 e^{\frac{1}{2}x(x+2)}$$
$$y(x) \to -1$$

3.17 problem 18

Internal problem ID [43]
Internal file name [`OUTPUT/43_Sunday_June_05_2022_01_34_00_AM_93942449/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y'x^2 - y^2 + x^2y^2 = -x^2 + 1}$$

3.17.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{(x^2-1)(-y^2-1)}{x^2} \end{aligned}$$

Where $f(x) = \frac{x^2-1}{x^2}$ and $g(y) = -y^2 - 1$. Integrating both sides gives

$$\frac{1}{-y^2-1}\,dy = \frac{x^2-1}{x^2}\,dx$$

$$\int \frac{1}{-y^2-1}\,dy = \int \frac{x^2-1}{x^2}\,dx$$

$$-\arctan(y) = x + \frac{1}{x} + c_1$$

Which results in

$$y = -\tan\left(\frac{c_1x + x^2 + 1}{x}\right)$$

Summary

The solution(s) found are the following

$$y = -\tan\left(\frac{c_1x + x^2 + 1}{x}\right) \tag{1}$$

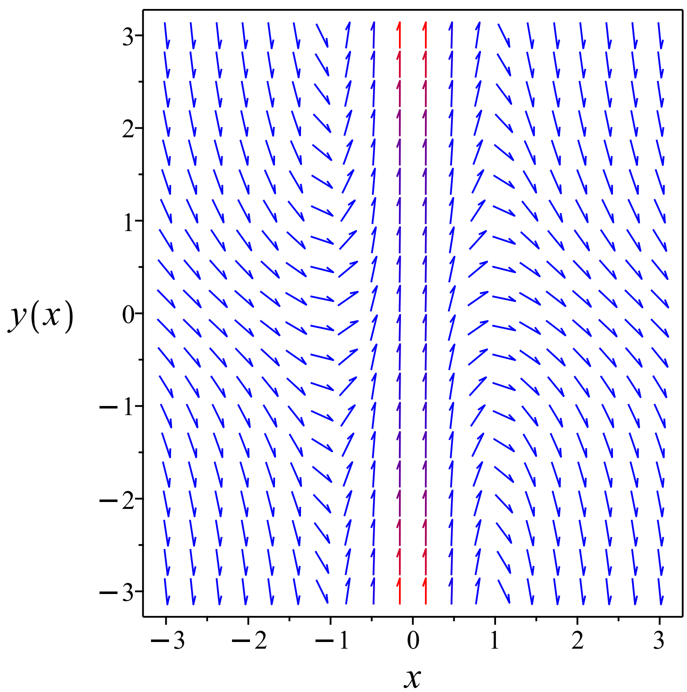


Figure 114: Slope field plot

Verification of solutions

$$y = -\tan\left(\frac{c_1x + x^2 + 1}{x}\right)$$

Verified OK.

3.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x^2y^2 + x^2 - y^2 - 1}{x^2} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 91: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2}{x^2-1} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2}{x^2-1}} dx
\end{aligned}
$$

Which results in

$$
S = x + \frac{1}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x^2y^2 + x^2 - y^2 - 1}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= 1 - \frac{1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x + \frac{1}{x} = -\arctan(y) + c_1$$

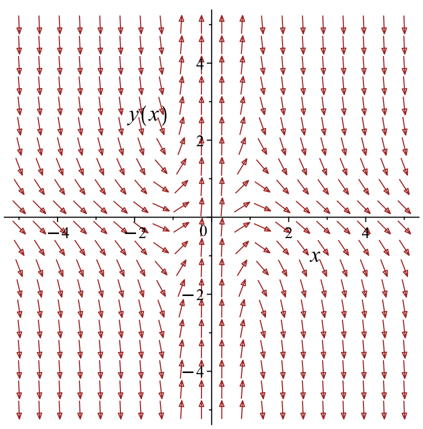
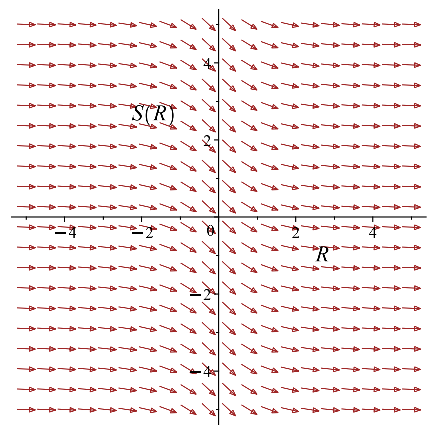
Which simplifies to

$$x + \frac{1}{x} = -\arctan(y) + c_1$$

Which gives

$$y = \tan\left(\frac{c_1 x - x^2 - 1}{x}\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x^2y^2+x^2-y^2-1}{x^2}$ 	$R = y$ $S = x + \frac{1}{x}$	$\frac{dS}{dR} = -\frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{c_1 x - x^2 - 1}{x}\right) \tag{1}$$

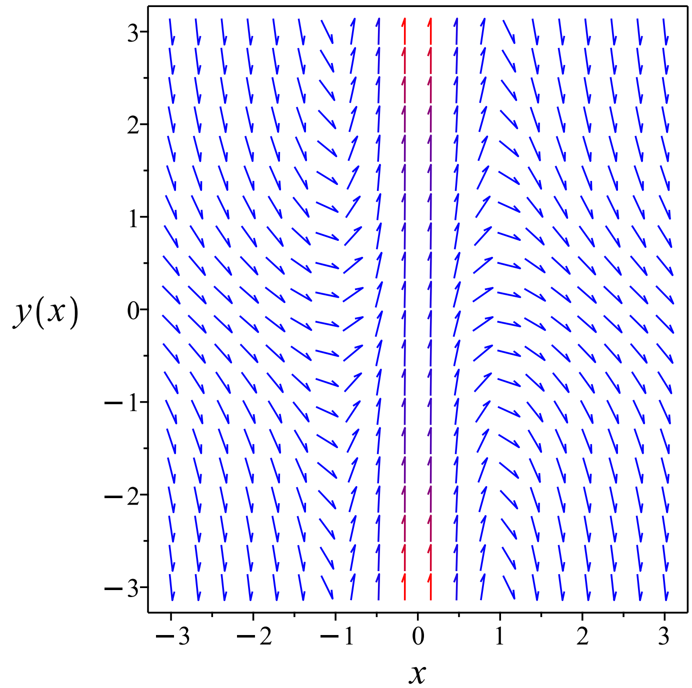


Figure 115: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{c_1 x - x^2 - 1}{x}\right)$$

Verified OK.

3.17.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{-y^2-1}\right)\mathrm{d}y = \left(\frac{x^2-1}{x^2}\right)\mathrm{d}x$$
$$\left(-\frac{x^2-1}{x^2}\right)\mathrm{d}x + \left(\frac{1}{-y^2-1}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x^2-1}{x^2}$$
$$N(x,y) = \frac{1}{-y^2-1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x^2-1}{x^2}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-y^2-1}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int M\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int -\frac{x^2-1}{x^2}\, \mathrm{d}x$$

$$\phi = -x - \frac{1}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-y^2-1}$. Therefore equation (4) becomes

$$\frac{1}{-y^2-1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y^2+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(-\frac{1}{y^2+1}\right) \mathrm{d}y$$

$$f(y) = -\arctan(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x - \frac{1}{x} - \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x - \frac{1}{x} - \arctan(y)$$

The solution becomes

$$y = -\tan\left(\frac{c_1 x + x^2 + 1}{x}\right)$$

Summary

The solution(s) found are the following

$$y = -\tan\left(\frac{c_1 x + x^2 + 1}{x}\right) \tag{1}$$

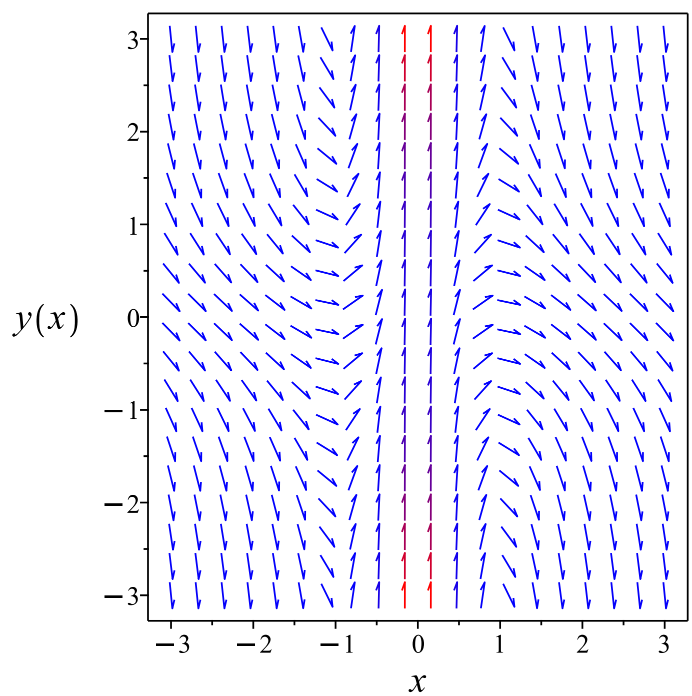


Figure 116: Slope field plot

Verification of solutions

$$y = -\tan\left(\frac{c_1 x + x^2 + 1}{x}\right)$$

Verified OK.

3.17.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{x^2 y^2 + x^2 - y^2 - 1}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -y^2 - 1 + \frac{y^2}{x^2} + \frac{1}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = -\frac{x^2-1}{x^2}$, $f_1(x) = 0$ and $f_2(x) = -\frac{x^2-1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{(x^2-1)u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= \frac{2x^2 - 2}{x^3} - \frac{2}{x} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= -\frac{(x^2-1)^3}{x^6} \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(x^2-1)\, u''(x)}{x^2} - \left(\frac{2x^2-2}{x^3} - \frac{2}{x}\right) u'(x) - \frac{(x^2-1)^3 u(x)}{x^6} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{x^2+1}{x}\right) + c_2 \cos\left(\frac{x^2+1}{x}\right)$$

The above shows that

$$u'(x) = \frac{(x^2-1)\left(c_1 \cos\left(\frac{x^2+1}{x}\right) - c_2 \sin\left(\frac{x^2+1}{x}\right)\right)}{x^2}$$

Using the above in (1) gives the solution

$$y = \frac{c_1 \cos\left(\frac{x^2+1}{x}\right) - c_2 \sin\left(\frac{x^2+1}{x}\right)}{c_1 \sin\left(\frac{x^2+1}{x}\right) + c_2 \cos\left(\frac{x^2+1}{x}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-\sin\left(\frac{x^2+1}{x}\right) + c_3 \cos\left(\frac{x^2+1}{x}\right)}{c_3 \sin\left(\frac{x^2+1}{x}\right) + \cos\left(\frac{x^2+1}{x}\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\sin\left(\frac{x^2+1}{x}\right) + c_3 \cos\left(\frac{x^2+1}{x}\right)}{c_3 \sin\left(\frac{x^2+1}{x}\right) + \cos\left(\frac{x^2+1}{x}\right)} \tag{1}$$

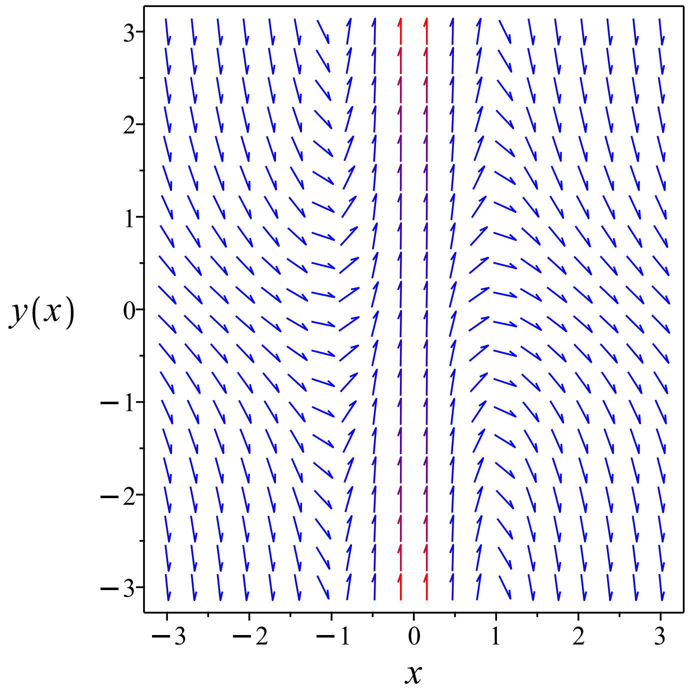


Figure 117: Slope field plot

Verification of solutions

$$y = \frac{-\sin\left(\frac{x^2+1}{x}\right) + c_3 \cos\left(\frac{x^2+1}{x}\right)}{c_3 \sin\left(\frac{x^2+1}{x}\right) + \cos\left(\frac{x^2+1}{x}\right)}$$

Verified OK.

3.17.5 Maple step by step solution

Let's solve

$y'x^2 - y^2 + x^2y^2 = -x^2 + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = -\frac{(x+1)(x-1)}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int -\frac{(x+1)(x-1)}{x^2} dx + c_1$

- Evaluate integral

  $\arctan(y) = c_1 - x - \frac{1}{x}$

- Solve for y

  $y = \tan\left(\frac{c_1 x - x^2 - 1}{x}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x) = 1-x^2+y(x)^2-x^2*y(x)^2,y(x), singsol=all)
```

$$y(x) = -\tan\left(\frac{c_1 x + x^2 + 1}{x}\right)$$

✓ Solution by Mathematica

Time used: 0.267 (sec). Leaf size: 17

```
DSolve[x^2*y'[x] == 1-x^2+y[x]^2-x^2*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\tan\left(x + \frac{1}{x} - c_1\right)$$

3.18 problem 19

Internal problem ID [44]
Internal file name [`OUTPUT/44_Sunday_June_05_2022_01_34_01_AM_61389824/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \mathrm{e}^{x} y = 0}$$

With initial conditions

$$[y(0) = 2\,\mathrm{e}]$$

3.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\mathrm{e}^{x} \\ q(x) &= 0 \end{aligned}$$

Hence the ode is

$$y' - e^x y = 0$$

The domain of $p(x) = -e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

3.18.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= e^x y \end{aligned}$$

Where $f(x) = e^x$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= e^x\, dx \\ \int \frac{1}{y}\, dy &= \int e^x\, dx \\ \ln(y) &= e^x + c_1 \\ y &= e^{e^x + c_1} \\ &= c_1 e^{e^x} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2\,e$ in the above solution gives an equation to solve for the constant of integration.

$$2\,e = e c_1$$

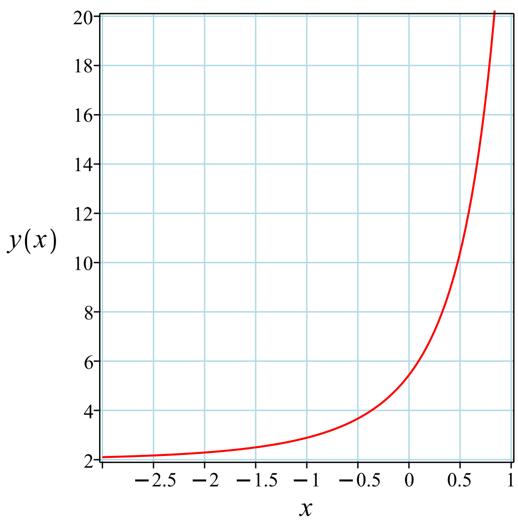
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

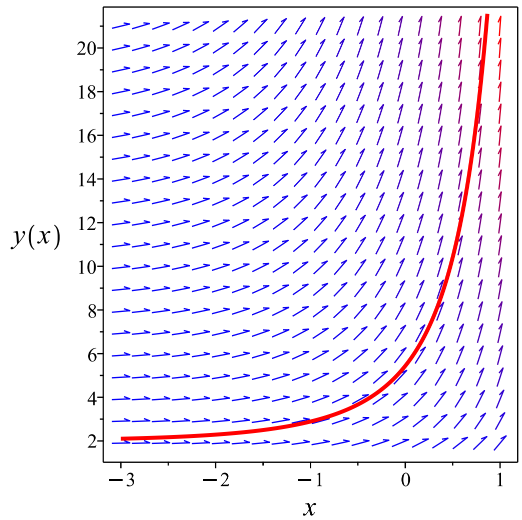
$$y = 2\,e^{e^x}$$

Summary
The solution(s) found are the following

$$y = 2\,e^{e^x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{e^{x}}$$

Verified OK.

3.18.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -e^{x}dx} \\ &= e^{-e^{x}}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}\mu y &= 0 \\ \frac{d}{dx}\left(e^{-e^{x}}y\right) &= 0\end{aligned}$$

Integrating gives

$$e^{-e^{x}}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-e^{x}}$ results in

$$y = c_1 e^{e^{x}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2\,\mathrm{e}$ in the above solution gives an equation to solve for the constant of integration.

$$2\,\mathrm{e} = \mathrm{e}c_1$$

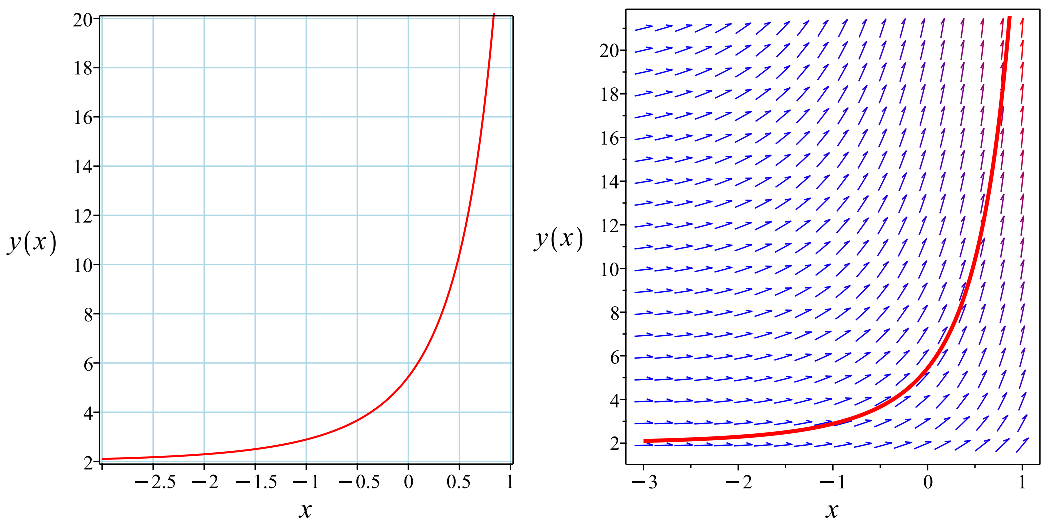
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

$$y = 2\,\mathrm{e}^{\mathrm{e}^x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 2\,\mathrm{e}^{\mathrm{e}^x} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,\mathrm{e}^{\mathrm{e}^x}$$

Verified OK.

3.18.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - e^{x} u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(x\,e^{x} - 1)}{x} \end{aligned}$$

Where $f(x) = \frac{x\,e^{x}-1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= \frac{x\,e^{x} - 1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int \frac{x\,e^{x} - 1}{x}\,dx \\ \ln(u) &= -\ln(x) + e^{x} + c_2 \\ u &= e^{-\ln(x)+e^{x}+c_2} \\ &= c_2 e^{-\ln(x)+e^{x}} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 e^{e^{x}}}{x}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= c_2 e^{e^{x}} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 2\,e$ in the above solution gives an equation to solve for the constant of integration.

$$2\,e = e c_2$$

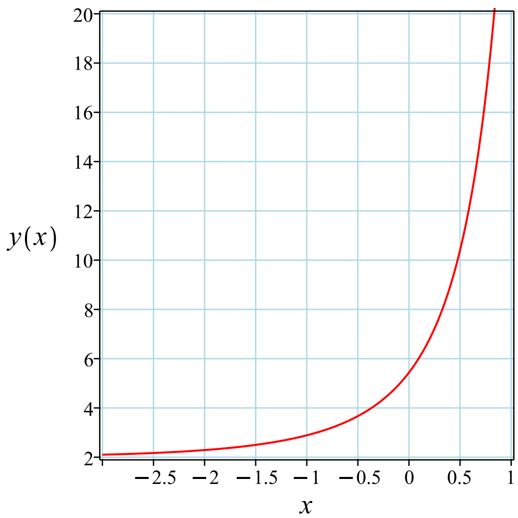
$$c_2 = 2$$

Substituting c_2 found above in the general solution gives

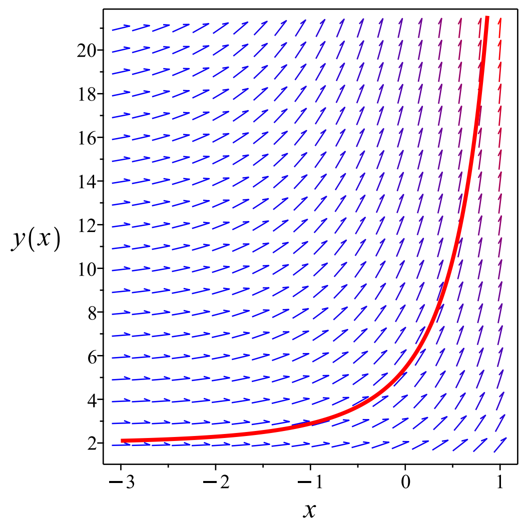
$$y = 2\,e^{e^{x}}$$

Summary
The solution(s) found are the following

$$y = 2\,\mathrm{e}^{\mathrm{e}^{x}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{\mathrm{e}^{x}}$$

Verified OK.

3.18.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \mathrm{e}^{x} y \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 94: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{e^x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\mathrm{e}^x}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{-\mathrm{e}^x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \mathrm{e}^x y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -y\, \mathrm{e}^{x-\mathrm{e}^x} \\ S_y &= \mathrm{e}^{-\mathrm{e}^x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-e^x} y = c_1$$

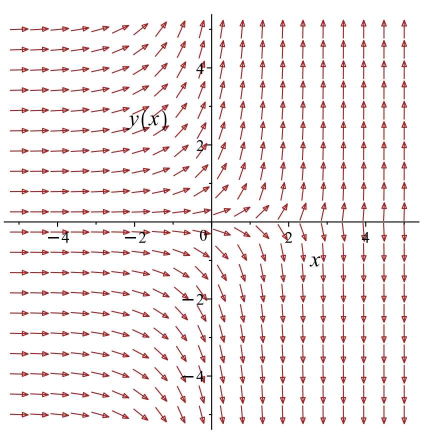
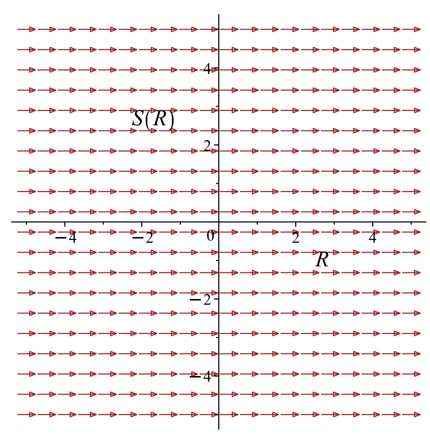
Which simplifies to

$$e^{-e^x} y = c_1$$

Which gives

$$y = c_1 e^{e^x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = e^x y$  	$R = x$ $S = e^{-e^x} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2e$ in the above solution gives an equation to solve for the constant of integration.

$$2e = ec_1$$

$$c_1 = 2$$

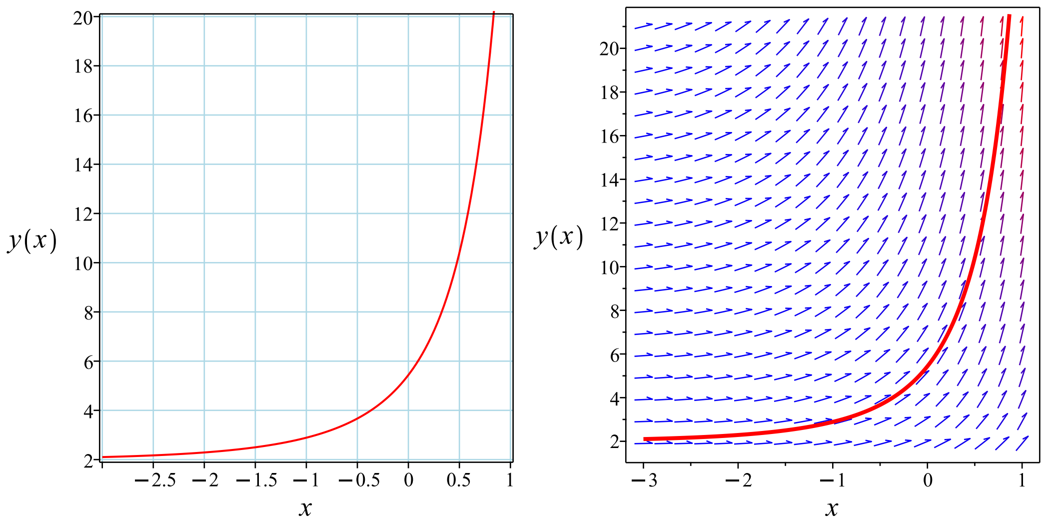
Substituting c_1 found above in the general solution gives

$$y = 2\,e^{e^x}$$

Summary

The solution(s) found are the following

$$y = 2\,e^{e^x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 2\,e^{e^x}$$

Verified OK.

3.18.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=(\mathrm{e}^{x})\,\mathrm{d}x\\ (-\mathrm{e}^{x})\,\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\mathrm{e}^{x}\\ N(x,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-\mathrm{e}^{x})\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\mathrm{e}^{x} \, \mathrm{d}x \\
\phi &= -\mathrm{e}^{x} + f(y)
\end{aligned}
\tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -e^{x} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -e^{x} + \ln(y)$$

The solution becomes

$$y = e^{e^{x}+c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 2\,e$ in the above solution gives an equation to solve for the constant of integration.

$$2\,e = e^{1+c_1}$$

$$c_1 = \ln(2)$$

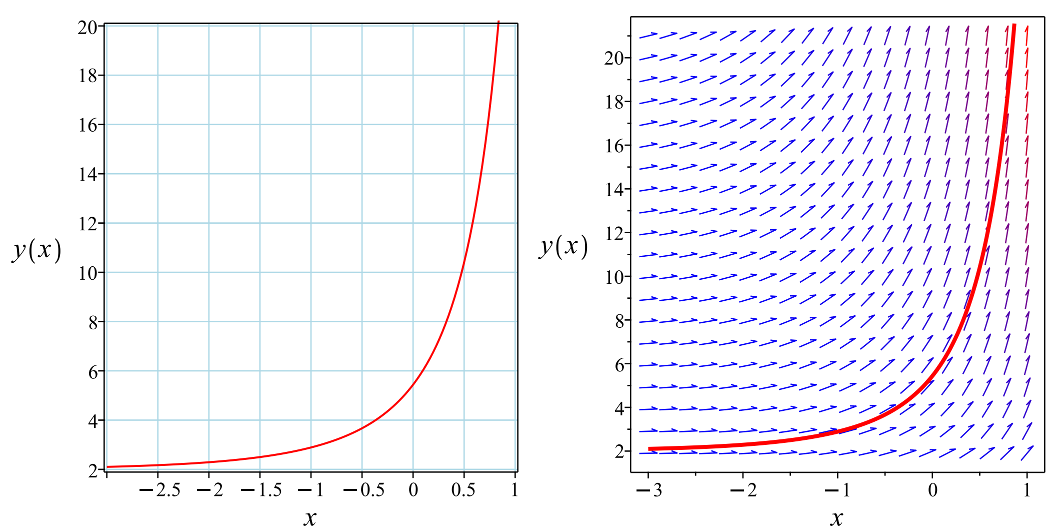
Substituting c_1 found above in the general solution gives

$$y = 2\,e^{e^{x}}$$

<u>Summary</u>

The solution(s) found are the following

$$y = 2\,e^{e^{x}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 2\,e^{e^{x}}$$

Verified OK.

3.18.7 Maple step by step solution

Let's solve

$[y' - e^{x}y = 0, y(0) = 2\,e]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = e^{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int e^{x}dx + c_1$

- Evaluate integral

  $\ln(y) = e^{x} + c_1$

- Solve for y

  $y = e^{e^{x}+c_1}$

- Use initial condition $y(0) = 2e$

  $2\,e = e^{1+c_1}$

- Solve for c_1

  $c_1 = \ln(2)$

- Substitute $c_1 = \ln(2)$ into general solution and simplify

  $y = 2\,e^{e^{x}}$

- Solution to the IVP

  $y = 2\,e^{e^{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([diff(y(x),x) = exp(x)*y(x),y(0) = 2*exp(1)],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{\mathrm{e}^{x}}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 12

```
DSolve[{y'[x] == Exp[x]*y[x],y[0]==2*Exp[1]},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2e^{e^x}$$

3.19 problem 20

Internal problem ID [45]
Internal file name [`OUTPUT/45_Sunday_June_05_2022_01_34_01_AM_89532822/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 3(1 + y^2)\, x^2 = 0}$$

With initial conditions

$$[y(0) = 1]$$

3.19.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x, y) \\ &= 3(y^2 + 1)\, x^2 \end{aligned}$$

The x domain of $f(x, y)$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x,y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(3(y^2+1)\,x^2\right)\\ &= 6y\,x^2\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 1$ is inside this domain. Therefore solution exists and is unique.

3.19.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= x^2\left(3y^2+3\right)\end{aligned}$$

Where $f(x) = x^2$ and $g(y) = 3y^2 + 3$. Integrating both sides gives

$$\begin{aligned}\frac{1}{3y^2+3}\,dy &= x^2\,dx\\ \int \frac{1}{3y^2+3}\,dy &= \int x^2\,dx\\ \frac{\arctan(y)}{3} &= \frac{x^3}{3} + c_1\end{aligned}$$

Which results in

$$y = \tan\left(x^3 + 3c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(3c_1)$$

$$c_1 = \frac{\pi}{12}$$

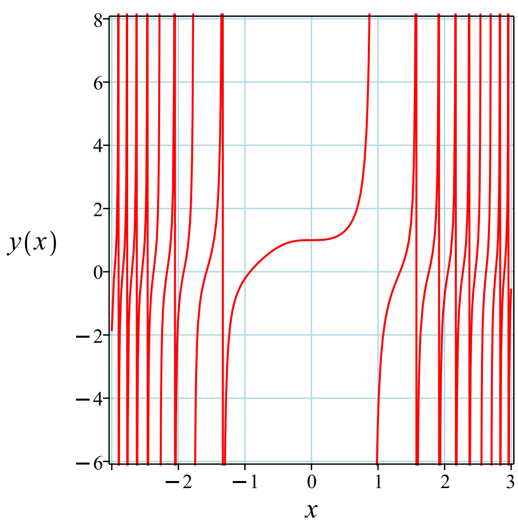
Substituting c_1 found above in the general solution gives

$$y = \tan\left(x^3 + \frac{\pi}{4}\right)$$

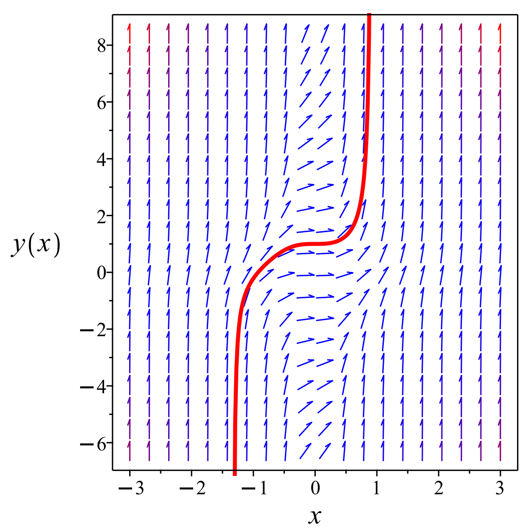
Summary

The solution(s) found are the following

$$y = \tan\left(x^3 + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \tan\left(x^3 + \frac{\pi}{4}\right)$$

Verified OK.

3.19.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 3(y^2+1)\,x^2 \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 97: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x^2} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x^2}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^3}{3}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = 3\left(y^2 + 1\right) x^2
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x^2 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3y^2+3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3R^2+3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\arctan(R)}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{x^3}{3} = \frac{\arctan(y)}{3} + c_1$$

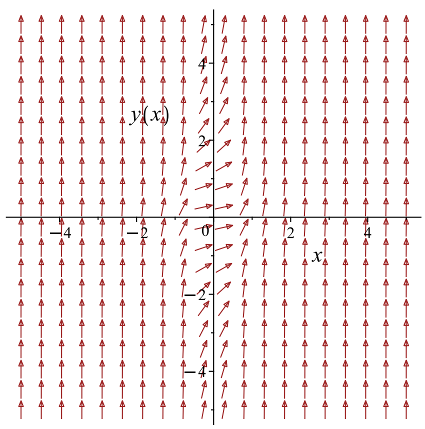
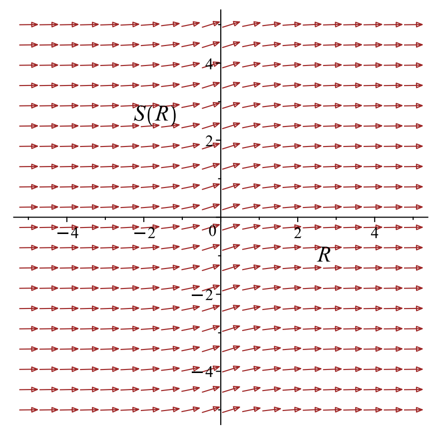
Which simplifies to

$$\frac{x^3}{3} = \frac{\arctan(y)}{3} + c_1$$

Which gives

$$y = -\tan\left(-x^3 + 3c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3(y^2+1)\,x^2$ 	$R = y$ $S = \frac{x^3}{3}$	$\frac{dS}{dR} = \frac{1}{3R^2+3}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -\tan(3c_1)$$

$$c_1 = -\frac{\pi}{12}$$

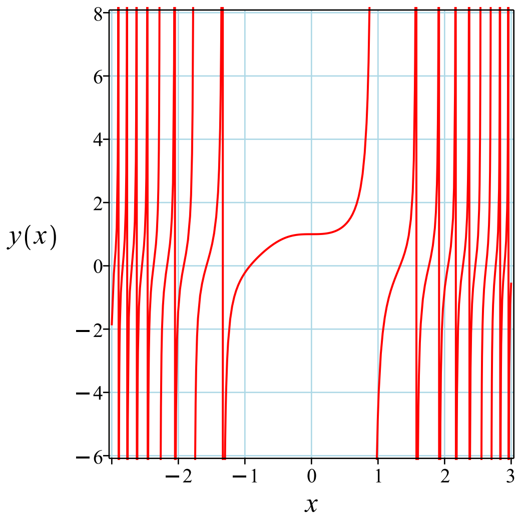
Substituting c_1 found above in the general solution gives

$$y = \tan\left(x^3 + \frac{\pi}{4}\right)$$

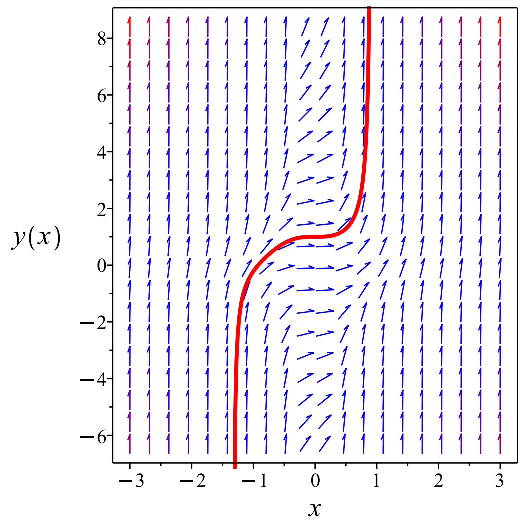
Summary

The solution(s) found are the following

$$y = \tan\left(x^3 + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \tan\left(x^3 + \frac{\pi}{4}\right)$$

Verified OK.

3.19.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{3y^2+3}\right)\mathrm{d}y &= \left(x^2\right)\mathrm{d}x\\ \left(-x^2\right)\mathrm{d}x + \left(\frac{1}{3y^2+3}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^2\\ N(x,y) &= \frac{1}{3y^2+3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{3y^2+3}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x^2 \, \mathrm{d}x$$
$$\phi = -\frac{x^3}{3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{3y^2+3}$. Therefore equation (4) becomes

$$\frac{1}{3y^2 + 3} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{3y^2 + 3}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int \left(\frac{1}{3y^2 + 3}\right) \mathrm{d}y$$
$$f(y) = \frac{\arctan(y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \frac{\arctan(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \frac{\arctan(y)}{3}$$

The solution becomes

$$y = \tan\left(x^3 + 3c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \tan(3c_1)$$

$$c_1 = \frac{\pi}{12}$$

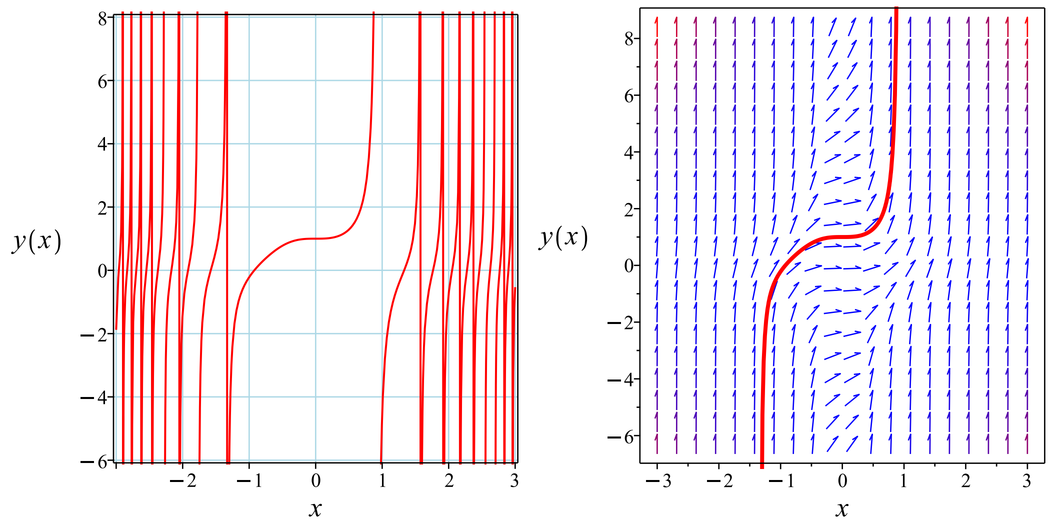
Substituting c_1 found above in the general solution gives

$$y = \tan\left(x^3 + \frac{\pi}{4}\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(x^3 + \frac{\pi}{4}\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \tan\left(x^3 + \frac{\pi}{4}\right)$$

Verified OK.

3.19.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 3(y^2 + 1)\, x^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 3x^2y^2 + 3x^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 3x^2$, $f_1(x) = 0$ and $f_2(x) = 3x^2$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{3x^2 u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 6x \\ f_1f_2 &= 0 \\ f_2^2f_0 &= 27x^6 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$3x^2u''(x) - 6xu'(x) + 27x^6u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(x^3\right) + c_2 \cos\left(x^3\right)$$

The above shows that

$$u'(x) = -3x^2\left(c_2 \sin\left(x^3\right) - c_1 \cos\left(x^3\right)\right)$$

Using the above in (1) gives the solution

$$y = \frac{c_2 \sin\left(x^3\right) - c_1 \cos\left(x^3\right)}{c_1 \sin\left(x^3\right) + c_2 \cos\left(x^3\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cos\left(x^3\right) + \sin\left(x^3\right)}{c_3 \sin\left(x^3\right) + \cos\left(x^3\right)}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = -c_3$$

$$c_3 = -1$$

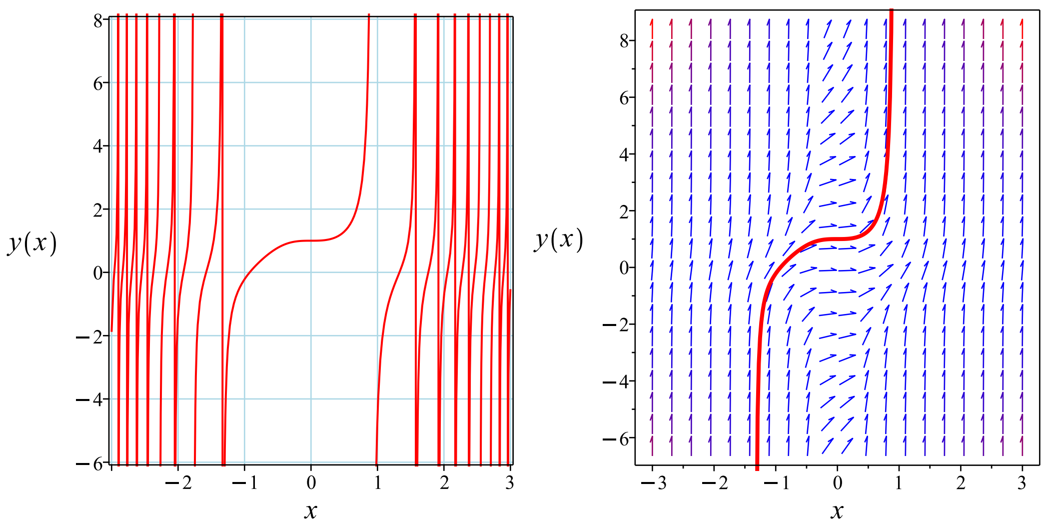
Substituting c_3 found above in the general solution gives

$$y = \frac{-\sin(x^3) - \cos(x^3)}{\sin(x^3) - \cos(x^3)}$$

Summary

The solution(s) found are the following

$$y = \frac{-\sin(x^3) - \cos(x^3)}{\sin(x^3) - \cos(x^3)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{-\sin(x^3) - \cos(x^3)}{\sin(x^3) - \cos(x^3)}$$

Verified OK.

3.19.6 Maple step by step solution

Let's solve

$[y' - 3(1 + y^2)\, x^2 = 0, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = 3x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int 3x^2 dx + c_1$

- Evaluate integral

  $\arctan(y) = x^3 + c_1$

- Solve for y

  $y = \tan(x^3 + c_1)$

- Use initial condition $y(0) = 1$

  $1 = \tan(c_1)$

- Solve for c_1

  $c_1 = \frac{\pi}{4}$

- Substitute $c_1 = \frac{\pi}{4}$ into general solution and simplify

  $y = \tan\left(x^3 + \frac{\pi}{4}\right)$

- Solution to the IVP

  $y = \tan\left(x^3 + \frac{\pi}{4}\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(x),x) = 3*x^2*(1+y(x)^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)$$

✓ Solution by Mathematica

Time used: 0.167 (sec). Leaf size: 15

```
DSolve[{y'[x]== 3*x^2*(1+y[x]^2),y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan \left(x^3 + \frac{\pi}{4}\right)$$

3.20 problem 21

Internal problem ID [46]
Internal file name [`OUTPUT/46_Sunday_June_05_2022_01_34_02_AM_95687917/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2yy' = \frac{x}{\sqrt{x^2-16}}}$$

With initial conditions

$$[y(5) = 2]$$

3.20.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{x}{2\sqrt{x^2-16}\,y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=2$ is

$$\{-\infty \leq x < -4, -4 < x < 4, 4 < x \leq \infty\}$$

And the point $x_0 = 5$ is inside this domain. The y domain of $f(x, y)$ when $x = 5$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x}{2\sqrt{x^2-16}\,y}\right) \\
&= -\frac{x}{2\sqrt{x^2-16}\,y^2}
\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 2$ is

$$\{-\infty \leq x < -4, -4 < x < 4, 4 < x \leq \infty\}$$

And the point $x_0 = 5$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 5$ is

$$\{y < 0 \lor 0 < y\}$$

And the point $y_0 = 2$ is inside this domain. Therefore solution exists and is unique.

3.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x, y) \\
&= f(x)g(y) \\
&= \frac{x}{2\sqrt{x^2-16}\,y}
\end{aligned}$$

Where $f(x) = \frac{x}{2\sqrt{x^2-16}}$ and $g(y) = \frac{1}{y}$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{\frac{1}{y}}\,dy &= \frac{x}{2\sqrt{x^2-16}}\,dx \\
\int \frac{1}{\frac{1}{y}}\,dy &= \int \frac{x}{2\sqrt{x^2-16}}\,dx \\
\frac{y^2}{2} &= \frac{(x-4)(x+4)}{2\sqrt{x^2-16}} + c_1
\end{aligned}$$

Which results in

$$y = \frac{\sqrt{\sqrt{x^2-16}\,\left(2c_1\sqrt{x^2-16}+x^2-16\right)}}{\sqrt{x^2-16}}$$

$$y = -\frac{\sqrt{\sqrt{x^2-16}\,\left(2c_1\sqrt{x^2-16}+x^2-16\right)}}{\sqrt{x^2-16}}$$

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = -\sqrt{2c_1+3}$$

Warning: Unable to solve for constant of integration. Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \sqrt{2c_1+3}$$

$$c_1 = \frac{1}{2}$$

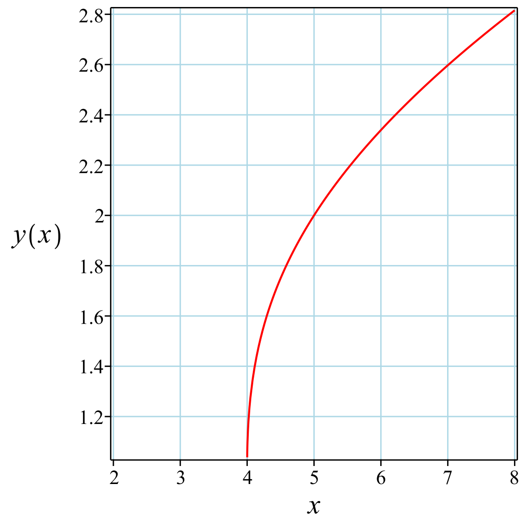
Substituting c_1 found above in the general solution gives

$$y = \frac{\sqrt{x^2-16+\sqrt{x^2-16}\,x^2-16\sqrt{x^2-16}}}{\sqrt{x^2-16}}$$

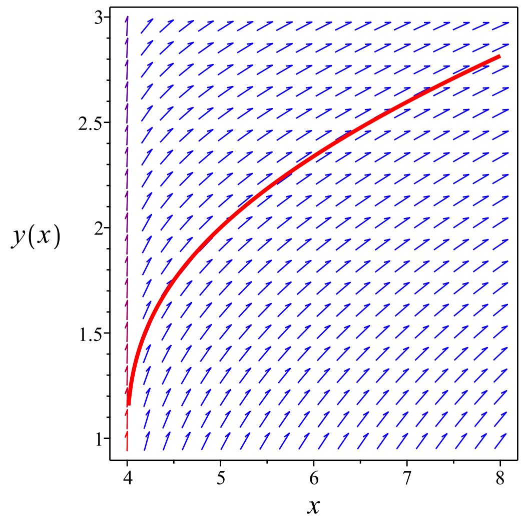
Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x^2-16+\sqrt{x^2-16}\,x^2-16\sqrt{x^2-16}}}{\sqrt{x^2-16}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x^2 - 16 + \sqrt{x^2 - 16}\, x^2 - 16\sqrt{x^2 - 16}}}{\sqrt{x^2 - 16}}$$

Verified OK.

3.20.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x}{2\sqrt{x^2 - 16}\, y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 100: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{2\sqrt{x^2-16}}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{2\sqrt{x^2-16}}{x}} dx \end{aligned}$$

Which results in

$$S = \frac{(x-4)(x+4)}{2\sqrt{x^2-16}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x}{2\sqrt{x^2-16}\, y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= \frac{x}{2\sqrt{x^2-16}} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = y \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R^2}{2} + c_1 \tag{4}$$

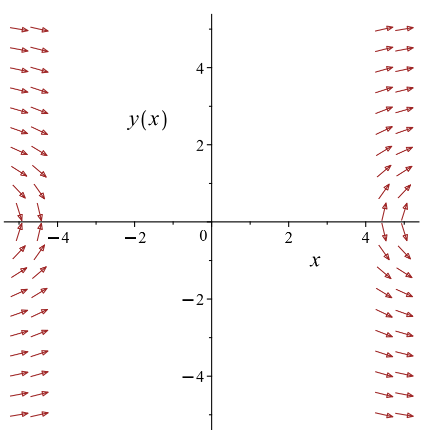
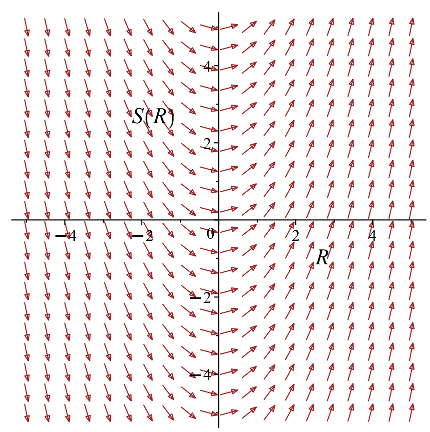
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\sqrt{x^2-16}}{2} = \frac{y^2}{2} + c_1$$

Which simplifies to

$$\frac{\sqrt{x^2-16}}{2} = \frac{y^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x}{2\sqrt{x^2-16}\, y}$ 	$R = y$ $S = \frac{\sqrt{x^2-16}}{2}$	$\frac{dS}{dR} = R$ 

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{3}{2} = 2 + c_1$$

$$
c_1 = -\frac{1}{2}
$$

Substituting c_1 found above in the general solution gives

$$
\frac{\sqrt{x^2-16}}{2} = \frac{y^2}{2} - \frac{1}{2}
$$

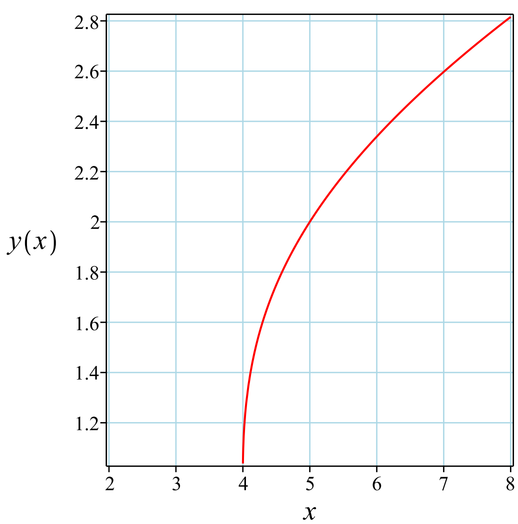
Solving for y from the above gives

$$
y = \sqrt{\sqrt{x^2-16}+1}
$$

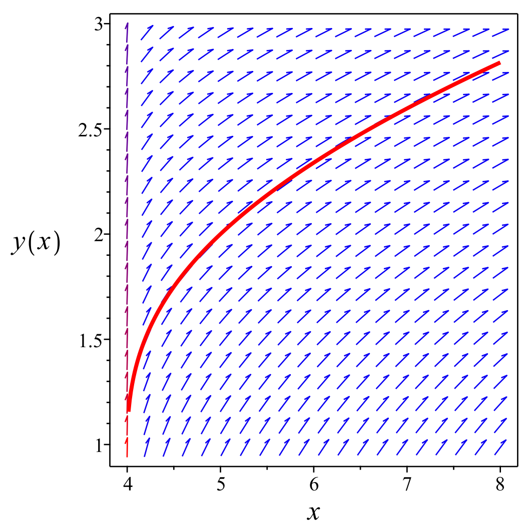
Summary

The solution(s) found are the following

$$
y = \sqrt{\sqrt{x^2-16}+1} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = \sqrt{\sqrt{x^2-16}+1}
$$

Verified OK.

3.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$(2y)\,\mathrm{d}y=\left(\frac{x}{\sqrt{x^2-16}}\right)\mathrm{d}x$$

$$\left(-\frac{x}{\sqrt{x^2-16}}\right)\mathrm{d}x+(2y)\,\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{x}{\sqrt{x^2-16}} \\
N(x,y) &= 2y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{\sqrt{x^2-16}}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2y) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{x}{\sqrt{x^2-16}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\sqrt{x^2-16} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = 2y$. Therefore equation (4) becomes

$$2y = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (2y)\,\mathrm{d}y$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^2 - \sqrt{x^2 - 16} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^2 - \sqrt{x^2 - 16}$$

Initial conditions are used to solve for c_1. Substituting $x = 5$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y^2 - \sqrt{x^2 - 16} = 1$$

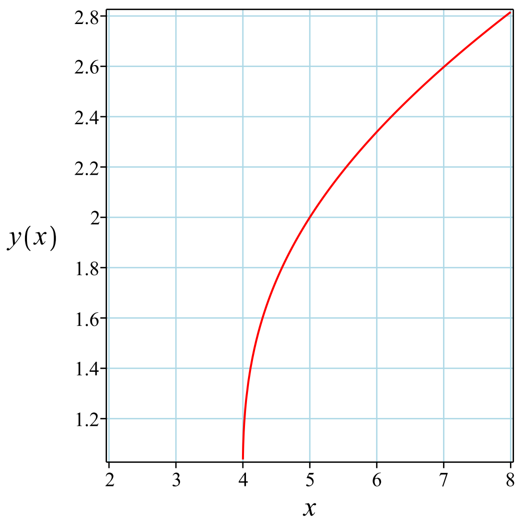
Solving for y from the above gives

$$y = \sqrt{\sqrt{x^2 - 16} + 1}$$

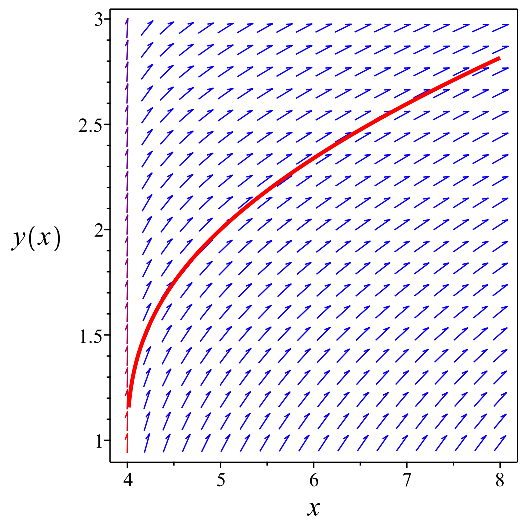
Summary

The solution(s) found are the following

$$y = \sqrt{\sqrt{x^2 - 16} + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \sqrt{\sqrt{x^2 - 16} + 1}$$

Verified OK.

3.20.5 Maple step by step solution

Let's solve

$\left[2yy' = \frac{x}{\sqrt{x^2-16}}, y(5) = 2\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int 2yy' dx = \int \frac{x}{\sqrt{x^2-16}} dx + c_1$

- Evaluate integral

  $y^2 = \frac{(x-4)(x+4)}{\sqrt{x^2-16}} + c_1$

- Solve for y

$$\left\{y = \frac{\sqrt{\sqrt{x^2-16}\left(c_1\sqrt{x^2-16}+x^2-16\right)}}{\sqrt{x^2-16}}, y = -\frac{\sqrt{\sqrt{x^2-16}\left(c_1\sqrt{x^2-16}+x^2-16\right)}}{\sqrt{x^2-16}}\right\}$$

- Use initial condition $y(5) = 2$

$$2 = \frac{\sqrt{9}\sqrt{\sqrt{9}\left(c_1\sqrt{9}+9\right)}}{9}$$

- Solve for c_1

$$c_1 = 1$$

- Substitute $c_1 = 1$ into general solution and simplify

$$y = \frac{\sqrt{\sqrt{x^2-16}\left(\sqrt{x^2-16}+x^2-16\right)}}{\sqrt{x^2-16}}$$

- Use initial condition $y(5) = 2$

$$2 = -\frac{\sqrt{9}\sqrt{\sqrt{9}\left(c_1\sqrt{9}+9\right)}}{9}$$

- Solution does not satisfy initial condition
- Solution to the IVP

$$y = \frac{\sqrt{\sqrt{x^2-16}\left(\sqrt{x^2-16}+x^2-16\right)}}{\sqrt{x^2-16}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 34

```
dsolve([2*y(x)*diff(y(x),x) = x/(x^2-16)^(1/2),y(5) = 2],y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{\sqrt{x^2-16}\,\left(x^2+\sqrt{x^2-16}-16\right)}}{\sqrt{x^2-16}}$$

✓ Solution by Mathematica

Time used: 1.931 (sec). Leaf size: 20

```
DSolve[{2*y[x]*y'[x] == x/(x^2-16)^(1/2),y[5]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt{\sqrt{x^2-16}+1}$$

3.21 problem 22

Internal problem ID [47]
Internal file name [`OUTPUT/47_Sunday_June_05_2022_01_34_03_AM_52437484/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + y - 4yx^3 = 0}$$

With initial conditions

$$[y(1) = -3]$$

3.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -4x^3 + 1$$
$$q(x) = 0$$

Hence the ode is

$$y' + (-4x^3 + 1)\, y = 0$$

The domain of $p(x) = -4x^3 + 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

3.21.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= y(4x^3 - 1) \end{aligned}$$

Where $f(x) = 4x^3 - 1$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= 4x^3 - 1\, dx \\ \int \frac{1}{y}\, dy &= \int 4x^3 - 1\, dx \\ \ln(y) &= x^4 + c_1 - x \\ y &= \mathrm{e}^{x^4 + c_1 - x} \\ &= c_1 \mathrm{e}^{x^4 - x} \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = c_1$$

$$c_1 = -3$$

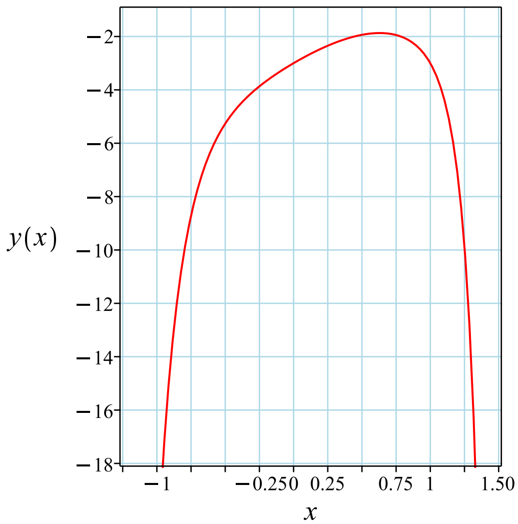
Substituting c_1 found above in the general solution gives

$$y = -3\, \mathrm{e}^{x^4 - x}$$

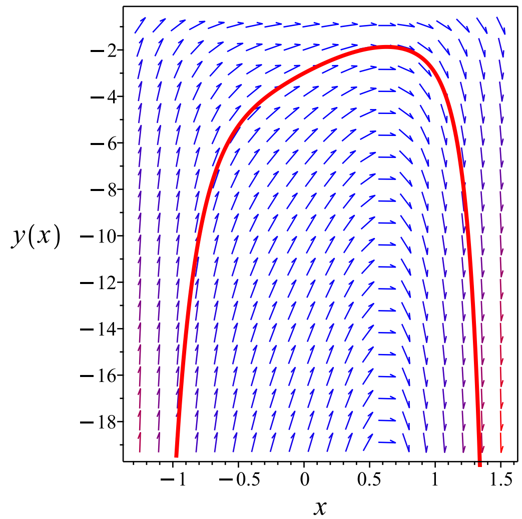
Summary

The solution(s) found are the following

$$y = -3\, \mathrm{e}^{x^4 - x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -3\,\mathrm{e}^{x^4-x}$$

Verified OK.

3.21.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int(-4x^3+1)dx}\\ &= \mathrm{e}^{-x^4+x}\end{aligned}$$

Which simplifies to

$$\mu = \mathrm{e}^{-x(x-1)(x^2+x+1)}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x(x-1)(x^2+x+1)}y\right) &= 0\end{aligned}$$

Integrating gives

$$\mathrm{e}^{-x(x-1)(x^2+x+1)}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{-x(x-1)(x^2+x+1)}$ results in

$$y = c_1 e^{x(x-1)(x^2+x+1)}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = c_1$$

$$c_1 = -3$$

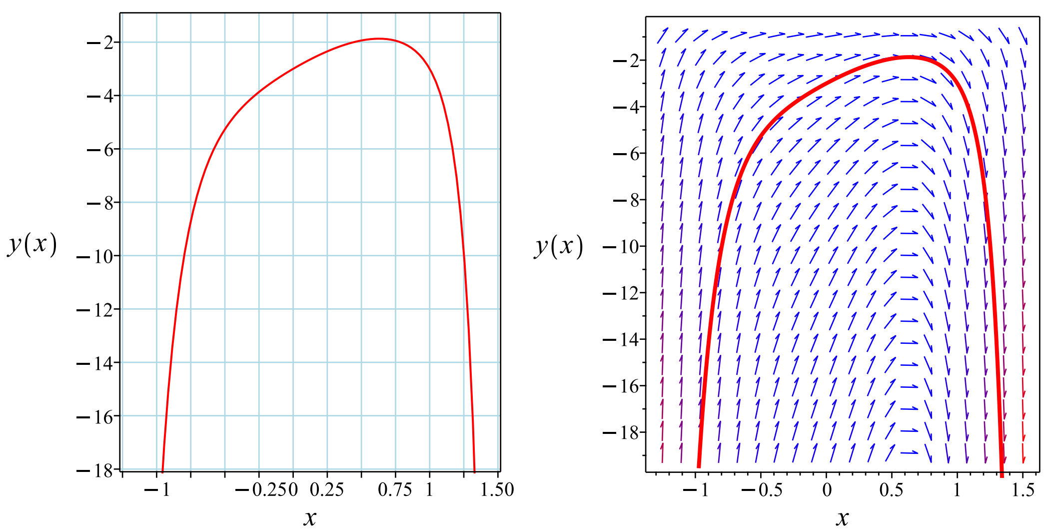
Substituting c_1 found above in the general solution gives

$$y = -3\,e^{x(x-1)(x^2+x+1)}$$

Summary

The solution(s) found are the following

$$y = -3\,e^{x(x-1)(x^2+x+1)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -3\,e^{x(x-1)(x^2+x+1)}$$

Verified OK.

3.21.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u'(x)\, x + u(x) + u(x)\, x - 4u(x)\, x^4 = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= \frac{u(4x^4 - x - 1)}{x}
\end{aligned}$$

Where $f(x) = \frac{4x^4 - x - 1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= \frac{4x^4 - x - 1}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{4x^4 - x - 1}{x}\, dx \\
\ln(u) &= x^4 - x - \ln(x) + c_2 \\
u &= \mathrm{e}^{x^4 - x - \ln(x) + c_2} \\
&= c_2 \mathrm{e}^{x^4 - x - \ln(x)}
\end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_2 \mathrm{e}^{x^4} \mathrm{e}^{-x}}{x}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= c_2 \mathrm{e}^{x^4} \mathrm{e}^{-x}
\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = c_2$$

$$c_2 = -3$$

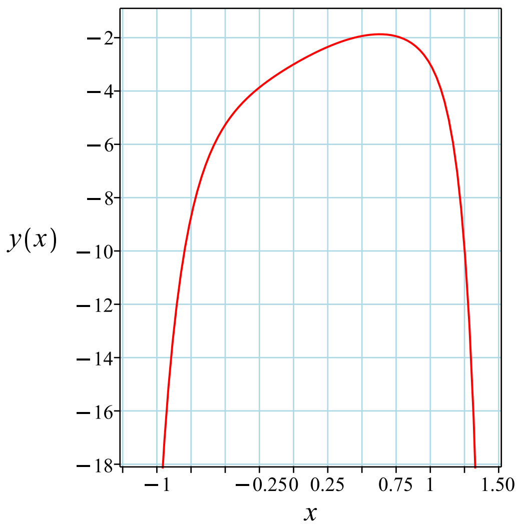
Substituting c_2 found above in the general solution gives

$$y = -3\, \mathrm{e}^{x(x-1)(x^2+x+1)}$$

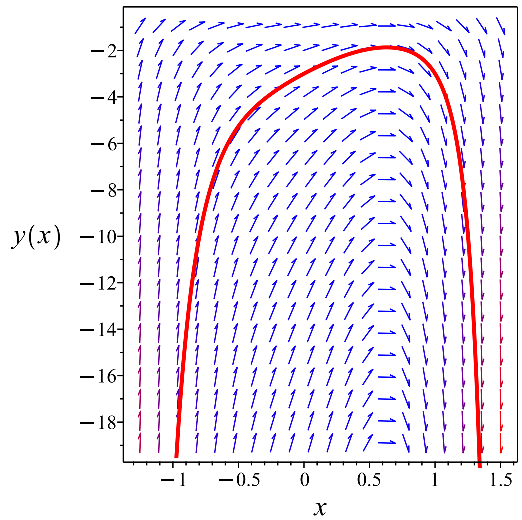
Summary

The solution(s) found are the following

$$
y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$
y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)}
$$

Verified OK.

3.21.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 4y\,x^3 - y \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 103: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x^4-x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^4 - x}} dy \end{aligned}$$

Which results in

$$S = e^{-x^4 + x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 4y\, x^3 - y$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \left(-4x^3 + 1\right) e^{-x(x-1)(x^2+x+1)} y \\ S_y &= e^{-x(x-1)(x^2+x+1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x(x-1)(x^2+x+1)} y = c_1$$

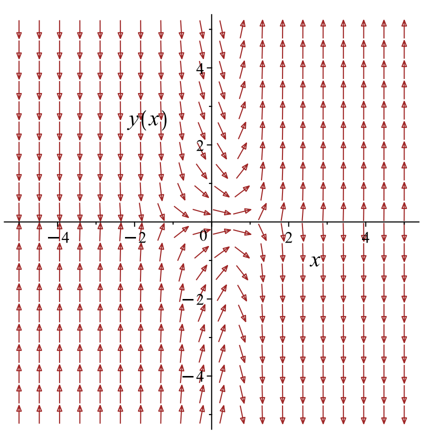
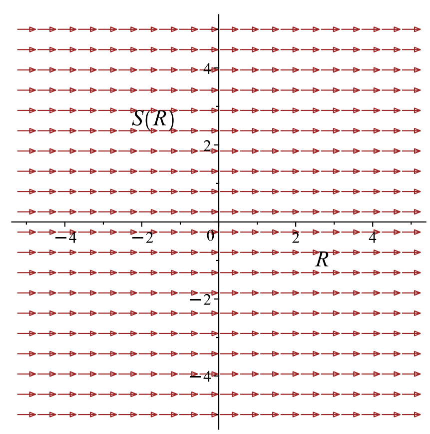
Which simplifies to

$$e^{-x(x-1)(x^2+x+1)} y = c_1$$

Which gives

$$y = c_1 e^{x(x-1)(x^2+x+1)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 4y\,x^3 - y$ 	$R = x$ $S = e^{-x(x-1)(x^2+x+1)} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = c_1$$

$$c_1 = -3$$

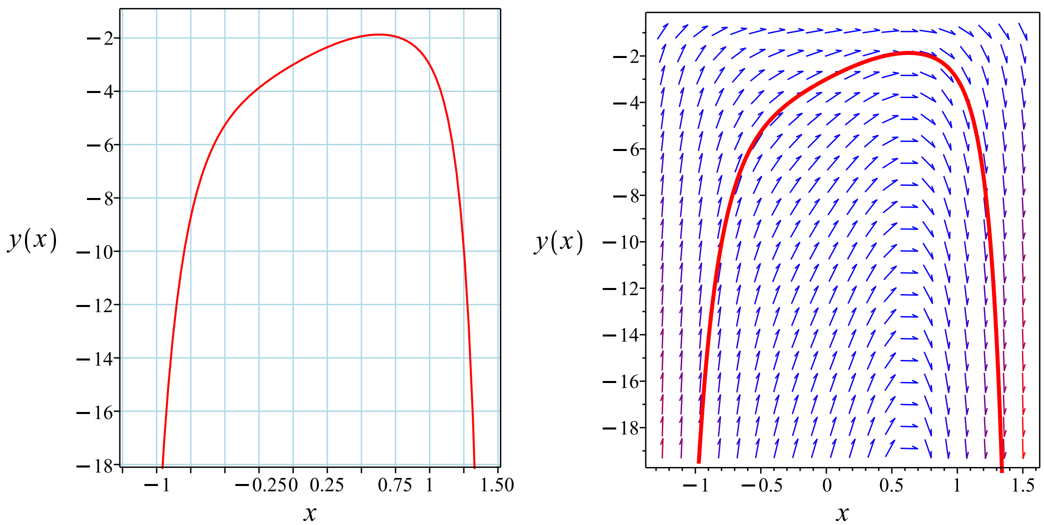
Substituting c_1 found above in the general solution gives

$$y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)}$$

Summary

The solution(s) found are the following

$$y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)}$$

Verified OK.

3.21.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{1}{y}\right) \mathrm{d} y &= \left(4 x^3-1\right) \mathrm{d} x \\
\left(-4 x^3+1\right) \mathrm{d} x+\left(\frac{1}{y}\right) \mathrm{d} y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -4 x^3+1 \\
N(x, y) &= \frac{1}{y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-4 x^3+1\right) \\
&= 0
\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -4x^3 + 1 \,\mathrm{d}x \\
\phi &= -x^4 + x + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^4 + x + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^4 + x + \ln(y)$$

The solution becomes

$$y = e^{x^4 + c_1 - x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -3$ in the above solution gives an equation to solve for the constant of integration.

$$-3 = e^{c_1}$$

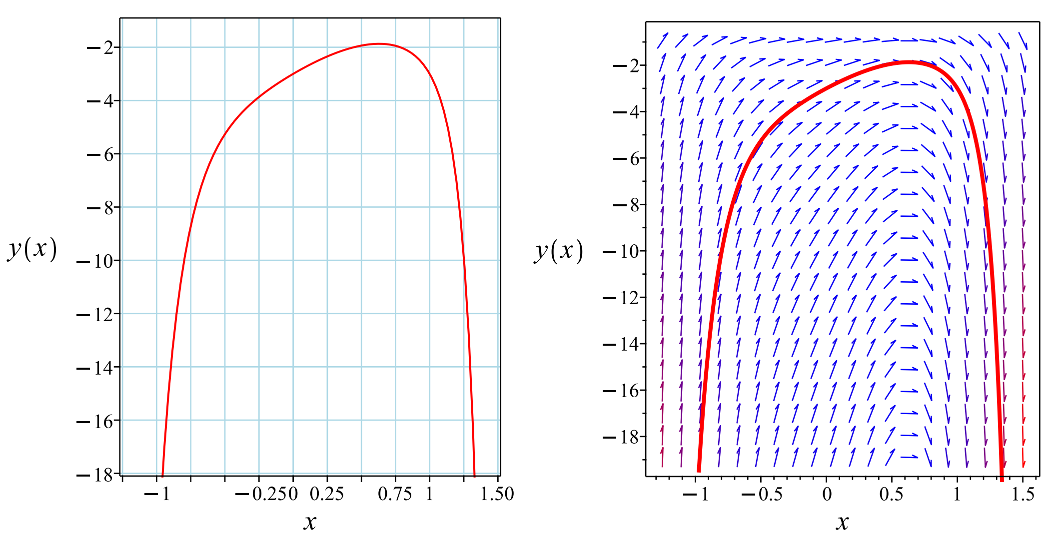
$$c_1 = \ln(3) + i\pi$$

Substituting c_1 found above in the general solution gives

$$y = -3\,e^{x(x-1)(x^2+x+1)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = -3\,e^{x(x-1)(x^2+x+1)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)}$$

Verified OK.

3.21.7 Maple step by step solution

Let's solve

$[y' + y - 4yx^3 = 0, y(1) = -3]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = 4x^3 - 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \left(4x^3 - 1\right)dx + c_1$

- Evaluate integral

  $\ln(y) = x^4 + c_1 - x$

- Solve for y

  $y = \mathrm{e}^{x^4+c_1-x}$

- Use initial condition $y(1) = -3$

  $-3 = \mathrm{e}^{c_1}$

- Solve for c_1

  $c_1 = \ln(3) + \mathrm{I}\pi$

- Substitute $c_1 = \ln(3) + \mathrm{I}\pi$ into general solution and simplify

  $y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)}$

- Solution to the IVP

  $y = -3\,\mathrm{e}^{x(x-1)\left(x^2+x+1\right)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(y(x),x) = -y(x)+4*x^3*y(x),y(1) = -3],y(x), singsol=all)
```

$$y(x) = -3\,e^{x(x-1)(x^2+x+1)}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 16

```
DSolve[{y'[x]== -y[x]+4*x^3*y[x],y[1]==-3},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -3e^{x^4-x}$$

3.22 problem 23

Internal problem ID [48]
Internal file name [`OUTPUT/48_Sunday_June_05_2022_01_34_03_AM_42337791/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - 2y = -1}$$

With initial conditions

$$[y(1) = 1]$$

3.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2$$
$$q(x) = -1$$

Hence the ode is

$$y' - 2y = -1$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

3.22.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-1+2y} dy = \int dx$$
$$\frac{\ln(-1+2y)}{2} = x + c_1$$

Raising both side to exponential gives

$$\sqrt{-1+2y} = e^{x+c_1}$$

Which simplifies to

$$\sqrt{-1+2y} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_2^2 e^2}{2} + \frac{1}{2}$$

$$c_2 = -e^{-2}e$$

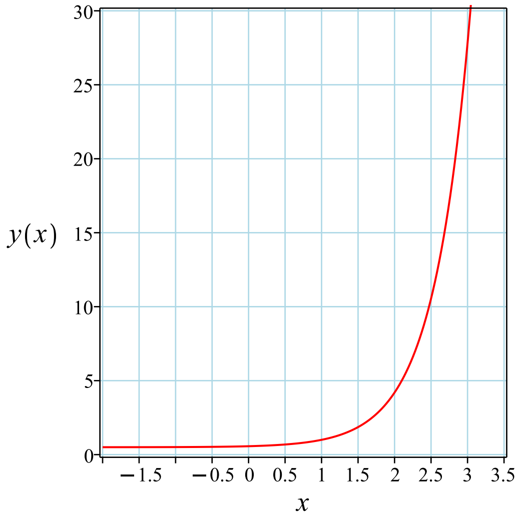
Substituting c_2 found above in the general solution gives

$$y = \frac{e^{2x-2}}{2} + \frac{1}{2}$$

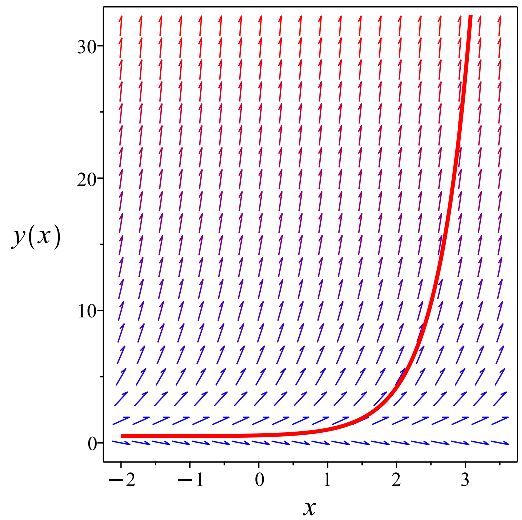
Summary

The solution(s) found are the following

$$y = \frac{e^{2x-2}}{2} + \frac{1}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{2x-2}}{2} + \frac{1}{2}$$

Verified OK.

3.22.3 Maple step by step solution

Let's solve

$[y' - 2y = -1, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-1+2y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-1+2y} dx = \int 1 dx + c_1$

- Evaluate integral

  $\frac{\ln(-1+2y)}{2} = x + c_1$

- Solve for y

$y = \frac{e^{2x+2c_1}}{2} + \frac{1}{2}$

- Use initial condition $y(1) = 1$

  $1 = \frac{e^{2+2c_1}}{2} + \frac{1}{2}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = \frac{e^{2x-2}}{2} + \frac{1}{2}$

- Solution to the IVP

  $y = \frac{e^{2x-2}}{2} + \frac{1}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([1+diff(y(x),x) = 2*y(x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{2} + \frac{e^{2x-2}}{2}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 18

```
DSolve[{1+y'[x] == 2*y[x],y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(e^{2x-2} + 1\right)$$

3.23 problem 24

Internal problem ID [49]
Internal file name [`OUTPUT/49_Sunday_June_05_2022_01_34_04_AM_52046128/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{\tan(x)\, y' - y = 0}$$

With initial conditions

$$\left[y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\right]$$

3.23.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y}{\tan(x)} \end{aligned}$$

Where $f(x) = \frac{1}{\tan(x)}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{1}{\tan(x)}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{1}{\tan(x)}\,dx \\
\ln(y) &= \ln(\sin(x)) + c_1 \\
y &= \mathrm{e}^{\ln(\sin(x))+c_1} \\
&= c_1 \sin(x)
\end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{2} = c_1$$

$$c_1 = \frac{\pi}{2}$$

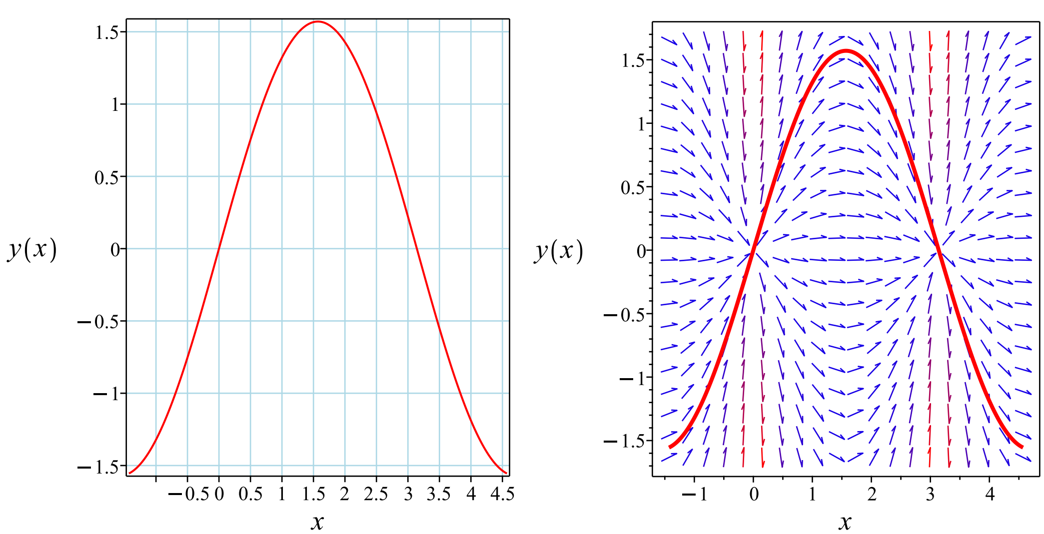
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi \sin(x)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\pi \sin(x)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\pi \sin (x)}{2}$$

Verified OK.

3.23.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int -\cot(x)dx}$$
$$= \frac{1}{\sin (x)}$$

Which simplifies to

$$\mu = \csc (x)$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}(\csc (x) y) = 0$$

Integrating gives

$$\csc (x) y = c_1$$

Dividing both sides by the integrating factor $\mu = \csc (x)$ results in

$$y = c_1 \sin (x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{2} = c_1$$

$$c_1 = \frac{\pi}{2}$$

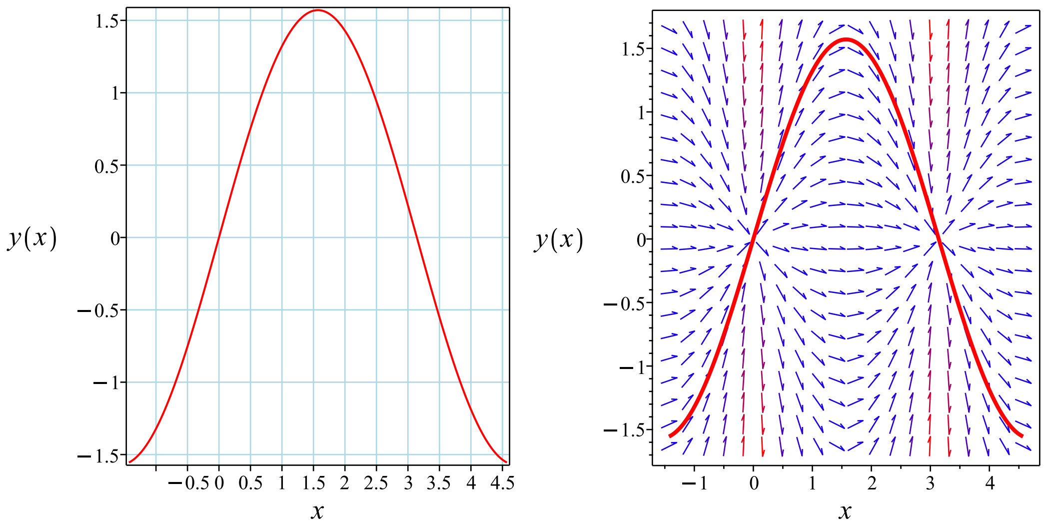
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi \sin (x)}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{\pi \sin(x)}{2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\pi \sin(x)}{2}$$

Verified OK.

3.23.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$\tan(x)\,(u'(x)\,x + u(x)) - u(x)\,x = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(\tan(x) - x)}{\tan(x)\,x} \end{aligned}$$

Where $f(x) = -\frac{\tan(x)-x}{x\tan(x)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{\tan(x)-x}{x\tan(x)}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{\tan(x)-x}{x\tan(x)}\,dx \\
\ln(u) &= \ln(\sin(x)) - \ln(x) + c_2 \\
u &= e^{\ln(\sin(x))-\ln(x)+c_2} \\
&= c_2 e^{\ln(\sin(x))-\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2\sin(x)}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= c_2\sin(x)
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$
\frac{\pi}{2} = c_2
$$

$$
c_2 = \frac{\pi}{2}
$$

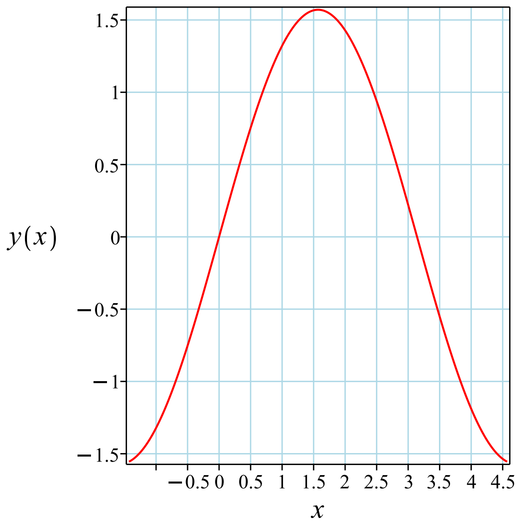
Substituting c_2 found above in the general solution gives

$$
y = \frac{\pi\sin(x)}{2}
$$

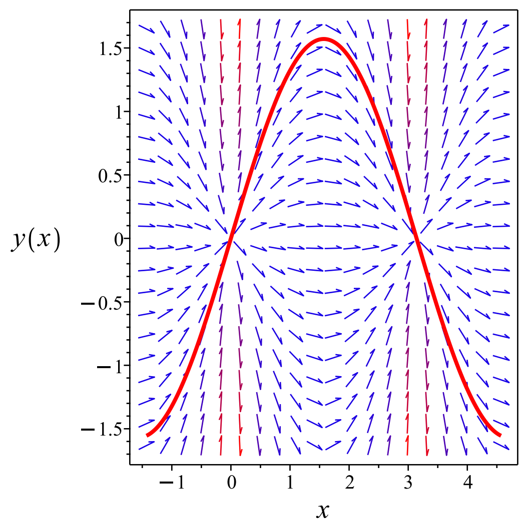
Summary

The solution(s) found are the following

$$
y = \frac{\pi\sin(x)}{2} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\pi \sin (x)}{2}$$

Verified OK.

3.23.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y}{\tan (x)}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 107: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int b f(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \sin(x)
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\sin(x)} dy \end{aligned}$$

Which results in

$$S = \frac{y}{\sin(x)}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{\tan(x)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\csc(x) \cot(x) y \\ S_y &= \csc(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\csc(x) y = c_1$$

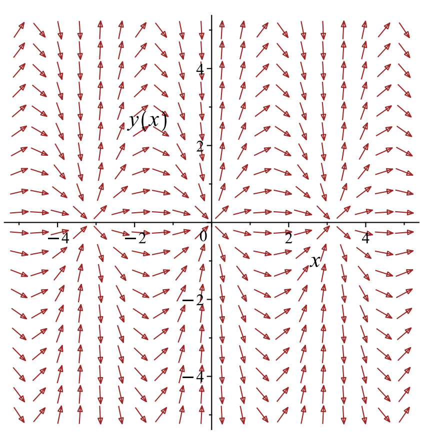
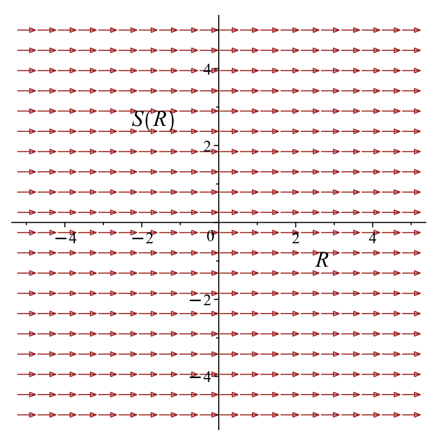
Which simplifies to

$$\csc(x) y = c_1$$

Which gives

$$y = \frac{c_1}{\csc(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{\tan(x)}$ 	$R = x$ $S = \csc(x) y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{2} = c_1$$

$$c_1 = \frac{\pi}{2}$$

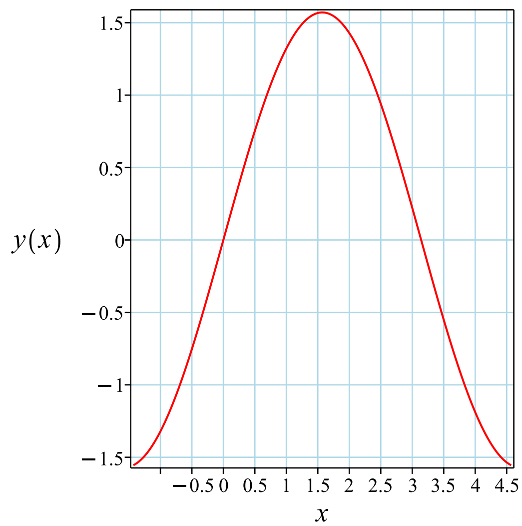
Substituting c_1 found above in the general solution gives

$$y = \frac{\pi \sin(x)}{2}$$

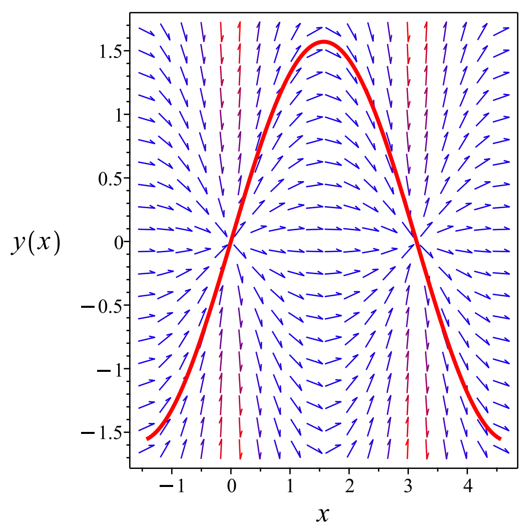
Summary

The solution(s) found are the following

$$y = \frac{\pi \sin(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi \sin(x)}{2}$$

Verified OK.

3.23.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right)\mathrm{d}y=\left(\frac{1}{\tan(x)}\right)\mathrm{d}x$$

$$\left(-\frac{1}{\tan(x)}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x,y) &= -\frac{1}{\tan(x)} \\
N(x,y) &= \frac{1}{y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\tan(x)}\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{1}{\tan(x)} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(\sin(x)) + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) + \ln(y)$$

The solution becomes

$$y = \mathrm{e}^{c_1} \sin(x)$$

Initial conditions are used to solve for c_1. Substituting $x = \frac{\pi}{2}$ and $y = \frac{\pi}{2}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{2} = \mathrm{e}^{c_1}$$

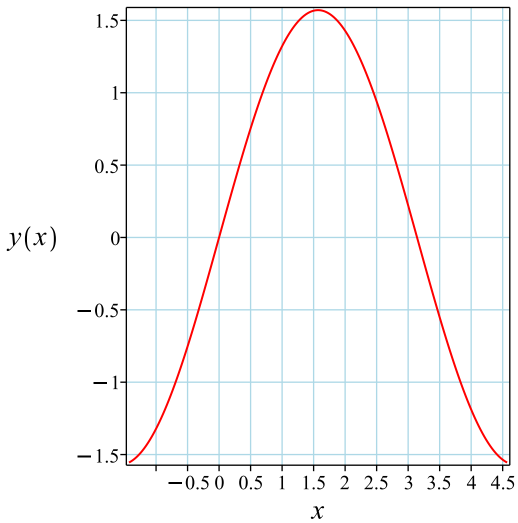
$$c_1 = \ln\left(\frac{\pi}{2}\right)$$

Substituting c_1 found above in the general solution gives

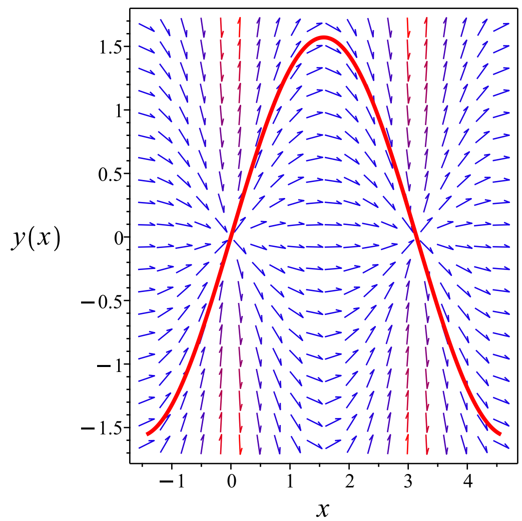
$$y = \frac{\pi \sin(x)}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{\pi \sin(x)}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{\pi \sin(x)}{2}$$

Verified OK.

3.23.6 Maple step by step solution

Let's solve

$\left[\tan(x) y' - y = 0, y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{1}{\tan(x)}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{1}{\tan(x)} dx + c_1$

- Evaluate integral

  $\ln(y) = \ln(\sin(x)) + c_1$
- Solve for y

  $y = e^{c_1} \sin(x)$
- Use initial condition $y\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

  $\frac{\pi}{2} = e^{c_1}$
- Solve for c_1

  $c_1 = \ln\left(\frac{\pi}{2}\right)$
- Substitute $c_1 = \ln\left(\frac{\pi}{2}\right)$ into general solution and simplify

  $y = \frac{\pi \sin(x)}{2}$
- Solution to the IVP

  $y = \frac{\pi \sin(x)}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 9

```
dsolve([tan(x)*diff(y(x),x) = y(x),y(1/2*Pi) = 1/2*Pi],y(x), singsol=all)
```

$$y(x) = \frac{\pi \sin(x)}{2}$$

✓ Solution by Mathematica

Time used: 0.04 (sec). Leaf size: 12

```
DSolve[{Tan[x]*y'[x] == y[x],y[Pi/2]==Pi/2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\pi\sin(x)$$

3.24 problem 25

Internal problem ID [50]
Internal file name [`OUTPUT/50_Sunday_June_05_2022_01_34_05_AM_66719473/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{-y + y'x - 2x^2y = 0}$$

With initial conditions

$$[y(1) = 1]$$

3.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{2x^2 + 1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y(2x^2+1)}{x} = 0$$

The domain of $p(x) = -\frac{2x^2+1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

3.24.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y(2x^2+1)}{x} \end{aligned}$$

Where $f(x) = \frac{2x^2+1}{x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y}\, dy &= \frac{2x^2+1}{x}\, dx \\ \int \frac{1}{y}\, dy &= \int \frac{2x^2+1}{x}\, dx \\ \ln(y) &= x^2 + \ln(x) + c_1 \\ y &= \mathrm{e}^{x^2+\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{x^2+\ln(x)} \end{aligned}$$

Which can be simplified to become

$$y = c_1 \mathrm{e}^{x^2} x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e} c_1$$

$$c_1 = \mathrm{e}^{-1}$$

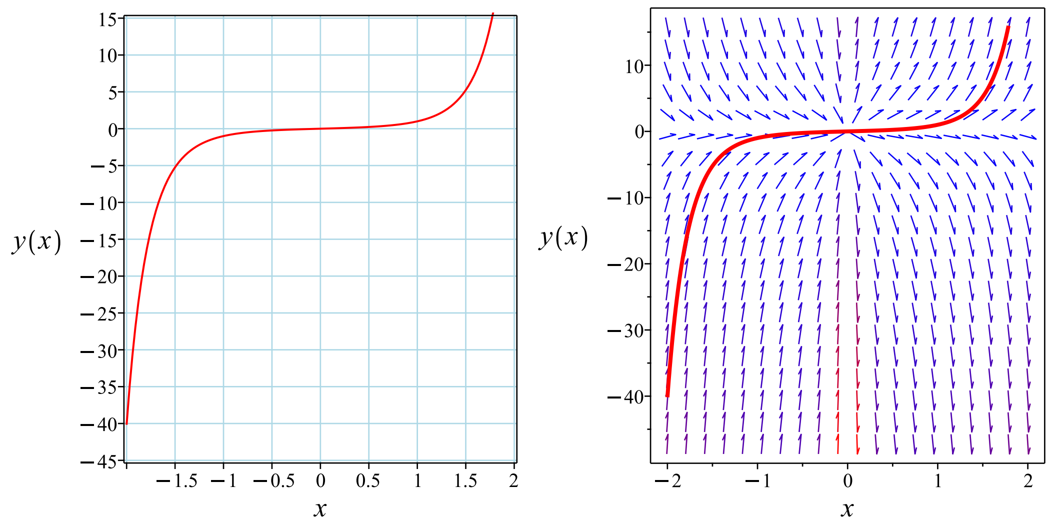
Substituting c_1 found above in the general solution gives

$$y = x\,\mathrm{e}^{(x-1)(x+1)}$$

Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{(x-1)(x+1)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{(x-1)(x+1)}$$

Verified OK.

3.24.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2x^2+1}{x}dx} \\ &= \mathrm{e}^{-x^2-\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{-x^2}}{x}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}\mu y = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{-x^2} y}{x}\right) = 0$$

Integrating gives

$$\frac{\mathrm{e}^{-x^2} y}{x} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{\mathrm{e}^{-x^2}}{x}$ results in

$$y = c_1 \mathrm{e}^{x^2} x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e} c_1$$

$$c_1 = \mathrm{e}^{-1}$$

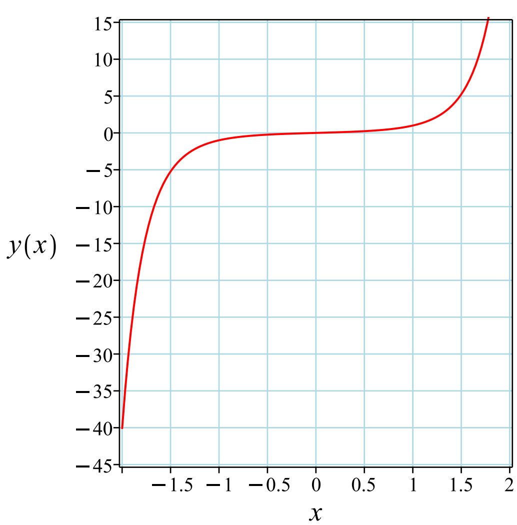
Substituting c_1 found above in the general solution gives

$$y = x\,\mathrm{e}^{(x-1)(x+1)}$$

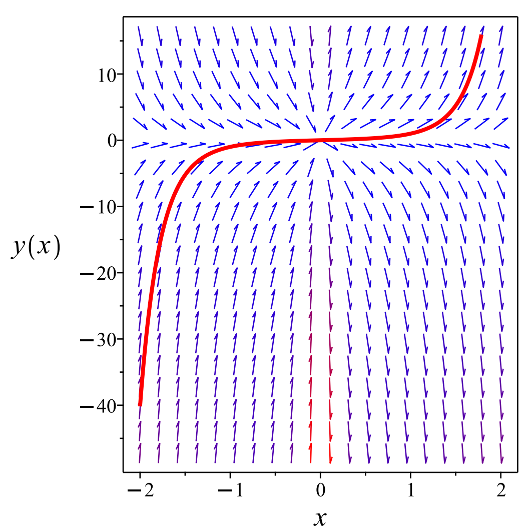
Summary

The solution(s) found are the following

$$y = x\,\mathrm{e}^{(x-1)(x+1)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x\,\mathrm{e}^{(x-1)(x+1)}$$

Verified OK.

3.24.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x + (u'(x)\,x + u(x))\,x - 2x^3u(x) = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= 2ux\end{aligned}$$

Where $f(x) = 2x$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= 2x\,dx\\ \int \frac{1}{u}\,du &= \int 2x\,dx\\ \ln(u) &= x^2 + c_2\\ u &= \mathrm{e}^{x^2+c_2}\\ &= c_2\mathrm{e}^{x^2}\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}y &= ux\\ &= xc_2\mathrm{e}^{x^2}\end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \mathrm{e}c_2$$

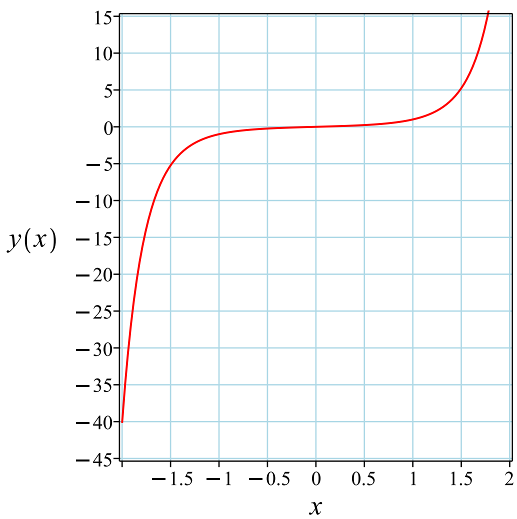
$$c_2 = \mathrm{e}^{-1}$$

Substituting c_2 found above in the general solution gives

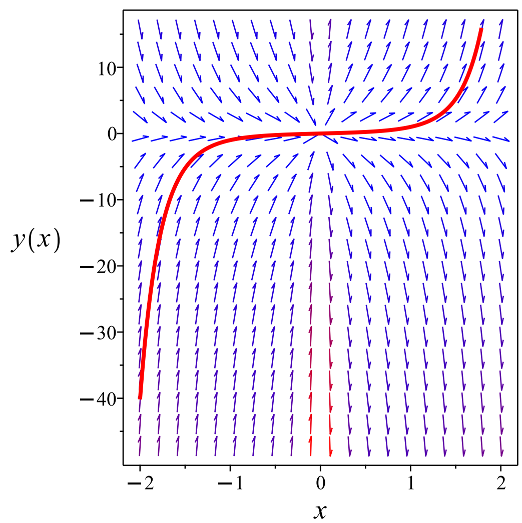
$$y = x\,\mathrm{e}^{(x-1)(x+1)}$$

Summary
The solution(s) found are the following

$$y = x \, e^{(x-1)(x+1)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x \, e^{(x-1)(x+1)}$$

Verified OK.

3.24.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(2x^2+1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 110: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{x^2+\ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^2+\ln(x)}} dy \end{aligned}$$

Which results in

$$S = \frac{e^{-x^2} y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(2x^2+1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y(-2x^2-1)\, e^{-x^2}}{x^2} \\ S_y &= \frac{e^{-x^2}}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{-x^2} y}{x} = c_1$$

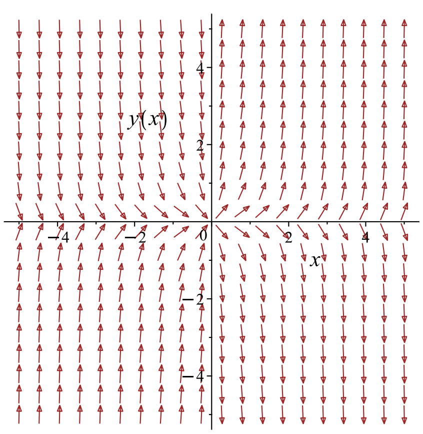
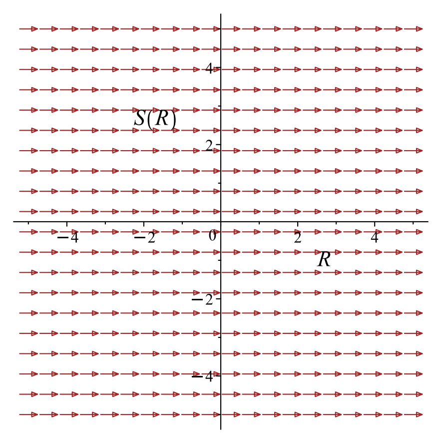
Which simplifies to

$$\frac{e^{-x^2} y}{x} = c_1$$

Which gives

$$y = c_1 e^{x^2} x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y(2x^2+1)}{x}$
</td><td>$R = x$
$S = \frac{e^{-x^2} y}{x}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e c_1$$

$$c_1 = e^{-1}$$

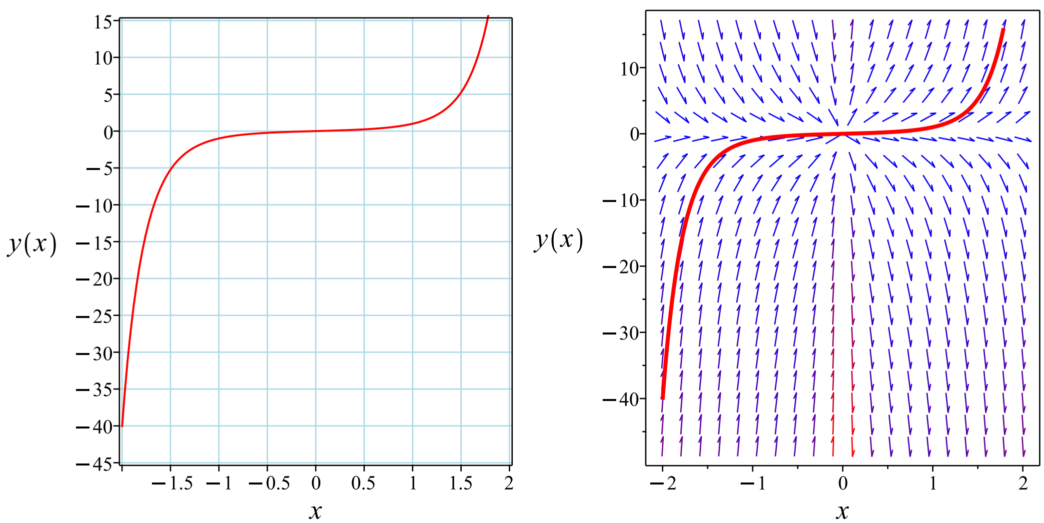
Substituting c_1 found above in the general solution gives

$$y = x\,e^{(x-1)(x+1)}$$

Summary

The solution(s) found are the following

$$y = x\,e^{(x-1)(x+1)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x\,e^{(x-1)(x+1)}$$

Verified OK.

3.24.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{2x^2+1}{x}\right)\mathrm{d}x\\ \left(-\frac{2x^2+1}{x}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{2x^2+1}{x}\\ N(x,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{2x^2+1}{x}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{2x^2+1}{x} \mathrm{d}x \\
\phi &= -x^2 - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2 - \ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2 - \ln(x) + \ln(y)$$

The solution becomes

$$y = e^{x^2+c_1} x$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = e^{1+c_1}$$

$$c_1 = -1$$

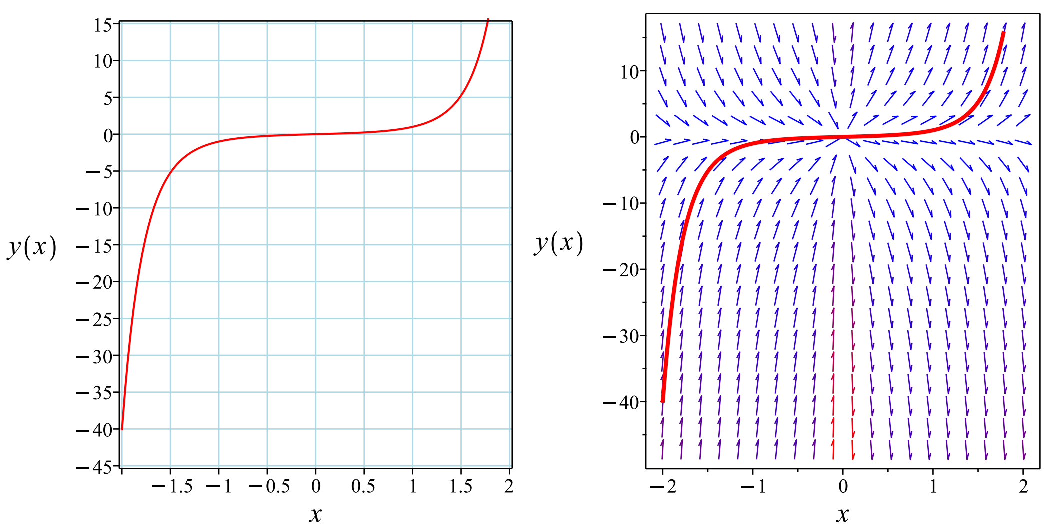
Substituting c_1 found above in the general solution gives

$$y = e^{x^2-1} x$$

Summary

The solution(s) found are the following

$$y = e^{x^2-1} x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{x^2-1}x$$

Verified OK.

3.24.7 Maple step by step solution

Let's solve

$[-y + y'x - 2x^2y = 0, y(1) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{2x^2+1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y}dx = \int \frac{2x^2+1}{x}dx + c_1$

- Evaluate integral

  $\ln(y) = x^2 + \ln(x) + c_1$

- Solve for y

  $y = \frac{x}{\mathrm{e}^{-x^2-c_1}}$

- Use initial condition $y(1) = 1$

  $1 = \frac{1}{\mathrm{e}^{-c_1-1}}$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = x\,\mathrm{e}^{(x-1)(x+1)}$

- Solution to the IVP

  $y = x\,\mathrm{e}^{(x-1)(x+1)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([-y(x)+x*diff(y(x),x) = 2*x^2*y(x),y(1) = 1],y(x), singsol=all)
```

$$y(x) = x\,\mathrm{e}^{(x-1)(x+1)}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 14

```
DSolve[{-y[x]+x*y'[x] == 2*x^2*y[x],y[1]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2-1}x$$

3.25 problem 26

Internal problem ID [51]
Internal file name [`OUTPUT/51_Sunday_June_05_2022_01_34_05_AM_60682592/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - 2xy^2 - 3x^2y^2 = 0}$$

With initial conditions

$$[y(1) = -1]$$

3.25.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 3x^2y^2 + 2x\,y^2 \end{aligned}$$

The x domain of $f(x,y)$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $f(x, y)$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(3x^2y^2 + 2x\,y^2) \\ &= 6y\,x^2 + 4yx\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 1$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = -1$ is inside this domain. Therefore solution exists and is unique.

3.25.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y) \\ &= f(x)g(y) \\ &= (3x^2 + 2x)\,y^2\end{aligned}$$

Where $f(x) = 3x^2 + 2x$ and $g(y) = y^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{y^2}\,dy &= 3x^2 + 2x\,dx \\ \int \frac{1}{y^2}\,dy &= \int 3x^2 + 2x\,dx \\ -\frac{1}{y} &= x^3 + x^2 + c_1\end{aligned}$$

Which results in

$$y = -\frac{1}{x^3 + x^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_1 + 2}$$

$$c_1 = -1$$

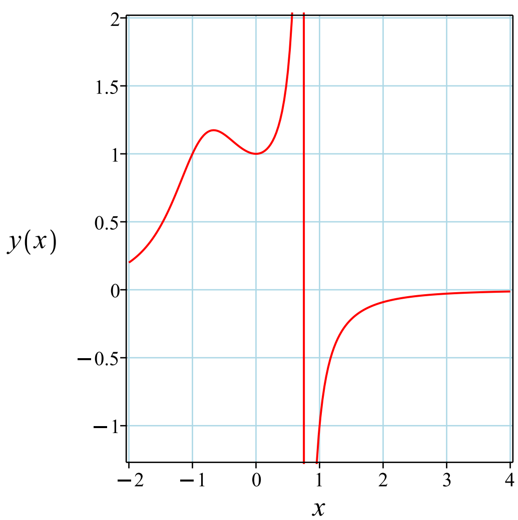
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{x^3 + x^2 - 1}$$

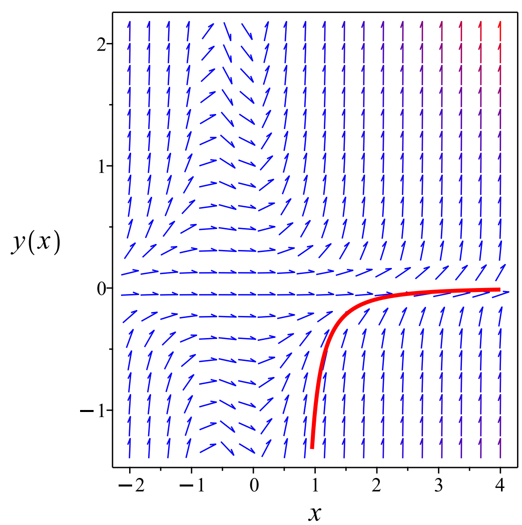
Summary

The solution(s) found are the following

$$y = -\frac{1}{x^3 + x^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{x^3 + x^2 - 1}$$

Verified OK.

3.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x^2y^2 + 2x\,y^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 113: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = \frac{1}{3x^2 + 2x}$$
$$\eta(x, y) = 0 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{1}{3x^2+2x}} dx \end{aligned}$$

Which results in

$$S = x^3 + x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 3x^2 y^2 + 2x\, y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 3x^2 + 2x \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2(x+1) = -\frac{1}{y} + c_1$$

Which simplifies to

$$x^2(x+1) = -\frac{1}{y} + c_1$$

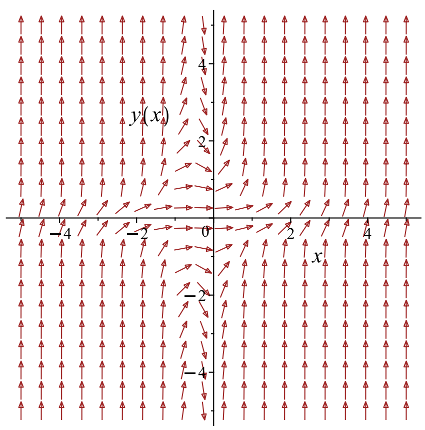
Which gives

$$y = \frac{1}{-x^3 - x^2 + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = 3x^2y^2 + 2x\,y^2$

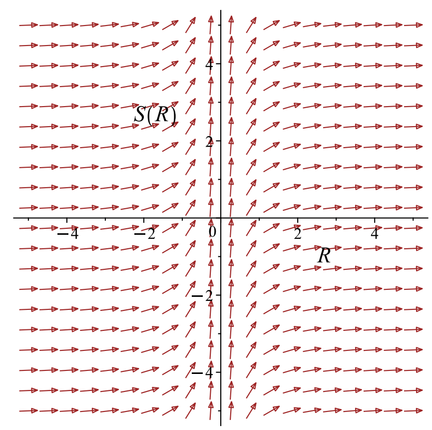


</td><td>

$R = y$
$S = x^2(x+1)$

</td><td>

$\frac{dS}{dR} = \frac{1}{R^2}$



</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = \frac{1}{c_1 - 2}$$

$$c_1 = 1$$

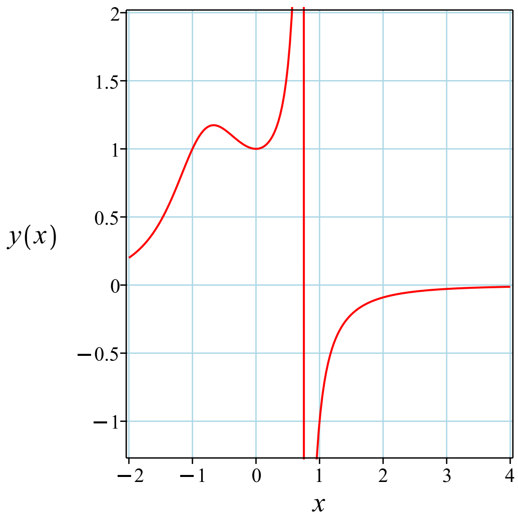
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{x^3 + x^2 - 1}$$

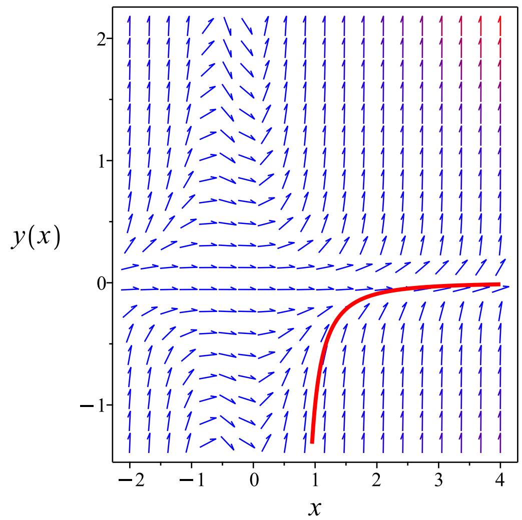
Summary

The solution(s) found are the following

$$y = -\frac{1}{x^3 + x^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -\frac{1}{x^3 + x^2 - 1}$$

Verified OK.

3.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^2}\right)\mathrm{d}y &= \left(3x^2+2x\right)\mathrm{d}x\\ \left(-3x^2-2x\right)\mathrm{d}x + \left(\frac{1}{y^2}\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -3x^2-2x\\ N(x,y) &= \frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x^2-2x\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -3x^2 - 2x \,\mathrm{d}x$$
$$\phi = -x^3 - x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$\frac{1}{y^2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y^2}$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int \left(\frac{1}{y^2}\right) \mathrm{d}y$$
$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 - x^2 - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 - x^2 - \frac{1}{y}$$

The solution becomes

$$y = -\frac{1}{x^3 + x^2 + c_1}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_1 + 2}$$

$$c_1 = -1$$

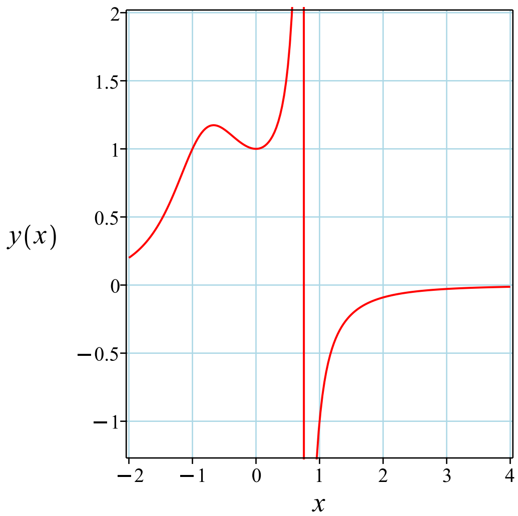
Substituting c_1 found above in the general solution gives

$$y = -\frac{1}{x^3 + x^2 - 1}$$

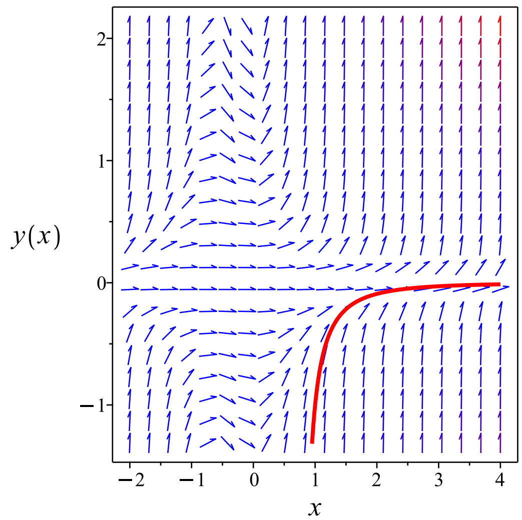
Summary

The solution(s) found are the following

$$y = -\frac{1}{x^3 + x^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{x^3 + x^2 - 1}$$

Verified OK.

3.25.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= 3x^2y^2 + 2x\, y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 3x^2y^2 + 2x\, y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = 3x^2 + 2x$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(3x^2 + 2x)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2u''(x) - (f_2' + f_1f_2)\,u'(x) + f_2^2f_0u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 6x + 2 \\ f_1f_2 &= 0 \\ f_2^2f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\left(3x^2 + 2x\right)u''(x) - (6x + 2)\,u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + x^2(x + 1)\,c_2$$

The above shows that

$$u'(x) = c_2x(3x + 2)$$

Using the above in (1) gives the solution

$$y = -\frac{c_2x(3x + 2)}{(3x^2 + 2x)\,(c_1 + x^2\,(x + 1)\,c_2)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{1}{x^3 + x^2 + c_3}$$

Initial conditions are used to solve for c_3. Substituting $x = 1$ and $y = -1$ in the above solution gives an equation to solve for the constant of integration.

$$-1 = -\frac{1}{c_3 + 2}$$

$$c_3 = -1$$

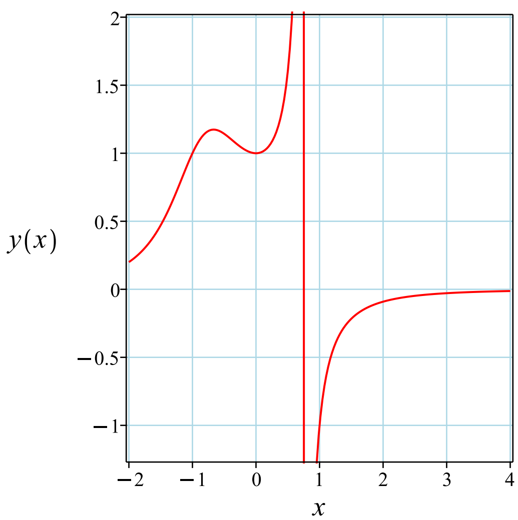
Substituting c_3 found above in the general solution gives

$$y = -\frac{1}{x^3 + x^2 - 1}$$

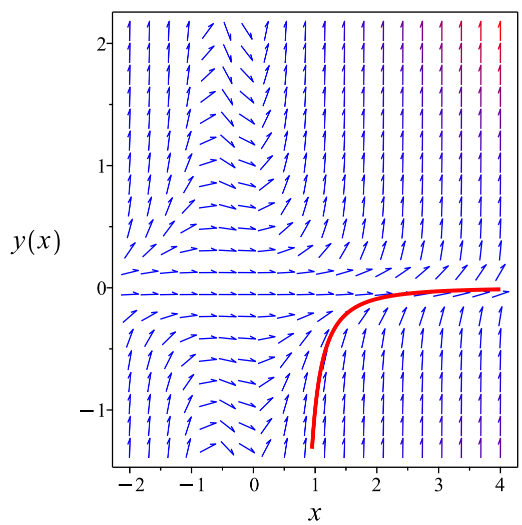
Summary

The solution(s) found are the following

$$y = -\frac{1}{x^3 + x^2 - 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{1}{x^3 + x^2 - 1}$$

Verified OK.

3.25.6 Maple step by step solution

Let's solve

$[y' - 2xy^2 - 3x^2y^2 = 0, y(1) = -1]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = x(3x+2)$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int x(3x+2)\, dx + c_1$
- Evaluate integral

  $-\frac{1}{y} = x^3 + x^2 + c_1$
- Solve for y

  $y = -\frac{1}{x^3+x^2+c_1}$
- Use initial condition $y(1) = -1$

  $-1 = -\frac{1}{c_1+2}$
- Solve for c_1

  $c_1 = -1$
- Substitute $c_1 = -1$ into general solution and simplify

  $y = -\frac{1}{x^3+x^2-1}$
- Solution to the IVP

  $y = -\frac{1}{x^3+x^2-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 16

```
dsolve([diff(y(x),x) = 2*x*y(x)^2+3*x^2*y(x)^2,y(1) = -1],y(x), singsol=all)
```

$$y(x) = -\frac{1}{x^3 + x^2 - 1}$$

✓ Solution by Mathematica

Time used: 0.118 (sec). Leaf size: 17

```
DSolve[{y'[x] == 2*x*y[x]^2+3*x^2*y[x]^2,y[1]==-1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{x^3 + x^2 - 1}$$

3.26 problem 27

Internal problem ID [52]
Internal file name [`OUTPUT/52_Sunday_June_05_2022_01_34_06_AM_21281142/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first order special form ID 1", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 6\,\mathrm{e}^{2x-y} = 0}$$

With initial conditions

$$[y(0) = 0]$$

3.26.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= 6\,\mathrm{e}^{2x-y} \end{aligned}$$

The x domain of $f(x,y)$ when $y=0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $f(x, y)$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(6\,\mathrm{e}^{2x-y}\right)\\ &= -6\,\mathrm{e}^{2x-y}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 0$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = 0$ is inside this domain. Therefore solution exists and is unique.

3.26.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x, y)\\ &= f(x)g(y)\\ &= 6\,\mathrm{e}^{2x}\mathrm{e}^{-y}\end{aligned}$$

Where $f(x) = 6\,\mathrm{e}^{2x}$ and $g(y) = \mathrm{e}^{-y}$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\mathrm{e}^{-y}}\,dy &= 6\,\mathrm{e}^{2x}\,dx\\ \int \frac{1}{\mathrm{e}^{-y}}\,dy &= \int 6\,\mathrm{e}^{2x}\,dx\\ \mathrm{e}^{y} &= 3\,\mathrm{e}^{2x} + c_1\end{aligned}$$

Which results in

$$y = \ln\left(3\,\mathrm{e}^{2x} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln (3 + c_1)$$

$$c_1 = -2$$

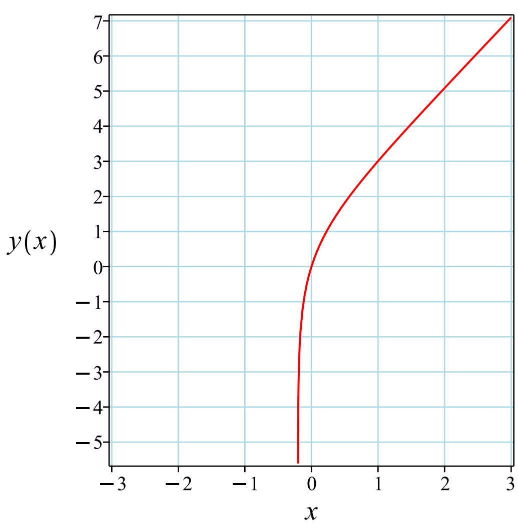
Substituting c_1 found above in the general solution gives

$$y = \ln \left(3 \, \mathrm{e}^{2x} - 2\right)$$

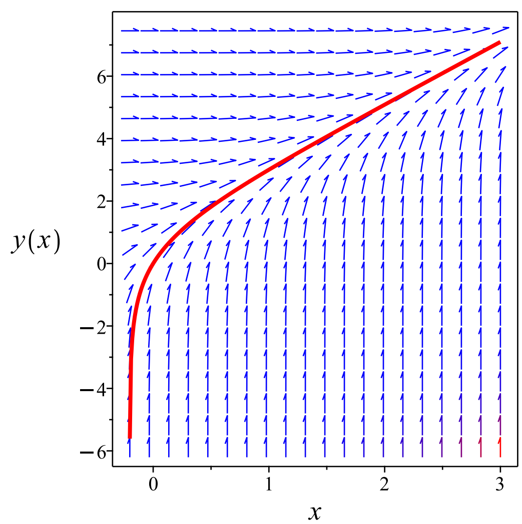
Summary

The solution(s) found are the following

$$y = \ln \left(3 \, \mathrm{e}^{2x} - 2\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln \left(3 \, \mathrm{e}^{2x} - 2\right)$$

Verified OK.

3.26.3 Solving as first order special form ID 1 ode

Writing the ode as

$$y' = 6\,e^{2x-y} \tag{1}$$

And using the substitution $u = e^y$ then

$$u' = y' e^y$$

The above shows that

$$\begin{aligned} y' &= u'(x)\,e^{-y} \\ &= \frac{u'(x)}{u} \end{aligned}$$

Substituting this in (1) gives

$$\frac{u'(x)}{u} = \frac{6\,e^{2x}}{u}$$

The above simplifies to

$$u'(x) = 6\,e^{2x} \tag{2}$$

Now ode (2) is solved for $u(x)$ Integrating both sides gives

$$\begin{aligned} u(x) &= \int 6\,e^{2x}\,dx \\ &= 3\,e^{2x} + c_1 \end{aligned}$$

Substituting the solution found for $u(x)$ in $u = e^y$ gives

$$\begin{aligned} y &= \ln(u(x)) \\ &= \ln\left(3\,e^{2x} + c_1\right) \\ &= \ln\left(3\,e^{2x} + c_1\right) \end{aligned}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(3 + c_1)$$

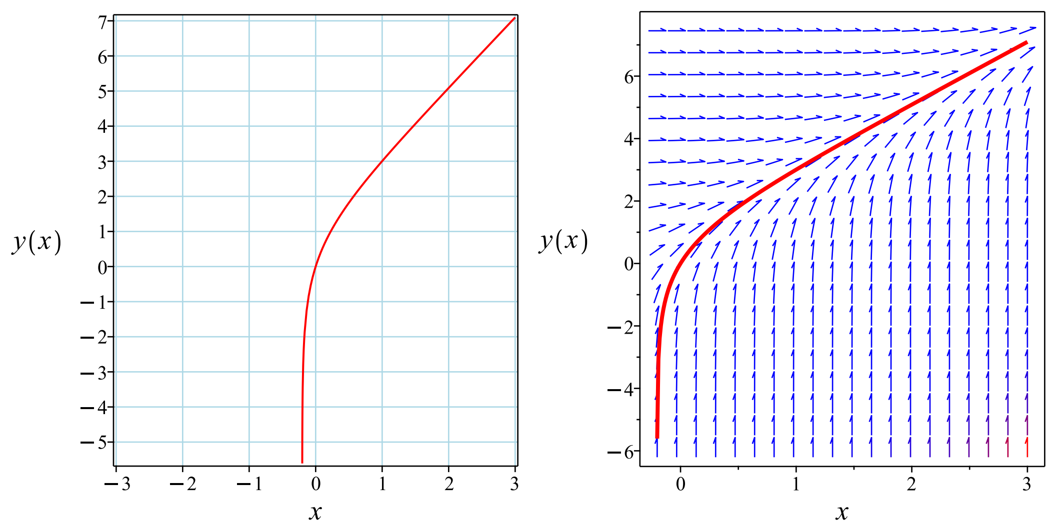
$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

$$y = \ln\left(3\,e^{2x}-2\right)$$

Summary
The solution(s) found are the following

$$y = \ln\left(3\,e^{2x}-2\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \ln\left(3\,e^{2x}-2\right)$$

Verified OK.

3.26.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 6\,e^{2x-y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 116: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{e^{-2x}}{6} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{\frac{e^{-2x}}{6}} dx \end{aligned}$$

Which results in

$$S = 3\, e^{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 6\, e^{2x-y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 0 \\ R_y &= 1 \\ S_x &= 6\, e^{2x} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{y} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$3\,e^{2x} = e^{y} + c_1$$

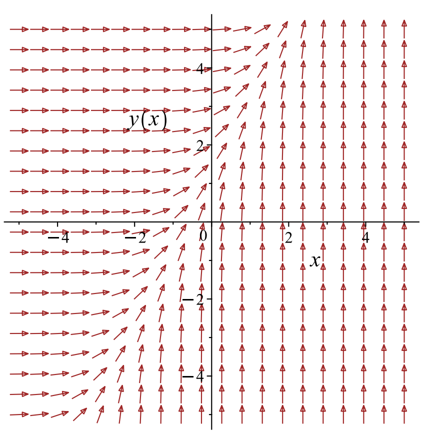
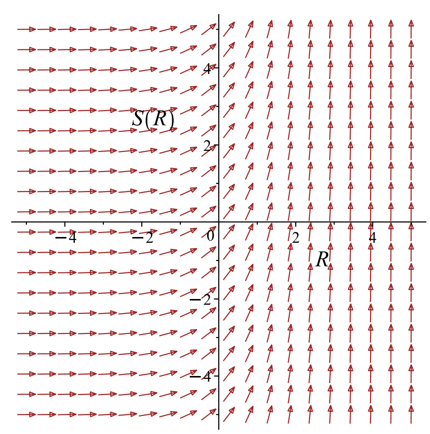
Which simplifies to

$$3\,e^{2x} = e^{y} + c_1$$

Which gives

$$y = \ln\left(3\,e^{2x} - c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 6\,e^{2x-y}$ 	$R = y$ $S = 3\,e^{2x}$	$\frac{dS}{dR} = e^{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(3 - c_1)$$

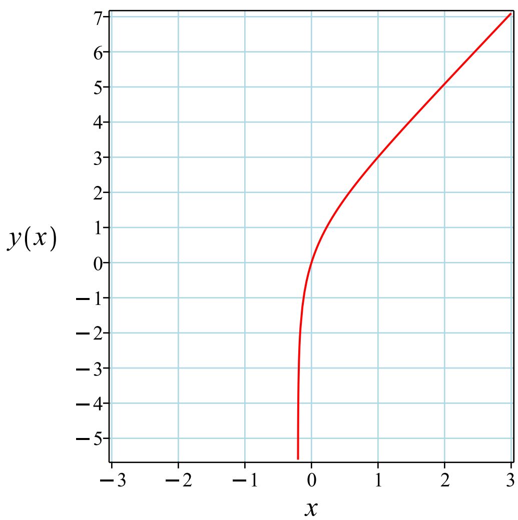
$$c_1 = 2$$

Substituting c_1 found above in the general solution gives

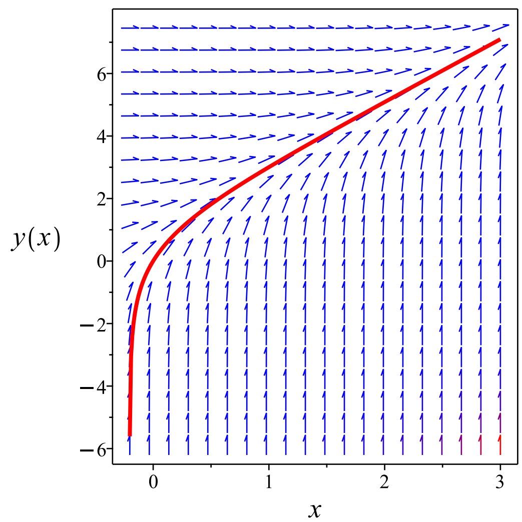
$$y = \ln\left(3\,e^{2x} - 2\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \ln\left(3\,e^{2x} - 2\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \ln\left(3\,e^{2x} - 2\right)$$

Verified OK.

3.26.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{y})\,\mathrm{d}y &= \left(6\,\mathrm{e}^{2x}\right)\mathrm{d}x \\ \left(-6\,\mathrm{e}^{2x}\right)\mathrm{d}x + (\mathrm{e}^{y})\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -6\,\mathrm{e}^{2x} \\ N(x,y) &= \mathrm{e}^{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-6\,\mathrm{e}^{2x}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{y}) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -6\,\mathrm{e}^{2x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -3\,\mathrm{e}^{2x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{y}$. Therefore equation (4) becomes

$$
\mathrm{e}^{y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \mathrm{e}^{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (\mathrm{e}^{y}) \, \mathrm{d}y \\
f(y) &= \mathrm{e}^{y} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3\,e^{2x} + e^{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3\,e^{2x} + e^{y}$$

The solution becomes

$$y = \ln\left(3\,e^{2x} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \ln(3 + c_1)$$

$$c_1 = -2$$

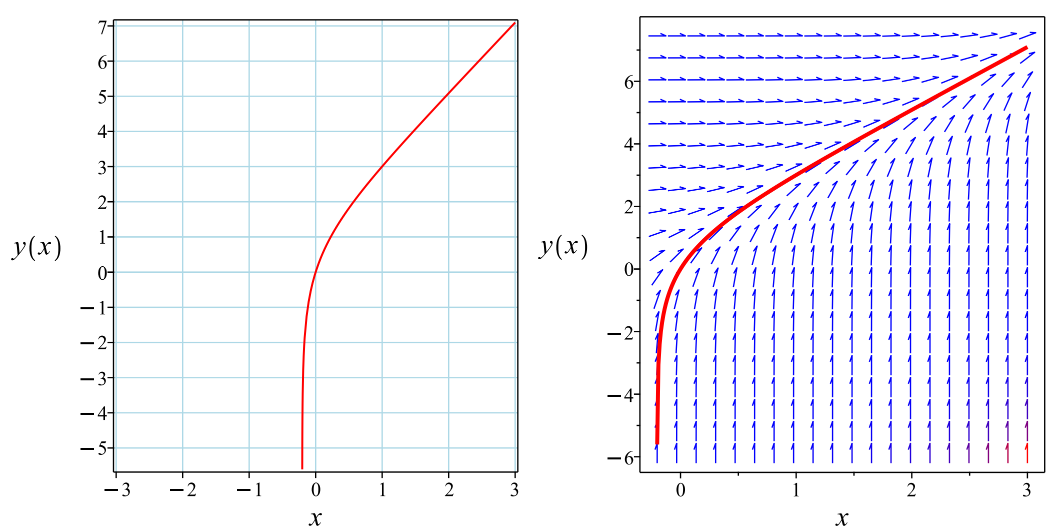
Substituting c_1 found above in the general solution gives

$$y = \ln\left(3\,e^{2x} - 2\right)$$

Summary

The solution(s) found are the following

$$y = \ln\left(3\,e^{2x} - 2\right) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \ln\left(3\,e^{2x} - 2\right)$$

Verified OK.

3.26.6 Maple step by step solution

Let's solve

$[y' - 6\,e^{2x-y} = 0, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $y' e^y = 6(e^x)^2$

- Integrate both sides with respect to x

  $\int y' e^y dx = \int 6(e^x)^2\,dx + c_1$

- Evaluate integral

  $e^y = 3(e^x)^2 + c_1$

- Solve for y

  $y = \ln\left(3(e^x)^2 + c_1\right)$

- Use initial condition $y(0) = 0$

  $0 = \ln(3 + c_1)$

- Solve for c_1

  $c_1 = -2$

- Substitute $c_1 = -2$ into general solution and simplify

  $y = \ln(3\,e^{2x} - 2)$

- Solution to the IVP

  $y = \ln(3\,e^{2x} - 2)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 13

```
dsolve([diff(y(x),x) = 6*exp(2*x-y(x)),y(0) = 0],y(x), singsol=all)
```

$$y(x) = \ln\left(-2 + 3\,\mathrm{e}^{2x}\right)$$

✓ Solution by Mathematica

Time used: 0.739 (sec). Leaf size: 15

```
DSolve[{y'[x] == 6*Exp[2*x-y[x]],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(3e^{2x} - 2\right)$$

3.27 problem 28

Internal problem ID [53]
Internal file name [`OUTPUT/53_Sunday_June_05_2022_01_34_06_AM_31221340/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.4. Separable equations. Page 43
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2\sqrt{x}\, y' - \cos(y)^2 = 0}$$

With initial conditions

$$\left[y(4) = \frac{\pi}{4}\right]$$

3.27.1 Existence and uniqueness analysis

This is non linear first order ODE. In canonical form it is written as

$$\begin{aligned} y' &= f(x,y) \\ &= \frac{\cos(y)^2}{2\sqrt{x}} \end{aligned}$$

The x domain of $f(x,y)$ when $y = \frac{\pi}{4}$ is

$$\{0 < x\}$$

And the point $x_0 = 4$ is inside this domain. The y domain of $f(x,y)$ when $x = 4$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Now we will look at the continuity of

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\cos(y)^2}{2\sqrt{x}}\right)\\ &= -\frac{\cos(y)\sin(y)}{\sqrt{x}}\end{aligned}$$

The x domain of $\frac{\partial f}{\partial y}$ when $y = \frac{\pi}{4}$ is

$$\{0 < x\}$$

And the point $x_0 = 4$ is inside this domain. The y domain of $\frac{\partial f}{\partial y}$ when $x = 4$ is

$$\{-\infty < y < \infty\}$$

And the point $y_0 = \frac{\pi}{4}$ is inside this domain. Therefore solution exists and is unique.

3.27.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}y' &= F(x,y)\\ &= f(x)g(y)\\ &= \frac{\cos(y)^2}{2\sqrt{x}}\end{aligned}$$

Where $f(x) = \frac{1}{2\sqrt{x}}$ and $g(y) = \cos(y)^2$. Integrating both sides gives

$$\begin{aligned}\frac{1}{\cos(y)^2}\,dy &= \frac{1}{2\sqrt{x}}\,dx\\ \int \frac{1}{\cos(y)^2}\,dy &= \int \frac{1}{2\sqrt{x}}\,dx\\ \tan(y) &= \sqrt{x} + c_1\end{aligned}$$

Which results in

$$y = \arctan\left(\sqrt{x} + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 4$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = \arctan\left(c_1 + 2\right)$$

$$c_1 = -1$$

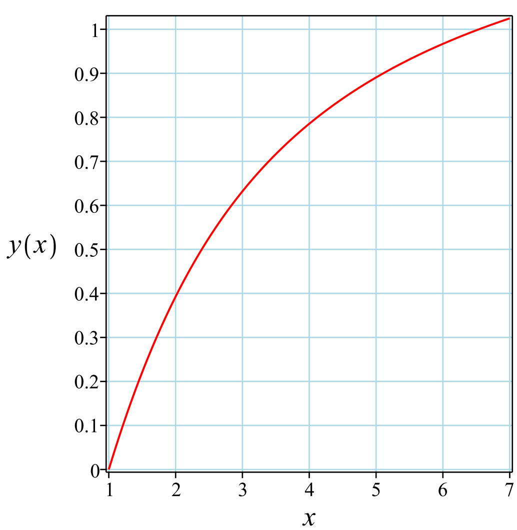
Substituting c_1 found above in the general solution gives

$$y = \arctan\left(\sqrt{x} - 1\right)$$

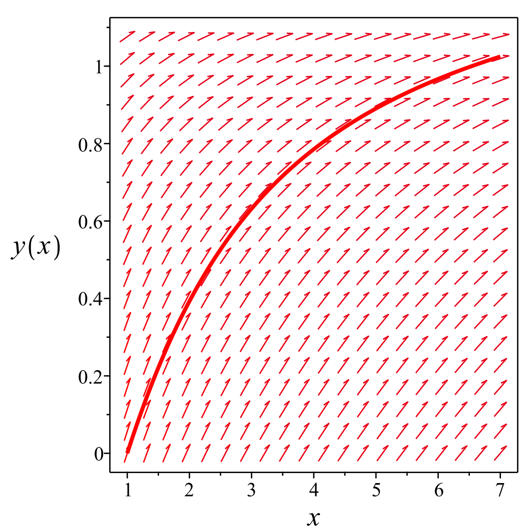
Summary

The solution(s) found are the following

$$y = \arctan\left(\sqrt{x} - 1\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arctan\left(\sqrt{x} - 1\right)$$

Verified OK.

3.27.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{\cos(y)^2}{2\sqrt{x}} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 119: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 2\sqrt{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{2\sqrt{x}} dx
\end{aligned}
$$

Which results in

$$
S = \sqrt{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{\cos(y)^2}{2\sqrt{x}}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{1}{2\sqrt{x}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sec(y)^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sec(R)^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \tan(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\sqrt{x} = \tan(y) + c_1
$$

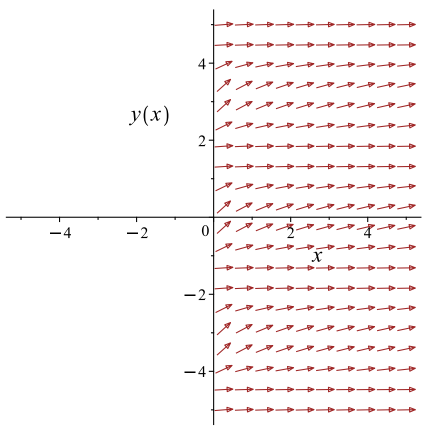
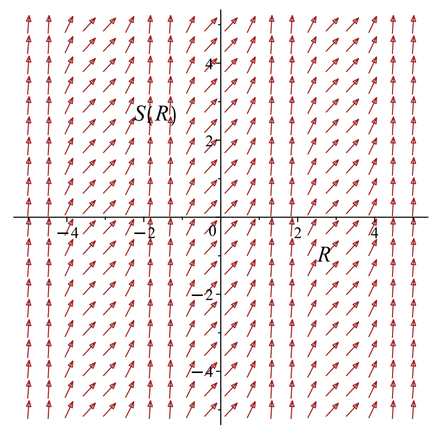
Which simplifies to

$$
\sqrt{x} = \tan(y) + c_1
$$

Which gives

$$
y = -\arctan\left(-\sqrt{x} + c_1\right)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\cos(y)^2}{2\sqrt{x}}$ 	$R = y$ $S = \sqrt{x}$	$\frac{dS}{dR} = \sec(R)^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 4$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$\frac{\pi}{4} = -\arctan(c_1 - 2)$$

$$c_1 = 1$$

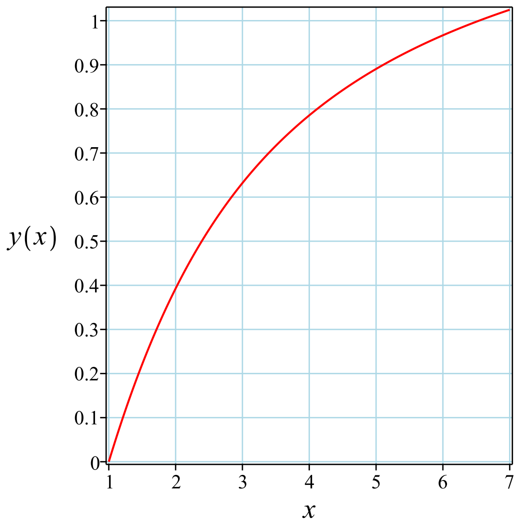
Substituting c_1 found above in the general solution gives

$$y = \arctan(\sqrt{x} - 1)$$

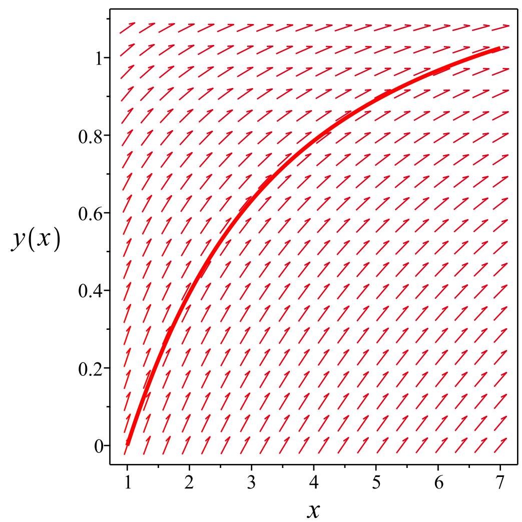
Summary

The solution(s) found are the following

$$y = \arctan(\sqrt{x} - 1) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arctan\left(\sqrt{x} - 1\right)$$

Verified OK.

3.27.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{2}{\cos(y)^2}\right)\mathrm{d}y = \left(\frac{1}{\sqrt{x}}\right)\mathrm{d}x$$
$$\left(-\frac{1}{\sqrt{x}}\right)\mathrm{d}x + \left(\frac{2}{\cos(y)^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{1}{\sqrt{x}}$$
$$N(x, y) = \frac{2}{\cos(y)^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{\sqrt{x}}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2}{\cos(y)^2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -\frac{1}{\sqrt{x}}\,\mathrm{d}x$$
$$\phi=-2\sqrt{x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=0+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{2}{\cos(y)^2}$. Therefore equation (4) becomes

$$\frac{2}{\cos\left(y\right)^{2}}=0+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{2}{\cos\left(y\right)^{2}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int \left(2\sec\left(y\right)^{2}\right)\mathrm{d}y$$
$$f(y)=2\tan\left(y\right)+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\sqrt{x} + 2\tan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\sqrt{x} + 2\tan(y)$$

Initial conditions are used to solve for c_1. Substituting $x = 4$ and $y = \frac{\pi}{4}$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_1$$

$$c_1 = -2$$

Substituting c_1 found above in the general solution gives

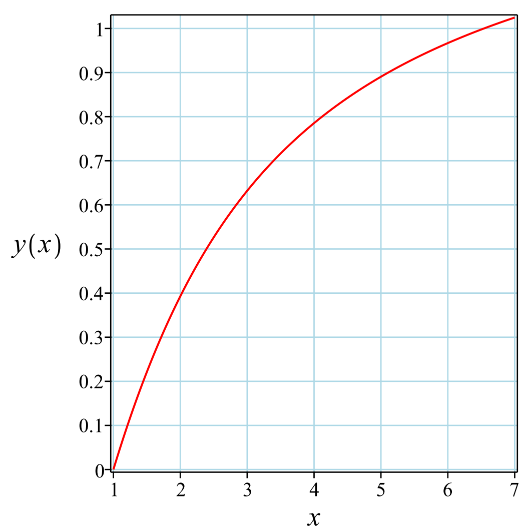
$$-2\sqrt{x} + 2\tan(y) = -2$$

Solving for y from the above gives

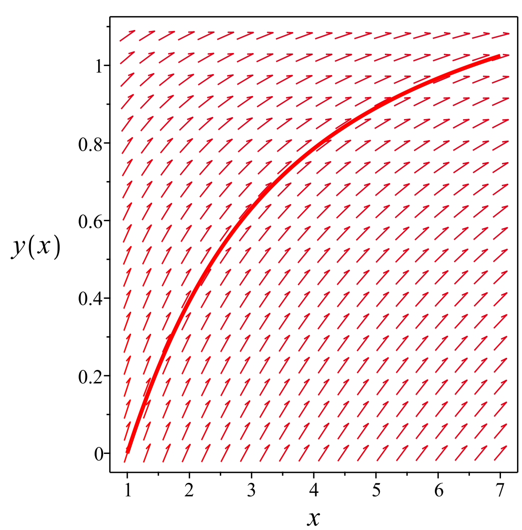
$$y = \arctan\left(\sqrt{x} - 1\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \arctan\left(\sqrt{x} - 1\right) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \arctan\left(\sqrt{x} - 1\right)$$

Verified OK.

3.27.5 Maple step by step solution

Let's solve

$\left[2\sqrt{x}\, y' - \cos\left(y\right)^2 = 0, y(4) = \frac{\pi}{4}\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\cos(y)^2} = \frac{1}{2\sqrt{x}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{\cos(y)^2} dx = \int \frac{1}{2\sqrt{x}} dx + c_1$

- Evaluate integral

  $\tan\left(y\right) = \sqrt{x} + c_1$

- Solve for y

  $y = \arctan\left(\sqrt{x} + c_1\right)$

- Use initial condition $y(4) = \frac{\pi}{4}$

  $\frac{\pi}{4} = \arctan\left(\sqrt{4} + c_1\right)$

- Solve for c_1

  $c_1 = -1$

- Substitute $c_1 = -1$ into general solution and simplify

  $y = \arctan\left(\sqrt{x} - 1\right)$

- Solution to the IVP

  $y = \arctan\left(\sqrt{x} - 1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.125 (sec). Leaf size: 10

```
dsolve([2*x^(1/2)*diff(y(x),x) = cos(y(x))^2,y(4) = 1/4*Pi],y(x), singsol=all)
```

$$y(x) = \arctan\left(-1+\sqrt{x}\right)$$

✓ Solution by Mathematica

Time used: 0.46 (sec). Leaf size: 17

```
DSolve[{2*x^(1/2)*y'[x] == Cos[y[x]]^2,y[4]==Pi/4},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\arctan\left(1-\sqrt{x}\right)$$

4 Section 1.5. Linear first order equations. Page 56

4.1 problem 1

Internal problem ID [54]
Internal file name [`OUTPUT/54_Sunday_June_05_2022_01_34_08_AM_49603279/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

```
[_quadrature]
```

$$\boxed{y' + y = 2}$$

With initial conditions

$$[y(0) = 0]$$

4.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = 2$$

Hence the ode is

$$y' + y = 2$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.1.2 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{-y+2} dy = \int dx$$
$$-\ln(-y+2) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{-y+2} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{-y+2} = c_2 e^x$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = \frac{-1 + 2c_2}{c_2}$$

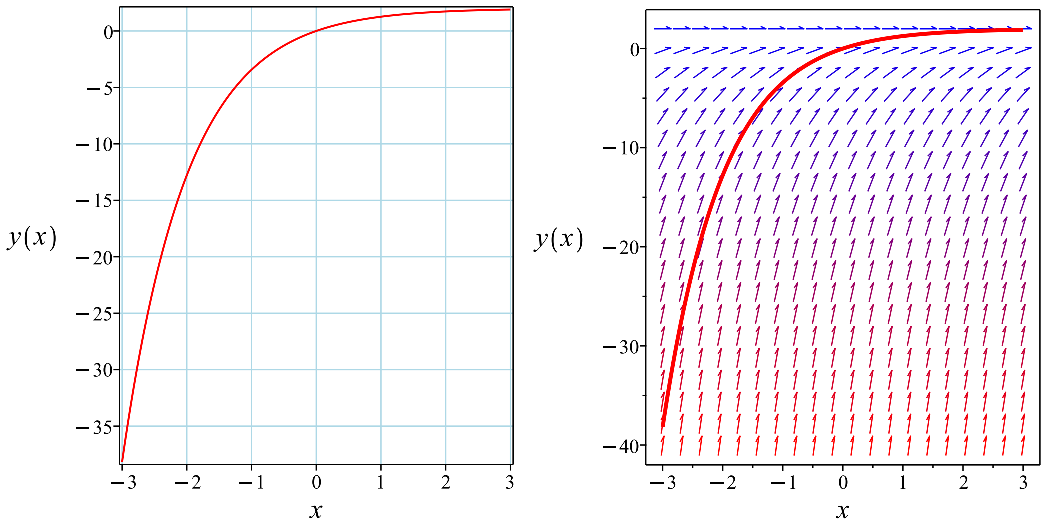
$$c_2 = \frac{1}{2}$$

Substituting c_2 found above in the general solution gives

$$y = -2\,e^{-x} + 2$$

Summary
The solution(s) found are the following

$$y = -2\,e^{-x} + 2 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2\,e^{-x} + 2$$

Verified OK.

4.1.3 Maple step by step solution

Let's solve

$[y' + y = 2, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{-y+2} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{-y+2}dx = \int 1dx + c_1$

- Evaluate integral

  $-\ln(-y + 2) = x + c_1$

- Solve for y

  $y = -e^{-x-c_1} + 2$

- Use initial condition $y(0) = 0$

  $0 = -e^{-c_1} + 2$
- Solve for c_1

  $c_1 = -\ln(2)$
- Substitute $c_1 = -\ln(2)$ into general solution and simplify

  $y = -2\,e^{-x} + 2$
- Solution to the IVP

  $y = -2\,e^{-x} + 2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([y(x)+diff(y(x),x) = 2,y(0) = 0],y(x), singsol=all)
```

$$y(x) = 2 - 2\,e^{-x}$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 14

```
DSolve[{y[x]+y'[x] == 2,y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 2 - 2e^{-x}$$

4.2 problem 2

Internal problem ID [55]
Internal file name [`OUTPUT/55_Sunday_June_05_2022_01_34_08_AM_17924870/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' - 2y = 3\,\mathrm{e}^{2x}}$$

With initial conditions

$$[y(0) = 0]$$

4.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2$$
$$q(x) = 3\,\mathrm{e}^{2x}$$

Hence the ode is

$$y' - 2y = 3\,\mathrm{e}^{2x}$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3\,\mathrm{e}^{2x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.2.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int (-2)dx}\\ &= \mathrm{e}^{-2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(3\,\mathrm{e}^{2x}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-2x}y\right) &= \left(\mathrm{e}^{-2x}\right)\left(3\,\mathrm{e}^{2x}\right)\\ \mathrm{d}\left(\mathrm{e}^{-2x}y\right) &= 3\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\mathrm{e}^{-2x}y &= \int 3\,\mathrm{d}x\\ \mathrm{e}^{-2x}y &= 3x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-2x}$ results in

$$y = 3x\,\mathrm{e}^{2x} + c_1 \mathrm{e}^{2x}$$

which simplifies to

$$y = \mathrm{e}^{2x}(3x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

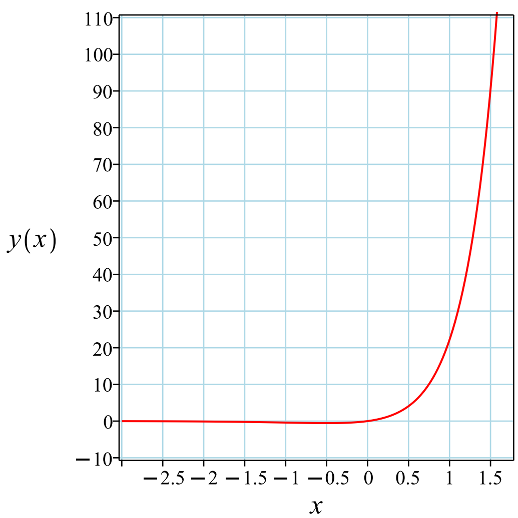
Substituting c_1 found above in the general solution gives

$$y = 3x\,\mathrm{e}^{2x}$$

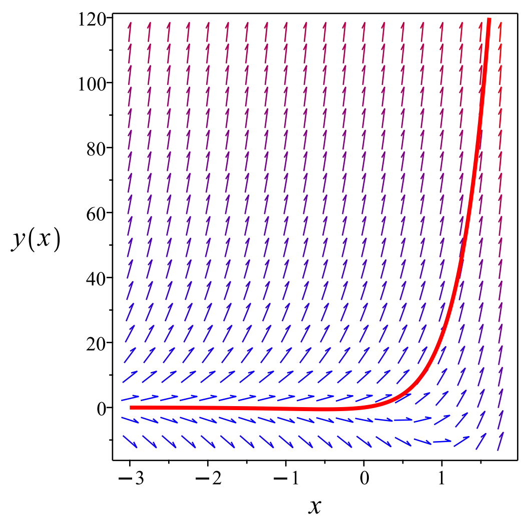
Summary

The solution(s) found are the following

$$y = 3x\,\mathrm{e}^{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3x\,\mathrm{e}^{2x}$$

Verified OK.

4.2.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 2y + 3\,\mathrm{e}^{2x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 123: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= e^{2x}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{e^{2x}} dy\end{aligned}$$

Which results in

$$S = e^{-2x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 2y + 3\,e^{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 0\\ S_x &= -2\,e^{-2x} y\\ S_y &= e^{-2x}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 3R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-2x} y = 3x + c_1$$

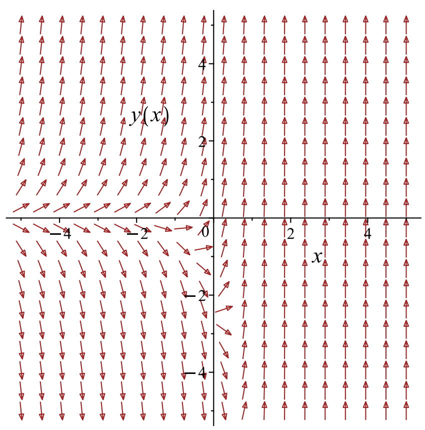
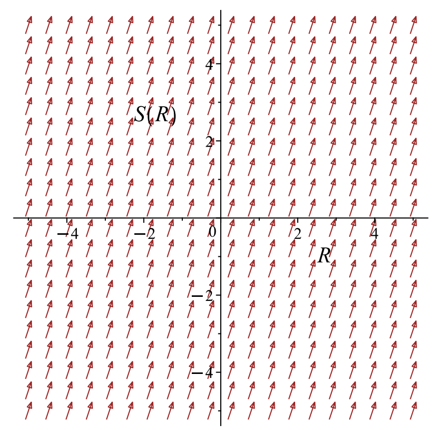
Which simplifies to

$$\mathrm{e}^{-2x} y = 3x + c_1$$

Which gives

$$y = \mathrm{e}^{2x}(3x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = 2y + 3\,\mathrm{e}^{2x}$

</td><td>$R = x$
$S = \mathrm{e}^{-2x} y$</td><td>$\frac{dS}{dR} = 3$

</td></tr>
</table>

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

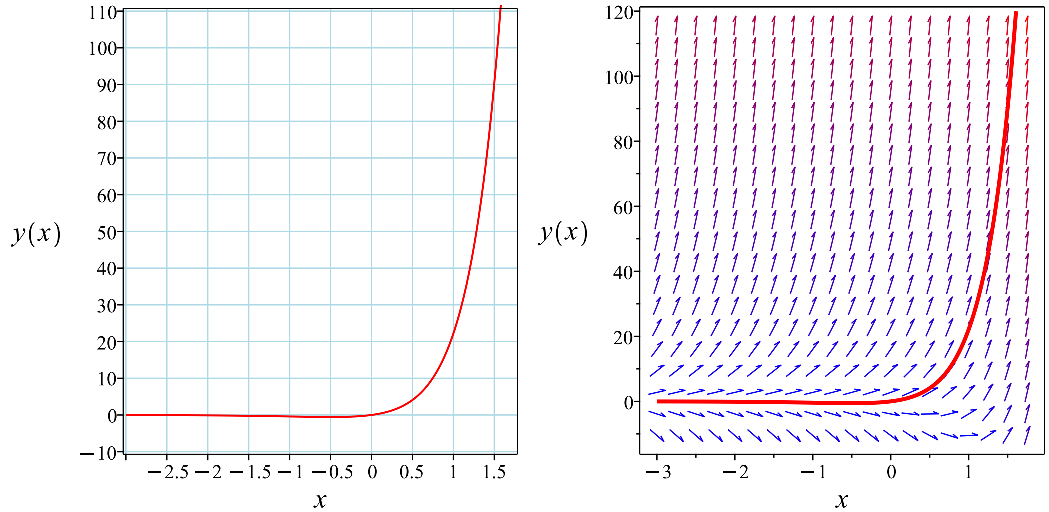
Substituting c_1 found above in the general solution gives

$$y = 3x\,\mathrm{e}^{2x}$$

Summary

The solution(s) found are the following

$$y = 3x\,\mathrm{e}^{2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = 3x\,\mathrm{e}^{2x}$$

Verified OK.

4.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\mathrm{d}y&=\left(2y+3\,\mathrm{e}^{2x}\right)\mathrm{d}x\\ \left(-2y-3\,\mathrm{e}^{2x}\right)\mathrm{d}x+\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-2y-3\,\mathrm{e}^{2x}\\ N(x,y)&=1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-2y-3\,\mathrm{e}^{2x}\right)\\ &=-2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(1)\\ &=0\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=1((-2)-(0))\\ &=-2\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -2 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2x} \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-2x}\left(-2y - 3\,\mathrm{e}^{2x}\right) \\
&= -2\,\mathrm{e}^{-2x} y - 3
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-2x}(1) \\
&= \mathrm{e}^{-2x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-2\,\mathrm{e}^{-2x} y - 3\right) + \left(\mathrm{e}^{-2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -2\,\mathrm{e}^{-2x} y - 3 \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -3x + \mathrm{e}^{-2x} y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-2x}$. Therefore equation (4) becomes

$$\mathrm{e}^{-2x} = \mathrm{e}^{-2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -3x + \mathrm{e}^{-2x} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3x + \mathrm{e}^{-2x} y$$

The solution becomes

$$y = \mathrm{e}^{2x}(3x + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1$$

$$c_1 = 0$$

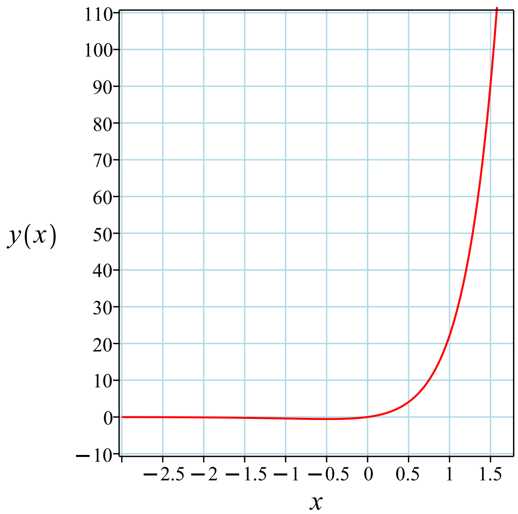
Substituting c_1 found above in the general solution gives

$$y = 3x\,\mathrm{e}^{2x}$$

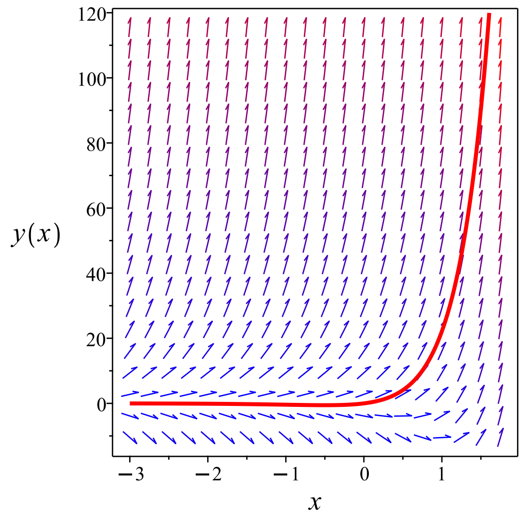
Summary

The solution(s) found are the following

$$y = 3x\,\mathrm{e}^{2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3x\,\mathrm{e}^{2x}$$

Verified OK.

4.2.5 Maple step by step solution

Let's solve

$[y' - 2y = 3\,\mathrm{e}^{2x}, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2y + 3\,\mathrm{e}^{2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - 2y = 3\,\mathrm{e}^{2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\left(y'-2y\right)=3\mu(x)\,\mathrm{e}^{2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\left(y'-2y\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x)=-2\mu(x)$

- Solve to find the integrating factor

$\mu(x)=\mathrm{e}^{-2x}$

- Integrate both sides with respect to x

$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 3\mu(x)\,\mathrm{e}^{2x}dx+c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y=\int 3\mu(x)\,\mathrm{e}^{2x}dx+c_1$

- Solve for y

$y=\frac{\int 3\mu(x)\mathrm{e}^{2x}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\mathrm{e}^{-2x}$

$y=\frac{\int 3\,\mathrm{e}^{2x}\mathrm{e}^{-2x}dx+c_1}{\mathrm{e}^{-2x}}$

- Evaluate the integrals on the rhs

$y=\frac{3x+c_1}{\mathrm{e}^{-2x}}$

- Simplify

$y=\mathrm{e}^{2x}(3x+c_1)$

- Use initial condition $y(0)=0$

$0=c_1$

- Solve for c_1

$c_1=0$

- Substitute $c_1=0$ into general solution and simplify

$y=3x\,\mathrm{e}^{2x}$

- Solution to the IVP

$y=3x\,\mathrm{e}^{2x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([-2*y(x)+diff(y(x),x) = 3*exp(2*x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = 3\,\mathrm{e}^{2x} x$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 13

```
DSolve[{-2*y[x]+y'[x] == 3*Exp[2*x],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3e^{2x} x$$

4.3 problem 3

Internal problem ID [56]
Internal file name [`OUTPUT/56_Sunday_June_05_2022_01_34_09_AM_7434653/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{3y + y' = 2x\,\mathrm{e}^{-3x}}$$

4.3.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 2x\,\mathrm{e}^{-3x} \end{aligned}$$

Hence the ode is

$$3y + y' = 2x\,\mathrm{e}^{-3x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int 3dx} \\ &= \mathrm{e}^{3x} \end{aligned}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(2x\,e^{-3x}\right)$$
$$\frac{d}{dx}\left(y\,e^{3x}\right) = \left(e^{3x}\right)\left(2x\,e^{-3x}\right)$$
$$d\left(y\,e^{3x}\right) = (2x)\,dx$$

Integrating gives

$$y\,e^{3x} = \int 2x\,dx$$
$$y\,e^{3x} = x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = e^{3x}$ results in

$$y = e^{-3x}x^2 + c_1 e^{-3x}$$

which simplifies to

$$y = e^{-3x}\left(x^2 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = e^{-3x}\left(x^2 + c_1\right) \tag{1}$$

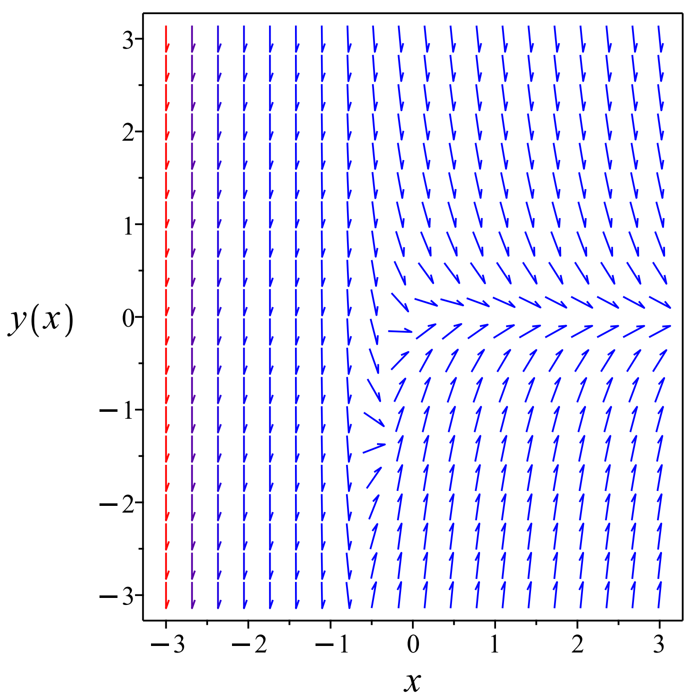


Figure 161: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^2 + c_1)$$

Verified OK.

4.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -(3y e^{3x} - 2x) e^{-3x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 126: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-3x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-3x}} dy \end{aligned}$$

Which results in

$$S = y\,e^{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\left(3y\,e^{3x} - 2x\right) e^{-3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\,e^{3x} \\ S_y &= e^{3x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{3x} = x^2 + c_1$$

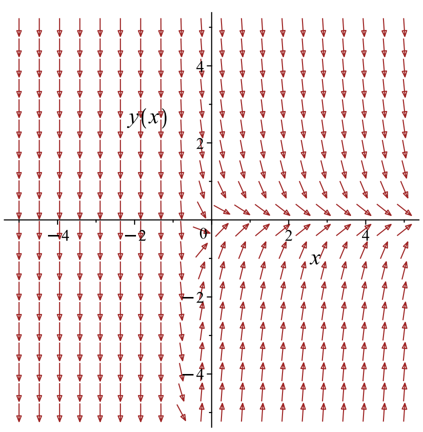
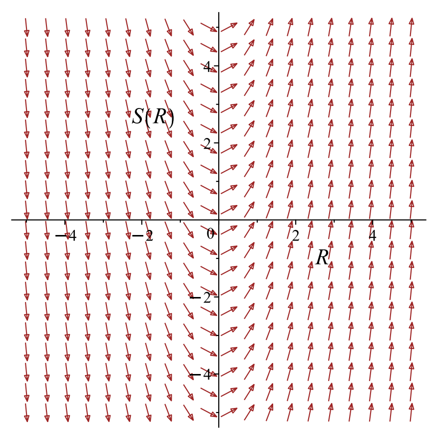
Which simplifies to

$$y\,\mathrm{e}^{3x} = x^2 + c_1$$

Which gives

$$y = \mathrm{e}^{-3x}\left(x^2 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -(3y\,\mathrm{e}^{3x} - 2x)\,\mathrm{e}^{-3x}$ 	$R = x$ $S = y\,\mathrm{e}^{3x}$	$\frac{dS}{dR} = 2R$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}\left(x^2 + c_1\right) \tag{1}$$

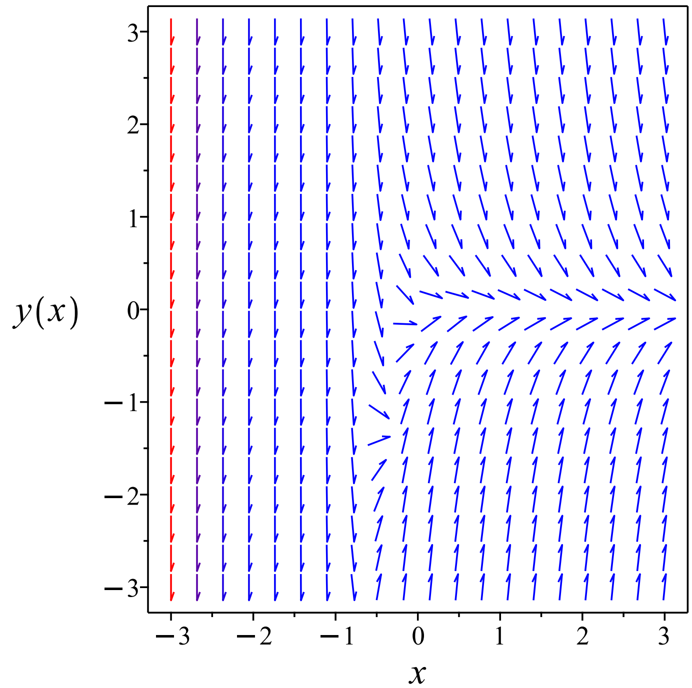


Figure 162: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^2 + c_1)$$

Verified OK.

4.3.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{3x})\,\mathrm{d}y &= (-3y\,\mathrm{e}^{3x} + 2x)\,\mathrm{d}x\\ (3y\,\mathrm{e}^{3x} - 2x)\,\mathrm{d}x + (\mathrm{e}^{3x})\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3y\,\mathrm{e}^{3x} - 2x\\ N(x,y) &= \mathrm{e}^{3x}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y\,\mathrm{e}^{3x} - 2x)\\ &= 3\,\mathrm{e}^{3x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^{3x})\\ &= 3\,\mathrm{e}^{3x}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 3y\,\mathrm{e}^{3x}-2x\,\mathrm{d}x$$
$$\phi=-x^2+y\,\mathrm{e}^{3x}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=\mathrm{e}^{3x}+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\mathrm{e}^{3x}$. Therefore equation (4) becomes

$$\mathrm{e}^{3x}=\mathrm{e}^{3x}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=0$$

Therefore

$$f(y)=c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi=-x^2+y\,\mathrm{e}^{3x}+c_1$$

But since ϕ itself is a constant function, then let $\phi=c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1=-x^2+y\,\mathrm{e}^{3x}$$

The solution becomes

$$y = e^{-3x}(x^2 + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(x^2 + c_1) \tag{1}$$

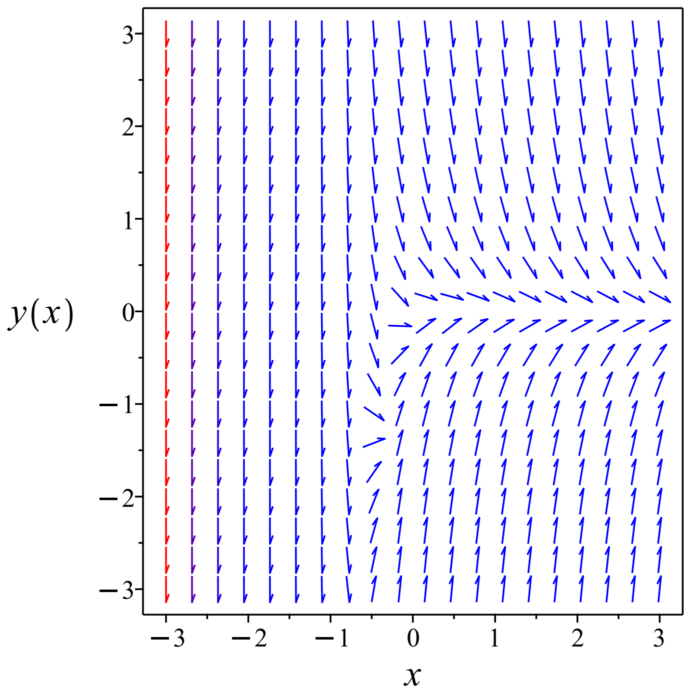


Figure 163: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^2 + c_1)$$

Verified OK.

4.3.4 Maple step by step solution

Let's solve

$3y + y' = \frac{2x}{e^{3x}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -3y + \frac{2x}{e^{3x}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $3y + y' = \frac{2x}{e^{3x}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(3y + y') = \frac{2\mu(x)x}{e^{3x}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(3y + y') = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 3\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = (e^{3x})^2\,e^{-3x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{2\mu(x)x}{e^{3x}} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{2\mu(x)x}{e^{3x}} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(x)x}{e^{3x}} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (e^{3x})^2\,e^{-3x}$

  $y = \frac{\int 2x\,e^{-3x} e^{3x} dx + c_1}{(e^{3x})^2 e^{-3x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1}{(e^{3x})^2 e^{-3x}}$

- Simplify

$$y = e^{-3x}(x^2 + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(3*y(x)+diff(y(x),x) = 2*x/exp(3*x),y(x), singsol=all)
```

$$y(x) = (x^2 + c_1) e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.051 (sec). Leaf size: 17

```
DSolve[3*y[x]+y'[x] == 2*x/Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(x^2 + c_1)$$

4.4 problem 4

Internal problem ID [57]
Internal file name [`OUTPUT/57_Sunday_June_05_2022_01_34_09_AM_17869309/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-2yx + y' = e^{x^2}}$$

4.4.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -2x$$
$$q(x) = e^{x^2}$$

Hence the ode is

$$-2yx + y' = e^{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -2x dx}$$
$$= e^{-x^2}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(e^{x^2}\right) \\
\frac{d}{dx}\left(e^{-x^2} y\right) &= \left(e^{-x^2}\right)\left(e^{x^2}\right) \\
d\left(e^{-x^2} y\right) &= dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x^2} y &= \int dx \\
e^{-x^2} y &= x + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$
y = e^{x^2} x + c_1 e^{x^2}
$$

which simplifies to

$$
y = e^{x^2}(x + c_1)
$$

Summary

The solution(s) found are the following

$$
y = e^{x^2}(x + c_1) \tag{1}
$$

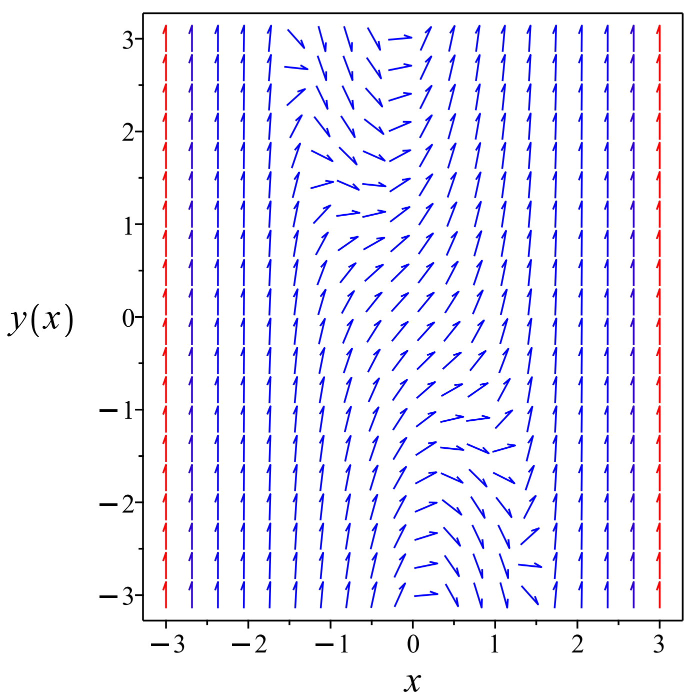


Figure 164: Slope field plot

Verification of solutions

$$y = e^{x^2}(x + c_1)$$

Verified OK.

4.4.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 2yx + e^{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 129: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^2}} dy \end{aligned}$$

Which results in

$$S = e^{-x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 2yx + e^{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -2x\, e^{-x^2} y \\ S_y &= e^{-x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x^2} y = x + c_1$$

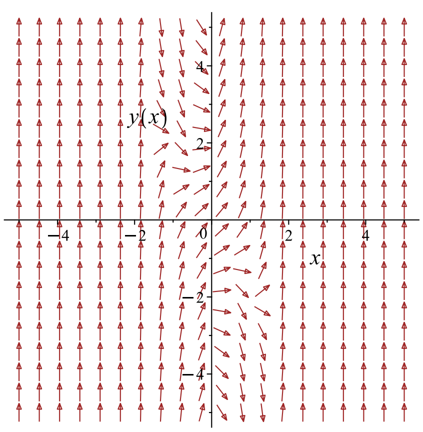
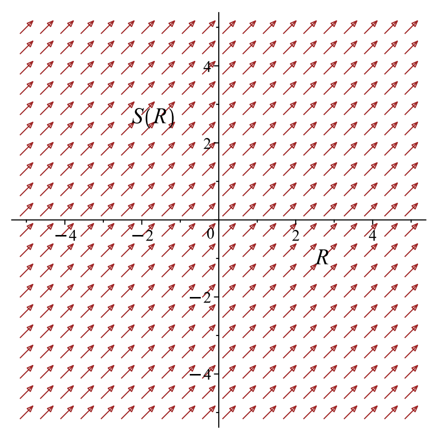
Which simplifies to

$$e^{-x^2} y = x + c_1$$

Which gives

$$y = e^{x^2} (x + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 2yx + e^{x^2}$ 	$R = x$ $S = e^{-x^2} y$	$\frac{dS}{dR} = 1$ 

Summary

The solution(s) found are the following

$$y = e^{x^2} (x + c_1) \tag{1}$$

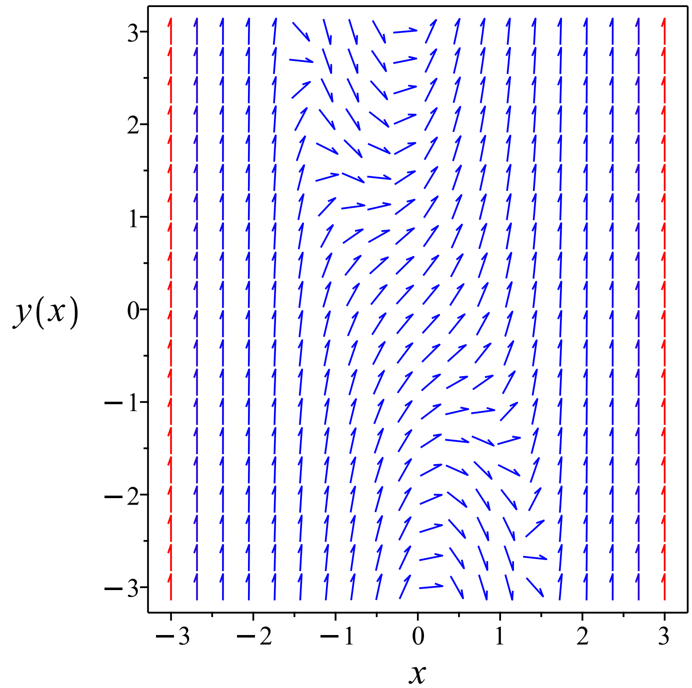


Figure 165: Slope field plot

Verification of solutions

$$y = e^{x^2}(x + c_1)$$

Verified OK.

4.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(2yx + \mathrm{e}^{x^2}\right)\mathrm{d}x$$
$$\left(-2yx - \mathrm{e}^{x^2}\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -2yx - \mathrm{e}^{x^2}$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-2yx - \mathrm{e}^{x^2}\right)$$
$$= -2x$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((-2x) - (0)) \\
&= -2x
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -2x \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-x^2} \\
&= \mathrm{e}^{-x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x^2}\left(-2yx - \mathrm{e}^{x^2}\right) \\
&= -2x\,\mathrm{e}^{-x^2} y - 1
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x^2}(1) \\
&= \mathrm{e}^{-x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-2x\,\mathrm{e}^{-x^2} y - 1\right) + \left(\mathrm{e}^{-x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int -2x\,\mathrm{e}^{-x^2} y - 1 \,\mathrm{d}x$$

$$\phi = -x + \mathrm{e}^{-x^2} y + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x^2} = \mathrm{e}^{-x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \mathrm{e}^{-x^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \mathrm{e}^{-x^2} y$$

The solution becomes

$$y = e^{x^2}(x + c_1)$$

Summary

The solution(s) found are the following

$$y = e^{x^2}(x + c_1) \tag{1}$$

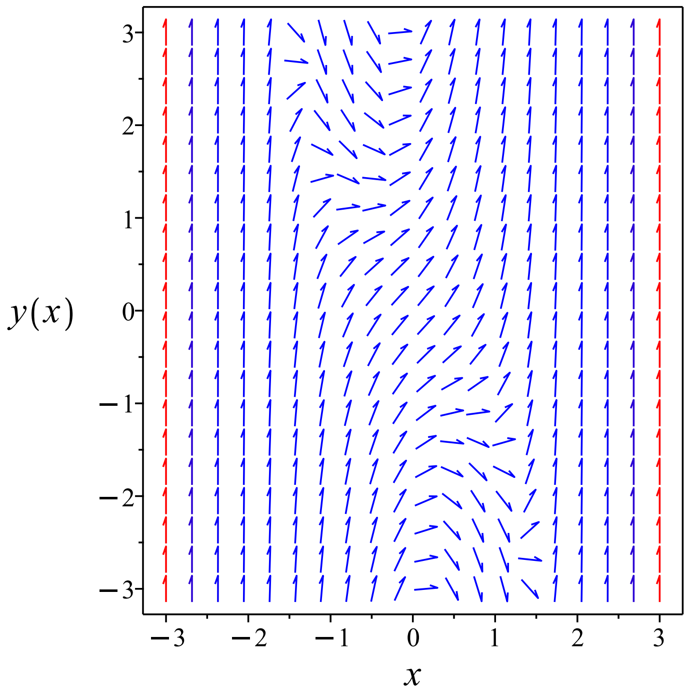


Figure 166: Slope field plot

Verification of solutions

$$y = e^{x^2}(x + c_1)$$

Verified OK.

4.4.4 Maple step by step solution

Let's solve

$-2yx + y' = \mathrm{e}^{x^2}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 2yx + \mathrm{e}^{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-2yx + y' = \mathrm{e}^{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\,(-2yx + y') = \mu(x)\,\mathrm{e}^{x^2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\,(-2yx + y') = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -2\mu(x)\,x$

- Solve to find the integrating factor

  $\mu(x) = \mathrm{e}^{-x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,\mathrm{e}^{x^2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \mu(x)\,\mathrm{e}^{x^2} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\mathrm{e}^{x^2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \mathrm{e}^{-x^2}$

  $y = \frac{\int \mathrm{e}^{x^2}\mathrm{e}^{-x^2} dx + c_1}{\mathrm{e}^{-x^2}}$

- Evaluate the integrals on the rhs

  $y = \frac{x + c_1}{\mathrm{e}^{-x^2}}$

- Simplify

$$y = e^{x^2}(x + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(-2*x*y(x)+diff(y(x),x) = exp(x^2),y(x), singsol=all)
```

$$y(x) = (c_1 + x)\, e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.041 (sec). Leaf size: 15

```
DSolve[-2*x*y[x]+y'[x] == Exp[x^2],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}(x + c_1)$$

4.5 problem 5

Internal problem ID [58]
Internal file name [`OUTPUT/58_Sunday_June_05_2022_01_34_09_AM_42143003/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + y'x = 3x}$$

With initial conditions

$$[y(1) = 5]$$

4.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = 3$$

Hence the ode is

$$y' + \frac{2y}{x} = 3$$

The domain of $p(x) = \frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.5.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x} dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(3) \\ \frac{d}{dx}(y\,x^2) &= (x^2)(3) \\ d(y\,x^2) &= (3x^2)\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^2 &= \int 3x^2\, dx \\ y\,x^2 &= x^3 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = x + \frac{c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 1 + c_1$$

$$c_1 = 4$$

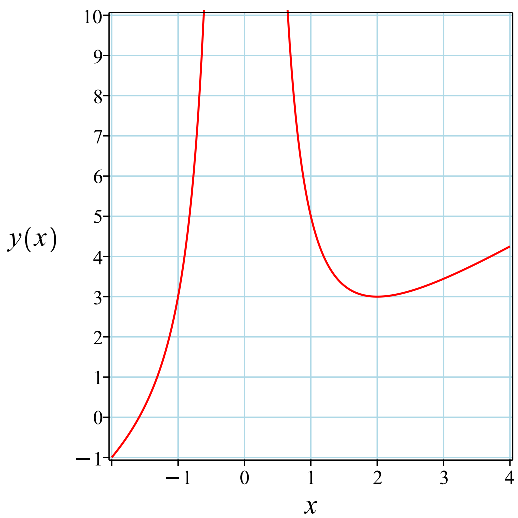
Substituting c_1 found above in the general solution gives

$$y = \frac{x^3 + 4}{x^2}$$

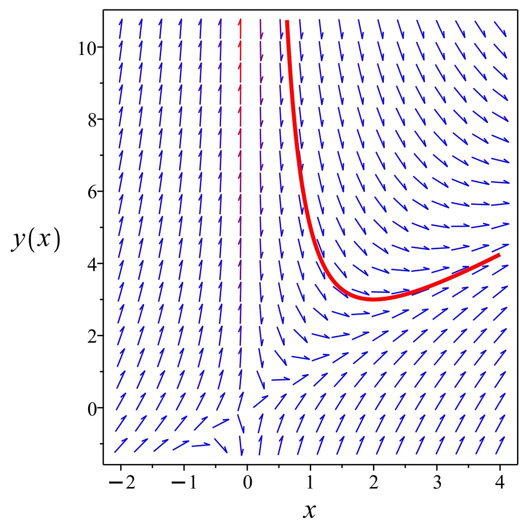
Summary

The solution(s) found are the following

$$y = \frac{x^3 + 4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^3 + 4}{x^2}$$

Verified OK.

4.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x + (u'(x)\,x + u(x))\,x = 3x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-3u+3}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -3u + 3$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-3u+3}\,du &= \frac{1}{x}\,dx \\ \int \frac{1}{-3u+3}\,du &= \int \frac{1}{x}\,dx \\ -\frac{\ln(u-1)}{3} &= \ln(x) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{(u-1)^{\frac{1}{3}}} = \mathrm{e}^{\ln(x)+c_2}$$

Which simplifies to

$$\frac{1}{(u-1)^{\frac{1}{3}}} = c_3 x$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= \frac{(c_3^3 \mathrm{e}^{3c_2} x^3 + 1)\,\mathrm{e}^{-3c_2}}{x^2 c_3^3} \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = \frac{\mathrm{e}^{-3c_2}\mathrm{e}^{3c_2} c_3^3 + \mathrm{e}^{-3c_2}}{c_3^3}$$

$$c_2 = -\frac{\ln(4c_3^3)}{3}$$

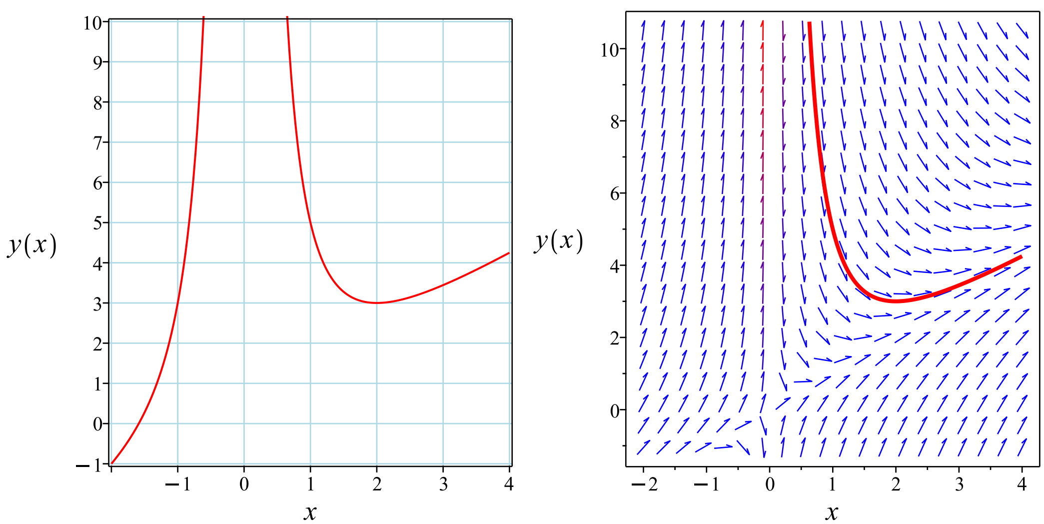
Substituting c_2 found above in the general solution gives

$$y = \frac{x^3+4}{x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3+4}{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x^3+4}{x^2}$$

Verified OK.

4.5.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2y-3x}{x}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 132: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{x^2} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\,x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y - 3x}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 2yx \\
S_y &= x^2
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 3x^2 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 3R^2
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = R^3 + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
x^2 y = x^3 + c_1
$$

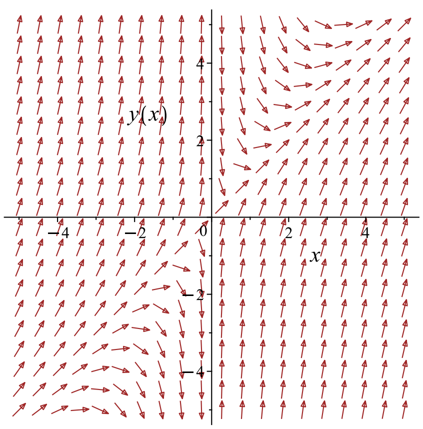
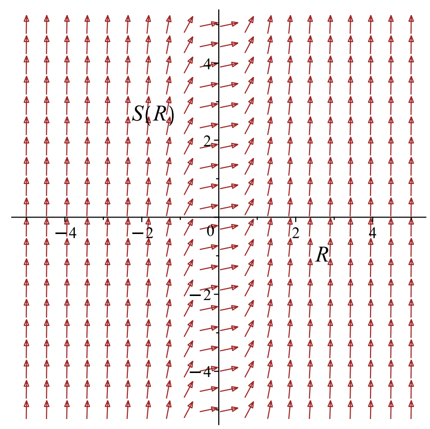
Which simplifies to

$$
x^2 y = x^3 + c_1
$$

Which gives

$$
y = \frac{x^3 + c_1}{x^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y-3x}{x}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = 3R^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 1 + c_1$$

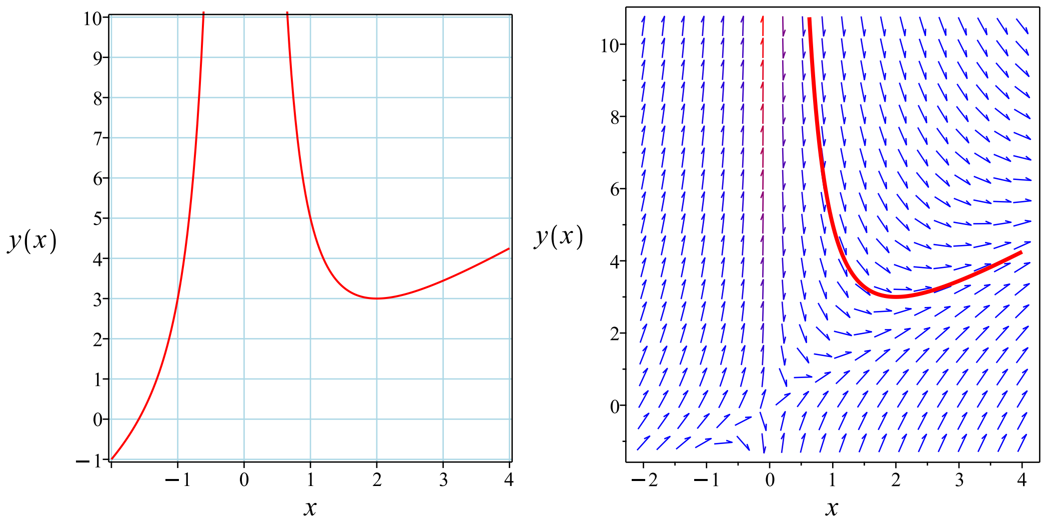
$$c_1 = 4$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x^3 + 4}{x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{x^3 + 4}{x^2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$
y = \frac{x^3 + 4}{x^2}
$$

Verified OK.

4.5.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$
M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x, y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\frac{\partial \phi}{\partial x} = M
$$
$$
\frac{\partial \phi}{\partial y} = N
$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=(-2y+3x)\,\mathrm{d}x\\ (2y-3x)\,\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=2y-3x\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(2y-3x)\\ &=2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\ &=1\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{x}((2)-(1))\\ &=\frac{1}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\ln(x)} \\ &= x\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x(2y - 3x) \\ &= x(2y - 3x)\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x(x) \\ &= x^2\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (x(2y - 3x)) + \left(x^2\right) \frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int x(2y - 3x) \, \mathrm{d}x\end{aligned}$$

$$\phi = -x^2(x - y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^2(x - y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^2(x - y)$$

The solution becomes

$$y = \frac{x^3 + c_1}{x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 1 + c_1$$

$$c_1 = 4$$

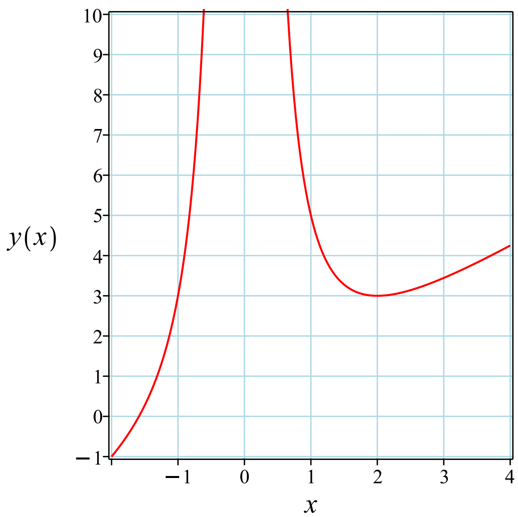
Substituting c_1 found above in the general solution gives

$$y = \frac{x^3 + 4}{x^2}$$

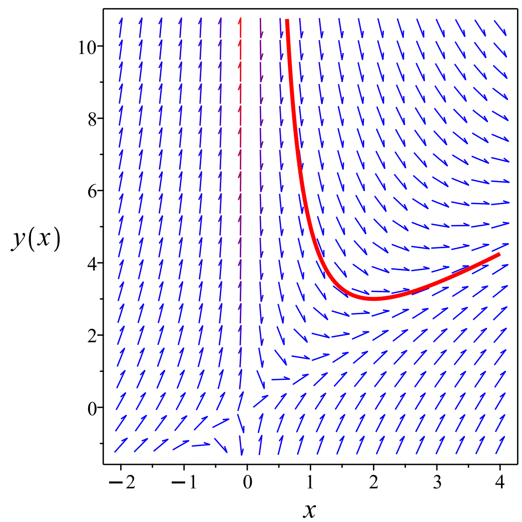
Summary

The solution(s) found are the following

$$y = \frac{x^3 + 4}{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{x^3 + 4}{x^2}$$

Verified OK.

4.5.6 Maple step by step solution

Let's solve

$[2y + y'x = 3x, y(1) = 5]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3 - \frac{2y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = 3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{2y}{x}\right)=3\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{2y}{x}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x)=x^2$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int 3\mu(x)\,dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int 3\mu(x)\,dx+c_1$

- Solve for y

  $y=\frac{\int 3\mu(x)dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=x^2$

  $y=\frac{\int 3x^2dx+c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y=\frac{x^3+c_1}{x^2}$

- Use initial condition $y(1)=5$

  $5=1+c_1$

- Solve for c_1

  $c_1=4$

- Substitute $c_1=4$ into general solution and simplify

  $y=\frac{x^3+4}{x^2}$

- Solution to the IVP

  $y=\frac{x^3+4}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 11

```
dsolve([2*y(x)+x*diff(y(x),x) = 3*x,y(1) = 5],y(x), singsol=all)
```

$$y(x) = x + \frac{4}{x^2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 12

```
DSolve[{2*y[x]+x*y'[x] == 3*x,y[1]==5},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{4}{x^2} + x$$

4.6 problem 6

Internal problem ID [59]
Internal file name [`OUTPUT/59_Sunday_June_05_2022_01_34_10_AM_18874532/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y + 2y'x = 10\sqrt{x}}$$

With initial conditions

$$[y(2) = 5]$$

4.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{5}{\sqrt{x}}$$

Hence the ode is

$$y' + \frac{y}{2x} = \frac{5}{\sqrt{x}}$$

The domain of $p(x) = \frac{1}{2x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = \frac{5}{\sqrt{x}}$ is

$$\{0 < x\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

4.6.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{5}{\sqrt{x}}\right) \\ \frac{d}{dx}(y\sqrt{x}) &= (\sqrt{x})\left(\frac{5}{\sqrt{x}}\right) \\ d(y\sqrt{x}) &= 5\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\sqrt{x} &= \int 5\,dx \\ y\sqrt{x} &= 5x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$y = 5\sqrt{x} + \frac{c_1}{\sqrt{x}}$$

which simplifies to

$$y = \frac{5x + c_1}{\sqrt{x}}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 5\sqrt{2} + \frac{\sqrt{2}\, c_1}{2}$$

$$c_1 = -5\left(\sqrt{2}-1\right)\sqrt{2}$$

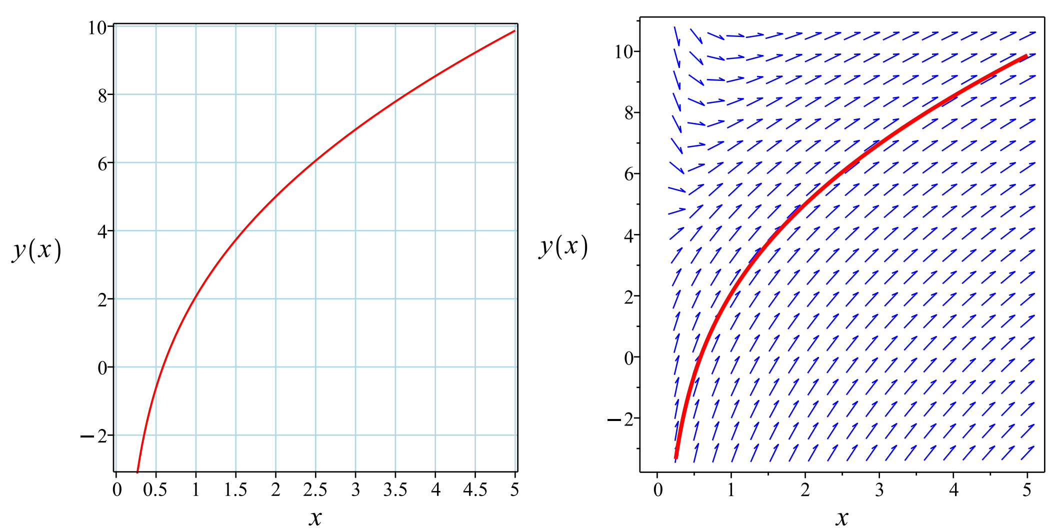
Substituting c_1 found above in the general solution gives

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}}$$

Verified OK.

4.6.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - 10\sqrt{x}}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 135: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sqrt{x}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{\sqrt{x}}} dy
\end{aligned}
$$

Which results in

$$
S = y\sqrt{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = -\frac{y - 10\sqrt{x}}{2x}
$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = \frac{y}{2\sqrt{x}}$$
$$S_y = \sqrt{x}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 5 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 5$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 5R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x}\, y = 5x + c_1$$

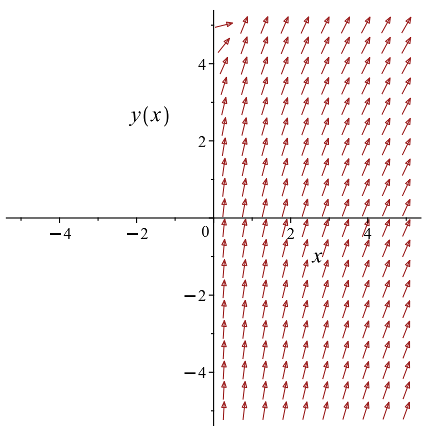
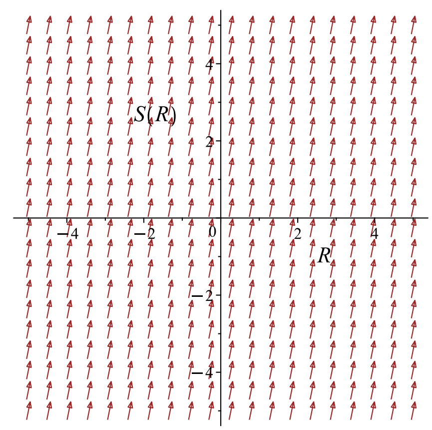
Which simplifies to

$$\sqrt{x}\, y = 5x + c_1$$

Which gives

$$y = \frac{5x + c_1}{\sqrt{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-10\sqrt{x}}{2x}$ 	$R = x$ $S = y\sqrt{x}$	$\frac{dS}{dR} = 5$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 5\sqrt{2} + \frac{\sqrt{2}\, c_1}{2}$$

$$c_1 = -5\left(\sqrt{2}-1\right)\sqrt{2}$$

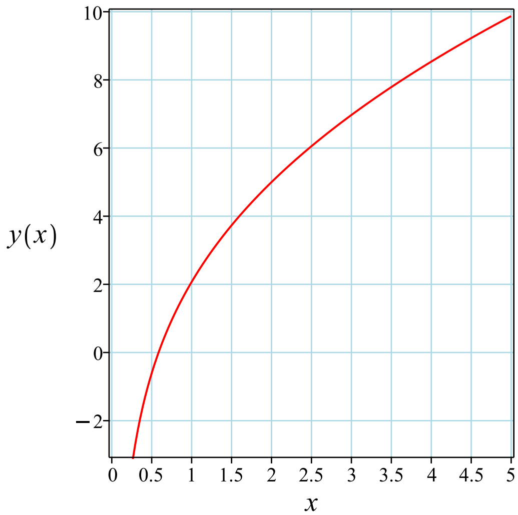
Substituting c_1 found above in the general solution gives

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}}$$

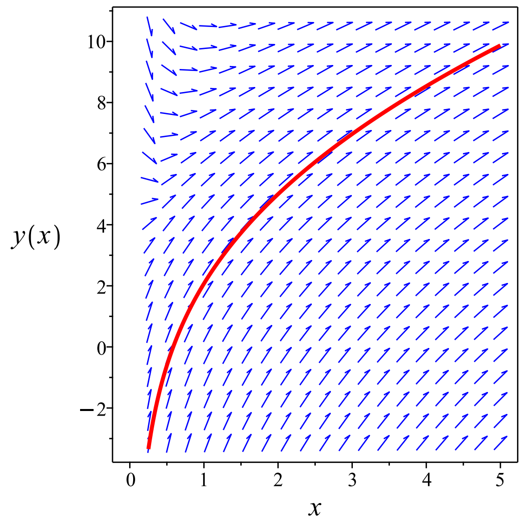
Summary

The solution(s) found are the following

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}}$$

Verified OK.

4.6.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2x)\,\mathrm{d}y &= \left(-y + 10\sqrt{x}\right)\mathrm{d}x \\ \left(y - 10\sqrt{x}\right)\mathrm{d}x + (2x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= y - 10\sqrt{x} \\ N(x, y) &= 2x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y - 10\sqrt{x}\right) \\ &= 1 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x) \\ &= 2 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2x}((1) - (2)) \\ &= -\frac{1}{2x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{2x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln(x)}{2}} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{x}}(y - 10\sqrt{x}) \\
&= \frac{y - 10\sqrt{x}}{\sqrt{x}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{x}}(2x) \\
&= 2\sqrt{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y - 10\sqrt{x}}{\sqrt{x}}\right) + \left(2\sqrt{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{y - 10\sqrt{x}}{\sqrt{x}} \, \mathrm{d}x$$

$$\phi = -10x + 2y\sqrt{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\sqrt{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\sqrt{x}$. Therefore equation (4) becomes

$$2\sqrt{x} = 2\sqrt{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -10x + 2y\sqrt{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -10x + 2y\sqrt{x}$$

The solution becomes

$$y = \frac{10x + c_1}{2\sqrt{x}}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = 5\sqrt{2} + \frac{\sqrt{2}\, c_1}{4}$$

$$c_1 = -10\left(\sqrt{2}-1\right)\sqrt{2}$$

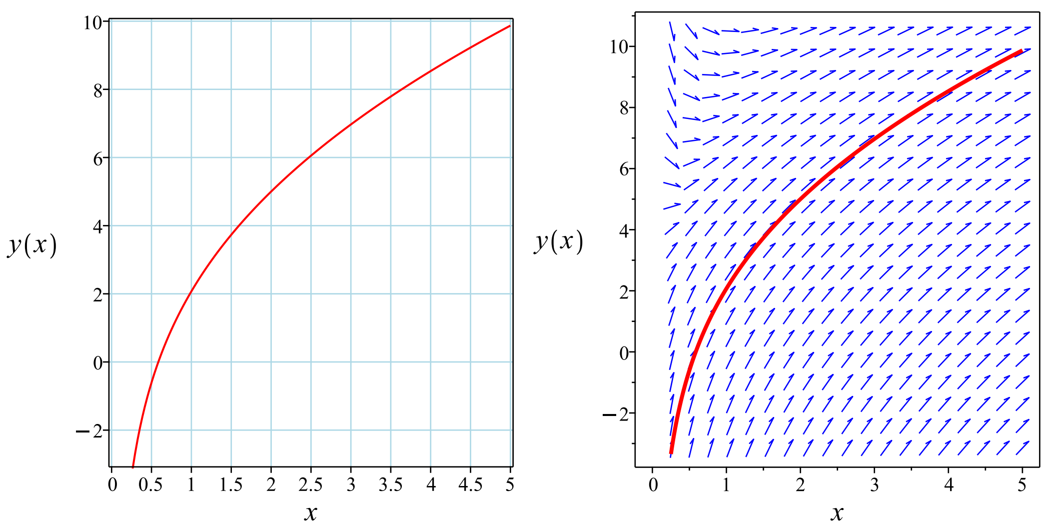
Substituting c_1 found above in the general solution gives

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{5x - 10 + 5\sqrt{2}}{\sqrt{x}}$$

Verified OK.

4.6.5 Maple step by step solution

Let's solve

$\left[y+2y'x=10\sqrt{x}, y(2)=5\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{y}{2x}+\frac{5}{\sqrt{x}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{y}{2x}=\frac{5}{\sqrt{x}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{y}{2x}\right)=\frac{5\mu(x)}{\sqrt{x}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y'+\frac{y}{2x}\right)=\mu'(x)\, y+\mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{2x}$

- Solve to find the integrating factor

  $\mu(x)=\sqrt{x}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\, y)\right)dx=\int\frac{5\mu(x)}{\sqrt{x}}dx+c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y=\int\frac{5\mu(x)}{\sqrt{x}}dx+c_1$

- Solve for y

  $y=\frac{\int\frac{5\mu(x)}{\sqrt{x}}dx+c_1}{\mu(x)}$

- Substitute $\mu(x)=\sqrt{x}$

  $y=\frac{\int 5dx+c_1}{\sqrt{x}}$

- Evaluate the integrals on the rhs

  $y=\frac{5x+c_1}{\sqrt{x}}$

- Use initial condition $y(2)=5$

$$5 = \frac{(10+c_1)\sqrt{2}}{2}$$

- Solve for c_1

$$c_1 = -5\left(\sqrt{2}-1\right)\sqrt{2}$$

- Substitute $c_1 = -5\left(\sqrt{2}-1\right)\sqrt{2}$ into general solution and simplify

$$y = \frac{5\left(x-2+\sqrt{2}\right)}{\sqrt{x}}$$

- Solution to the IVP

$$y = \frac{5\left(x-2+\sqrt{2}\right)}{\sqrt{x}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([y(x)+2*x*diff(y(x),x) = 10*x^(1/2),y(2) = 5],y(x), singsol=all)
```

$$y(x) = \frac{-10+5\sqrt{2}+5x}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 20

```
DSolve[{y[x]+2*x*y'[x]== 10*x^(1/2),y[2]==5},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5\left(x+\sqrt{2}-2\right)}{\sqrt{x}}$$

4.7 problem 7

Internal problem ID [60]
Internal file name [`OUTPUT/60_Sunday_June_05_2022_01_34_11_AM_22228658/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y + 2y'x = 10\sqrt{x}}$$

4.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{2x}$$
$$q(x) = \frac{5}{\sqrt{x}}$$

Hence the ode is

$$y' + \frac{y}{2x} = \frac{5}{\sqrt{x}}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{2x}dx} \\ &= \sqrt{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{5}{\sqrt{x}}\right) \\ \frac{d}{dx}(y\sqrt{x}) &= (\sqrt{x})\left(\frac{5}{\sqrt{x}}\right) \\ d(y\sqrt{x}) &= 5\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\sqrt{x} &= \int 5\,dx \\ y\sqrt{x} &= 5x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{x}$ results in

$$y = 5\sqrt{x} + \frac{c_1}{\sqrt{x}}$$

which simplifies to

$$y = \frac{5x + c_1}{\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{5x + c_1}{\sqrt{x}} \tag{1}$$

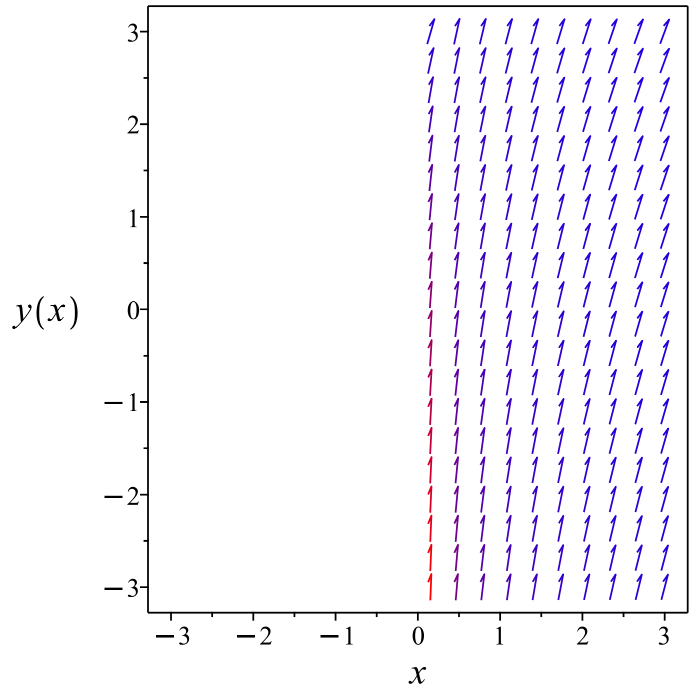


Figure 174: Slope field plot

Verification of solutions

$$
y = \frac{5x + c_1}{\sqrt{x}}
$$

Verified OK.

4.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
y' = -\frac{y - 10\sqrt{x}}{2x}
$$
$$
y' = \omega(x, y)
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 138: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sqrt{x}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{x}}} dy \end{aligned}$$

Which results in

$$S = y\sqrt{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - 10\sqrt{x}}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{2\sqrt{x}} \\ S_y &= \sqrt{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 5 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 5$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 5R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{x}\, y = 5x + c_1$$

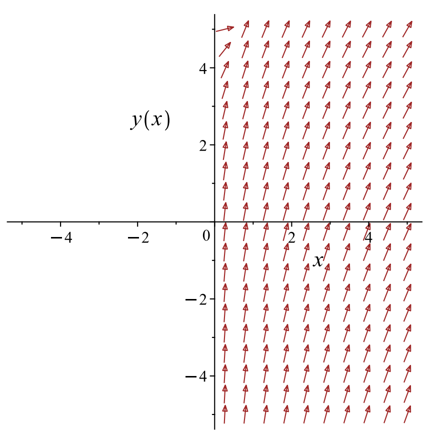
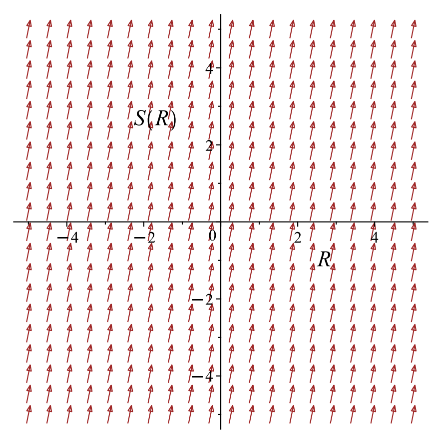
Which simplifies to

$$\sqrt{x}\, y = 5x + c_1$$

Which gives

$$y = \frac{5x + c_1}{\sqrt{x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y-10\sqrt{x}}{2x}$ 	$R = x$ $S = y\sqrt{x}$	$\frac{dS}{dR} = 5$ 

Summary
The solution(s) found are the following

$$y = \frac{5x + c_1}{\sqrt{x}} \tag{1}$$

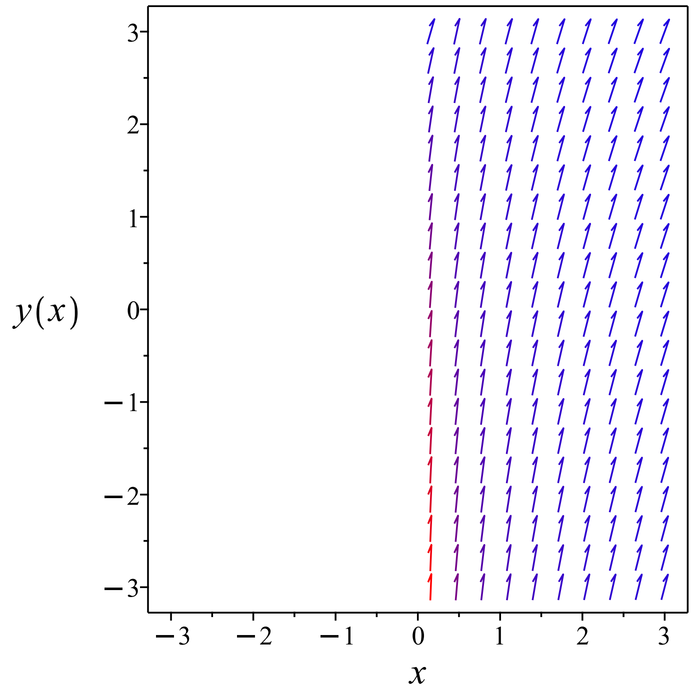


Figure 175: Slope field plot

Verification of solutions

$$y = \frac{5x + c_1}{\sqrt{x}}$$

Verified OK.

4.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(2x)\,\mathrm{d}y = \left(-y + 10\sqrt{x}\right)\mathrm{d}x$$
$$\left(y - 10\sqrt{x}\right)\mathrm{d}x + (2x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - 10\sqrt{x}$$
$$N(x, y) = 2x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y - 10\sqrt{x}\right)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(2x)$$
$$= 2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2x}((1) - (2)) \\ &= -\frac{1}{2x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{1}{2x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{\ln(x)}{2}} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{\sqrt{x}}(y - 10\sqrt{x}) \\ &= \frac{y - 10\sqrt{x}}{\sqrt{x}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{\sqrt{x}}(2x) \\ &= 2\sqrt{x} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y - 10\sqrt{x}}{\sqrt{x}}\right) + \left(2\sqrt{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int \frac{y - 10\sqrt{x}}{\sqrt{x}} \, dx$$
$$\phi = -10x + 2y\sqrt{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2\sqrt{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2\sqrt{x}$. Therefore equation (4) becomes

$$2\sqrt{x} = 2\sqrt{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -10x + 2y\sqrt{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -10x + 2y\sqrt{x}$$

The solution becomes

$$y = \frac{10x + c_1}{2\sqrt{x}}$$

Summary

The solution(s) found are the following

$$y = \frac{10x + c_1}{2\sqrt{x}} \tag{1}$$

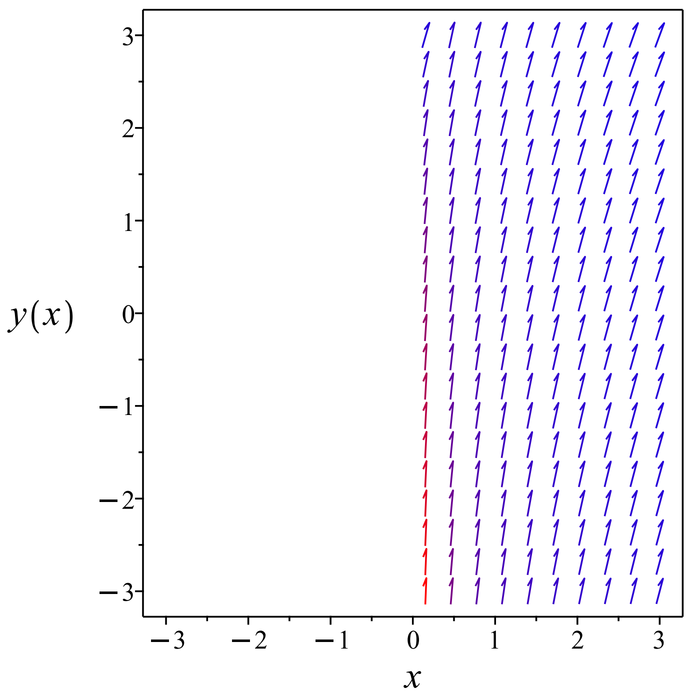


Figure 176: Slope field plot

Verification of solutions

$$y = \frac{10x + c_1}{2\sqrt{x}}$$

Verified OK.

4.7.4 Maple step by step solution

Let's solve

$y + 2y'x = 10\sqrt{x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{2x} + \frac{5}{\sqrt{x}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{2x} = \frac{5}{\sqrt{x}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{2x}\right) = \frac{5\mu(x)}{\sqrt{x}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{2x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{2x}$

- Solve to find the integrating factor

  $\mu(x) = \sqrt{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{5\mu(x)}{\sqrt{x}} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{5\mu(x)}{\sqrt{x}} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{5\mu(x)}{\sqrt{x}} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sqrt{x}$

  $y = \frac{\int 5 dx + c_1}{\sqrt{x}}$

- Evaluate the integrals on the rhs

  $y = \frac{5x + c_1}{\sqrt{x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(y(x)+2*x*diff(y(x),x) = 10*x^(1/2),y(x), singsol=all)
```

$$y(x) = \frac{5x + c_1}{\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 17

```
DSolve[y[x]+2*x*y'[x] == 10*x^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5x + c_1}{\sqrt{x}}$$

4.8 problem 8

Internal problem ID [61]
Internal file name [`OUTPUT/61_Sunday_June_05_2022_01_34_11_AM_24442165/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y + 3y'x = 12x}$$

4.8.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{3x} \\ q(x) &= 4 \end{aligned}$$

Hence the ode is

$$y' + \frac{y}{3x} = 4$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{3x}dx}$$
$$= x^{\frac{1}{3}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(4)$$
$$\frac{d}{dx}\left(x^{\frac{1}{3}}y\right) = \left(x^{\frac{1}{3}}\right)(4)$$
$$d\left(x^{\frac{1}{3}}y\right) = \left(4x^{\frac{1}{3}}\right) dx$$

Integrating gives

$$x^{\frac{1}{3}}y = \int 4x^{\frac{1}{3}} dx$$
$$x^{\frac{1}{3}}y = 3x^{\frac{4}{3}} + c_1$$

Dividing both sides by the integrating factor $\mu = x^{\frac{1}{3}}$ results in

$$y = 3x + \frac{c_1}{x^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = 3x + \frac{c_1}{x^{\frac{1}{3}}} \tag{1}$$

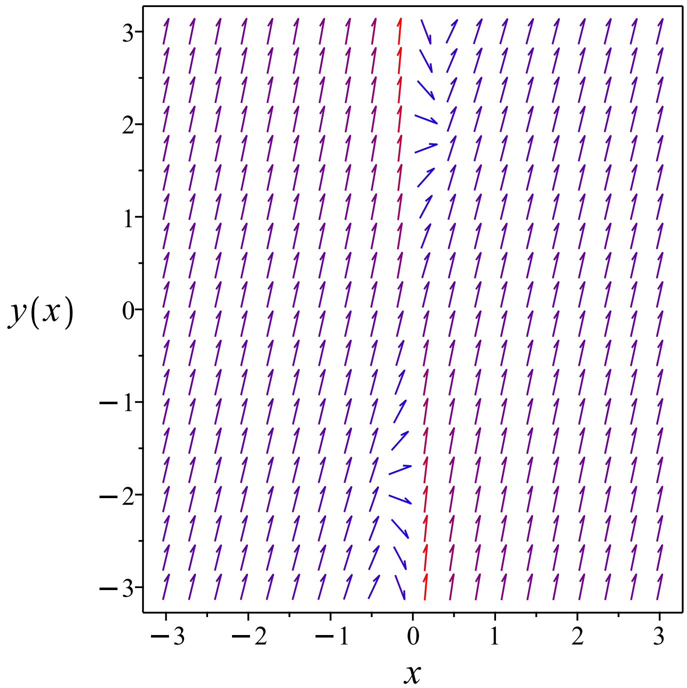


Figure 177: Slope field plot

Verification of solutions

$$y = 3x + \frac{c_1}{x^{\frac{1}{3}}}$$

Verified OK.

4.8.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x + 3(u'(x)\,x + u(x))\,x = 12x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{-\frac{4u}{3} + 4}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = -\frac{4u}{3} + 4$. Integrating both sides gives

$$\frac{1}{-\frac{4u}{3} + 4}\,du = \frac{1}{x}\,dx$$

$$
\int \frac{1}{-\frac{4u}{3}+4}\, du = \int \frac{1}{x}\, dx
$$

$$
-\frac{3\ln\left(u-3\right)}{4} = \ln\left(x\right)+c_2
$$

Raising both side to exponential gives

$$
\frac{1}{\left(u-3\right)^{\frac{3}{4}}} = \mathrm{e}^{\ln(x)+c_2}
$$

Which simplifies to

$$
\frac{1}{\left(u-3\right)^{\frac{3}{4}}} = c_3 x
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= x\left(3+\left(\frac{\mathrm{e}^{-c_2}}{c_3 x}\right)^{\frac{4}{3}}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = x\left(3+\left(\frac{\mathrm{e}^{-c_2}}{c_3 x}\right)^{\frac{4}{3}}\right) \tag{1}
$$

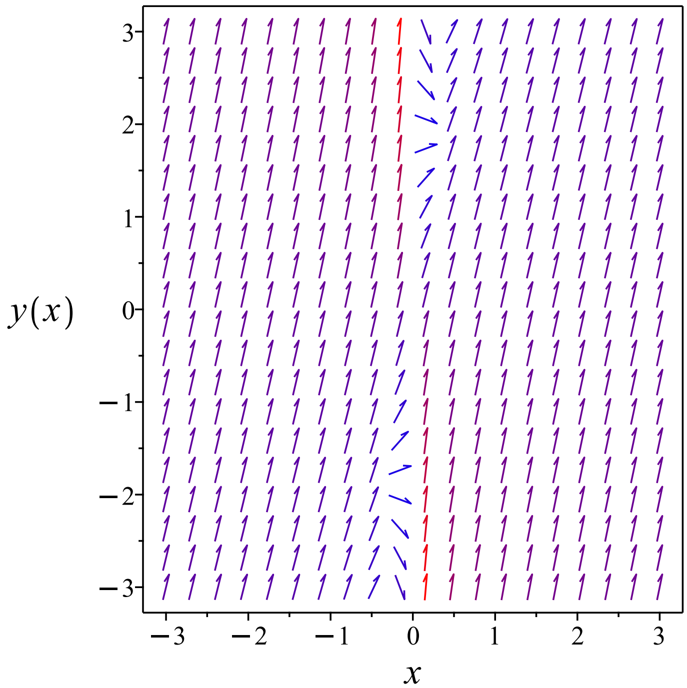


Figure 178: Slope field plot

Verification of solutions

$$y = x\left(3 + \left(\frac{e^{-c_2}}{c_3 x}\right)^{\frac{4}{3}}\right)$$

Verified OK.

4.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y - 12x}{3x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 141: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^{\frac{1}{3}}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^{\frac{1}{3}}}} dy \end{aligned}$$

Which results in

$$S = x^{\frac{1}{3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y - 12x}{3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{3x^{\frac{2}{3}}} \\ S_y &= x^{\frac{1}{3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4x^{\frac{1}{3}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4R^{\frac{1}{3}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3R^{\frac{4}{3}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^{\frac{1}{3}} y = 3x^{\frac{4}{3}} + c_1$$

Which simplifies to

$$x^{\frac{1}{3}} y = 3x^{\frac{4}{3}} + c_1$$

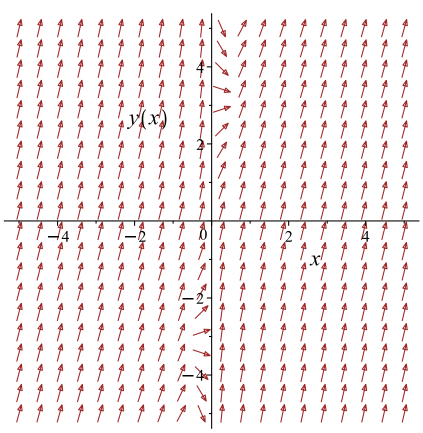
Which gives

$$y = \frac{3x^{\frac{4}{3}} + c_1}{x^{\frac{1}{3}}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{y-12x}{3x}$



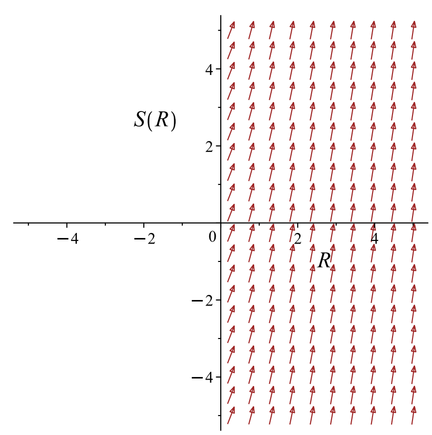
</td><td>

$R = x$

$S = x^{\frac{1}{3}} y$

</td><td>

$\frac{dS}{dR} = 4R^{\frac{1}{3}}$



</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = \frac{3x^{\frac{4}{3}} + c_1}{x^{\frac{1}{3}}} \tag{1}$$

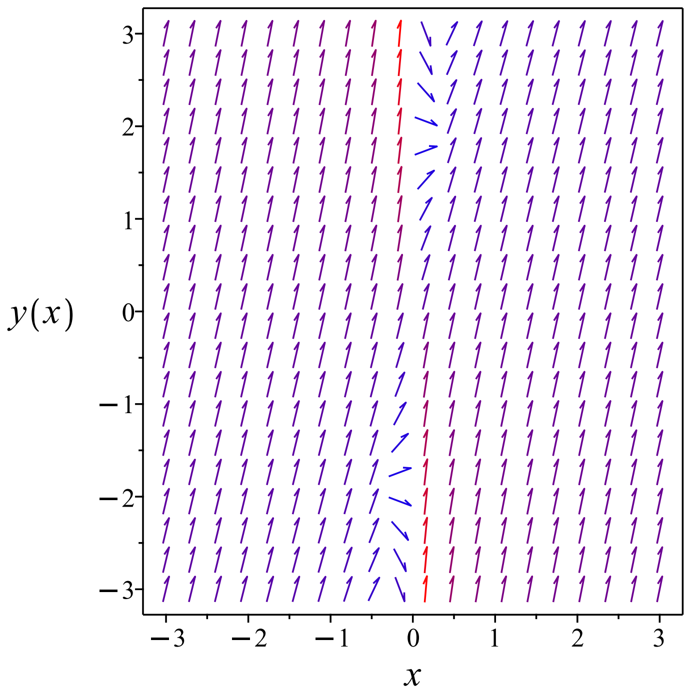


Figure 179: Slope field plot

Verification of solutions

$$y = \frac{3x^{\frac{4}{3}} + c_1}{x^{\frac{1}{3}}}$$

Verified OK.

4.8.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x)\,\mathrm{d}y &= (-y+12x)\,\mathrm{d}x\\ (y-12x)\,\mathrm{d}x + (3x)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= y-12x\\ N(x,y) &= 3x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y-12x)\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x)\\ &= 3\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{3x}((1)-(3))\\ &= -\frac{2}{3x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int -\frac{2}{3x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-\frac{2\ln(x)}{3}}\\ &= \frac{1}{x^{\frac{2}{3}}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^{\frac{2}{3}}}(y-12x)\\ &= \frac{y-12x}{x^{\frac{2}{3}}}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^{\frac{2}{3}}}(3x) \\
&= 3x^{\frac{1}{3}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{y-12x}{x^{\frac{2}{3}}}\right) + \left(3x^{\frac{1}{3}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x &= \int \frac{y-12x}{x^{\frac{2}{3}}}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = (-9x+3y)\,x^{\frac{1}{3}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial\phi}{\partial y} = 3x^{\frac{1}{3}} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = 3x^{\frac{1}{3}}$. Therefore equation (4) becomes

$$
3x^{\frac{1}{3}} = 3x^{\frac{1}{3}} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = (-9x + 3y)\, x^{\frac{1}{3}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = (-9x + 3y)\, x^{\frac{1}{3}}$$

The solution becomes

$$y = \frac{9x^{\frac{4}{3}} + c_1}{3x^{\frac{1}{3}}}$$

Summary

The solution(s) found are the following

$$y = \frac{9x^{\frac{4}{3}} + c_1}{3x^{\frac{1}{3}}} \tag{1}$$

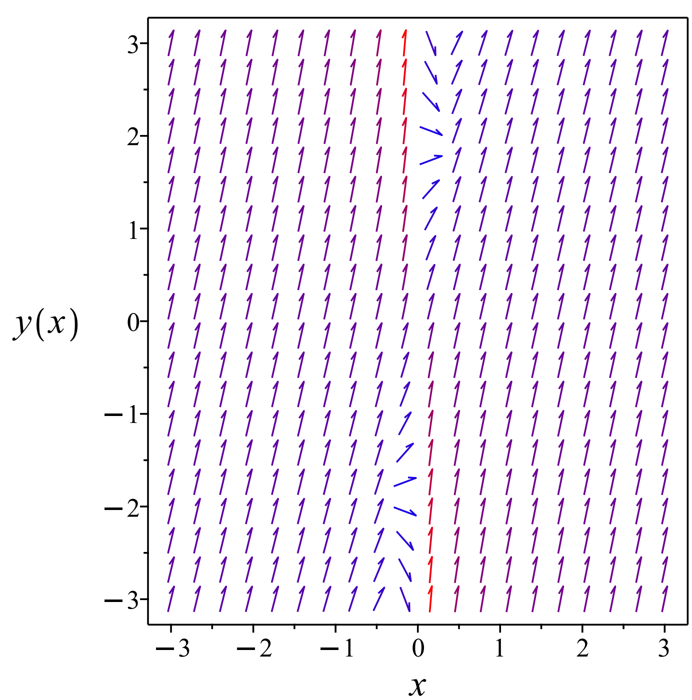


Figure 180: Slope field plot

Verification of solutions

$$y = \frac{9x^{\frac{4}{3}} + c_1}{3x^{\frac{1}{3}}}$$

Verified OK.

4.8.5 Maple step by step solution

Let's solve

$y + 3y'x = 12x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 4 - \frac{y}{3x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{3x} = 4$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{3x}\right) = 4\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{3x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{3x}$

- Solve to find the integrating factor

  $\mu(x) = x^{\frac{1}{3}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 4\mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 4\mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int 4\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^{\frac{1}{3}}$

$$y = \frac{\int 4x^{\frac{1}{3}}dx + c_1}{x^{\frac{1}{3}}}$$

- Evaluate the integrals on the rhs

$$y = \frac{3x^{\frac{4}{3}} + c_1}{x^{\frac{1}{3}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(y(x)+3*x*diff(y(x),x) = 12*x,y(x), singsol=all)
```

$$y(x) = 3x + \frac{c_1}{x^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 17

```
DSolve[y[x]+3*x*y'[x] == 12*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x + \frac{c_1}{\sqrt[3]{x}}$$

4.9 problem 9

Internal problem ID [62]
Internal file name [`OUTPUT/62_Sunday_June_05_2022_01_34_12_AM_84258183/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-y + y'x = x}$$

With initial conditions

$$[y(1) = 7]$$

4.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = 1$$

Hence the ode is

$$y' - \frac{y}{x} = 1$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.9.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= \mu \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \frac{1}{x} \\ d\left(\frac{y}{x}\right) &= \frac{1}{x}dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{1}{x}\,dx \\ \frac{y}{x} &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\ln(x) + c_1 x$$

which simplifies to

$$y = x(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 7$ in the above solution gives an equation to solve for the constant of integration.

$$7 = c_1$$

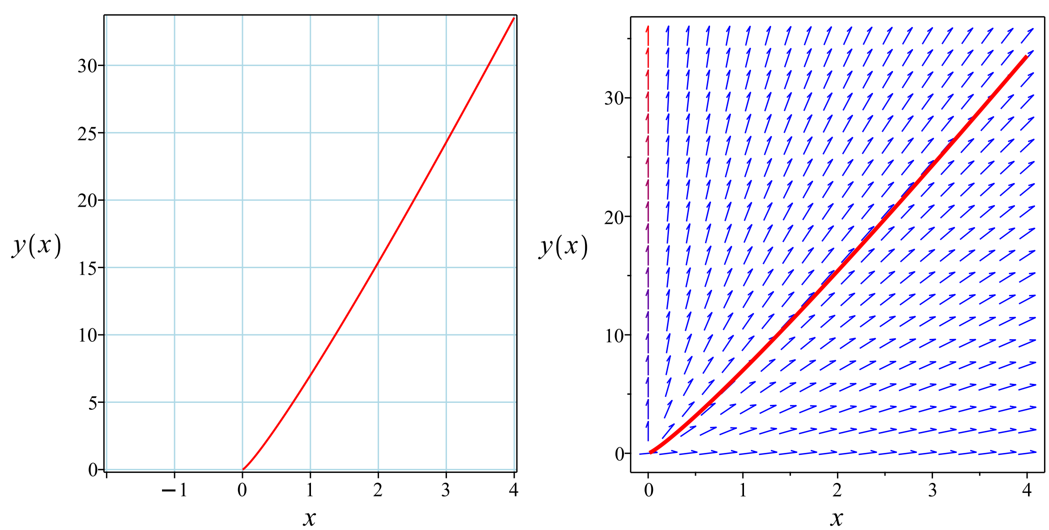
$$c_1 = 7$$

Substituting c_1 found above in the general solution gives

$$y = x\ln(x) + 7x$$

Summary
The solution(s) found are the following

$$y = x\ln(x) + 7x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x\ln(x) + 7x$$

Verified OK.

4.9.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$-u(x)\,x + (u'(x)\,x + u(x))\,x = x$$

Integrating both sides gives

$$\begin{aligned} u(x) &= \int \frac{1}{x}\,\mathrm{d}x \\ &= \ln(x) + c_2 \end{aligned}$$

Therefore the solution y is

$$\begin{aligned} y &= ux \\ &= x(\ln(x) + c_2) \end{aligned}$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 7$ in the above solution gives an equation to solve for the constant of integration.

$$7 = c_2$$

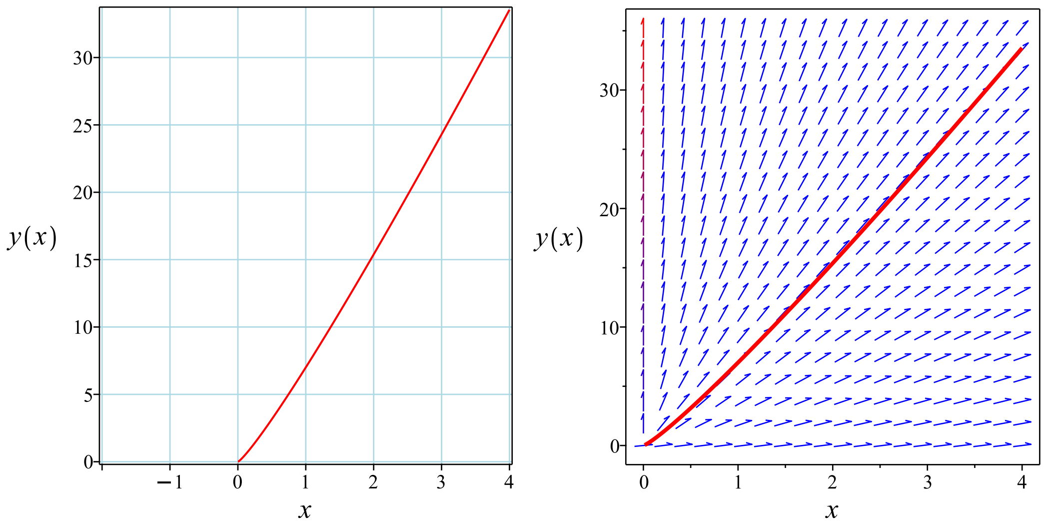
$$c_2 = 7$$

Substituting c_2 found above in the general solution gives

$$y = x\ln(x) + 7x$$

Summary
The solution(s) found are the following

$$y = x\ln(x) + 7x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x \ln(x) + 7x$$

Verified OK.

4.9.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x+y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 144: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x+y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y}{x^2} \\ S_y &= \frac{1}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x} = \ln(x) + c_1$$

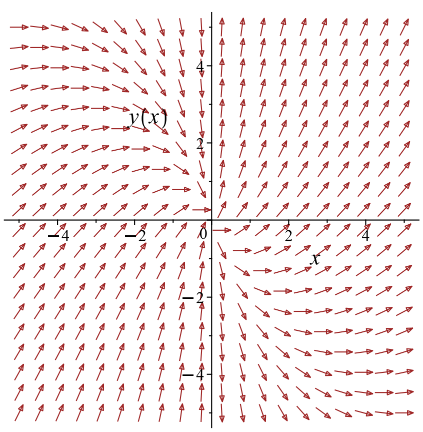
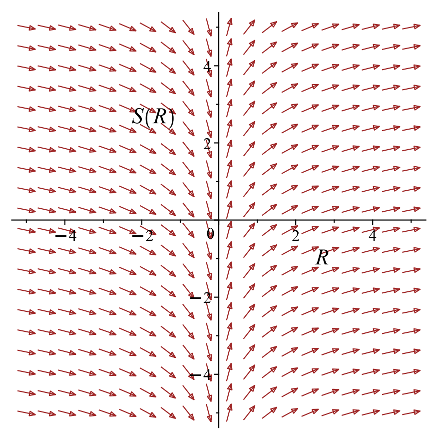
Which simplifies to

$$\frac{y}{x} = \ln(x) + c_1$$

Which gives

$$y = x(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x+y}{x}$ 	$R = x$ $S = \frac{y}{x}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 7$ in the above solution gives an equation to solve for the constant of integration.

$$7 = c_1$$

$$c_1 = 7$$

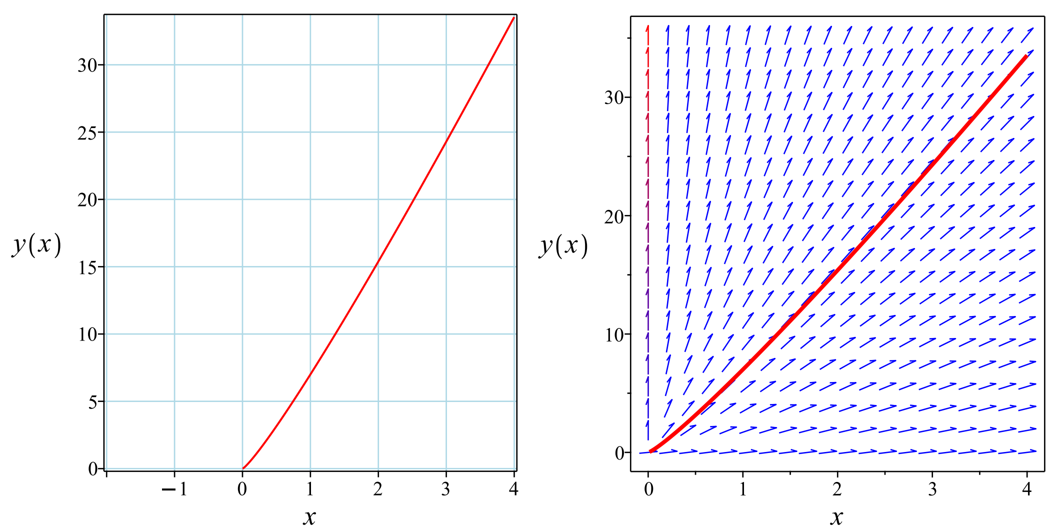
Substituting c_1 found above in the general solution gives

$$y = x \ln(x) + 7x$$

Summary

The solution(s) found are the following

$$y = x \ln(x) + 7x \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x \ln(x) + 7x$$

Verified OK.

4.9.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = (x+y)\,\mathrm{d}x$$
$$(-x-y)\,\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -x-y$$
$$N(x,y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x-y)$$
$$= -1$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-1) - (1)) \\
&= -\frac{2}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{2}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(x)} \\
&= \frac{1}{x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^2}(-x - y) \\
&= \frac{-x - y}{x^2}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(x) \\
&= \frac{1}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x-y}{x^2}\right) + \left(\frac{1}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{-x-y}{x^2} \,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + \frac{y}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x}$. Therefore equation (4) becomes

$$
\frac{1}{x} = \frac{1}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{y}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{y}{x}$$

The solution becomes

$$y = x(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 7$ in the above solution gives an equation to solve for the constant of integration.

$$7 = c_1$$

$$c_1 = 7$$

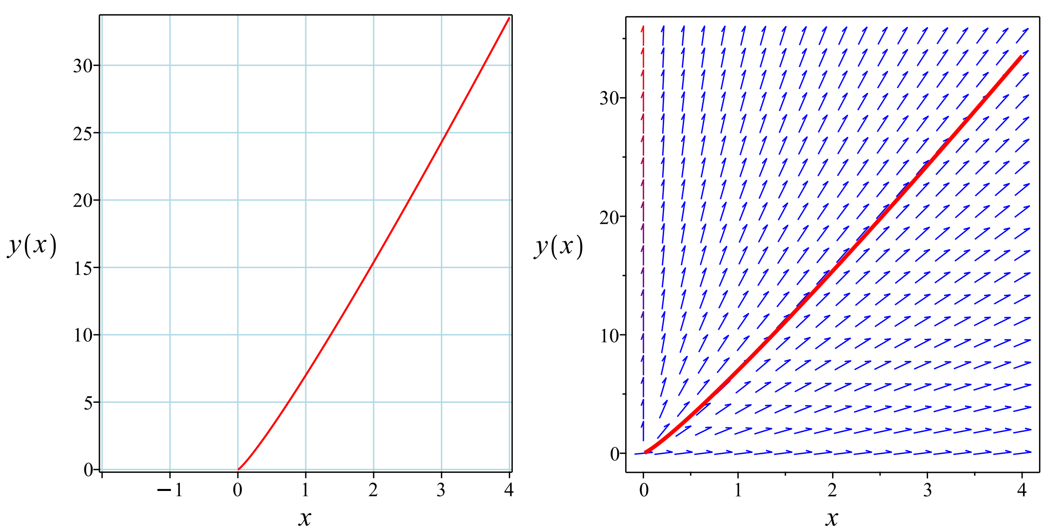
Substituting c_1 found above in the general solution gives

$$y = x\ln(x) + 7x$$

Summary

The solution(s) found are the following

$$y = x\ln(x) + 7x \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = x \ln(x) + 7x$$

Verified OK.

4.9.6 Maple step by step solution

Let's solve

$[-y + y'x = x, y(1) = 7]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 1 + \frac{y}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{y}{x} = 1$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{y}{x}\right) = \mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' - \frac{y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x}$

  $y = x\left(\int \frac{1}{x}dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x(\ln(x) + c_1)$
- Use initial condition $y(1) = 7$

  $7 = c_1$
- Solve for c_1

  $c_1 = 7$
- Substitute $c_1 = 7$ into general solution and simplify

  $y = (\ln(x) + 7)\,x$
- Solution to the IVP

  $y = (\ln(x) + 7)\,x$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 10

```
dsolve([-y(x)+x*diff(y(x),x) = x,y(1) = 7],y(x), singsol=all)
```

$$y(x) = (\ln(x) + 7)\,x$$

✓ Solution by Mathematica
Time used: 0.021 (sec). Leaf size: 11

```
DSolve[{-y[x]+x*y'[x]== x,y[1]==7},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x(\log(x) + 7)$$

4.10 problem 10

Internal problem ID [63]
Internal file name [`OUTPUT/63_Sunday_June_05_2022_01_34_12_AM_21495753/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{-3y + 2y'x = 9x^3}$$

4.10.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{2x}$$
$$q(x) = \frac{9x^2}{2}$$

Hence the ode is

$$y' - \frac{3y}{2x} = \frac{9x^2}{2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{2x}dx}$$
$$= \frac{1}{x^{\frac{3}{2}}}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{9x^2}{2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x^{\frac{3}{2}}}\right) = \left(\frac{1}{x^{\frac{3}{2}}}\right)\left(\frac{9x^2}{2}\right)$$
$$d\left(\frac{y}{x^{\frac{3}{2}}}\right) = \left(\frac{9\sqrt{x}}{2}\right) dx$$

Integrating gives

$$\frac{y}{x^{\frac{3}{2}}} = \int \frac{9\sqrt{x}}{2} dx$$
$$\frac{y}{x^{\frac{3}{2}}} = 3x^{\frac{3}{2}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{3}{2}}}$ results in

$$y = 3x^3 + x^{\frac{3}{2}}c_1$$

Summary
The solution(s) found are the following

$$y = 3x^3 + x^{\frac{3}{2}}c_1 \tag{1}$$

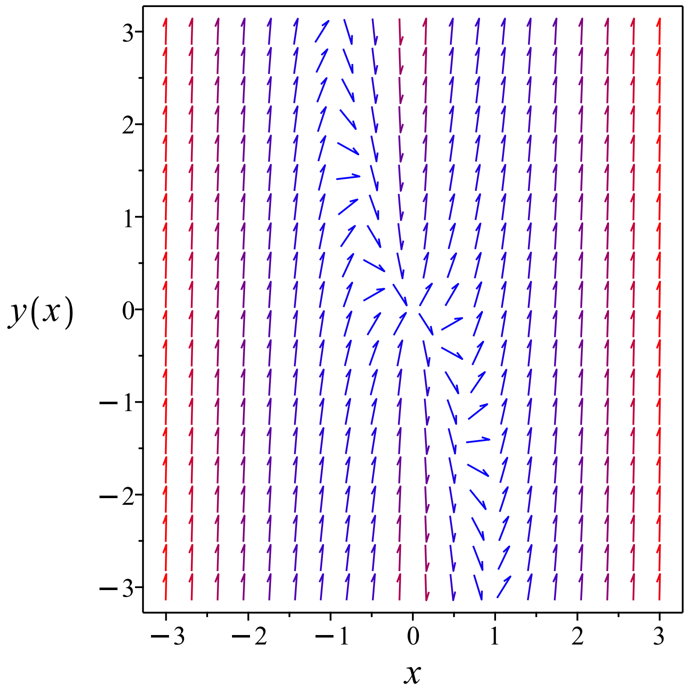


Figure 185: Slope field plot

<u>Verification of solutions</u>

$$y = 3x^3 + x^{\frac{3}{2}} c_1$$

Verified OK.

4.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{\frac{9x^3}{2} + \frac{3y}{2}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 147: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^{\frac{3}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^{\frac{3}{2}}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^{\frac{3}{2}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{\frac{9x^3}{2} + \frac{3y}{2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y}{2x^{\frac{5}{2}}} \\ S_y &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{9\sqrt{x}}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{9\sqrt{R}}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3R^{\frac{3}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^{\frac{3}{2}}} = 3x^{\frac{3}{2}} + c_1$$

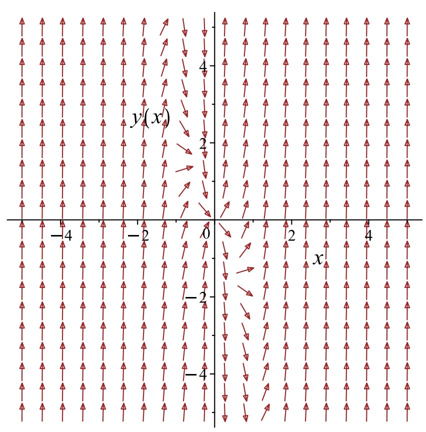
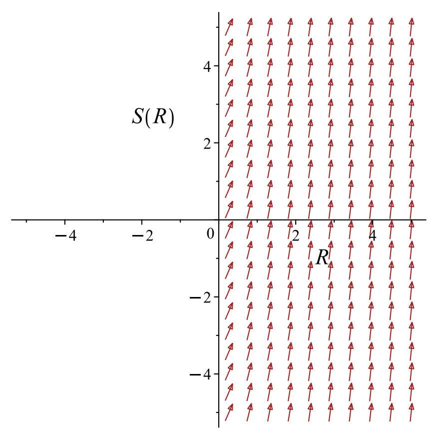
Which simplifies to

$$\frac{y}{x^{\frac{3}{2}}} = 3x^{\frac{3}{2}} + c_1$$

Which gives

$$y = 3x^3 + x^{\frac{3}{2}} c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\frac{9x^3}{2} + \frac{3y}{2}}{x}$ 	$R = x$ $S = \frac{y}{x^{\frac{3}{2}}}$	$\frac{dS}{dR} = \frac{9\sqrt{R}}{2}$ 

Summary

The solution(s) found are the following

$$y = 3x^3 + x^{\frac{3}{2}} c_1 \tag{1}$$

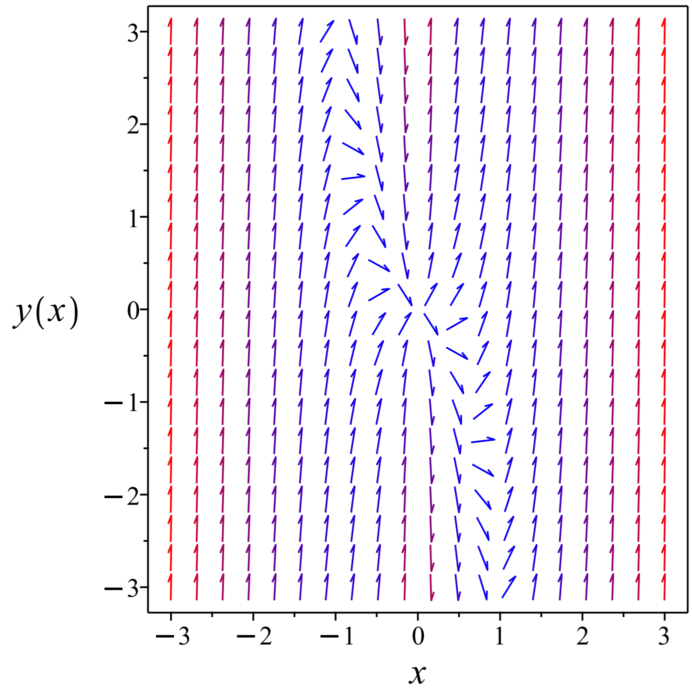


Figure 186: Slope field plot

Verification of solutions

$$y = 3x^3 + x^{\frac{3}{2}} c_1$$

Verified OK.

4.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(2x) \,\mathrm{d}y &= \left(9x^3 + 3y\right) \mathrm{d}x \\
\left(-9x^3 - 3y\right) \mathrm{d}x + (2x) \,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -9x^3 - 3y \\
N(x, y) &= 2x
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-9x^3 - 3y\right) \\
&= -3
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2x) \\
&= 2
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{2x}((-3) - (2)) \\ &= -\frac{5}{2x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{5}{2x} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-\frac{5\ln(x)}{2}} \\ &= \frac{1}{x^{\frac{5}{2}}} \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= \frac{1}{x^{\frac{5}{2}}}\left(-9x^3 - 3y\right) \\ &= \frac{-9x^3 - 3y}{x^{\frac{5}{2}}} \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= \frac{1}{x^{\frac{5}{2}}}(2x) \\ &= \frac{2}{x^{\frac{3}{2}}} \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-9x^3 - 3y}{x^{\frac{5}{2}}}\right) + \left(\frac{2}{x^{\frac{3}{2}}}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{-9x^3 - 3y}{x^{\frac{5}{2}}} \,\mathrm{d}x$$

$$\phi = \frac{-6x^3 + 2y}{x^{\frac{3}{2}}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2}{x^{\frac{3}{2}}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{x^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$\frac{2}{x^{\frac{3}{2}}} = \frac{2}{x^{\frac{3}{2}}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-6x^3 + 2y}{x^{\frac{3}{2}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-6x^3 + 2y}{x^{\frac{3}{2}}}$$

The solution becomes

$$y = \frac{x^{\frac{3}{2}} c_1}{2} + 3x^3$$

Summary
The solution(s) found are the following

$$y = \frac{x^{\frac{3}{2}} c_1}{2} + 3x^3 \tag{1}$$

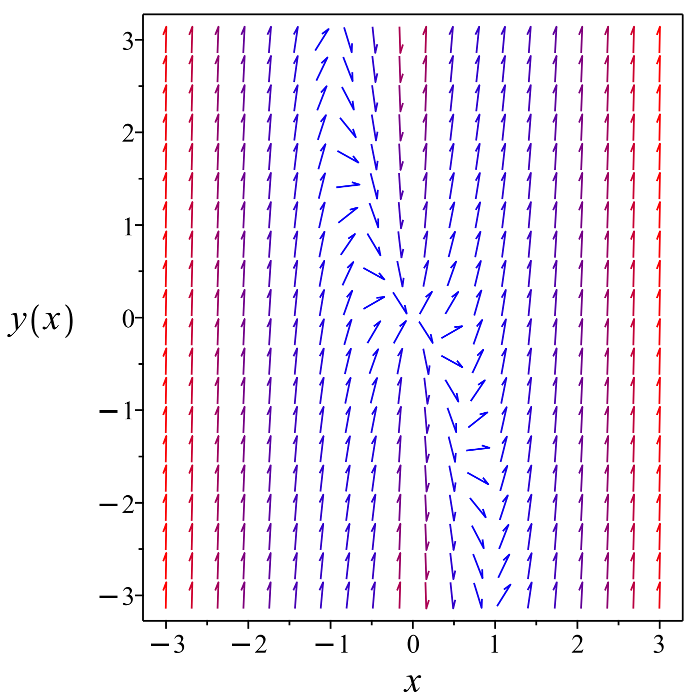


Figure 187: Slope field plot

Verification of solutions

$$y = \frac{x^{\frac{3}{2}} c_1}{2} + 3x^3$$

Verified OK.

4.10.4 Maple step by step solution

Let's solve

$-3y + 2y'x = 9x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{2x} + \frac{9x^2}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{3y}{2x} = \frac{9x^2}{2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{3y}{2x}\right) = \frac{9\mu(x)x^2}{2}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{3y}{2x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{3\mu(x)}{2x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^{\frac{3}{2}}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{9\mu(x)x^2}{2} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{9\mu(x)x^2}{2} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{9\mu(x)x^2}{2} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^{\frac{3}{2}}}$

  $y = x^{\frac{3}{2}}\left(\int \frac{9\sqrt{x}}{2} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^{\frac{3}{2}}\left(3x^{\frac{3}{2}} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(-3*y(x)+2*x*diff(y(x),x) = 9*x^3,y(x), singsol=all)
```

$$y(x) = 3x^3 + x^{\frac{3}{2}}c_1$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 19

```
DSolve[-3*y[x]+2*x*y'[x] == 9*x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x^3 + c_1x^{3/2}$$

4.11 problem 11

Internal problem ID [64]
Internal file name [`OUTPUT/64_Sunday_June_05_2022_01_34_13_AM_35107389/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y + y'x - 3yx = 0}$$

With initial conditions

$$[y(1) = 0]$$

4.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3x - 1}{x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y(3x-1)}{x} = 0$$

The domain of $p(x) = -\frac{3x-1}{x}$ is

$$\{x < 0 \vee 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. Hence solution exists and is unique.

4.11.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y(3x-1)}{x} \end{aligned}$$

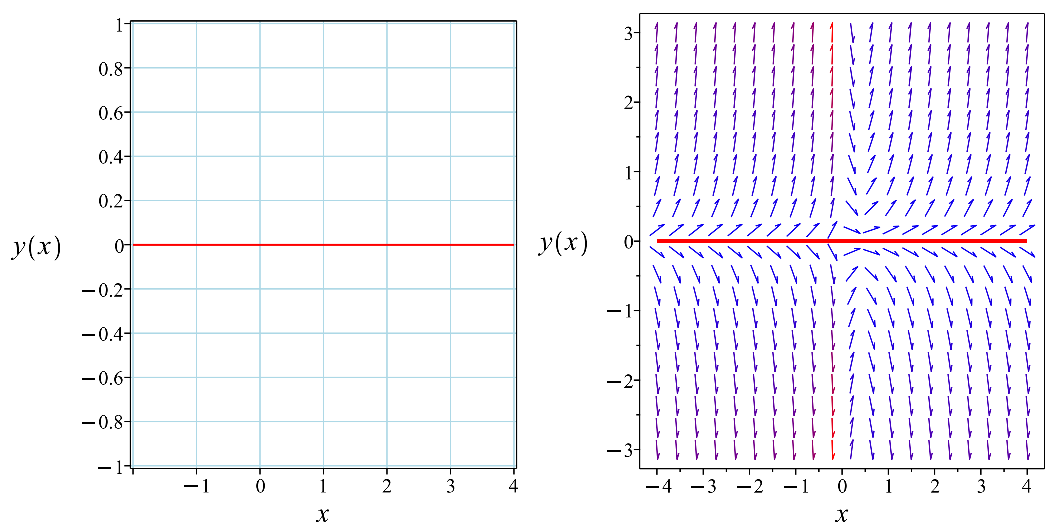
Where $f(x) = \frac{3x-1}{x}$ and $g(y) = y$. Since unique solution exists and $g(y)$ evaluated at $y_0 = 0$ is zero, then the solution is

$$\begin{aligned} y &= y_0 \\ &= 0 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = 0 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.11.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{3x-1}{x}dx} \\ &= \mathrm{e}^{-3x+\ln(x)}\end{aligned}$$

Which simplifies to

$$\mu = x\,\mathrm{e}^{-3x}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mu y &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(x\,\mathrm{e}^{-3x}y\right) &= 0\end{aligned}$$

Integrating gives

$$x\,\mathrm{e}^{-3x}y = c_1$$

Dividing both sides by the integrating factor $\mu = x\,\mathrm{e}^{-3x}$ results in

$$y = \frac{c_1\mathrm{e}^{3x}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1\mathrm{e}^3$$

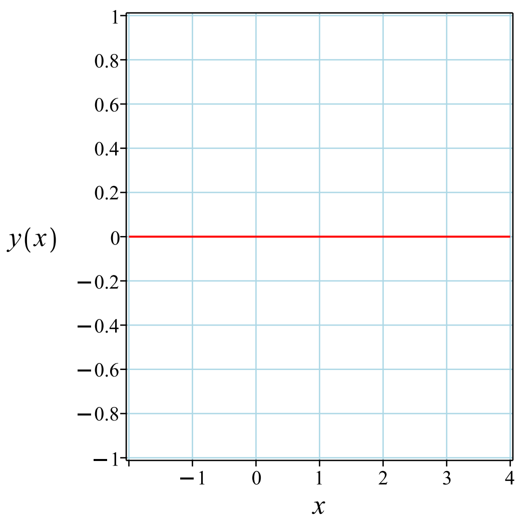
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

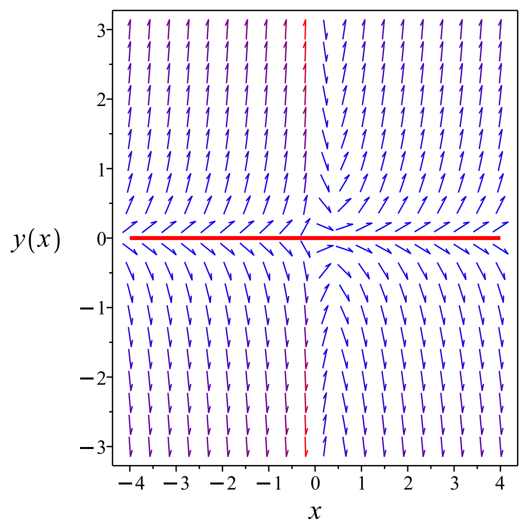
$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.11.4 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$u(x)\, x + (u'(x)\, x + u(x))\, x - 3u(x)\, x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(3x-2)}{x} \end{aligned}$$

Where $f(x) = \frac{3x-2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= \frac{3x-2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{3x-2}{x}\,dx \\
\ln(u) &= 3x - 2\ln(x) + c_2 \\
u &= \mathrm{e}^{3x-2\ln(x)+c_2} \\
&= c_2 \mathrm{e}^{3x-2\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{3x}}{x^2}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= \frac{c_2 \mathrm{e}^{3x}}{x}
\end{aligned}
$$

Initial conditions are used to solve for c_2. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = c_2 \mathrm{e}^3
$$

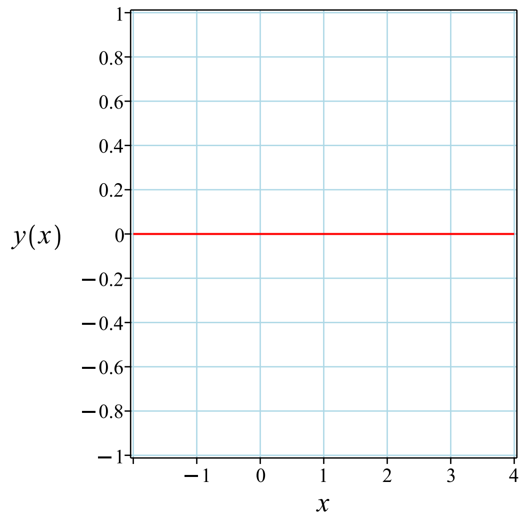
$$
c_2 = 0
$$

Substituting c_2 found above in the general solution gives

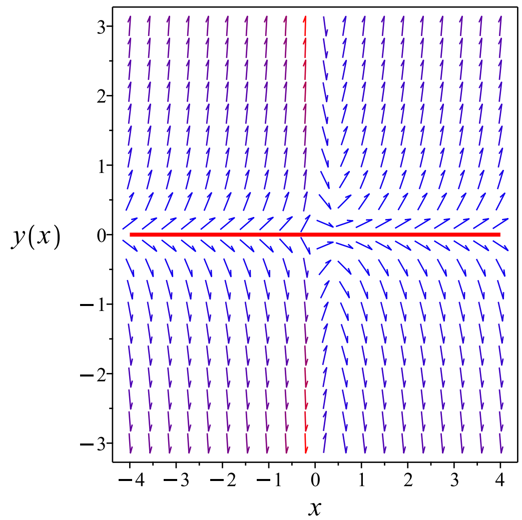
$$
y = 0
$$

Summary
The solution(s) found are the following

$$
y = 0 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.11.5 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(3x - 1)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 150: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{3x - \ln(x)}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{3x-\ln(x)}} dy \end{aligned}$$

Which results in

$$S = x\,e^{-3x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(3x - 1)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= e^{-3x} y(-3x + 1) \\ S_y &= x\,e^{-3x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x\,\mathrm{e}^{-3x} y = c_1$$

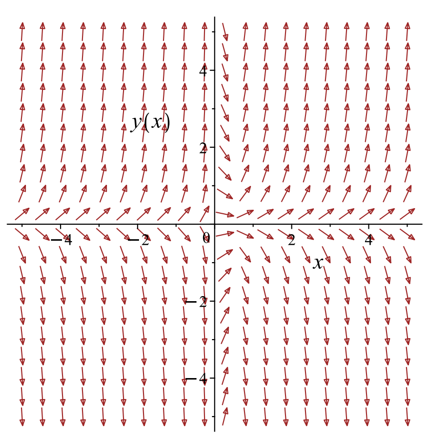
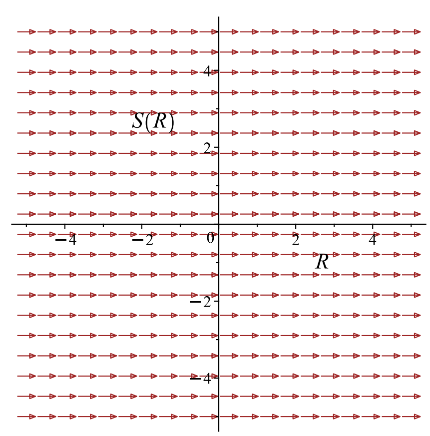
Which simplifies to

$$x\,\mathrm{e}^{-3x} y = c_1$$

Which gives

$$y = \frac{c_1 \mathrm{e}^{3x}}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(3x-1)}{x}$ 	$R = x$ $S = x\,\mathrm{e}^{-3x} y$	$\frac{dS}{dR} = 0$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 \mathrm{e}^3$$

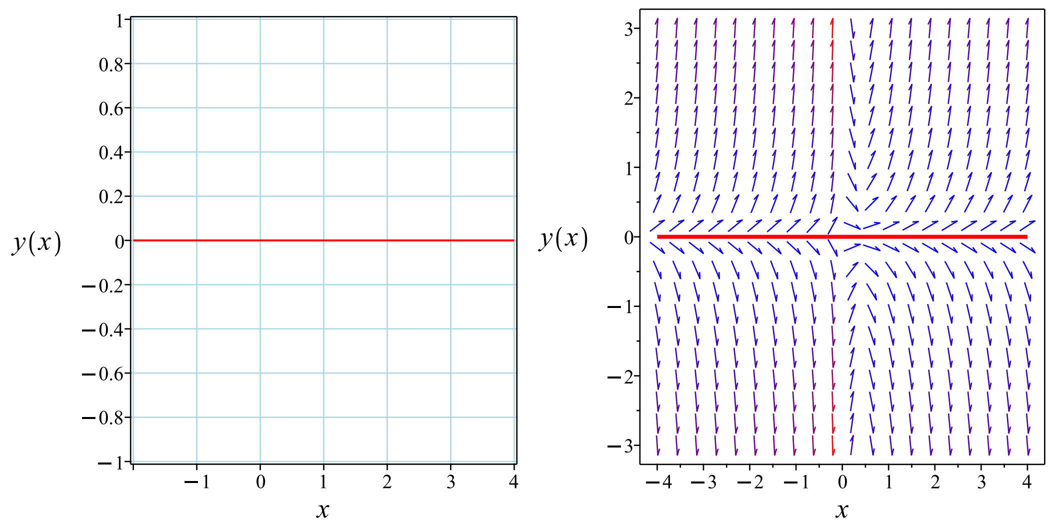
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = 0$$

Summary
The solution(s) found are the following

$$y = 0 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 0$$

Verified OK.

4.11.6 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y}\right)\mathrm{d}y&=\left(\frac{3x-1}{x}\right)\mathrm{d}x\\ \left(-\frac{3x-1}{x}\right)\mathrm{d}x+\left(\frac{1}{y}\right)\mathrm{d}y&=0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-\frac{3x-1}{x}\\ N(x,y)&=\frac{1}{y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-\frac{3x-1}{x}\right)\\ &=0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1) \\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{3x-1}{x} \,\mathrm{d}x \\
\phi &= -3x + \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$
\frac{1}{y} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(\frac{1}{y}\right) \mathrm{d}y \\
f(y) &= \ln(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -3x + \ln(x) + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3x + \ln(x) + \ln(y)$$

The solution becomes

$$y = \frac{e^{3x+c_1}}{x}$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^{3+c_1}$$

Unable to solve for constant of integration. Warning: Unable to solve for constant of integration.

Verification of solutions N/A

4.11.7 Maple step by step solution

Let's solve

$[y + y'x - 3yx = 0, y(1) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{3x-1}{x}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{3x-1}{x} dx + c_1$

- Evaluate integral

  $\ln(y) = 3x - \ln(x) + c_1$

- Solve for y

$$y = \frac{e^{3x+c_1}}{x}$$

- Use initial condition $y(1) = 0$

$$0 = e^{3+c_1}$$

- Solution does not satisfy initial condition

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 5

```
dsolve([y(x)+x*diff(y(x),x) = 3*x*y(x),y(1) = 0],y(x), singsol=all)
```

$$y(x) = 0$$

✓ Solution by Mathematica

Time used: 0.001 (sec). Leaf size: 6

```
DSolve[{y[x]+x*y'[x] == 3*x*y[x],y[1]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 0$$

4.12 problem 12

Internal problem ID [65]
Internal file name [`OUTPUT/65_Sunday_June_05_2022_01_34_13_AM_88553762/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{3y + y'x = 2x^5}$$

With initial conditions

$$[y(2) = 1]$$

4.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = 2x^4$$

Hence the ode is

$$y' + \frac{3y}{x} = 2x^4$$

The domain of $p(x) = \frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = 2x^4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

4.12.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{3}{x}dx} \\ &= x^3\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(2x^4\right) \\ \frac{d}{dx}\left(y\,x^3\right) &= \left(x^3\right)\left(2x^4\right) \\ d\left(y\,x^3\right) &= \left(2x^7\right)\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^3 &= \int 2x^7\,dx \\ y\,x^3 &= \frac{x^8}{4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$y = \frac{x^5}{4} + \frac{c_1}{x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{8} + 8$$

$$c_1 = -56$$

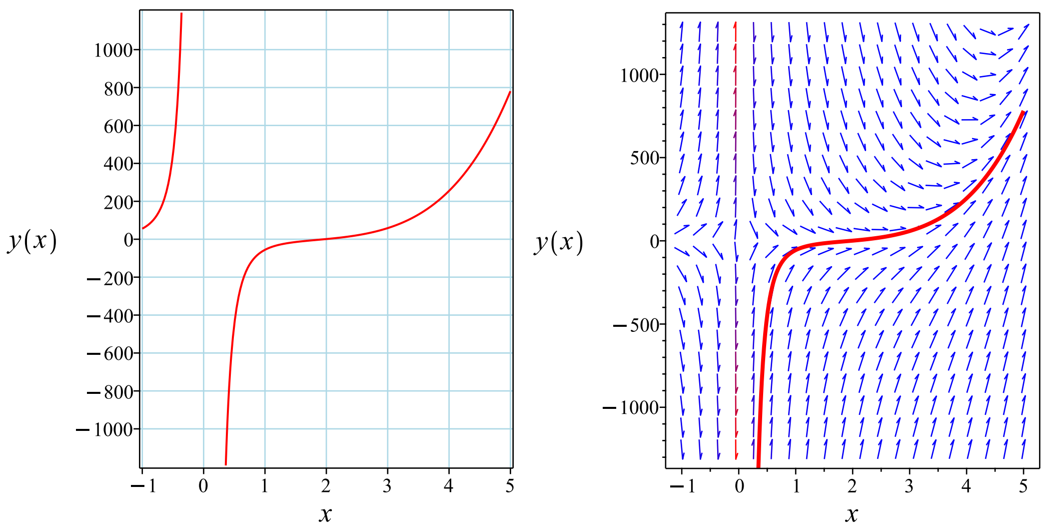
Substituting c_1 found above in the general solution gives

$$y = \frac{x^8 - 224}{4x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{x^8 - 224}{4x^3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x^8 - 224}{4x^3}$$

Verified OK.

4.12.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-2x^5 + 3y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 153: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{x^3}} dy
\end{aligned}
$$

Which results in

$$
S = y\, x^3
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{-2x^5 + 3y}{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 3y\,x^2 \\
S_y &= x^3
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 2x^7 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 2R^7
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{R^8}{4} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
yx^3 = \frac{x^8}{4} + c_1
$$

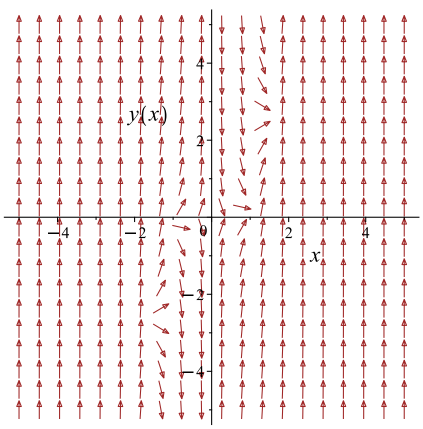
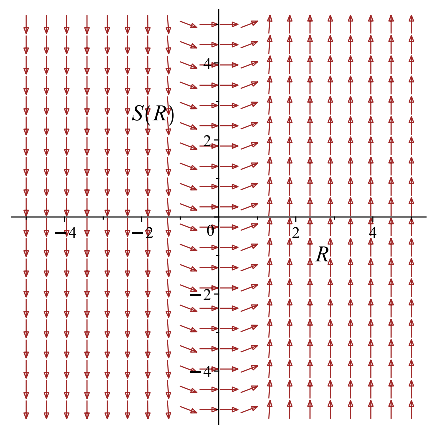
Which simplifies to

$$
yx^3 = \frac{x^8}{4} + c_1
$$

Which gives

$$
y = \frac{x^8 + 4c_1}{4x^3}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{-2x^5+3y}{x}$ 	$R = x$ $S = y\,x^3$	$\frac{dS}{dR} = 2R^7$ 

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{8} + 8$$

$$c_1 = -56$$

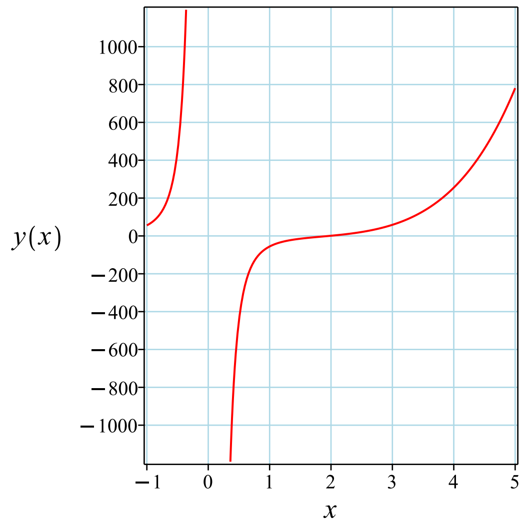
Substituting c_1 found above in the general solution gives

$$y = \frac{x^8 - 224}{4x^3}$$

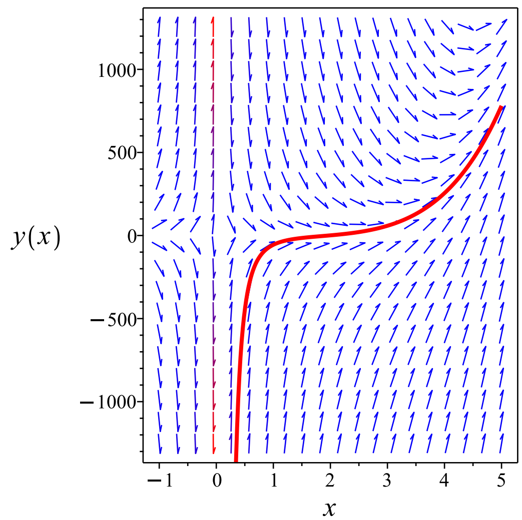
Summary

The solution(s) found are the following

$$y = \frac{x^8 - 224}{4x^3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^8 - 224}{4x^3}$$

Verified OK.

4.12.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (x)\,\mathrm{d}y &= \left(2x^5 - 3y\right)\mathrm{d}x \\ \left(-2x^5 + 3y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= -2x^5 + 3y \\ N(x,y) &= x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2x^5 + 3y\right) \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((3) - (1)) \\ &= \frac{2}{x} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int \frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2\ln(x)} \\ &= x^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= x^2(-2x^5 + 3y) \\ &= -2x^7 + 3y\,x^2\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= x^2(x) \\ &= x^3\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(-2x^7 + 3y\,x^2\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -2x^7 + 3y\,x^2\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{1}{4}x^8 + y\,x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$x^3 = x^3 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{4}x^8 + y\,x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{4}x^8 + y\,x^3$$

The solution becomes

$$y = \frac{x^8 + 4c_1}{4x^3}$$

Initial conditions are used to solve for c_1. Substituting $x = 2$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{c_1}{8} + 8$$

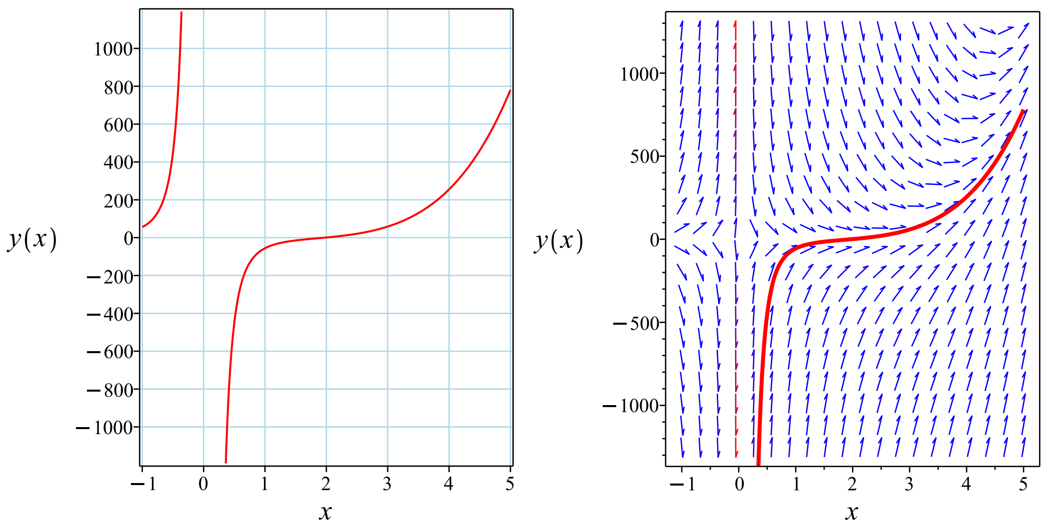
$$c_1 = -56$$

Substituting c_1 found above in the general solution gives

$$y = \frac{x^8 - 224}{4x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{x^8 - 224}{4x^3} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{x^8 - 224}{4x^3}$$

Verified OK.

4.12.5 Maple step by step solution

Let's solve

$[3y + y'x = 2x^5, y(2) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{3y}{x} + 2x^4$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = 2x^4$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = 2\mu(x)\, x^4$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x^3$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 2\mu(x)\, x^4 dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 2\mu(x)\, x^4 dx + c_1$
- Solve for y

  $y = \frac{\int 2\mu(x) x^4 dx + c_1}{\mu(x)}$
- Substitute $\mu(x) = x^3$

  $y = \frac{\int 2x^7 dx + c_1}{x^3}$
- Evaluate the integrals on the rhs

  $y = \frac{\frac{x^8}{4} + c_1}{x^3}$
- Simplify

  $y = \frac{x^8 + 4c_1}{4x^3}$
- Use initial condition $y(2) = 1$

  $1 = \frac{c_1}{8} + 8$
- Solve for c_1

  $c_1 = -56$
- Substitute $c_1 = -56$ into general solution and simplify

  $y = \frac{x^8 - 224}{4x^3}$
- Solution to the IVP

  $y = \frac{x^8 - 224}{4x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([3*y(x)+x*diff(y(x),x) = 2*x^5,y(2) = 1],y(x), singsol=all)
```

$$y(x) = \frac{x^8 - 224}{4x^3}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 17

```
DSolve[{3*y[x]+x*y'[x] == 2*x^5,y[2]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^8 - 224}{4x^3}$$

4.13 problem 13

Internal problem ID [66]
Internal file name [`OUTPUT/66_Sunday_June_05_2022_01_34_14_AM_18474770/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{y' + y = e^x}$$

With initial conditions

$$[y(0) = 1]$$

4.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 1$$
$$q(x) = e^x$$

Hence the ode is

$$y' + y = e^{x}$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = e^{x}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.13.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 1dx} \\ &= e^{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(e^{x}\right) \\ \frac{d}{dx}(e^{x} y) &= (e^{x})\left(e^{x}\right) \\ d(e^{x} y) &= e^{2x}\, dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{x} y &= \int e^{2x}\, dx \\ e^{x} y &= \frac{e^{2x}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$y = \frac{e^{-x} e^{2x}}{2} + c_1 e^{-x}$$

which simplifies to

$$y = \frac{e^{x}}{2} + c_1 e^{-x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

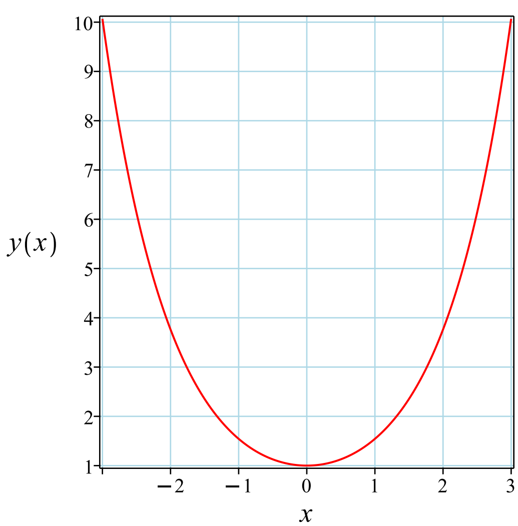
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

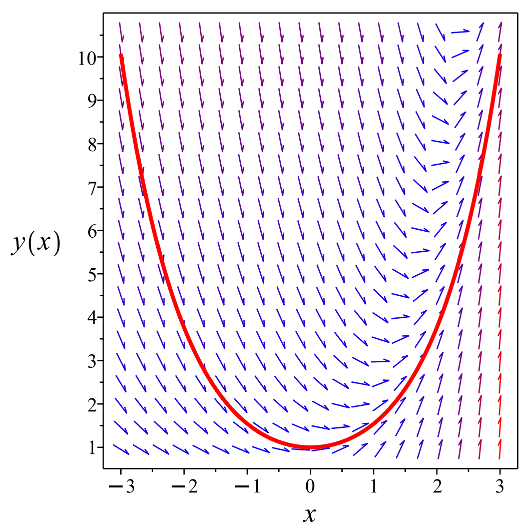
$$y = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{e^x}{2} + \frac{e^{-x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

Verified OK.

4.13.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y + e^x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 156: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\mathrm{e}^{-x}} dy
\end{aligned}
$$

Which results in

$$
S = \mathrm{e}^{x} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -y + \mathrm{e}^{x}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \mathrm{e}^{x} y \\
S_y &= \mathrm{e}^{x}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{2R}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^x y = \frac{e^{2x}}{2} + c_1$$

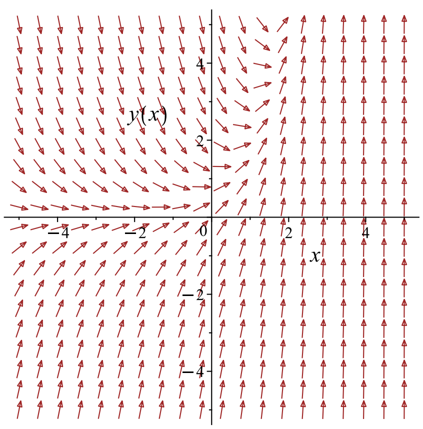
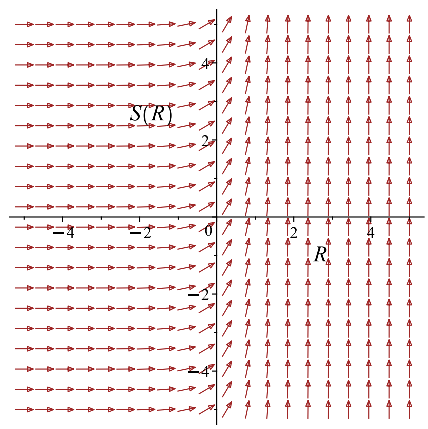
Which simplifies to

$$e^x y = \frac{e^{2x}}{2} + c_1$$

Which gives

$$y = \frac{(e^{2x} + 2c_1)\, e^{-x}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y + e^{x}$  	$R = x$ $S = e^{x} y$	$\frac{dS}{dR} = e^{2R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

$$c_1 = \frac{1}{2}$$

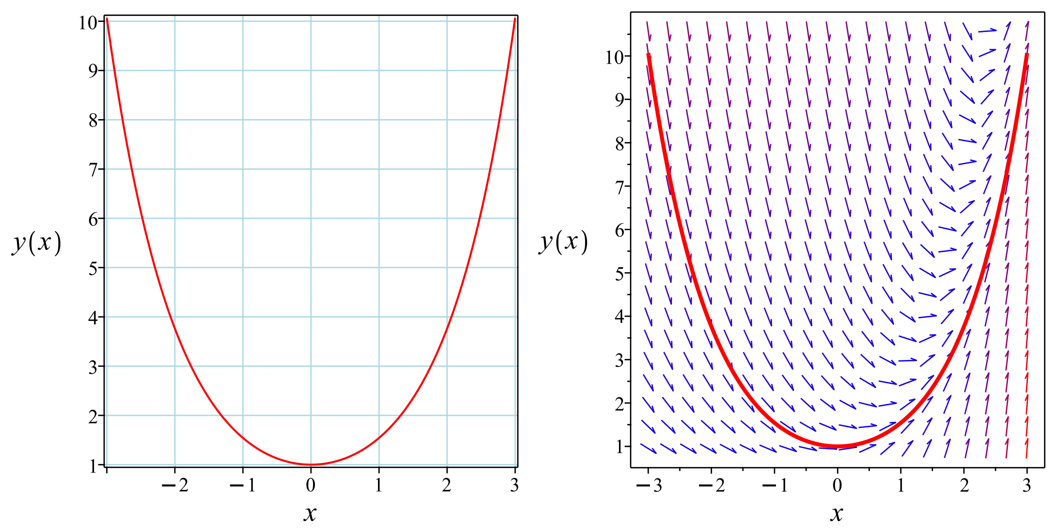
Substituting c_1 found above in the general solution gives

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

Verified OK.

4.13.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\mathrm{d}y &= (-y + \mathrm{e}^x)\,\mathrm{d}x \\
(y - \mathrm{e}^x)\,\mathrm{d}x + \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= y - \mathrm{e}^x \\
N(x, y) &= 1
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \mathrm{e}^x) \\
&= 1
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((1) - (0)) \\
&= 1
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int 1 \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{x} \\
&= \mathrm{e}^{x}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{x}(y - \mathrm{e}^{x}) \\
&= (y - \mathrm{e}^{x})\,\mathrm{e}^{x}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{x}(1) \\
&= \mathrm{e}^{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
((y - \mathrm{e}^{x})\,\mathrm{e}^{x}) + (\mathrm{e}^{x})\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int (y - \mathrm{e}^{x})\,\mathrm{e}^{x} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x}$. Therefore equation (4) becomes

$$\mathrm{e}^{x} = \mathrm{e}^{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{x} y - \frac{\mathrm{e}^{2x}}{2}$$

The solution becomes

$$y = \frac{(\mathrm{e}^{2x} + 2c_1)\,\mathrm{e}^{-x}}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{2} + c_1$$

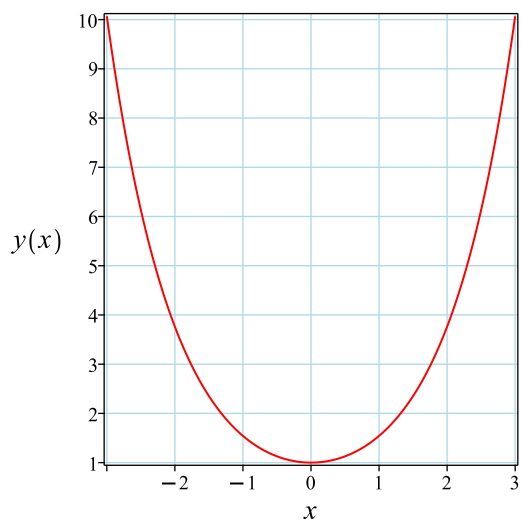
$$c_1 = \frac{1}{2}$$

Substituting c_1 found above in the general solution gives

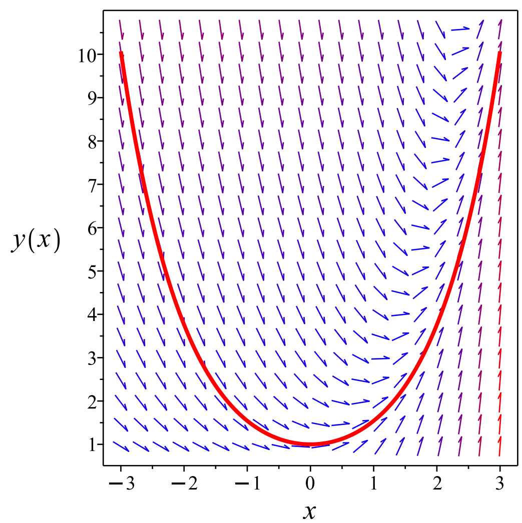
$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$$

Verified OK.

4.13.5 Maple step by step solution

Let's solve

$[y' + y = e^{x}, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$y' = -y + e^{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$y' + y = e^{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$\mu(x)\,(y' + y) = \mu(x)\,e^{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$\mu(x)\,(y' + y) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

$\mu'(x) = \mu(x)$

- Solve to find the integrating factor

$\mu(x) = e^{x}$

- Integrate both sides with respect to x

$\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \mu(x)\,e^{x} dx + c_1$

- Evaluate the integral on the lhs

$\mu(x)\,y = \int \mu(x)\,e^{x} dx + c_1$

- Solve for y

$y = \frac{\int \mu(x) e^{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = e^{x}$

$y = \frac{\int (e^{x})^{2} dx + c_1}{e^{x}}$

- Evaluate the integrals on the rhs

$y = \frac{\frac{(e^{x})^{2}}{2} + c_1}{e^{x}}$

- Simplify

$y = \frac{e^{x}}{2} + c_1 e^{-x}$

- Use initial condition $y(0) = 1$

$1 = \frac{1}{2} + c_1$

- Solve for c_1

$c_1 = \frac{1}{2}$

- Substitute $c_1 = \frac{1}{2}$ into general solution and simplify

$y = \frac{e^{x}}{2} + \frac{e^{-x}}{2}$

- Solution to the IVP

  $y = \frac{e^x}{2} + \frac{e^{-x}}{2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([y(x)+diff(y(x),x) = exp(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{e^x}{2} + \frac{e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.035 (sec). Leaf size: 21

```
DSolve[{y[x]+y'[x] == Exp[x],y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-x}(e^{2x} + 1)$$

4.14 problem 14

Internal problem ID [67]
Internal file name [`OUTPUT/67_Sunday_June_05_2022_01_34_14_AM_3504027/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-3y + y'x = x^3}$$

With initial conditions

$$[y(1) = 10]$$

4.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' - \frac{3y}{x} = x^2$$

The domain of $p(x) = -\frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = x^2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

4.14.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(x^2\right) \\ \mathrm{d}\left(\frac{y}{x^3}\right) &= \frac{1}{x}\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^3} &= \int \frac{1}{x}\,\mathrm{d}x \\ \frac{y}{x^3} &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = \ln(x)\, x^3 + c_1 x^3$$

which simplifies to

$$y = x^3(\ln(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = c_1$$

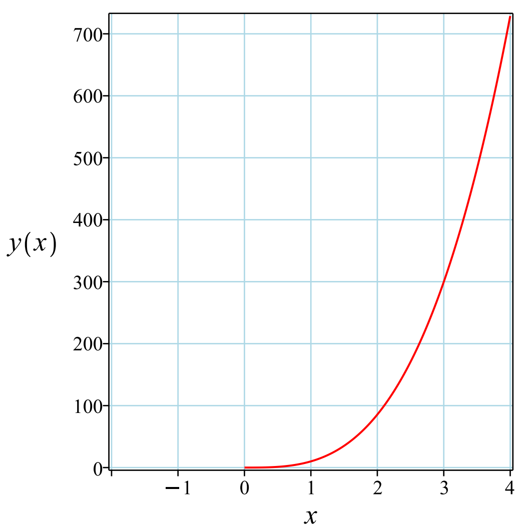
$$c_1 = 10$$

Substituting c_1 found above in the general solution gives

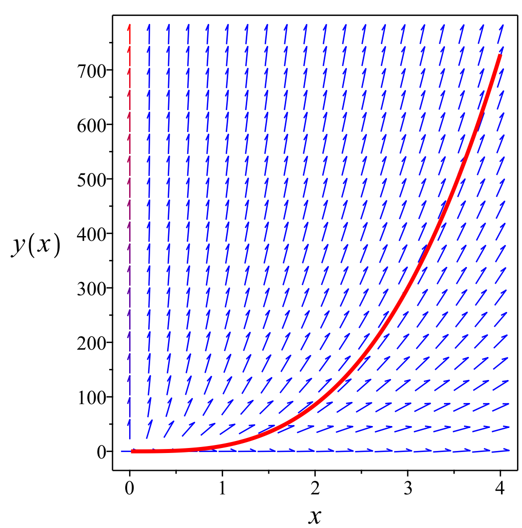
$$y = \ln(x)\, x^3 + 10x^3$$

Summary
The solution(s) found are the following

$$y = \ln(x)\, x^3 + 10x^3 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x^3 + 10x^3$$

Verified OK.

4.14.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^3 + 3y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 159: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x^3 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^3} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 + 3y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{3y}{x^4} \\
S_y &= \frac{1}{x^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \ln(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x^3} = \ln(x) + c_1
$$

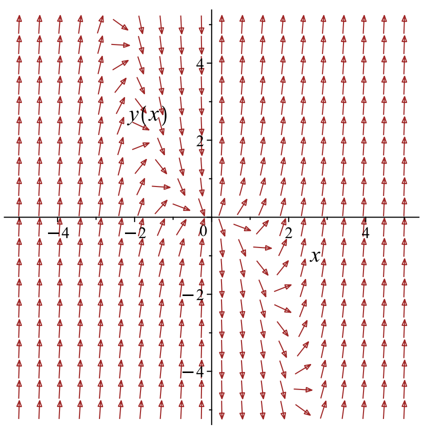
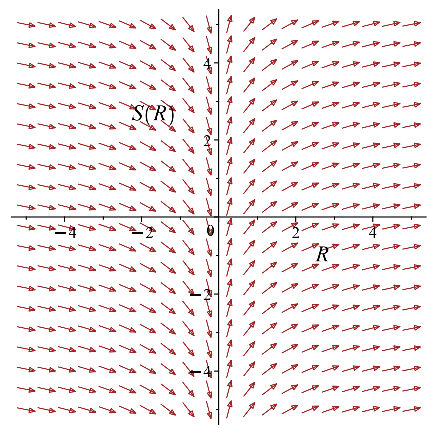
Which simplifies to

$$
\frac{y}{x^3} = \ln(x) + c_1
$$

Which gives

$$
y = x^3(\ln(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+3y}{x}$ 	$R = x$ $S = \frac{y}{x^3}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = c_1$$

$$c_1 = 10$$

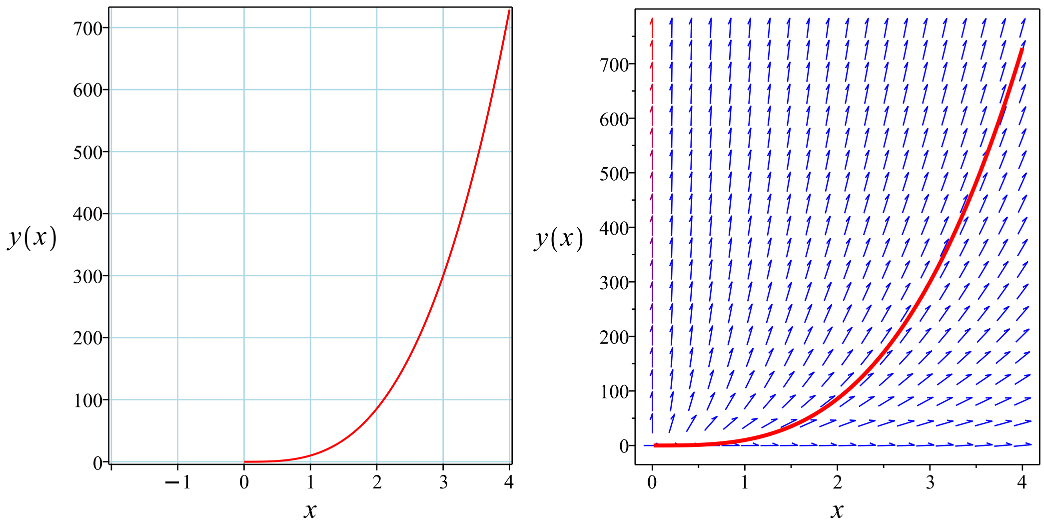
Substituting c_1 found above in the general solution gives

$$y = \ln(x) x^3 + 10x^3$$

Summary

The solution(s) found are the following

$$y = \ln(x) x^3 + 10x^3 \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x^3 + 10x^3$$

Verified OK.

4.14.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y &= \left(x^3+3y\right)\mathrm{d}x\\ \left(-x^3-3y\right)\mathrm{d}x + (x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^3-3y\\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3-3y\right)\\ &= -3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((-3)-(1))\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x}\\ &= e^{\int -\frac{4}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)}\\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^4}(-x^3 - 3y)\\ &= \frac{-x^3 - 3y}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N\\ &= \frac{1}{x^4}(x)\\ &= \frac{1}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \left(\frac{-x^3 - 3y}{x^4}\right) + \left(\frac{1}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{-x^3 - 3y}{x^4} \,\mathrm{d}x \\
\phi &= \frac{y}{x^3} - \ln(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{1}{x^3} + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^3}$. Therefore equation (4) becomes

$$
\frac{1}{x^3} = \frac{1}{x^3} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \frac{y}{x^3} - \ln(x) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \frac{y}{x^3} - \ln(x)
$$

The solution becomes

$$
y = x^3(\ln(x) + c_1)
$$

Initial conditions are used to solve for c_1. Substituting $x = 1$ and $y = 10$ in the above solution gives an equation to solve for the constant of integration.

$$10 = c_1$$

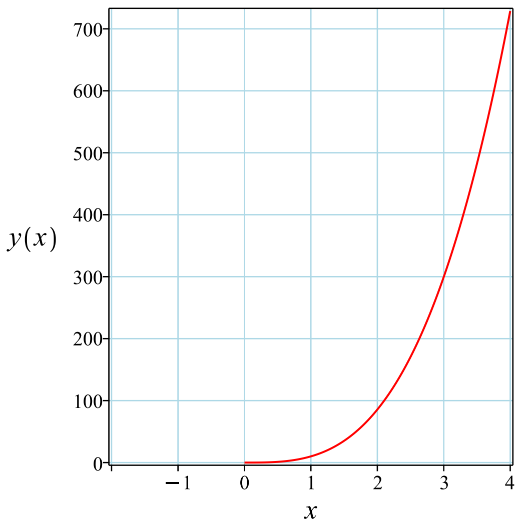
$$c_1 = 10$$

Substituting c_1 found above in the general solution gives

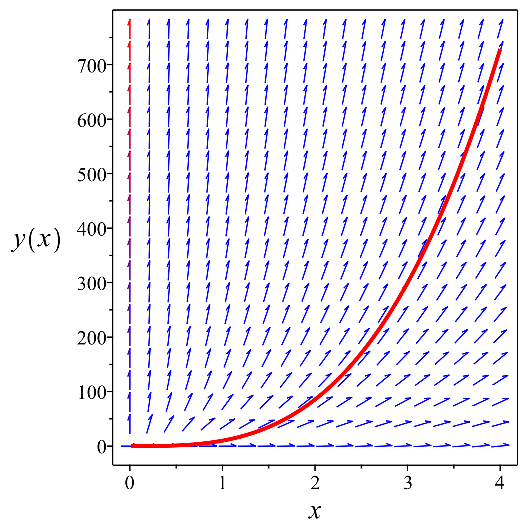
$$y = \ln(x)\, x^3 + 10x^3$$

Summary
The solution(s) found are the following

$$y = \ln(x)\, x^3 + 10x^3 \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \ln(x)\, x^3 + 10x^3$$

Verified OK.

4.14.5 Maple step by step solution

Let's solve

$[-3y + y'x = x^3, y(1) = 10]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{3y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{3y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{3y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^3}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^3}$

  $y = x^3\left(\int \frac{1}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^3(\ln(x) + c_1)$

- Use initial condition $y(1) = 10$

  $10 = c_1$

- Solve for c_1

  $c_1 = 10$
- Substitute $c_1 = 10$ into general solution and simplify

  $y = (\ln(x) + 10)\, x^3$
- Solution to the IVP

  $y = (\ln(x) + 10)\, x^3$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([-3*y(x)+x*diff(y(x),x) = x^3,y(1) = 10],y(x), singsol=all)
```

$$y(x) = (\ln(x) + 10)\, x^3$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 13

```
DSolve[{-3*y[x]+x*y'[x] == x^3,y[1]==10},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3(\log(x) + 10)$$

4.15 problem 15

Internal problem ID [68]
Internal file name [`OUTPUT/68_Sunday_June_05_2022_01_34_15_AM_99518507/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2yx + y' = x}$$

With initial conditions

$$[y(0) = -2]$$

4.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = 2x$$
$$q(x) = x$$

Hence the ode is

$$2yx + y' = x$$

The domain of $p(x) = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.15.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x(1 - 2y) \end{aligned}$$

Where $f(x) = x$ and $g(y) = 1 - 2y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1 - 2y}\, dy &= x\, dx \\ \int \frac{1}{1 - 2y}\, dy &= \int x\, dx \\ -\frac{\ln(1 - 2y)}{2} &= \frac{x^2}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{\sqrt{1 - 2y}} = e^{\frac{x^2}{2} + c_1}$$

Which simplifies to

$$\frac{1}{\sqrt{1 - 2y}} = c_2 e^{\frac{x^2}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{e^{-2c_1} e^{2c_1} c_2^2 - e^{-2c_1}}{2c_2^2}$$

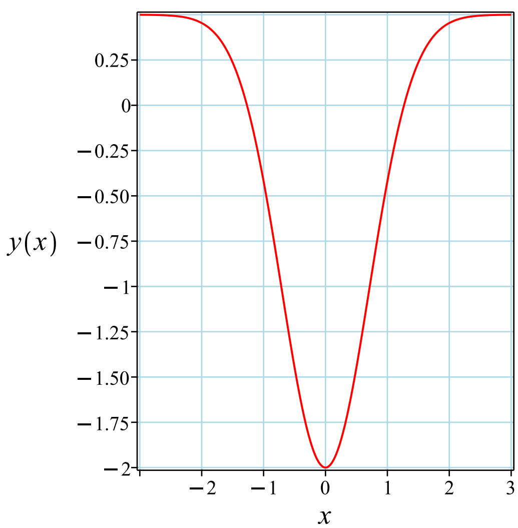
$$c_1 = -\frac{\ln(5c_2^2)}{2}$$

Substituting c_1 found above in the general solution gives

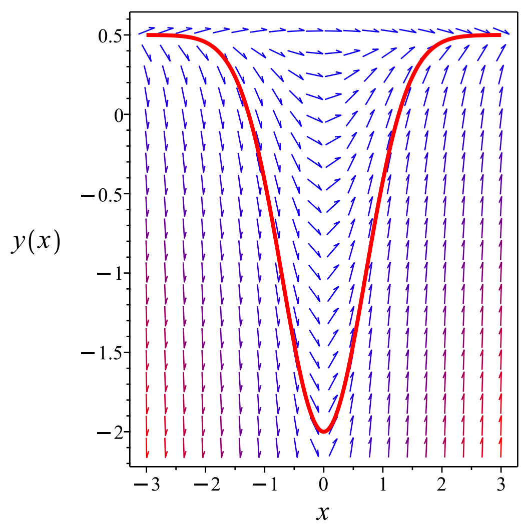
$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

Verified OK.

4.15.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2x dx} \\ &= e^{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)(x) \\ \frac{d}{dx}\left(e^{x^2} y\right) &= \left(e^{x^2}\right)(x) \\ d\left(e^{x^2} y\right) &= \left(e^{x^2} x\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{x^2} y &= \int e^{x^2} x \, dx \\ e^{x^2} y &= \frac{e^{x^2}}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{x^2}$ results in

$$y = \frac{e^{x^2} e^{-x^2}}{2} + c_1 e^{-x^2}$$

which simplifies to

$$y = \frac{1}{2} + c_1 e^{-x^2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{1}{2} + c_1$$

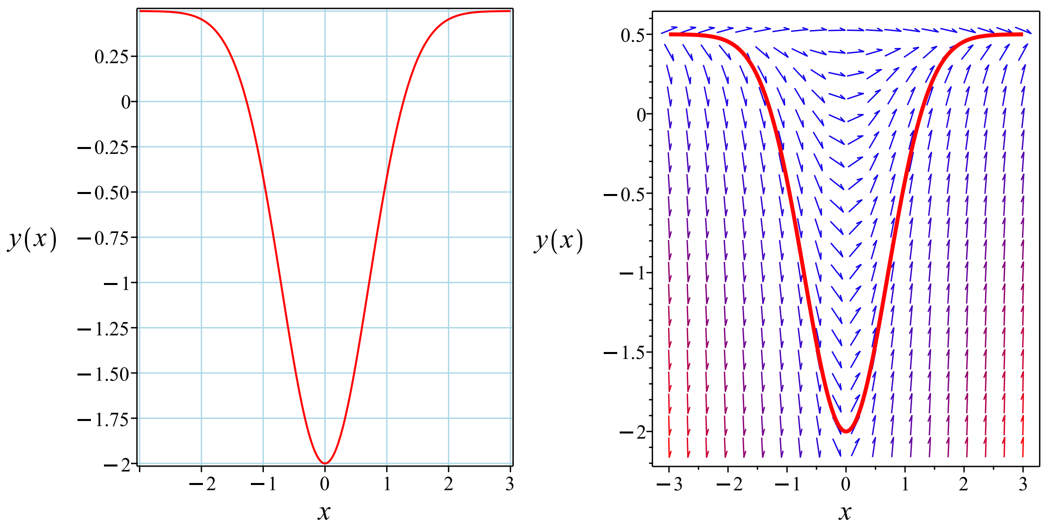
$$c_1 = -\frac{5}{2}$$

Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} - \frac{5 e^{-x^2}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

Verified OK.

4.15.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -2yx + x$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 162: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-x^2}} dy \end{aligned}$$

Which results in

$$S = e^{x^2} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -2yx + x$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2x\, e^{x^2} y \\ S_y &= e^{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = e^{x^2} x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = e^{R^2} R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{e^{R^2}}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{x^2} y = \frac{e^{x^2}}{2} + c_1$$

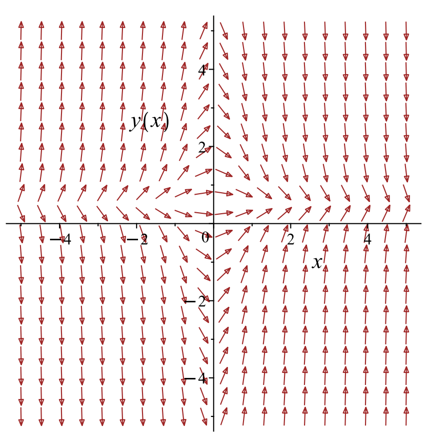
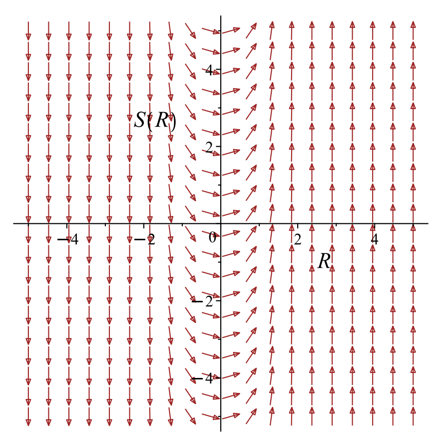
Which simplifies to

$$e^{x^2} y = \frac{e^{x^2}}{2} + c_1$$

Which gives

$$y = \frac{\left(e^{x^2} + 2c_1\right) e^{-x^2}}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -2yx + x$ 	$R = x$ $S = e^{x^2} y$	$\frac{dS}{dR} = e^{R^2} R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{1}{2} + c_1$$

$$c_1 = -\frac{5}{2}$$

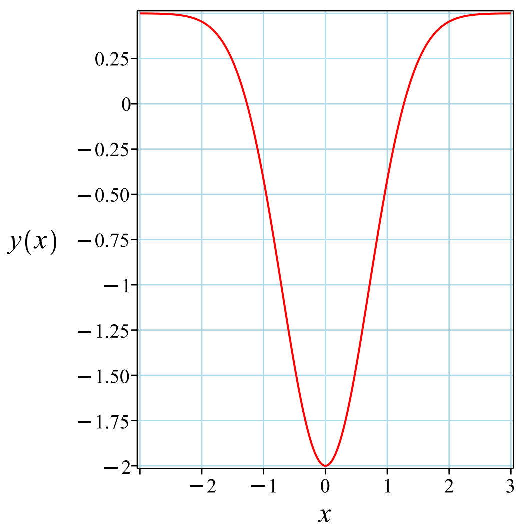
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} - \frac{5\,\mathrm{e}^{-x^2}}{2}$$

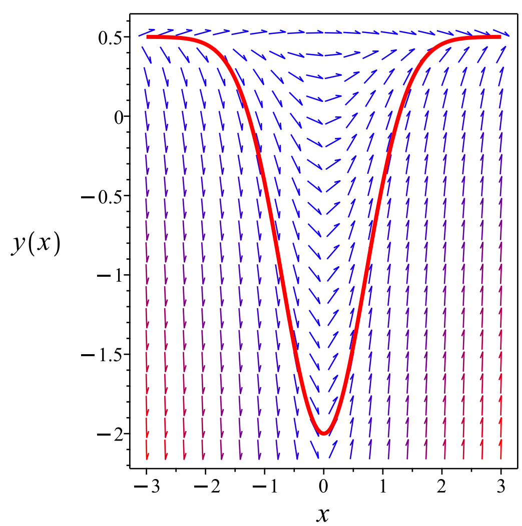
Summary

The solution(s) found are the following

$$y = \frac{1}{2} - \frac{5\,\mathrm{e}^{-x^2}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} - \frac{5\,\mathrm{e}^{-x^2}}{2}$$

Verified OK.

4.15.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{1-2y}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(\frac{1}{1-2y}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x \\
N(x, y) &= \frac{1}{1-2y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-2y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^2}{2} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \frac{1}{1-2y}$. Therefore equation (4) becomes

$$\frac{1}{1-2y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{-1+2y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{-1+2y}\right) \mathrm{d}y$$

$$f(y) = -\frac{\ln(-1+2y)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - \frac{\ln(-1+2y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(-1+2y)}{2}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-x^2-2c_1}}{2} + \frac{1}{2}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = \frac{\mathrm{e}^{-2c_1}}{2} + \frac{1}{2}$$

$$c_1 = -\frac{\ln(5)}{2} - \frac{i\pi}{2}$$

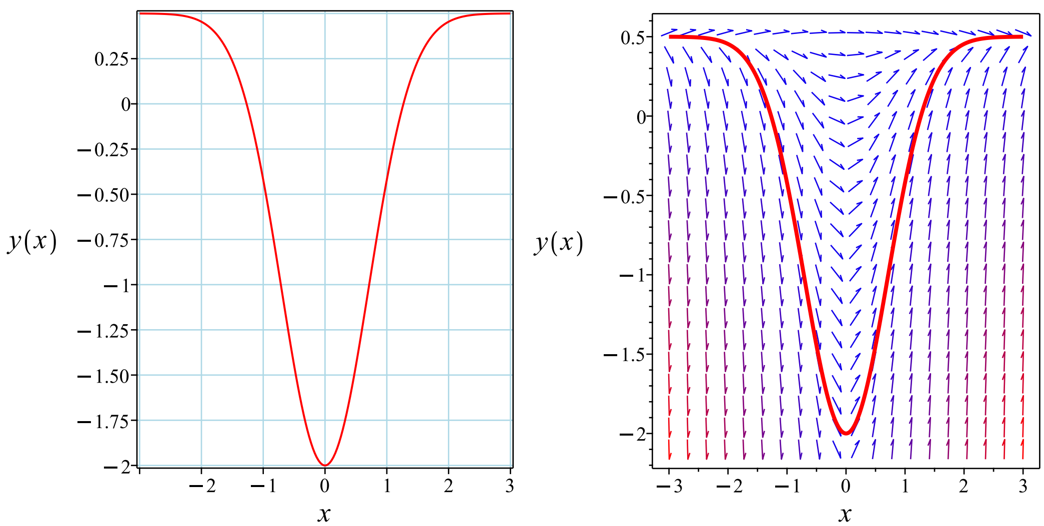
Substituting c_1 found above in the general solution gives

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

Verified OK.

4.15.6 Maple step by step solution

Let's solve

$[2yx + y' = x, y(0) = -2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$$\frac{y'}{-1+2y} = -x$$

- Integrate both sides with respect to x

$$\int \frac{y'}{-1+2y}dx = \int -xdx + c_1$$

- Evaluate integral

$$\frac{\ln(-1+2y)}{2} = -\frac{x^2}{2} + c_1$$

- Solve for y

$$y = \frac{e^{-x^2+2c_1}}{2} + \frac{1}{2}$$

- Use initial condition $y(0) = -2$

$$-2 = \frac{e^{2c_1}}{2} + \frac{1}{2}$$

- Solve for c_1

$$c_1 = \frac{\ln(5)}{2} + \frac{I\pi}{2}$$

- Substitute $c_1 = \frac{\ln(5)}{2} + \frac{I\pi}{2}$ into general solution and simplify

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

- Solution to the IVP

$$y = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 14

```
dsolve([2*x*y(x)+diff(y(x),x) = x,y(0) = -2],y(x), singsol=all)
```

$$y(x) = \frac{1}{2} - \frac{5\,e^{-x^2}}{2}$$

✓ Solution by Mathematica

Time used: 0.039 (sec). Leaf size: 20

```
DSolve[{2*x*y[x]+y'[x] == x,y[0]==-2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} - \frac{5e^{-x^2}}{2}$$

4.16 problem 16

Internal problem ID [69]
Internal file name [`OUTPUT/69_Sunday_June_05_2022_01_34_15_AM_62449306/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - \cos(x)(1-y) = 0}$$

With initial conditions

$$[y(\pi) = 2]$$

4.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \cos(x)$$
$$q(x) = \cos(x)$$

Hence the ode is

$$y' + \cos(x)\, y = \cos(x)$$

The domain of $p(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is inside this domain. The domain of $q(x) = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = \pi$ is also inside this domain. Hence solution exists and is unique.

4.16.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= \cos(x)\,(1 - y) \end{aligned}$$

Where $f(x) = \cos(x)$ and $g(y) = 1 - y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{1-y}\, dy &= \cos(x)\, dx \\ \int \frac{1}{1-y}\, dy &= \int \cos(x)\, dx \\ -\ln(y-1) &= \sin(x) + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{y-1} = e^{\sin(x)+c_1}$$

Which simplifies to

$$\frac{1}{y-1} = c_2 e^{\sin(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = \frac{e^{-c_1} e^{c_1} c_2 + e^{-c_1}}{c_2}$$

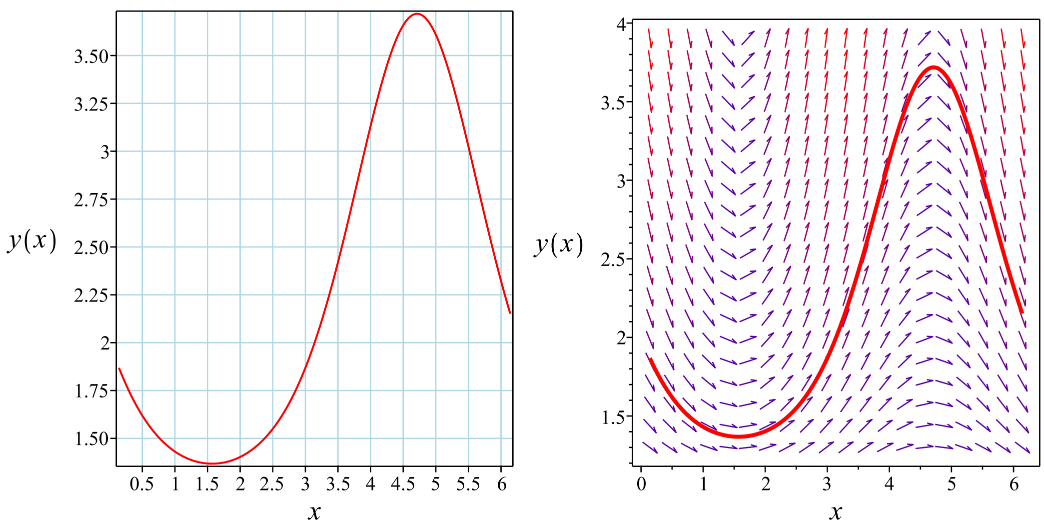
$$c_1 = -\ln(c_2)$$

Substituting c_1 found above in the general solution gives

$$y = 1 + e^{-\sin(x)}$$

Summary
The solution(s) found are the following

$$y = 1 + e^{-\sin(x)} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 + e^{-\sin(x)}$$

Verified OK.

4.16.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int \cos(x)dx}$$
$$= e^{\sin(x)}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(\cos(x))$$
$$\frac{d}{dx}\left(e^{\sin(x)}y\right) = \left(e^{\sin(x)}\right)(\cos(x))$$
$$d\left(e^{\sin(x)}y\right) = \left(\cos(x)\, e^{\sin(x)}\right)\, dx$$

Integrating gives

$$e^{\sin(x)}y = \int \cos(x)\, e^{\sin(x)}\, dx$$
$$e^{\sin(x)}y = e^{\sin(x)} + c_1$$

Dividing both sides by the integrating factor $\mu = e^{\sin(x)}$ results in

$$y = e^{-\sin(x)}e^{\sin(x)} + c_1 e^{-\sin(x)}$$

which simplifies to

$$y = 1 + c_1 e^{-\sin(x)}$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 + c_1$$

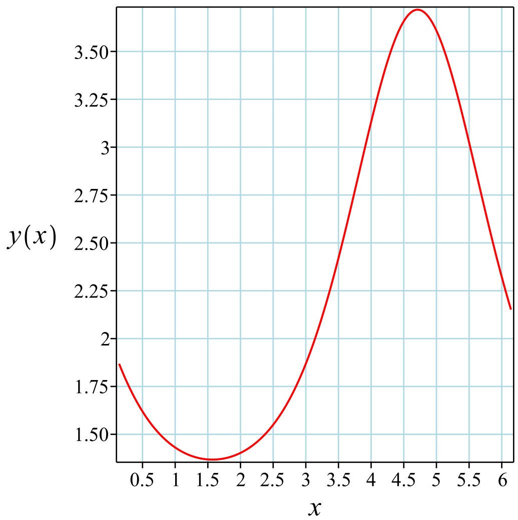
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

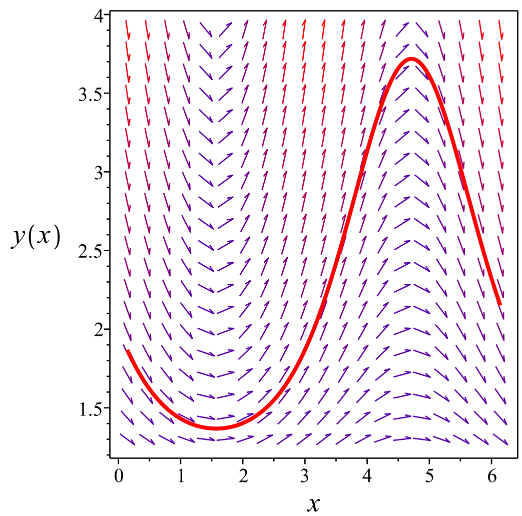
$$y = 1 + e^{-\sin(x)}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 1 + e^{-\sin(x)} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 1 + e^{-\sin(x)}$$

Verified OK.

4.16.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\cos(x)(y-1)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 165: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= \mathrm{e}^{-\sin(x)} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\sin(x)} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\cos(x)(y - 1)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x) \mathrm{e}^{\sin(x)} y \\ S_y &= \mathrm{e}^{\sin(x)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \mathrm{e}^{\sin(x)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R) \mathrm{e}^{\sin(R)}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{\sin(R)} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{\sin(x)} y = e^{\sin(x)} + c_1$$

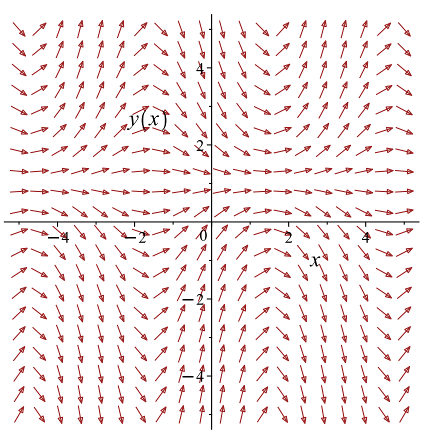
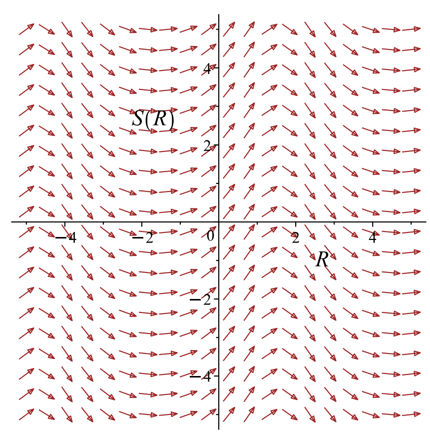
Which simplifies to

$$e^{\sin(x)} y = e^{\sin(x)} + c_1$$

Which gives

$$y = e^{-\sin(x)} \left(e^{\sin(x)} + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\cos(x)(y-1)$ 	$R = x$ $S = e^{\sin(x)} y$	$\frac{dS}{dR} = \cos(R) e^{\sin(R)}$ 

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = 1 + c_1$$

$$c_1 = 1$$

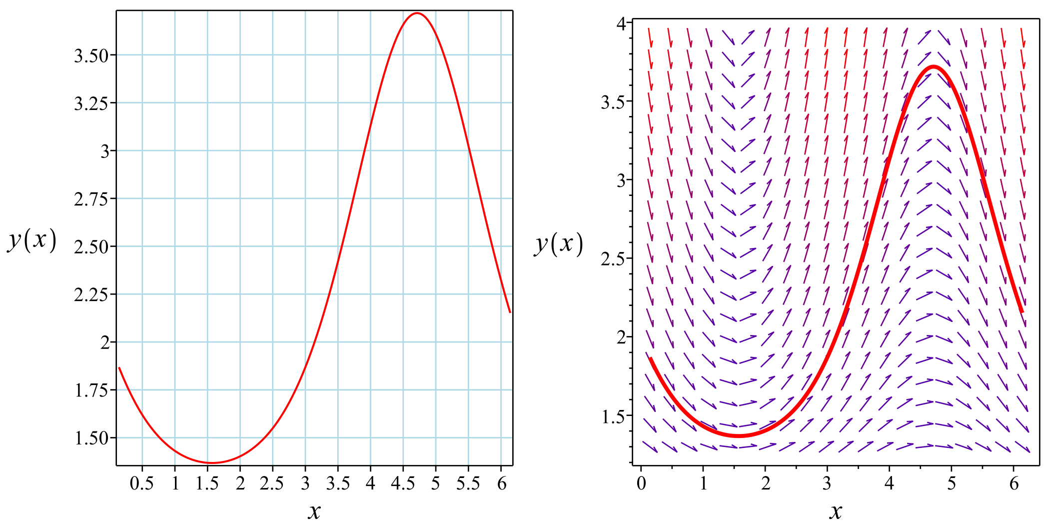
Substituting c_1 found above in the general solution gives

$$y = 1 + e^{-\sin(x)}$$

Summary

The solution(s) found are the following

$$y = 1 + e^{-\sin(x)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 + e^{-\sin(x)}$$

Verified OK.

4.16.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned} \left(\frac{1}{1-y}\right) \mathrm{d} y &= (\cos(x)) \mathrm{d} x \\ (-\cos(x)) \mathrm{d} x+\left(\frac{1}{1-y}\right) \mathrm{d} y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -\cos(x) \\ N(x, y) &= \frac{1}{1-y} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cos(x)) \\ &= 0 \end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{1-y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\cos(x)\,\mathrm{d}x\end{aligned}$$

$$\phi = -\sin(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{1-y}$. Therefore equation (4) becomes

$$\frac{1}{1-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -\frac{1}{y-1}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(-\frac{1}{y-1}\right)\mathrm{d}y\\ f(y) &= -\ln(y-1) + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) - \ln(y-1)$$

The solution becomes

$$y = e^{-\sin(x)-c_1} + 1$$

Initial conditions are used to solve for c_1. Substituting $x = \pi$ and $y = 2$ in the above solution gives an equation to solve for the constant of integration.

$$2 = e^{-c_1} + 1$$

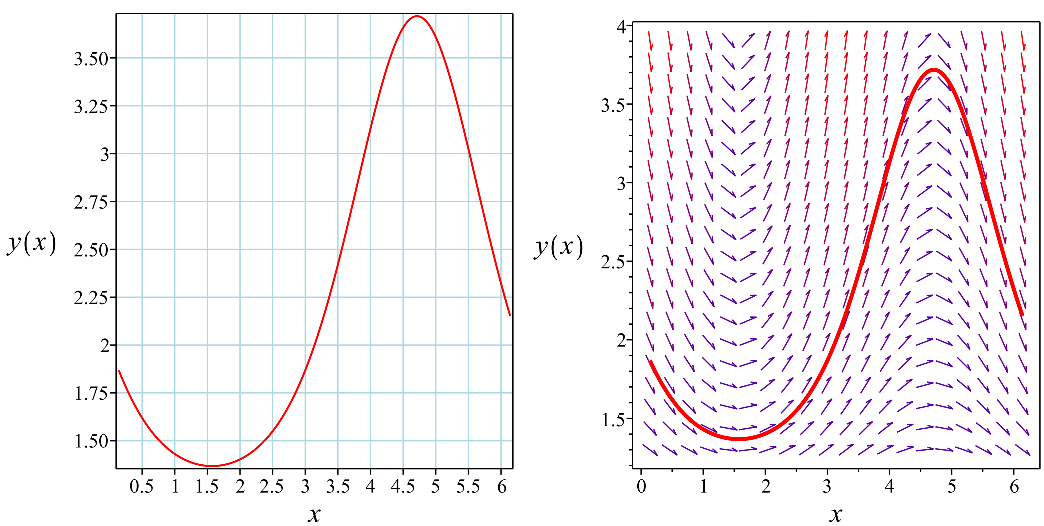
$$c_1 = 0$$

Substituting c_1 found above in the general solution gives

$$y = 1 + e^{-\sin(x)}$$

Summary
The solution(s) found are the following

$$y = 1 + e^{-\sin(x)} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 1 + e^{-\sin(x)}$$

Verified OK.

4.16.6 Maple step by step solution

Let's solve

$[y' - \cos(x)(1 - y) = 0, y(\pi) = 2]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1-y} = \cos(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{1-y} dx = \int \cos(x) dx + c_1$

- Evaluate integral

  $-\ln(1 - y) = \sin(x) + c_1$

- Solve for y

  $y = -e^{-\sin(x) - c_1} + 1$

- Use initial condition $y(\pi) = 2$

  $2 = -e^{-c_1} + 1$

- Solve for c_1

  $c_1 = -\mathrm{I}\pi$

- Substitute $c_1 = -\mathrm{I}\pi$ into general solution and simplify

  $y = 1 + e^{-\sin(x)}$

- Solution to the IVP

  $y = 1 + e^{-\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve([diff(y(x),x) = cos(x)*(1-y(x)),y(Pi) = 2],y(x), singsol=all)
```

$$y(x) = 1 + e^{-\sin(x)}$$

✓ Solution by Mathematica

Time used: 0.05 (sec). Leaf size: 13

```
DSolve[{y'[x] == Cos[x]*(1-y[x]),y[Pi]==2},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\sin(x)} + 1$$

4.17 problem 17

Internal problem ID [70]
Internal file name [`OUTPUT/70_Sunday_June_05_2022_01_34_16_AM_9186813/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_linear]
```

$$\boxed{y + (x+1)\,y' = \cos(x)}$$

With initial conditions

$$[y(0) = 1]$$

4.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x+1}$$
$$q(x) = \frac{\cos(x)}{x+1}$$

Hence the ode is

$$y' + \frac{y}{x+1} = \frac{\cos(x)}{x+1}$$

The domain of $p(x) = \frac{1}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{\cos(x)}{x+1}$ is

$$\{x < -1 \lor -1 < x\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.17.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{1}{x+1} dx} \\
&= x + 1
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{\cos(x)}{x+1}\right) \\
\frac{d}{dx}((x+1)\, y) &= (x+1)\left(\frac{\cos(x)}{x+1}\right) \\
d((x+1)\, y) &= \cos(x)\; dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
(x+1)\, y &= \int \cos(x)\; dx \\
(x+1)\, y &= \sin(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x + 1$ results in

$$y = \frac{\sin(x)}{x+1} + \frac{c_1}{x+1}$$

which simplifies to

$$y = \frac{\sin(x) + c_1}{x + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

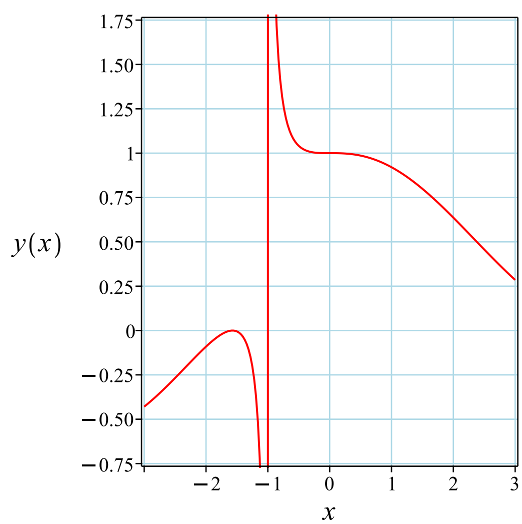
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

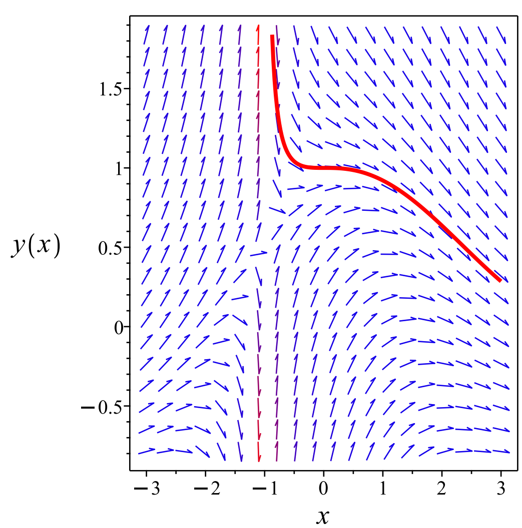
$$y = \frac{\sin(x) + 1}{x + 1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sin(x) + 1}{x + 1} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{\sin(x) + 1}{x + 1}$$

Verified OK.

4.17.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-y+\cos(x)}{x+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 168: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{x+1} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x+1}} dy \end{aligned}$$

Which results in

$$S = (x+1)\,y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y + \cos(x)}{x+1}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= y \\
S_y &= x + 1
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cos(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cos(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \sin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
(x + 1)\, y = \sin(x) + c_1
$$

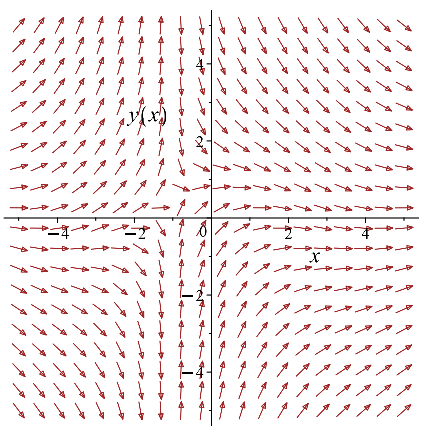
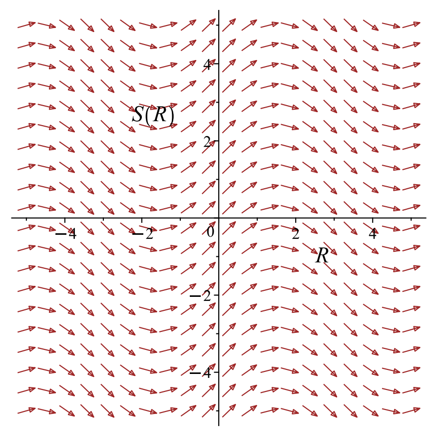
Which simplifies to

$$
(x + 1)\, y = \sin(x) + c_1
$$

Which gives

$$
y = \frac{\sin(x) + c_1}{x + 1}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+\cos(x)}{x+1}$ 	$R = x$ $S = (x + 1)\, y$	$\frac{dS}{dR} = \cos(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

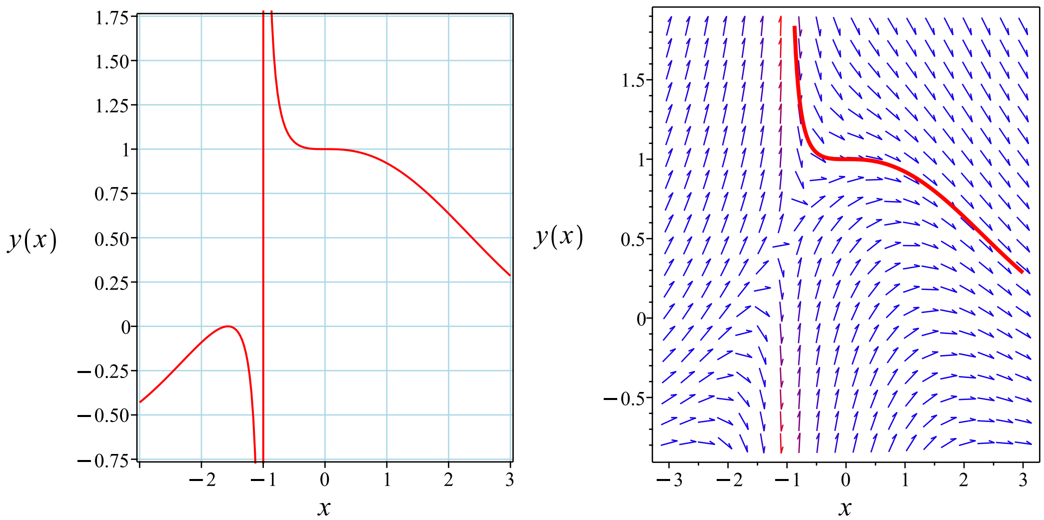
$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = \frac{\sin(x) + 1}{x + 1}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\sin(x) + 1}{x + 1} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + 1}{x + 1}$$

Verified OK.

4.17.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x+1)\,\mathrm{d}y &= (-y + \cos(x))\,\mathrm{d}x\\ (y - \cos(x))\,\mathrm{d}x + (x+1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= y - \cos(x)\\ N(x, y) &= x + 1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y - \cos(x))\\ &= 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x + 1)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int y - \cos(x) \, \mathrm{d}x \\
\phi &= yx - \sin(x) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x + 1$. Therefore equation (4) becomes

$$
x + 1 = x + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 1
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (1) \, \mathrm{d}y \\
f(y) &= y + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = yx - \sin(x) + y + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = yx - \sin(x) + y
$$

The solution becomes

$$y = \frac{\sin(x) + c_1}{x + 1}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

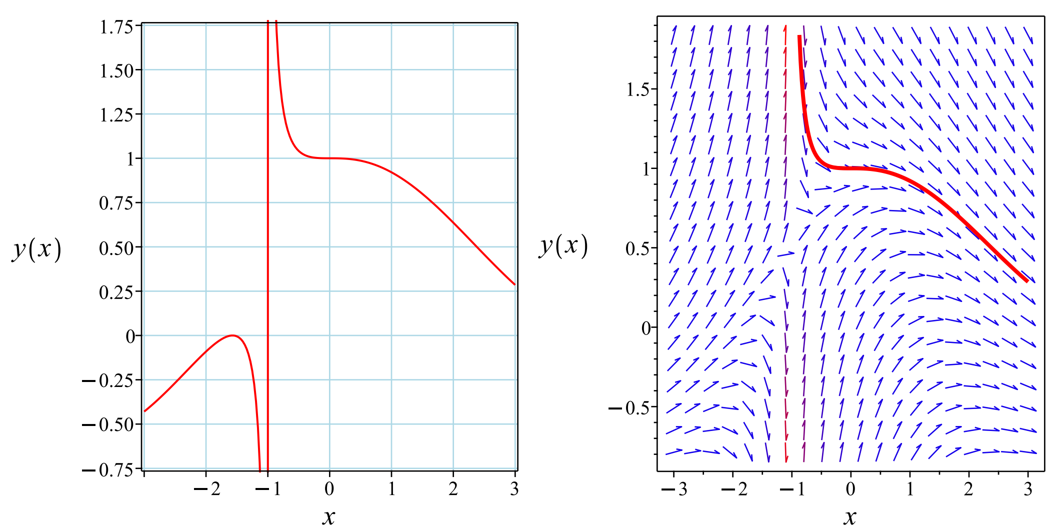
Substituting c_1 found above in the general solution gives

$$y = \frac{\sin(x) + 1}{x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x) + 1}{x + 1} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\sin(x) + 1}{x + 1}$$

Verified OK.

4.17.5 Maple step by step solution

Let's solve

$[y + (x+1)\, y' = \cos(x)\,, y(0) = 1]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x+1} + \frac{\cos(x)}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x+1} = \frac{\cos(x)}{x+1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x+1}\right) = \frac{\mu(x)\cos(x)}{x+1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{y}{x+1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x+1}$

- Solve to find the integrating factor

  $\mu(x) = x + 1$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)\cos(x)}{x+1} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)\cos(x)}{x+1} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)\cos(x)}{x+1} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x + 1$

  $y = \frac{\int \cos(x) dx + c_1}{x+1}$

- Evaluate the integrals on the rhs

  $y = \frac{\sin(x) + c_1}{x+1}$

- Use initial condition $y(0) = 1$

$1 = c_1$

- Solve for c_1

  $c_1 = 1$

- Substitute $c_1 = 1$ into general solution and simplify

  $y = \frac{\sin(x)+1}{x+1}$

- Solution to the IVP

  $y = \frac{\sin(x)+1}{x+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([y(x)+(1+x)*diff(y(x),x) = cos(x),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{\sin(x) + 1}{x + 1}$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 15

```
DSolve[{y[x]+(1+x)*y'[x] == Cos[x],y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sin(x) + 1}{x + 1}$$

4.18 problem 18

Internal problem ID [71]
Internal file name [`OUTPUT/71_Sunday_June_05_2022_01_34_17_AM_1048459/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y'x - 2y = x^3 \cos(x)}$$

4.18.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= x^2 \cos(x) \end{aligned}$$

Hence the ode is

$$y' - \frac{2y}{x} = x^2 \cos(x)$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{2}{x}dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(x^2\cos(x)\right) \\
\frac{d}{dx}\left(\frac{y}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(x^2\cos(x)\right) \\
d\left(\frac{y}{x^2}\right) &= \cos(x)\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x^2} &= \int \cos(x)\, dx \\
\frac{y}{x^2} &= \sin(x) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$
y = \sin(x)\, x^2 + c_1 x^2
$$

which simplifies to

$$
y = x^2(\sin(x) + c_1)
$$

Summary

The solution(s) found are the following

$$
y = x^2(\sin(x) + c_1) \tag{1}
$$

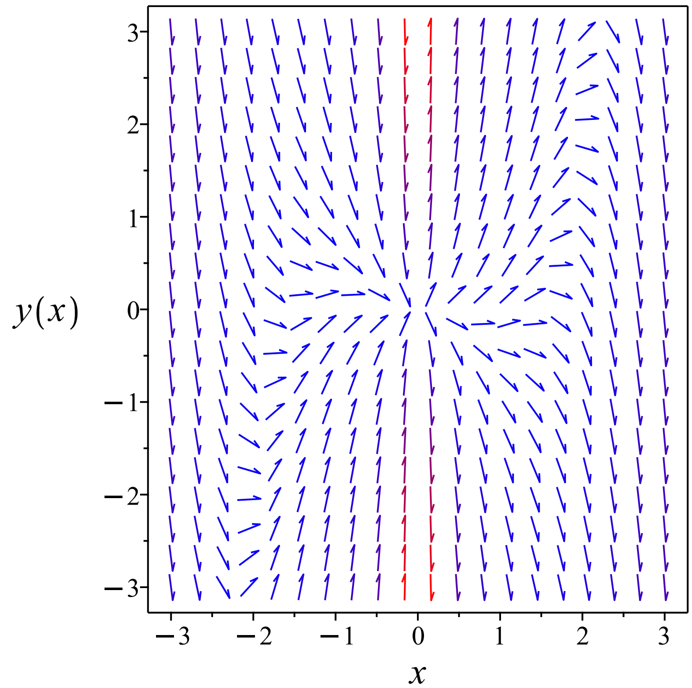


Figure 212: Slope field plot

<u>Verification of solutions</u>

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

4.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 \cos(x) + 2y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 171: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= x^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 \cos(x) + 2y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2y}{x^3} \\ S_y &= \frac{1}{x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \cos(x) \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \cos(R)$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \sin(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2} = \sin(x) + c_1$$

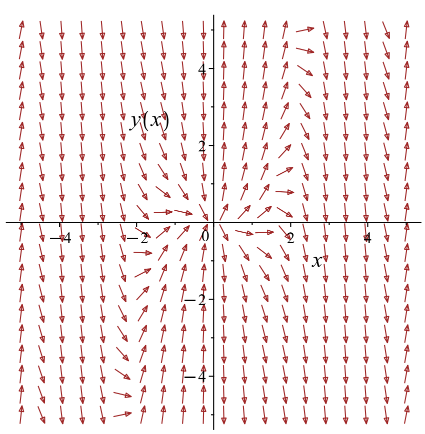
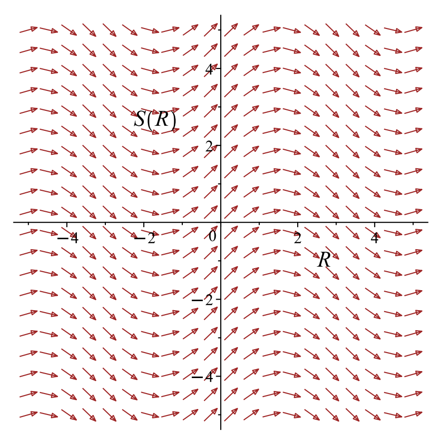
Which simplifies to

$$\frac{y}{x^2} = \sin(x) + c_1$$

Which gives

$$y = x^2(\sin(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{x^3\cos(x)+2y}{x}$

</td><td>$R = x$
$S = \frac{y}{x^2}$</td><td>$\frac{dS}{dR} = \cos(R)$

</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

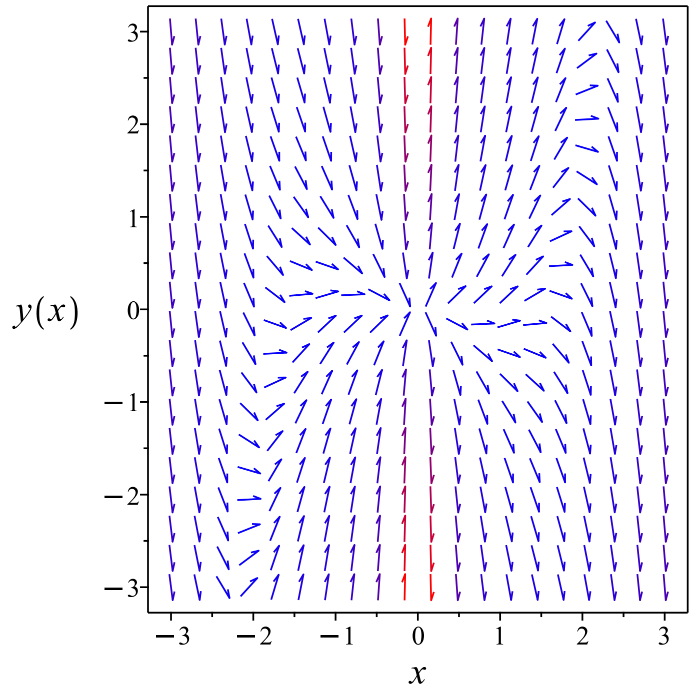


Figure 213: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

4.18.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x) \, \mathrm{d}y &= \left(x^3 \cos(x) + 2y\right) \mathrm{d}x \\ \left(-x^3 \cos(x) - 2y\right) \mathrm{d}x + (x) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^3 \cos(x) - 2y$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^3 \cos(x) - 2y\right) \\ &= -2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x) \\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x}((-2) - (1)) \\
&= -\frac{3}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{3}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^3}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3}\left(-x^3 \cos(x) - 2y\right) \\
&= \frac{-x^3 \cos(x) - 2y}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(x) \\
&= \frac{1}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^3 \cos(x) - 2y}{x^3}\right) + \left(\frac{1}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^3 \cos(x) - 2y}{x^3} \, \mathrm{d}x$$

$$\phi = -\sin(x) + \frac{y}{x^2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^2}$. Therefore equation (4) becomes

$$\frac{1}{x^2} = \frac{1}{x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) + \frac{y}{x^2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) + \frac{y}{x^2}$$

The solution becomes

$$y = x^2(\sin(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x^2(\sin(x) + c_1) \tag{1}$$

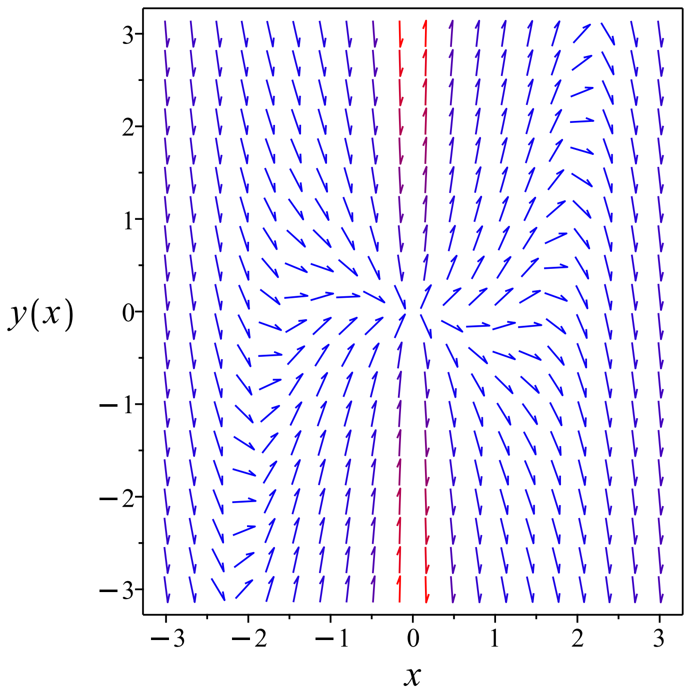


Figure 214: Slope field plot

Verification of solutions

$$y = x^2(\sin(x) + c_1)$$

Verified OK.

4.18.4 Maple step by step solution

Let's solve

$y'x - 2y = x^3 \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{2y}{x} + x^2 \cos(x)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{2y}{x} = x^2 \cos(x)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' - \frac{2y}{x}\right) = \mu(x)\, x^2 \cos(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' - \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^2}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 \cos(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 \cos(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x) x^2 \cos(x) dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^2}$

  $y = x^2 \left(\int \cos(x)\, dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^2 (\sin(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x*diff(y(x),x) = x^3*cos(x)+2*y(x),y(x), singsol=all)
```

$$y(x) = (\sin(x) + c_1)\, x^2$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 14

```
DSolve[x*y'[x]== x^3*Cos[x]+2*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(\sin(x) + c_1)$$

4.19 problem 19

Internal problem ID [72]
Internal file name [`OUTPUT/72_Sunday_June_05_2022_01_34_17_AM_13474884/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y\cot(x) + y' = \cos(x)}$$

4.19.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= \cot(x) \\ q(x) &= \cos(x) \end{aligned}$$

Hence the ode is

$$y\cot(x) + y' = \cos(x)$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \cot(x)dx} \\ &= \sin(x) \end{aligned}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(\cos(x)) \\
\frac{d}{dx}(\sin(x)\, y) &= (\sin(x))(\cos(x)) \\
d(\sin(x)\, y) &= \left(\frac{\sin(2x)}{2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\sin(x)\, y &= \int \frac{\sin(2x)}{2}\, dx \\
\sin(x)\, y &= -\frac{\cos(2x)}{4} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \sin(x)$ results in

$$
y = -\frac{\csc(x)\cos(2x)}{4} + c_1 \csc(x)
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\csc(x)\cos(2x)}{4} + c_1 \csc(x) \tag{1}
$$

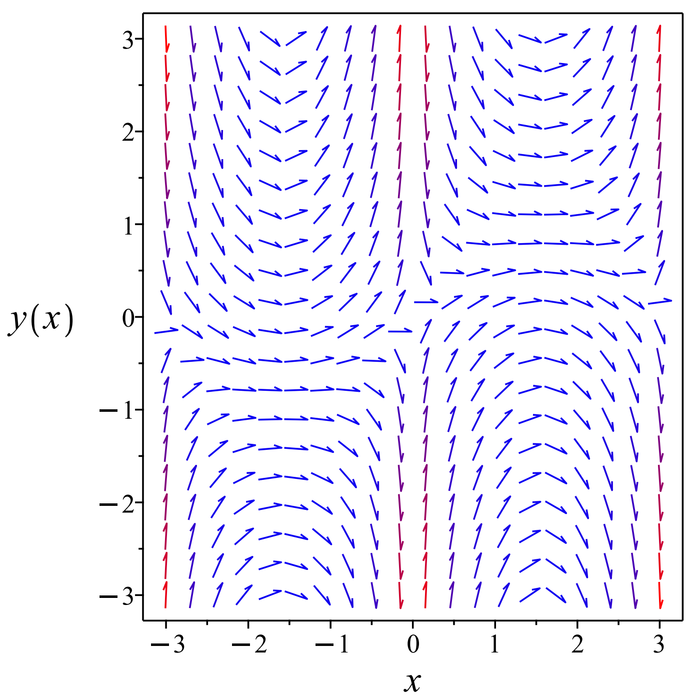


Figure 215: Slope field plot

Verification of solutions

$$y = -\frac{\csc(x)\cos(2x)}{4} + c_1 \csc(x)$$

Verified OK.

4.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -y\cot(x) + \cos(x)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 174: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 0$$
$$\eta(x,y) = \frac{1}{\sin(x)} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sin(x)}} dy \end{aligned}$$

Which results in

$$S = \sin(x)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -y \cot(x) + \cos(x)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \cos(x)\, y \\ S_y &= \sin(x) \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{\sin(2x)}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{\sin(2R)}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\cos(2R)}{4} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sin(x)\, y = -\frac{\cos(2x)}{4} + c_1$$

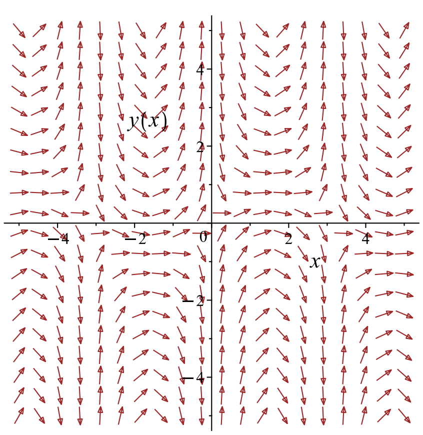
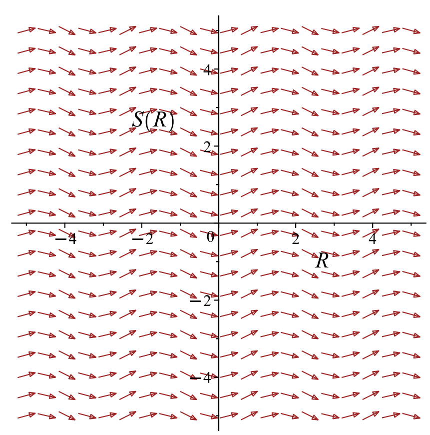
Which simplifies to

$$\sin(x)\, y = -\frac{\cos(2x)}{4} + c_1$$

Which gives

$$y = -\frac{\cos(2x) - 4c_1}{4 \sin(x)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -y\cot(x) + \cos(x)$ 	$R = x$ $S = \sin(x)\, y$	$\frac{dS}{dR} = \frac{\sin(2R)}{2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{\cos(2x) - 4c_1}{4 \sin(x)} \tag{1}$$

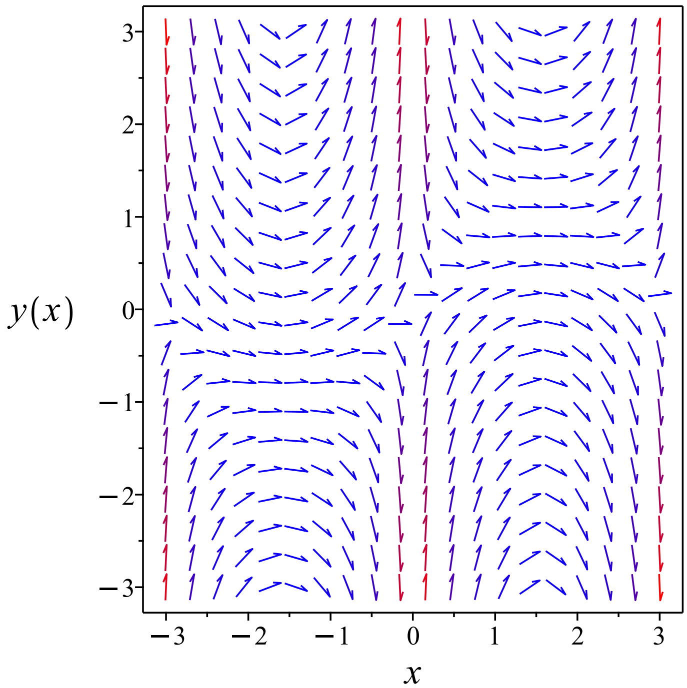


Figure 216: Slope field plot

Verification of solutions

$$y = -\frac{\cos(2x) - 4c_1}{4\sin(x)}$$

Verified OK.

4.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = (-y \cot(x) + \cos(x))\,\mathrm{d}x$$
$$(y \cot(x) - \cos(x))\,\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y \cot(x) - \cos(x)$$
$$N(x, y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y \cot(x) - \cos(x)) \\ &= \cot(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1) \\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= 1((\cot(x)) - (0)) \\
&= \cot(x)
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \cot(x) \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(\sin(x))} \\
&= \sin(x)
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \sin(x)\,(y\cot(x) - \cos(x)) \\
&= \cos(x)\,(-\sin(x) + y)
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \sin(x)\,(1) \\
&= \sin(x)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(\cos(x)\,(-\sin(x) + y)) + (\sin(x))\,\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \cos(x)(-\sin(x)+y) \mathrm{d}x \\
\phi &= -\frac{\sin(x)(-2y+\sin(x))}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \sin(x) + f'(y) \qquad (4)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(x)$. Therefore equation (4) becomes

$$\sin(x) = \sin(x) + f'(y) \qquad (5)$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\sin(x)(-2y+\sin(x))}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\sin(x)(-2y+\sin(x))}{2}$$

The solution becomes

$$y = \frac{\sin(x)^2 + 2c_1}{2\sin(x)}$$

Summary

The solution(s) found are the following

$$y = \frac{\sin(x)^2 + 2c_1}{2\sin(x)} \tag{1}$$

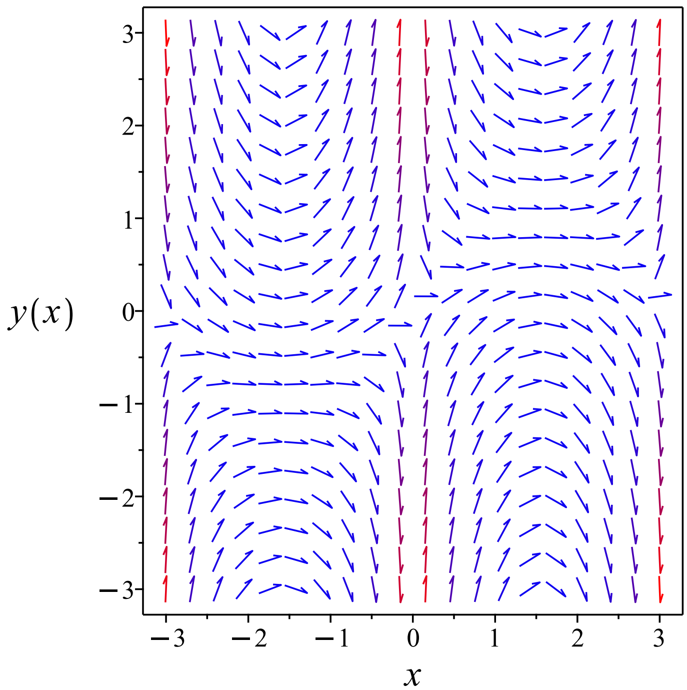


Figure 217: Slope field plot

Verification of solutions

$$y = \frac{\sin(x)^2 + 2c_1}{2\sin(x)}$$

Verified OK.

4.19.4 Maple step by step solution

Let's solve

$y\cot(x) + y' = \cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

$$y' = -y \cot(x) + \cos(x)$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y \cot(x) + y' = \cos(x)$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x) \left(y \cot(x) + y'\right) = \mu(x) \cos(x)$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

$$\mu(x) \left(y \cot(x) + y'\right) = \mu'(x)\, y + \mu(x)\, y'$$

- Isolate $\mu'(x)$

$$\mu'(x) = \mu(x) \cot(x)$$

- Solve to find the integrating factor

$$\mu(x) = \sin(x)$$

- Integrate both sides with respect to x

$$\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \cos(x)\, dx + c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\, y = \int \mu(x) \cos(x)\, dx + c_1$$

- Solve for y

$$y = \frac{\int \mu(x) \cos(x) dx + c_1}{\mu(x)}$$

- Substitute $\mu(x) = \sin(x)$

$$y = \frac{\int \cos(x) \sin(x) dx + c_1}{\sin(x)}$$

- Evaluate the integrals on the rhs

$$y = \frac{\frac{\sin(x)^2}{2} + c_1}{\sin(x)}$$

- Simplify

$$y = \frac{\sin(x)}{2} + c_1 \csc(x)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(cot(x)*y(x)+diff(y(x),x) = cos(x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(2\cos\left(x\right)^{2} - 4c_{1} - 1\right)\csc\left(x\right)}{4}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 19

```
DSolve[Cot[x]*y[x]+y'[x] == Cos[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}\cos(x)\cot(x) + c_1\csc(x)$$

4.20 problem 20

Internal problem ID [73]
Internal file name [`OUTPUT/73_Sunday_June_05_2022_01_34_18_AM_15979896/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - y - yx = x + 1}$$

With initial conditions

$$[y(0) = 0]$$

4.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -x - 1$$
$$q(x) = x + 1$$

Hence the ode is

$$y' + (-x - 1)\, y = x + 1$$

The domain of $p(x) = -x - 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = x + 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.20.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= (y + 1)\,(x + 1) \end{aligned}$$

Where $f(x) = x + 1$ and $g(y) = y + 1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{y + 1}\, dy &= x + 1\, dx \\ \int \frac{1}{y + 1}\, dy &= \int x + 1\, dx \\ \ln(y + 1) &= \frac{1}{2}x^2 + x + c_1 \end{aligned}$$

Raising both side to exponential gives

$$y + 1 = e^{\frac{1}{2}x^2 + x + c_1}$$

Which simplifies to

$$y + 1 = c_2 e^{\frac{1}{2}x^2 + x}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_2 e^{c_1} - 1$$

$$c_1 = -\ln(c_2)$$

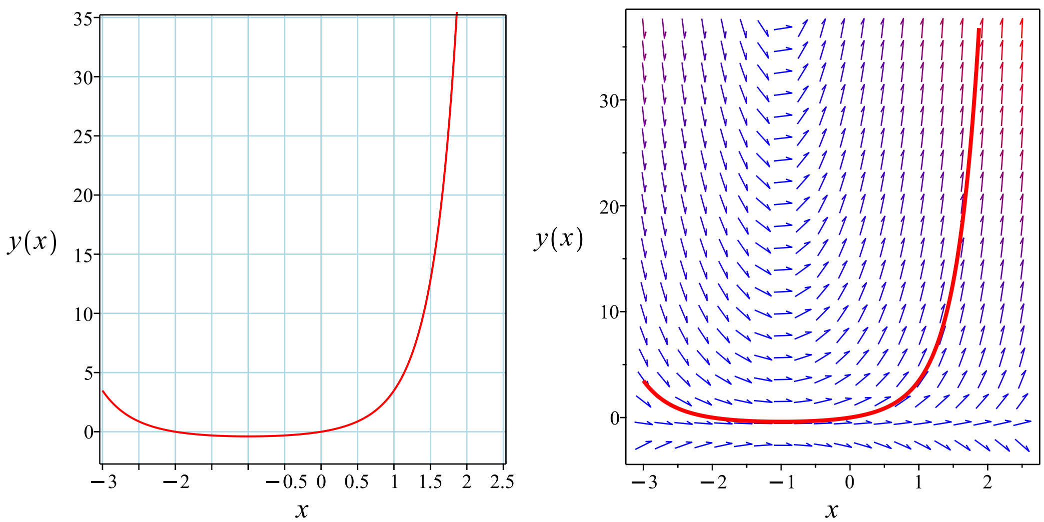
Substituting c_1 found above in the general solution gives

$$y = e^{\frac{x(2+x)}{2}} - 1$$

Summary

The solution(s) found are the following

$$y = e^{\frac{x(2+x)}{2}} - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{\frac{x(2+x)}{2}} - 1$$

Verified OK.

4.20.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\mu = e^{\int(-x-1)dx}$$
$$= e^{-\frac{1}{2}x^2 - x}$$

Which simplifies to

$$\mu = e^{-\frac{x(2+x)}{2}}$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)(x+1) \\
\frac{d}{dx}\left(e^{-\frac{x(2+x)}{2}} y\right) &= \left(e^{-\frac{x(2+x)}{2}}\right)(x+1) \\
d\left(e^{-\frac{x(2+x)}{2}} y\right) &= \left((x+1)\,e^{-\frac{x(2+x)}{2}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-\frac{x(2+x)}{2}} y &= \int (x+1)\,e^{-\frac{x(2+x)}{2}}\, dx \\
e^{-\frac{x(2+x)}{2}} y &= -e^{-\frac{x(2+x)}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-\frac{x(2+x)}{2}}$ results in

$$
y = -e^{\frac{x(2+x)}{2}} e^{-\frac{x(2+x)}{2}} + c_1 e^{\frac{x(2+x)}{2}}
$$

which simplifies to

$$
y = -1 + c_1 e^{\frac{x(2+x)}{2}}
$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$
0 = c_1 - 1
$$

$$
c_1 = 1
$$

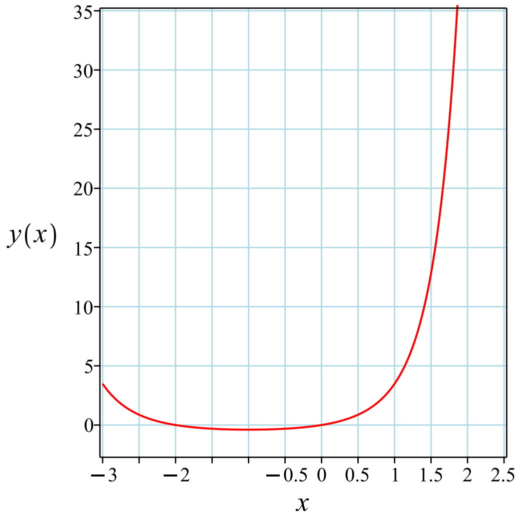
Substituting c_1 found above in the general solution gives

$$
y = e^{\frac{x(2+x)}{2}} - 1
$$

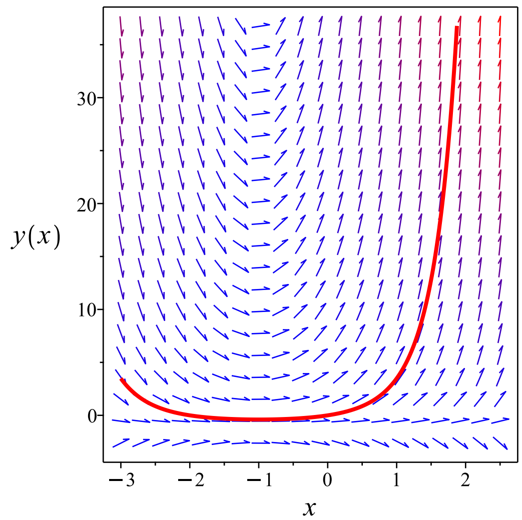
Summary

The solution(s) found are the following

$$
y = e^{\frac{x(2+x)}{2}} - 1 \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{\frac{x(2+x)}{2}} - 1$$

Verified OK.

4.20.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = yx + x + y + 1$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 177: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{\frac{1}{2}x^2+x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{\frac{1}{2}x^2+x}} dy \end{aligned}$$

Which results in

$$S = e^{-\frac{1}{2}x^2-x} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = yx + x + y + 1$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -(x+1)\, e^{-\frac{x(2+x)}{2}} y \\ S_y &= e^{-\frac{x(2+x)}{2}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = (x+1)\, e^{-\frac{x(2+x)}{2}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = (R+1)\, e^{-\frac{R(2+R)}{2}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\mathrm{e}^{-\frac{R(2+R)}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-\frac{x(2+x)}{2}} y = -\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1$$

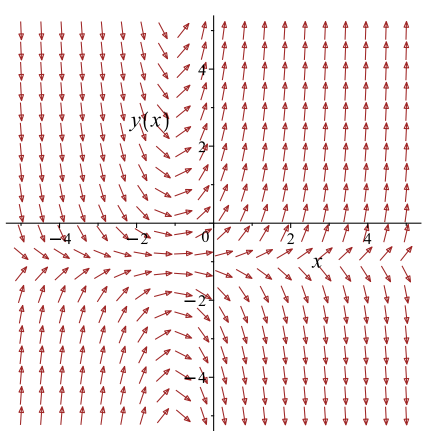
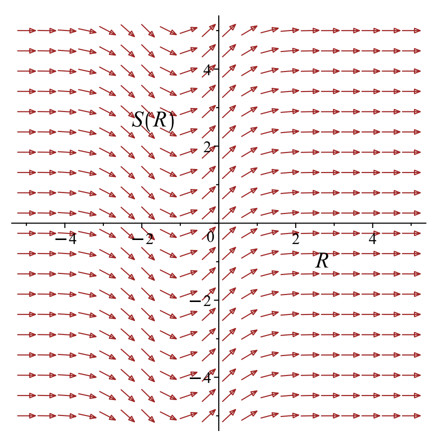
Which simplifies to

$$\mathrm{e}^{-\frac{x(2+x)}{2}} y = -\mathrm{e}^{-\frac{x(2+x)}{2}} + c_1$$

Which gives

$$y = -\left(\mathrm{e}^{-\frac{x(2+x)}{2}} - c_1\right) \mathrm{e}^{\frac{x(2+x)}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = yx + x + y + 1$ 	$R = x$ $S = \mathrm{e}^{-\frac{x(2+x)}{2}} y$	$\frac{dS}{dR} = (R + 1)\,\mathrm{e}^{-\frac{R(2+R)}{2}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = c_1 - 1$$

$$c_1 = 1$$

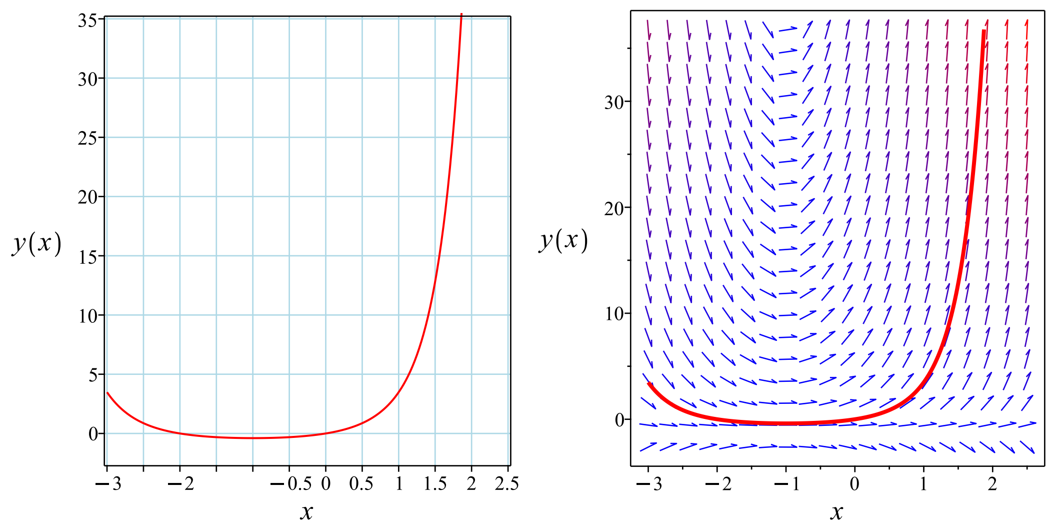
Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{\frac{x(2+x)}{2}} - 1$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\frac{x(2+x)}{2}} - 1 \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \mathrm{e}^{\frac{x(2+x)}{2}} - 1$$

Verified OK.

4.20.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y+1}\right)\mathrm{d}y &= (x+1)\,\mathrm{d}x \\ (-x-1)\,\mathrm{d}x + \left(\frac{1}{y+1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x - 1 \\ N(x, y) &= \frac{1}{y+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x-1) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y+1}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x = \int -x-1\,\mathrm{d}x$$
$$\phi = -\frac{1}{2}x^2 - x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y+1}$. Therefore equation (4) becomes

$$\frac{1}{y+1} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y+1}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y+1}\right)\mathrm{d}y$$
$$f(y) = \ln(y+1) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{2} - x + \ln(y+1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - x + \ln(y+1)$$

The solution becomes

$$y = e^{\frac{1}{2}x^2+x+c_1} - 1$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = e^{c_1} - 1$$

$$c_1 = 0$$

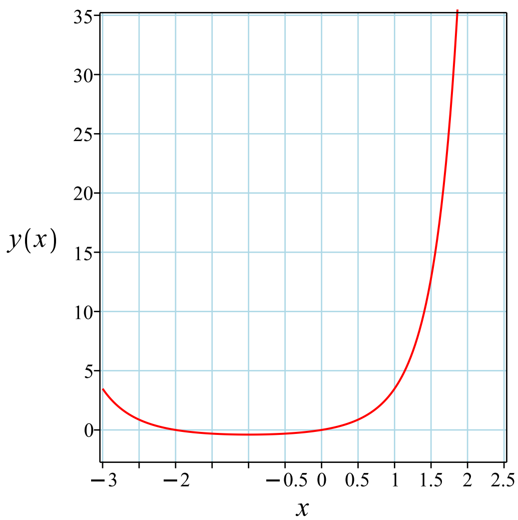
Substituting c_1 found above in the general solution gives

$$y = e^{\frac{1}{2}x^2+x} - 1$$

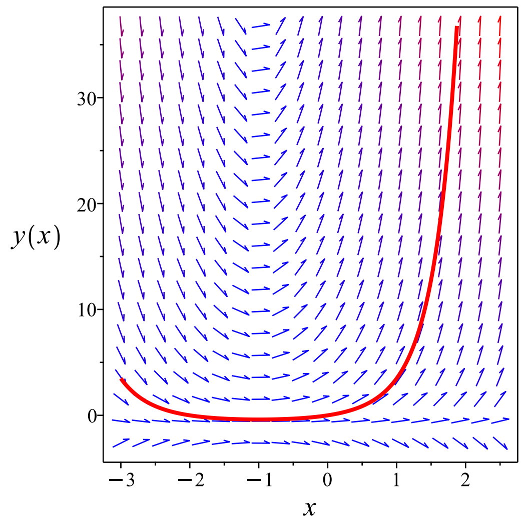
Summary

The solution(s) found are the following

$$y = e^{\frac{1}{2}x^2+x} - 1 \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{\frac{1}{2}x^2+x} - 1$$

Verified OK.

4.20.6 Maple step by step solution

Let's solve

$[y' - y - yx = x + 1, y(0) = 0]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = x + 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y} dx = \int (x + 1)\, dx + c_1$

- Evaluate integral

  $\ln(1 + y) = \frac{1}{2}x^2 + x + c_1$

- Solve for y

  $y = e^{\frac{1}{2}x^2+x+c_1} - 1$

- Use initial condition $y(0) = 0$

  $0 = e^{c_1} - 1$

- Solve for c_1

  $c_1 = 0$

- Substitute $c_1 = 0$ into general solution and simplify

  $y = e^{\frac{x(2+x)}{2}} - 1$

- Solution to the IVP

  $y = e^{\frac{x(2+x)}{2}} - 1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 13

```
dsolve([diff(y(x),x) = 1+x+y(x)+x*y(x),y(0) = 0],y(x), singsol=all)
```

$$y(x) = -1 + e^{\frac{x(2+x)}{2}}$$

✓ Solution by Mathematica
Time used: 0.028 (sec). Leaf size: 17

```
DSolve[{y'[x]== 1+x+y[x]+x*y[x],y[0]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{\frac{1}{2}x(x+2)} - 1$$

4.21 problem 21

Internal problem ID [74]
Internal file name [`OUTPUT/74_Sunday_June_05_2022_01_34_18_AM_23906152/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-3y + y'x = x^4 \cos(x)}$$

With initial conditions

$$[y(2\pi) = 0]$$

4.21.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = x^3 \cos(x)$$

Hence the ode is

$$y' - \frac{3y}{x} = x^3 \cos(x)$$

The domain of $p(x) = -\frac{3}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 2\pi$ is inside this domain. The domain of $q(x) = x^3 \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 2\pi$ is also inside this domain. Hence solution exists and is unique.

4.21.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(x^3 \cos(x)\right) \\ \frac{d}{dx}\left(\frac{y}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(x^3 \cos(x)\right) \\ d\left(\frac{y}{x^3}\right) &= \cos(x)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x^3} &= \int \cos(x)\, dx \\ \frac{y}{x^3} &= \sin(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = x^3 \sin(x) + c_1 x^3$$

which simplifies to

$$y = x^3(\sin(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 2\pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 8\pi^3 c_1$$

$$c_1 = 0$$

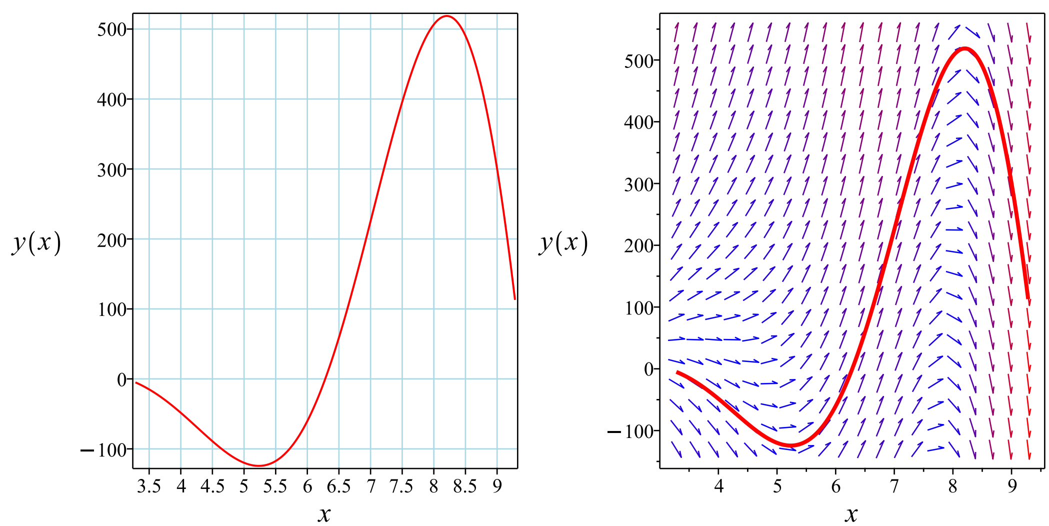
Substituting c_1 found above in the general solution gives

$$y = x^3 \sin(x)$$

Summary

The solution(s) found are the following

$$y = x^3 \sin(x) \tag{1}$$



(a) Solution plot　　(b) Slope field plot

Verification of solutions

$$y = x^3 \sin(x)$$

Verified OK.

4.21.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x^4 \cos(x) + 3y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 180: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 0\\ \eta(x,y) &= x^3\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy\\ &= \int \frac{1}{x^3} dy\end{aligned}$$

Which results in

$$S = \frac{y}{x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^4 \cos(x) + 3y}{x}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{3y}{x^4} \\
S_y &= \frac{1}{x^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \cos(x) \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \cos(R)
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \sin(R) + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{y}{x^3} = \sin(x) + c_1
$$

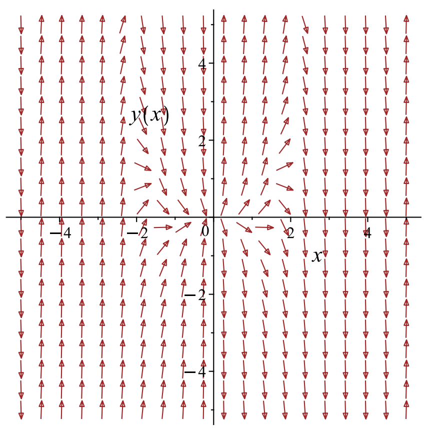
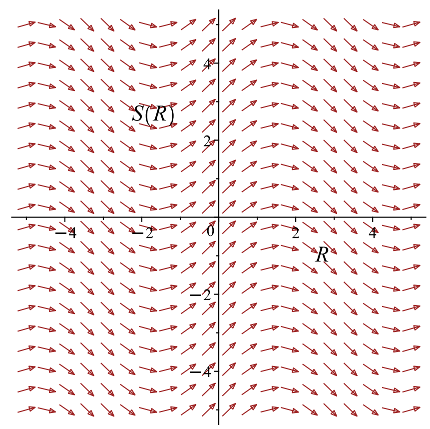
Which simplifies to

$$
\frac{y}{x^3} = \sin(x) + c_1
$$

Which gives

$$
y = x^3(\sin(x) + c_1)
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^4 \cos(x)+3y}{x}$ 	$R = x$ $S = \frac{y}{x^3}$	$\frac{dS}{dR} = \cos(R)$ 

Initial conditions are used to solve for c_1. Substituting $x = 2\pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 8\pi^3 c_1$$

$$c_1 = 0$$

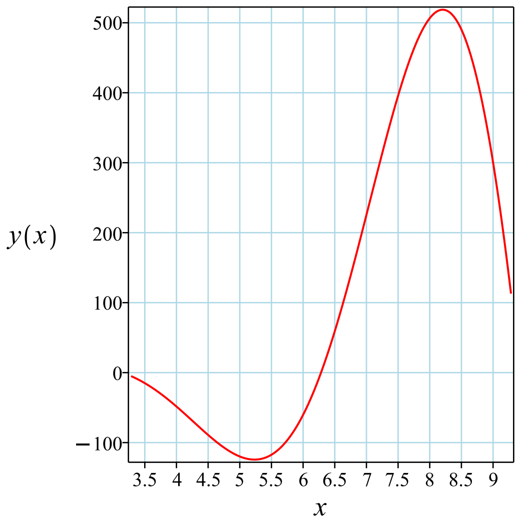
Substituting c_1 found above in the general solution gives

$$y = x^3 \sin(x)$$

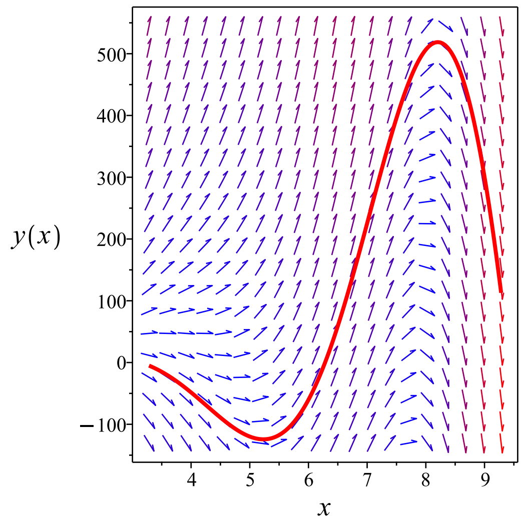
Summary

The solution(s) found are the following

$$y = x^3 \sin(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = x^3 \sin(x)$$

Verified OK.

4.21.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x)\,\mathrm{d}y&=\left(x^4\cos(x)+3y\right)\mathrm{d}x\\ \left(-x^4\cos(x)-3y\right)\mathrm{d}x+(x)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^4\cos(x)-3y\\ N(x,y)&=x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(-x^4\cos(x)-3y\right)\\ &=-3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(x)\\ &=1\end{aligned}$$

Since $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A&=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &=\frac{1}{x}((-3)-(1))\\ &=-\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int -\frac{4}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(-x^4\cos\left(x\right) - 3y\right) \\ &= \frac{-x^4\cos\left(x\right) - 3y}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^4}(x) \\ &= \frac{1}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x^4\cos\left(x\right) - 3y}{x^4}\right) + \left(\frac{1}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x^4 \cos(x) - 3y}{x^4} \mathrm{d}x$$
$$\phi = -\sin(x) + \frac{y}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{1}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{x^3}$. Therefore equation (4) becomes

$$\frac{1}{x^3} = \frac{1}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\sin(x) + \frac{y}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\sin(x) + \frac{y}{x^3}$$

The solution becomes

$$y = x^3(\sin(x) + c_1)$$

Initial conditions are used to solve for c_1. Substituting $x = 2\pi$ and $y = 0$ in the above solution gives an equation to solve for the constant of integration.

$$0 = 8\pi^3 c_1$$

$$c_1 = 0$$

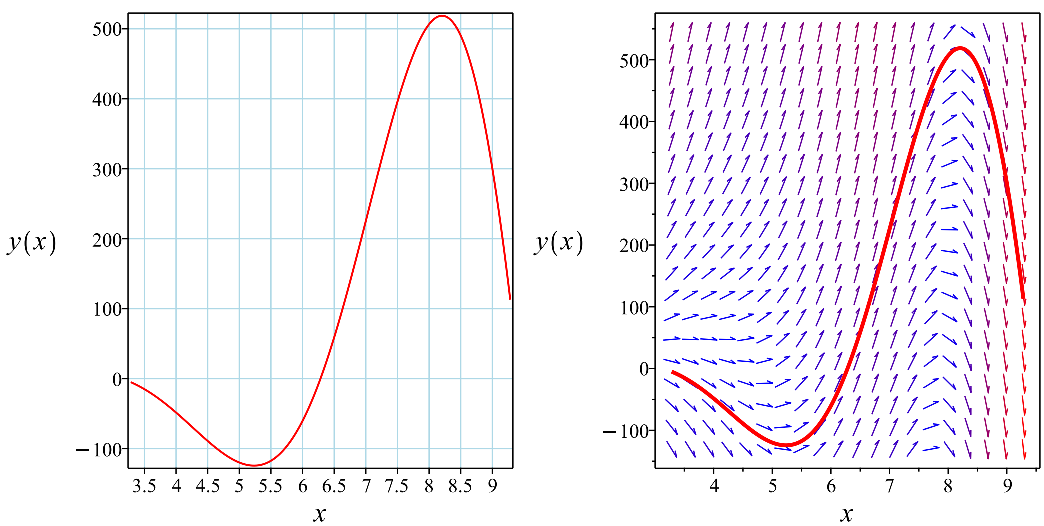
Substituting c_1 found above in the general solution gives

$$y = x^3 \sin(x)$$

Summary

The solution(s) found are the following

$$y = x^3 \sin(x) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = x^3 \sin(x)$$

Verified OK.

4.21.5 Maple step by step solution

Let's solve

$\left[-3y+y'x=x^{4} \cos \left(x\right), y \left(2 \pi \right)=0\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=\frac{3y}{x}+x^{3} \cos \left(x\right)$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'-\frac{3y}{x}=x^{3} \cos \left(x\right)$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y'-\frac{3y}{x}\right)=\mu(x) x^{3} \cos \left(x\right)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x) \left(y'-\frac{3y}{x}\right)=\mu'(x) y+\mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x)=-\frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x)=\frac{1}{x^{3}}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx=\int \mu(x) x^{3} \cos \left(x\right) dx+c_{1}$

- Evaluate the integral on the lhs

  $\mu(x) y=\int \mu(x) x^{3} \cos \left(x\right) dx+c_{1}$

- Solve for y

  $y=\frac{\int \mu(x) x^{3} \cos(x) dx+c_{1}}{\mu(x)}$

- Substitute $\mu(x)=\frac{1}{x^{3}}$

  $y=x^{3}\left(\int \cos \left(x\right) dx+c_{1}\right)$

- Evaluate the integrals on the rhs

  $y=x^{3}(\sin \left(x\right)+c_{1})$

- Use initial condition $y(2\pi)=0$

  $0=8\pi^{3}c_{1}$

- Solve for c_1

  $c_1 = 0$
- Substitute $c_1 = 0$ into general solution and simplify

  $y = x^3 \sin(x)$
- Solution to the IVP

  $y = x^3 \sin(x)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 10

```
dsolve([x*diff(y(x),x) = x^4*cos(x)+3*y(x),y(2*Pi) = 0],y(x), singsol=all)
```

$$y(x) = \sin(x) x^3$$

✓ Solution by Mathematica

Time used: 0.037 (sec). Leaf size: 11

```
DSolve[{x*y'[x] == x^4*Cos[x]+3*y[x],y[2*Pi]==0},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3 \sin(x)$$

4.22 problem 22

Internal problem ID [75]
Internal file name [`OUTPUT/75_Sunday_June_05_2022_01_34_19_AM_60896489/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{-2yx + y' = 3x^2 \mathrm{e}^{x^2}}$$

With initial conditions

$$[y(0) = 5]$$

4.22.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= 3x^2 \mathrm{e}^{x^2} \end{aligned}$$

Hence the ode is

$$-2yx + y' = 3x^2 e^{x^2}$$

The domain of $p(x) = -2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3x^2 e^{x^2}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.22.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -2x dx} \\ &= e^{-x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(3x^2 e^{x^2}\right) \\ \frac{d}{dx}\left(e^{-x^2} y\right) &= \left(e^{-x^2}\right)\left(3x^2 e^{x^2}\right) \\ d\left(e^{-x^2} y\right) &= \left(3x^2\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{-x^2} y &= \int 3x^2 \, dx \\ e^{-x^2} y &= x^3 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$y = e^{x^2} x^3 + c_1 e^{x^2}$$

which simplifies to

$$y = e^{x^2}\left(x^3 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = c_1$$

$$c_1 = 5$$

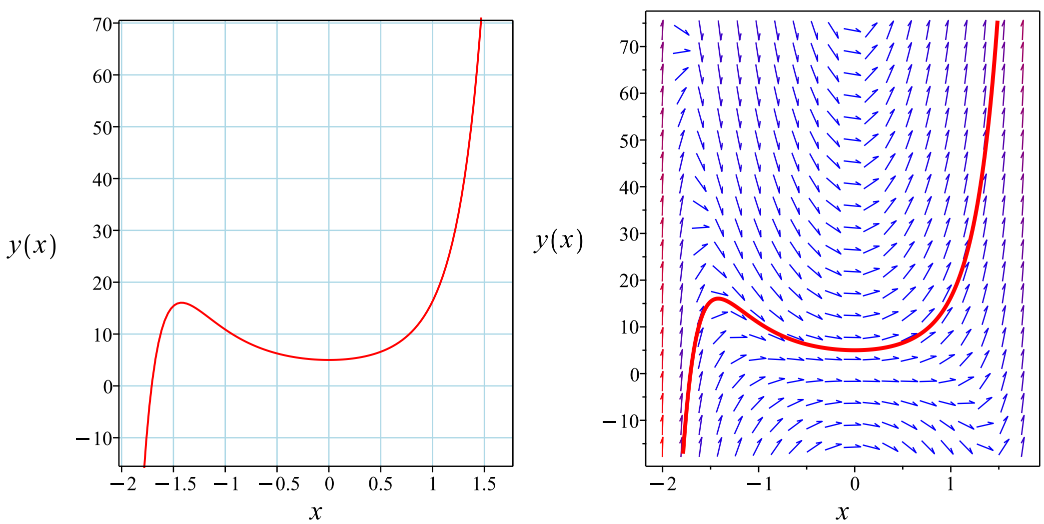
Substituting c_1 found above in the general solution gives

$$y = e^{x^2}(x^3 + 5)$$

Summary

The solution(s) found are the following

$$y = e^{x^2}(x^3 + 5) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{x^2}(x^3 + 5)$$

Verified OK.

4.22.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= 3x^2 e^{x^2} + 2yx \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 183: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{e^{x^2}} dy
\end{aligned}
$$

Which results in

$$
S = e^{-x^2} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = 3x^2 e^{x^2} + 2yx
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -2x\, e^{-x^2} y \\
S_y &= e^{-x^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$e^{-x^2} y = x^3 + c_1$$

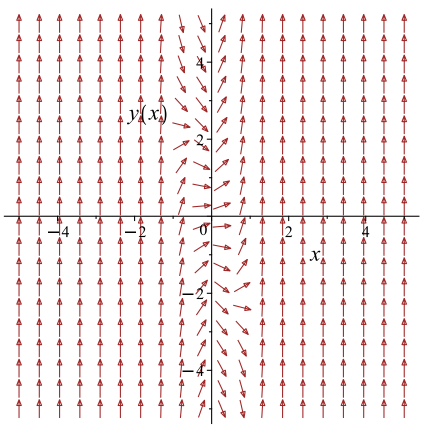
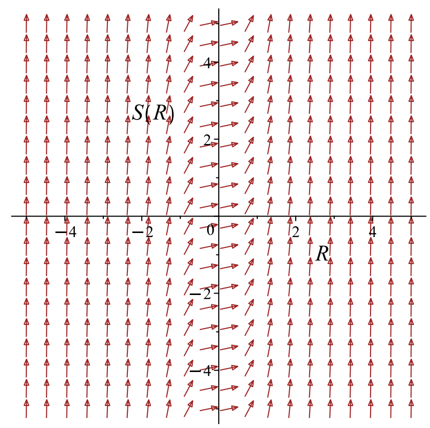
Which simplifies to

$$e^{-x^2} y = x^3 + c_1$$

Which gives

$$y = e^{x^2} \left(x^3 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x^2 \mathrm{e}^{x^2} + 2yx$ 	$R = x$ $S = \mathrm{e}^{-x^2} y$	$\frac{dS}{dR} = 3R^2$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = c_1$$

$$c_1 = 5$$

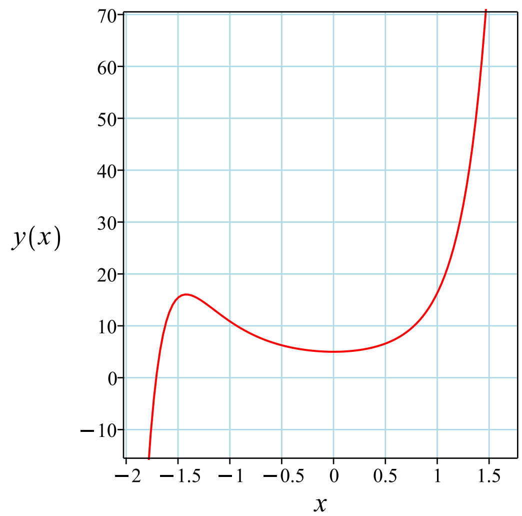
Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{x^2} x^3 + 5\,\mathrm{e}^{x^2}$$

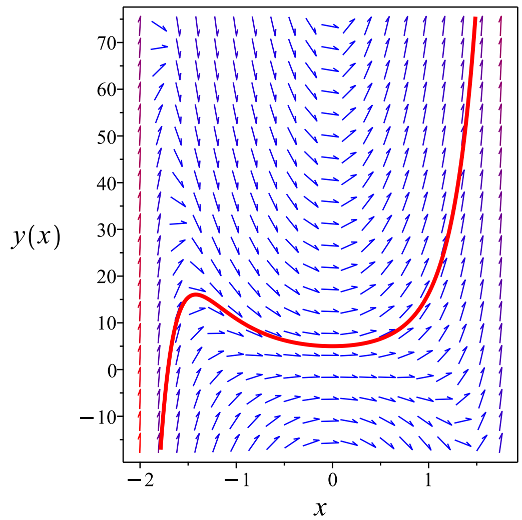
Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^2} x^3 + 5\,\mathrm{e}^{x^2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{x^2}x^3 + 5\,e^{x^2}$$

Verified OK.

4.22.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\mathrm{d}y = \left(3x^2\mathrm{e}^{x^2} + 2yx\right)\mathrm{d}x$$

$$\left(-3x^2\mathrm{e}^{x^2} - 2yx\right)\mathrm{d}x + \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -3x^2\mathrm{e}^{x^2} - 2yx$$
$$N(x,y) = 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x^2\mathrm{e}^{x^2} - 2yx\right)\\ &= -2x\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(1)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= 1((-2x) - (0))\\ &= -2x\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -2x \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-x^2} \\
&= \mathrm{e}^{-x^2}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \mathrm{e}^{-x^2}\left(-3x^2\mathrm{e}^{x^2} - 2yx\right) \\
&= -2x\,\mathrm{e}^{-x^2}y - 3x^2
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \mathrm{e}^{-x^2}(1) \\
&= \mathrm{e}^{-x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-2x\,\mathrm{e}^{-x^2}y - 3x^2\right) + \left(\mathrm{e}^{-x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int \overline{M} \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -2x\,\mathrm{e}^{-x^2}y - 3x^2 \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -x^3 + \mathrm{e}^{-x^2}y + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{-x^2}$. Therefore equation (4) becomes

$$\mathrm{e}^{-x^2} = \mathrm{e}^{-x^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 + \mathrm{e}^{-x^2} y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 + \mathrm{e}^{-x^2} y$$

The solution becomes

$$y = \mathrm{e}^{x^2} \left(x^3 + c_1\right)$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 5$ in the above solution gives an equation to solve for the constant of integration.

$$5 = c_1$$

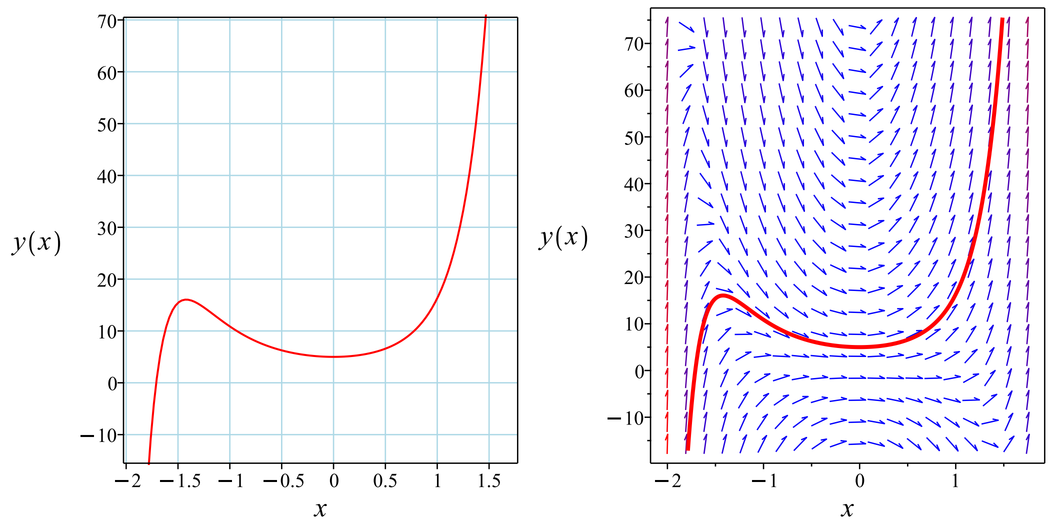
$$c_1 = 5$$

Substituting c_1 found above in the general solution gives

$$y = \mathrm{e}^{x^2} x^3 + 5\,\mathrm{e}^{x^2}$$

Summary
The solution(s) found are the following

$$y = e^{x^2} x^3 + 5\, e^{x^2} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{x^2} x^3 + 5\, e^{x^2}$$

Verified OK.

4.22.5 Maple step by step solution

Let's solve

$\left[-2yx + y' = 3x^2 e^{x^2}, y(0) = 5\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3x^2 e^{x^2} + 2yx$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $-2yx + y' = 3x^2 e^{x^2}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(-2yx+y'\right)=3\mu(x)\,x^{2}\mathrm{e}^{x^{2}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}\left(\mu(x)\,y\right)$

$$\mu(x)\left(-2yx+y'\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=-2\mu(x)\,x$$

- Solve to find the integrating factor

$$\mu(x)=\mathrm{e}^{-x^{2}}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}\left(\mu(x)\,y\right)\right)dx=\int 3\mu(x)\,x^{2}\mathrm{e}^{x^{2}}dx+c_{1}$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int 3\mu(x)\,x^{2}\mathrm{e}^{x^{2}}dx+c_{1}$$

- Solve for y

$$y=\frac{\int 3\mu(x)x^{2}\mathrm{e}^{x^{2}}dx+c_{1}}{\mu(x)}$$

- Substitute $\mu(x)=\mathrm{e}^{-x^{2}}$

$$y=\frac{\int 3x^{2}\mathrm{e}^{x^{2}}\mathrm{e}^{-x^{2}}dx+c_{1}}{\mathrm{e}^{-x^{2}}}$$

- Evaluate the integrals on the rhs

$$y=\frac{x^{3}+c_{1}}{\mathrm{e}^{-x^{2}}}$$

- Simplify

$$y=\mathrm{e}^{x^{2}}\left(x^{3}+c_{1}\right)$$

- Use initial condition $y(0)=5$

$$5=c_{1}$$

- Solve for c_{1}

$$c_{1}=5$$

- Substitute $c_{1}=5$ into general solution and simplify

$$y=\mathrm{e}^{x^{2}}\left(x^{3}+5\right)$$

- Solution to the IVP

$$y=\mathrm{e}^{x^{2}}\left(x^{3}+5\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(y(x),x) = 3*exp(x^2)*x^2+2*x*y(x),y(0) = 5],y(x), singsol=all)
```

$$y(x) = \left(x^3 + 5\right) e^{x^2}$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 16

```
DSolve[{y'[x] == 3*Exp[x^2]*x^2+2*x*y[x],y[0]==5},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{x^2}\left(x^3 + 5\right)$$

4.23 problem 23

Internal problem ID [76]
Internal file name [`OUTPUT/76_Sunday_June_05_2022_01_34_20_AM_39126778/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{(-3+2x)\,y+y'x=4x^4}$$

4.23.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=-\frac{-2x+3}{x}$$
$$q(x)=4x^3$$

Hence the ode is

$$y'-\frac{(-2x+3)\,y}{x}=4x^3$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-2x+3}{x}dx}$$
$$= e^{2x-3\ln(x)}$$

Which simplifies to

$$\mu = \frac{e^{2x}}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(4x^3\right)$$
$$\frac{d}{dx}\left(\frac{e^{2x}y}{x^3}\right) = \left(\frac{e^{2x}}{x^3}\right)\left(4x^3\right)$$
$$d\left(\frac{e^{2x}y}{x^3}\right) = \left(4\,e^{2x}\right)\,dx$$

Integrating gives

$$\frac{e^{2x}y}{x^3} = \int 4\,e^{2x}\,dx$$
$$\frac{e^{2x}y}{x^3} = 2\,e^{2x} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{e^{2x}}{x^3}$ results in

$$y = 2x^3e^{-2x}e^{2x} + c_1x^3e^{-2x}$$

which simplifies to

$$y = x^3\left(2 + c_1e^{-2x}\right)$$

Summary

The solution(s) found are the following

$$y = x^3\left(2 + c_1e^{-2x}\right) \tag{1}$$

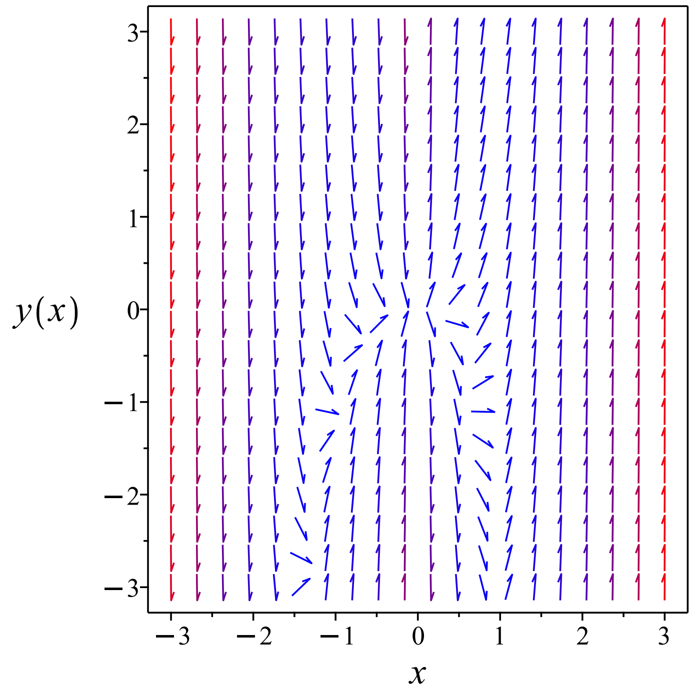


Figure 228: Slope field plot

Verification of solutions

$$y = x^3\left(2 + c_1 \mathrm{e}^{-2x}\right)$$

Verified OK.

4.23.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{-4x^4 + 2yx - 3y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 186: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{-2x+3\ln(x)}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-2x+3\ln(x)}} dy\end{aligned}$$

Which results in

$$S = \frac{\mathrm{e}^{2x} y}{x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-4x^4 + 2yx - 3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\mathrm{e}^{2x} y(-3+2x)}{x^4} \\ S_y &= \frac{\mathrm{e}^{2x}}{x^3}\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 4\,\mathrm{e}^{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 4\,\mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\,e^{2R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{2x} y}{x^3} = 2\,e^{2x} + c_1$$

Which simplifies to

$$\frac{e^{2x} y}{x^3} = 2\,e^{2x} + c_1$$

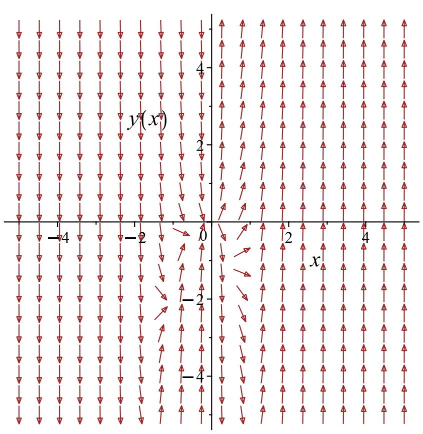
Which gives

$$y = x^3 \left(2\,e^{2x} + c_1\right) e^{-2x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{-4x^4+2yx-3y}{x}$



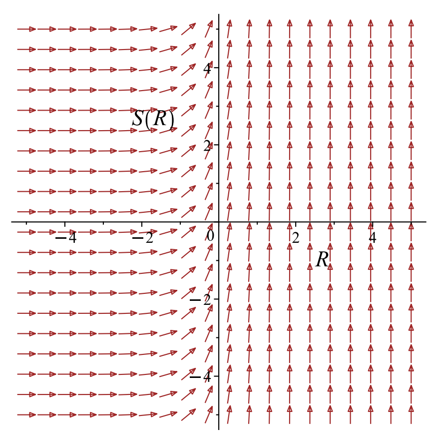
</td><td>

$R = x$

$S = \frac{e^{2x} y}{x^3}$

</td><td>

$\frac{dS}{dR} = 4\,e^{2R}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = x^3 \left(2\,e^{2x} + c_1\right) e^{-2x} \tag{1}$$

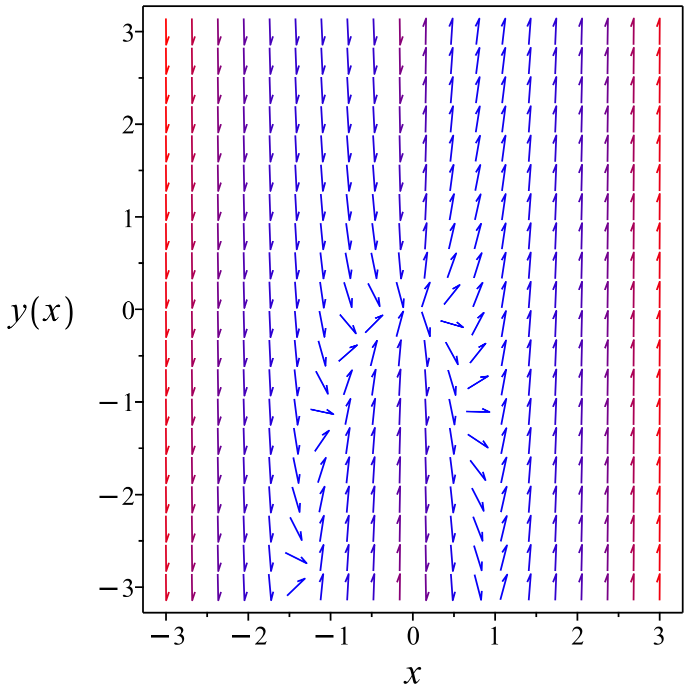


Figure 229: Slope field plot

Verification of solutions

$$y = x^3 \left(2\,e^{2x} + c_1\right) e^{-2x}$$

Verified OK.

4.23.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x) \, \mathrm{d}y = \left(-(-3 + 2x) \, y + 4x^4\right) \mathrm{d}x$$
$$\left((-3 + 2x) \, y - 4x^4\right) \mathrm{d}x + (x) \, \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = (-3 + 2x) \, y - 4x^4$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left((-3 + 2x) \, y - 4x^4\right) \\ &= -3 + 2x \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x} (x) \\ &= 1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x}((-3+2x) - (1)) \\ &= \frac{-4+2x}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \,\mathrm{d}x} \\ &= e^{\int \frac{-4+2x}{x} \,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2x - 4\ln(x)} \\ &= \frac{\mathrm{e}^{2x}}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{\mathrm{e}^{2x}}{x^4}((-3+2x)\,y - 4x^4) \\ &= \frac{\mathrm{e}^{2x}(-4x^4 + 2yx - 3y)}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{\mathrm{e}^{2x}}{x^4}(x) \\ &= \frac{\mathrm{e}^{2x}}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{\mathrm{e}^{2x}(-4x^4 + 2yx - 3y)}{x^4}\right) + \left(\frac{\mathrm{e}^{2x}}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{\mathrm{e}^{2x}(-4x^4 + 2yx - 3y)}{x^4} \mathrm{d}x$$

$$\phi = \frac{(-2x^3 + y)\,\mathrm{e}^{2x}}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{2x}}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{2x}}{x^3}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{2x}}{x^3} = \frac{\mathrm{e}^{2x}}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{(-2x^3 + y)\,\mathrm{e}^{2x}}{x^3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{(-2x^3 + y)\,\mathrm{e}^{2x}}{x^3}$$

The solution becomes

$$y = x^3 \left(2\,e^{2x} + c_1\right) e^{-2x}$$

Summary

The solution(s) found are the following

$$y = x^3 \left(2\,e^{2x} + c_1\right) e^{-2x} \tag{1}$$

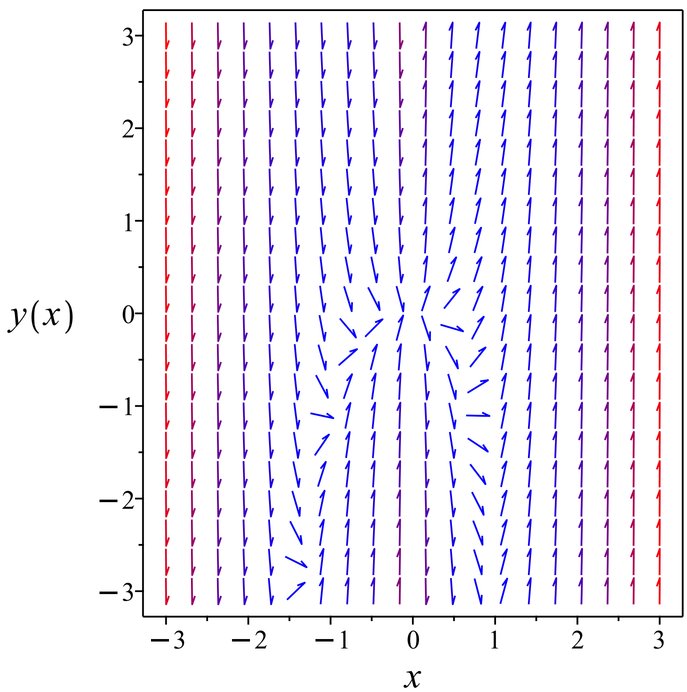


Figure 230: Slope field plot

Verification of solutions

$$y = x^3 \left(2\,e^{2x} + c_1\right) e^{-2x}$$

Verified OK.

4.23.4 Maple step by step solution

Let's solve

$(-3+2x)\, y + y' x = 4x^4$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{(-3+2x)y}{x} + 4x^3$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{(-3+2x)y}{x} = 4x^3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{(-3+2x)y}{x}\right) = 4\mu(x)\, x^3$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{(-3+2x)y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)(-3+2x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{e^{2x}}{x^3}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 4\mu(x)\, x^3 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 4\mu(x)\, x^3 dx + c_1$

- Solve for y

  $y = \frac{\int 4\mu(x) x^3 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{e^{2x}}{x^3}$

  $y = \frac{x^3 \left(\int 4\, e^{2x} dx + c_1\right)}{e^{2x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^3 \left(2\, e^{2x} + c_1\right)}{e^{2x}}$

- Simplify

$$y = x^3(2 + c_1 e^{-2x})$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((-3+2*x)*y(x)+x*diff(y(x),x) = 4*x^4,y(x), singsol=all)
```

$$y(x) = x^3(2 + e^{-2x}c_1)$$

✓ Solution by Mathematica

Time used: 0.032 (sec). Leaf size: 19

```
DSolve[(-3+2*x)*y[x]+x*y'[x] == 4*x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow x^3(2 + c_1 e^{-2x})$$

4.24 problem 24

Internal problem ID [77]
Internal file name [`OUTPUT/77_Sunday_June_05_2022_01_34_20_AM_50448999/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3yx + (x^2 + 4)\, y' = x}$$

With initial conditions

$$[y(0) = 1]$$

4.24.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3x}{x^2 + 4}$$
$$q(x) = \frac{x}{x^2 + 4}$$

Hence the ode is

$$y' + \frac{3xy}{x^2+4} = \frac{x}{x^2+4}$$

The domain of $p(x) = \frac{3x}{x^2+4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{x}{x^2+4}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.24.2 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(-3y+1)}{x^2+4} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+4}$ and $g(y) = -3y+1$. Integrating both sides gives

$$\begin{aligned} \frac{1}{-3y+1}\,dy &= \frac{x}{x^2+4}\,dx \\ \int \frac{1}{-3y+1}\,dy &= \int \frac{x}{x^2+4}\,dx \\ -\frac{\ln(-3y+1)}{3} &= \frac{\ln(x^2+4)}{2} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = \mathrm{e}^{\frac{\ln(x^2+4)}{2}+c_1}$$

Which simplifies to

$$\frac{1}{(-3y+1)^{\frac{1}{3}}} = c_2\sqrt{x^2+4}$$

Which can be simplified to become

$$y = \frac{\left(c_2^3 e^{3c_1} (x^2+4)^{\frac{3}{2}} - 1\right) e^{-3c_1}}{3c_2^3 (x^2+4)^{\frac{3}{2}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{8\, e^{-3c_1} e^{3c_1} c_2^3 - e^{-3c_1}}{24c_2^3}$$

$$c_1 = -\frac{\ln(-16c_2^3)}{3}$$

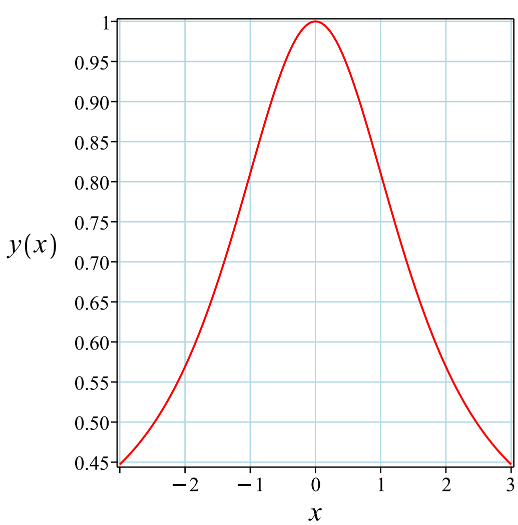
Substituting c_1 found above in the general solution gives

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3 (x^2+4)^{\frac{3}{2}}}$$

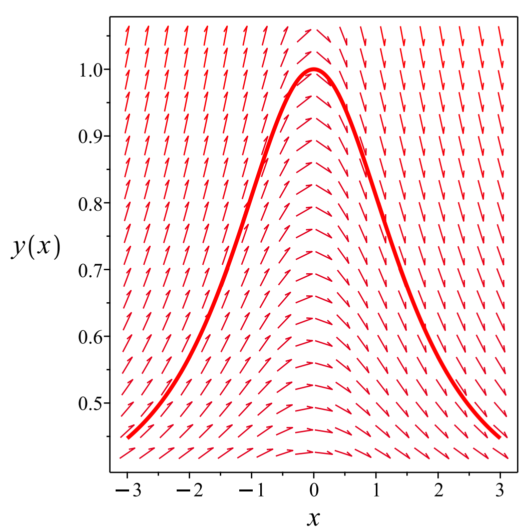
Summary

The solution(s) found are the following

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3 (x^2+4)^{\frac{3}{2}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3(x^2+4)^{\frac{3}{2}}}$$

Verified OK.

4.24.3 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{3x}{x^2+4} dx} \\
&= (x^2+4)^{\frac{3}{2}}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{x}{x^2+4}\right) \\
\frac{d}{dx}\left((x^2+4)^{\frac{3}{2}} y\right) &= \left((x^2+4)^{\frac{3}{2}}\right)\left(\frac{x}{x^2+4}\right) \\
d\left((x^2+4)^{\frac{3}{2}} y\right) &= \left(\sqrt{x^2+4}\, x\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
(x^2+4)^{\frac{3}{2}} y &= \int \sqrt{x^2+4}\, x \, dx \\
(x^2+4)^{\frac{3}{2}} y &= \frac{(x^2+4)^{\frac{3}{2}}}{3} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = (x^2+4)^{\frac{3}{2}}$ results in

$$y = \frac{1}{3} + \frac{c_1}{(x^2+4)^{\frac{3}{2}}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{3} + \frac{c_1}{8}$$

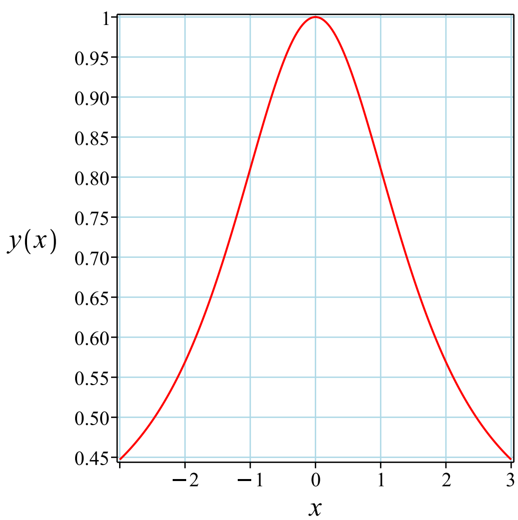
$$c_1 = \frac{16}{3}$$

Substituting c_1 found above in the general solution gives

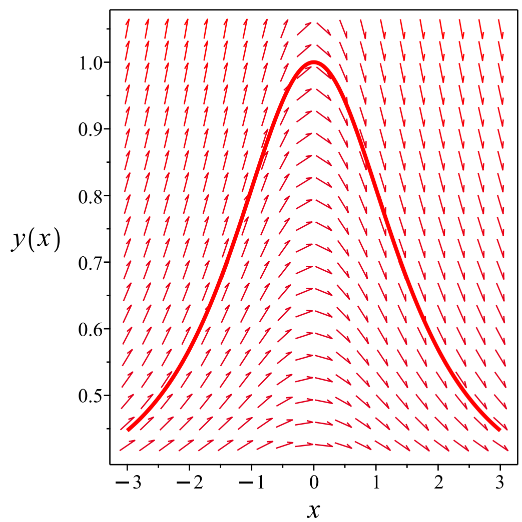
$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}}$$

Verified OK.

4.24.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{x(3y-1)}{x^2+4} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 189: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{1}{\left(x^2+4\right)^{\frac{3}{2}}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{1}{(x^2+4)^{\frac{3}{2}}}} dy
\end{aligned}
$$

Which results in

$$
S = \left(x^2+4\right)^{\frac{3}{2}} y
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{x(3y-1)}{x^2+4}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= 3\sqrt{x^2+4}\,yx \\
S_y &= \left(x^2+4\right)^{\frac{3}{2}}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \sqrt{x^2+4}\,x \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \sqrt{R^2+4}\,R
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{(R^2+4)^{\frac{3}{2}}}{3} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\left(x^2+4\right)^{\frac{3}{2}} y = \frac{(x^2+4)^{\frac{3}{2}}}{3} + c_1
$$

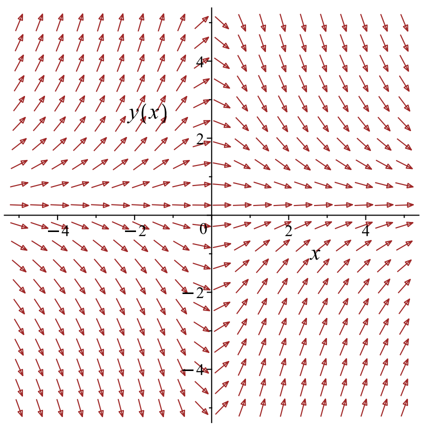
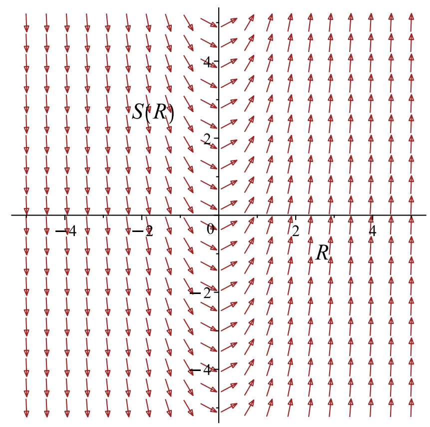
Which simplifies to

$$
\left(x^2+4\right)^{\frac{3}{2}} \left(y - \frac{1}{3}\right) - c_1 = 0
$$

Which gives

$$
y = \frac{(x^2+4)^{\frac{3}{2}} + 3c_1}{3\,(x^2+4)^{\frac{3}{2}}}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x(3y-1)}{x^2+4}$ 	$R = x$ $S = (x^2+4)^{\frac{3}{2}} y$	$\frac{dS}{dR} = \sqrt{R^2+4}\, R$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{1}{3} + \frac{c_1}{8}$$

$$c_1 = \frac{16}{3}$$

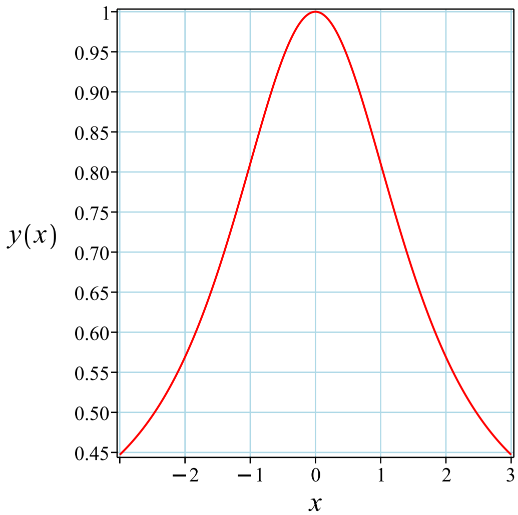
Substituting c_1 found above in the general solution gives

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}}$$

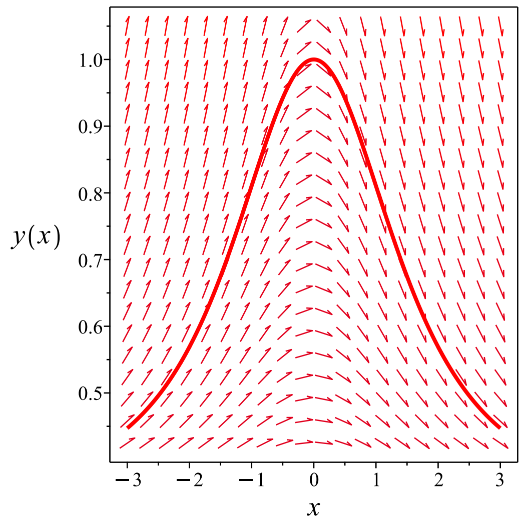
Summary

The solution(s) found are the following

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}}$$

Verified OK.

4.24.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{-3y+1}\right)\mathrm{d}y &= \left(\frac{x}{x^2+4}\right)\mathrm{d}x \\ \left(-\frac{x}{x^2+4}\right)\mathrm{d}x + \left(\frac{1}{-3y+1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{x}{x^2+4} \\ N(x, y) &= \frac{1}{-3y+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+4}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{-3y+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x}{x^2+4} \, \mathrm{d}x \\
\phi &= -\frac{\ln(x^2+4)}{2} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{-3y+1}$. Therefore equation (4) becomes

$$
\frac{1}{-3y+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{3y-1}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{3y-1}\right) \mathrm{d}y$$
$$f(y) = -\frac{\ln(3y-1)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+4)}{2} - \frac{\ln(3y-1)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+4)}{2} - \frac{\ln(3y-1)}{3}$$

The solution becomes

$$y = \frac{\mathrm{e}^{-\frac{3\ln\left(x^2+4\right)}{2} - 3c_1}}{3} + \frac{1}{3}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = \frac{\mathrm{e}^{-3c_1}}{24} + \frac{1}{3}$$

$$c_1 = -\frac{4\ln(2)}{3}$$

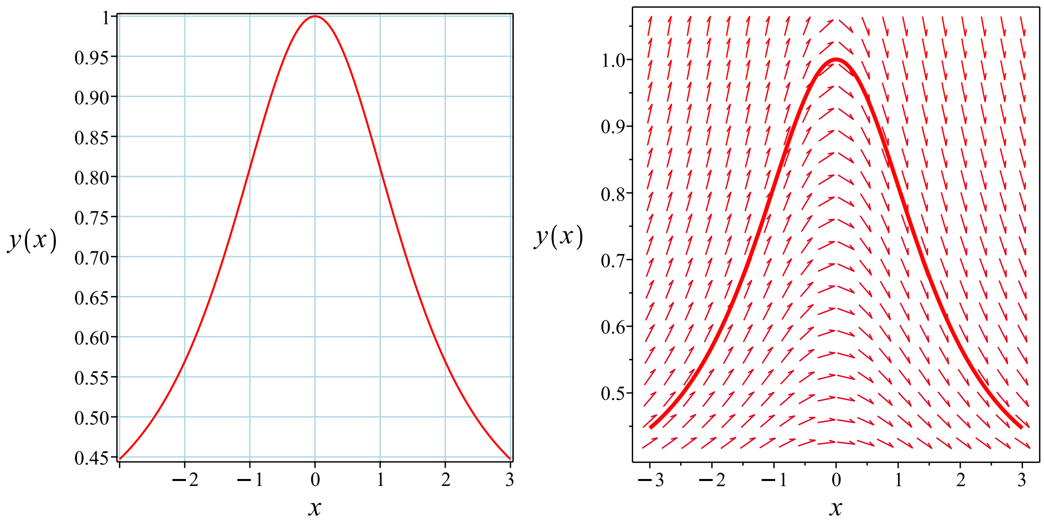
Substituting c_1 found above in the general solution gives

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$y = \frac{(x^2+4)^{\frac{3}{2}} + 16}{3\,(x^2+4)^{\frac{3}{2}}} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{\left(x^2+4\right)^{\frac{3}{2}}+16}{3\left(x^2+4\right)^{\frac{3}{2}}}$$

Verified OK.

4.24.6 Maple step by step solution

Let's solve

$\left[3yx+\left(x^2+4\right)y'=x, y(0)=1\right]$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{3y-1}=-\frac{x}{x^2+4}$

- Integrate both sides with respect to x

  $\int \frac{y'}{3y-1}dx=\int -\frac{x}{x^2+4}dx+c_1$

- Evaluate integral

  $\frac{\ln(3y-1)}{3}=-\frac{\ln(x^2+4)}{2}+c_1$

- Solve for y

$$y = \frac{e^{-\frac{3\ln(x^2+4)}{2}+3c_1}}{3} + \frac{1}{3}$$

- Use initial condition $y(0) = 1$

$$1 = \frac{e^{-3\ln(2)+3c_1}}{3} + \frac{1}{3}$$

- Solve for c_1

$$c_1 = \frac{4\ln(2)}{3}$$

- Substitute $c_1 = \frac{4\ln(2)}{3}$ into general solution and simplify

$$y = \frac{1}{3} + \frac{16}{3(x^2+4)^{\frac{3}{2}}}$$

- Solution to the IVP

$$y = \frac{1}{3} + \frac{16}{3(x^2+4)^{\frac{3}{2}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 15

```
dsolve([3*x*y(x)+(x^2+4)*diff(y(x),x) = x,y(0) = 1],y(x), singsol=all)
```

$$y(x) = \frac{1}{3} + \frac{16}{3(x^2+4)^{\frac{3}{2}}}$$

✓ Solution by Mathematica
Time used: 0.036 (sec). Leaf size: 22

```
DSolve[{3*x*y[x]+(x^2+4)*y'[x] == x,y[0]==1},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{16}{3(x^2+4)^{3/2}} + \frac{1}{3}$$

4.25 problem 25

Internal problem ID [78]
Internal file name [`OUTPUT/78_Sunday_June_05_2022_01_34_21_AM_53165874/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.5. Linear first order equations. Page 56
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{3yx^3 + (x^2+1)\,y' = 6x\,\mathrm{e}^{-\frac{3x^2}{2}}}$$

With initial conditions

$$[y(0) = 1]$$

4.25.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3x^3}{x^2+1}$$
$$q(x) = \frac{6x\,\mathrm{e}^{-\frac{3x^2}{2}}}{x^2+1}$$

Hence the ode is

$$y' + \frac{3x^3 y}{x^2+1} = \frac{6x\,\mathrm{e}^{-\frac{3x^2}{2}}}{x^2+1}$$

The domain of $p(x) = \frac{3x^3}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{6x\,\mathrm{e}^{-\frac{3x^2}{2}}}{x^2+1}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

4.25.2 Solving as linear ode

Entering Linear first order ODE solver. The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{3x^3}{x^2+1}dx} \\ &= \mathrm{e}^{\frac{3x^2}{2} - \frac{3\ln\left(x^2+1\right)}{2}}\end{aligned}$$

Which simplifies to

$$\mu = \frac{\mathrm{e}^{\frac{3x^2}{2}}}{\left(x^2+1\right)^{\frac{3}{2}}}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{6x\,\mathrm{e}^{-\frac{3x^2}{2}}}{x^2+1}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^{\frac{3x^2}{2}}y}{\left(x^2+1\right)^{\frac{3}{2}}}\right) &= \left(\frac{\mathrm{e}^{\frac{3x^2}{2}}}{\left(x^2+1\right)^{\frac{3}{2}}}\right)\left(\frac{6x\,\mathrm{e}^{-\frac{3x^2}{2}}}{x^2+1}\right) \\ \mathrm{d}\left(\frac{\mathrm{e}^{\frac{3x^2}{2}}y}{\left(x^2+1\right)^{\frac{3}{2}}}\right) &= \left(\frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\frac{e^{\frac{3x^2}{2}} y}{\left(x^2+1\right)^{\frac{3}{2}}} = \int \frac{6x}{\left(x^2+1\right)^{\frac{5}{2}}}\, dx$$

$$\frac{e^{\frac{3x^2}{2}} y}{\left(x^2+1\right)^{\frac{3}{2}}} = -\frac{2}{\left(x^2+1\right)^{\frac{3}{2}}} + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{e^{\frac{3x^2}{2}}}{\left(x^2+1\right)^{\frac{3}{2}}}$ results in

$$y = -2\, e^{-\frac{3x^2}{2}} + c_1 \left(x^2+1\right)^{\frac{3}{2}} e^{-\frac{3x^2}{2}}$$

which simplifies to

$$y = \left(\left(x^2+1\right)^{\frac{3}{2}} c_1 - 2\right) e^{-\frac{3x^2}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 2$$

$$c_1 = 3$$

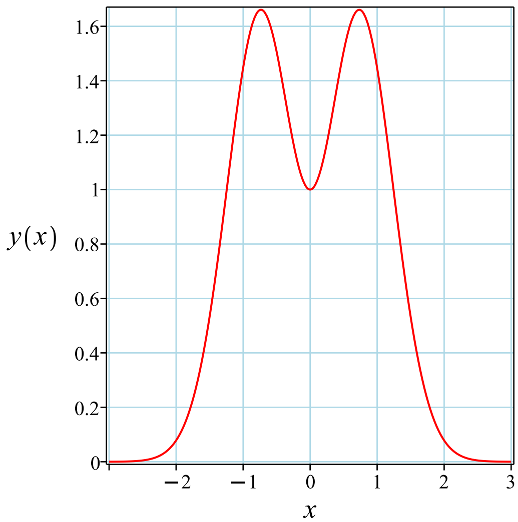
Substituting c_1 found above in the general solution gives

$$y = 3\left(x^2+1\right)^{\frac{3}{2}} e^{-\frac{3x^2}{2}} - 2\, e^{-\frac{3x^2}{2}}$$

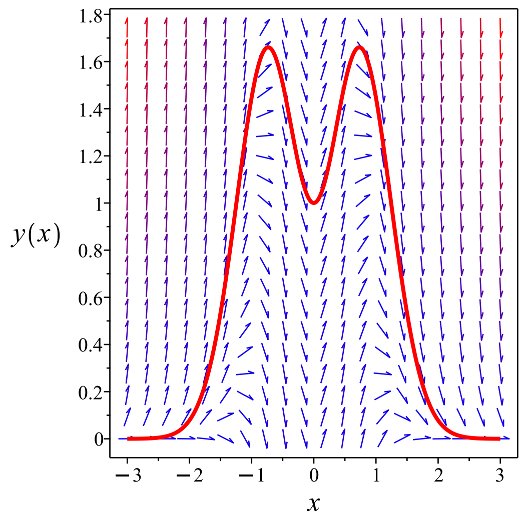
Summary

The solution(s) found are the following

$$y = 3\left(x^2+1\right)^{\frac{3}{2}} e^{-\frac{3x^2}{2}} - 2\, e^{-\frac{3x^2}{2}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\left(x^2+1\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{3x^2}{2}} - 2\,\mathrm{e}^{-\frac{3x^2}{2}}$$

Verified OK.

4.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{3x\left(y\,\mathrm{e}^{\frac{3x^2}{2}}x^2-2\right)\mathrm{e}^{-\frac{3x^2}{2}}}{x^2+1} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 192: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{-\frac{3x^2}{2}+\frac{3\ln\left(x^2+1\right)}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{-\frac{3x^2}{2} + \frac{3\ln(x^2+1)}{2}}} dy \end{aligned}$$

Which results in

$$S = e^{\frac{3x^2}{2} - \frac{3\ln(x^2+1)}{2}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x\left(y\, e^{\frac{3x^2}{2}} x^2 - 2\right) e^{-\frac{3x^2}{2}}}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{3y\, e^{\frac{3x^2}{2}} x^3}{(x^2+1)^{\frac{5}{2}}} \\ S_y &= \frac{e^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{6x}{(x^2+1)^{\frac{5}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{6R}{(R^2+1)^{\frac{5}{2}}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{2}{(R^2+1)^{\frac{3}{2}}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{e^{\frac{3x^2}{2}} y}{(x^2+1)^{\frac{3}{2}}} = -\frac{2}{(x^2+1)^{\frac{3}{2}}} + c_1$$

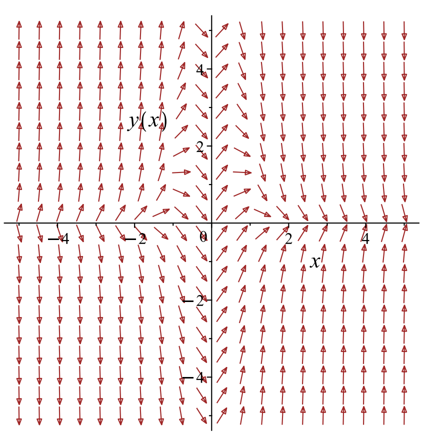
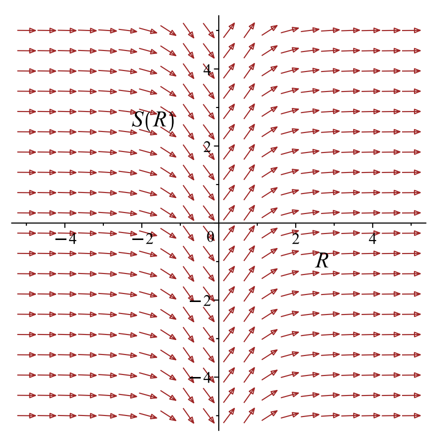
Which simplifies to

$$\frac{e^{\frac{3x^2}{2}} y}{(x^2+1)^{\frac{3}{2}}} = -\frac{2}{(x^2+1)^{\frac{3}{2}}} + c_1$$

Which gives

$$y = \left((x^2+1)^{\frac{3}{2}} c_1 - 2\right) e^{-\frac{3x^2}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x\left(y\,e^{\frac{3x^2}{2}} x^2-2\right) e^{-\frac{3x^2}{2}}}{x^2+1}$ 	$R = x$ $S = \frac{e^{\frac{3x^2}{2}} y}{(x^2+1)^{\frac{3}{2}}}$	$\frac{dS}{dR} = \frac{6R}{(R^2+1)^{\frac{5}{2}}}$ 

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1 - 2$$

$$c_1 = 3$$

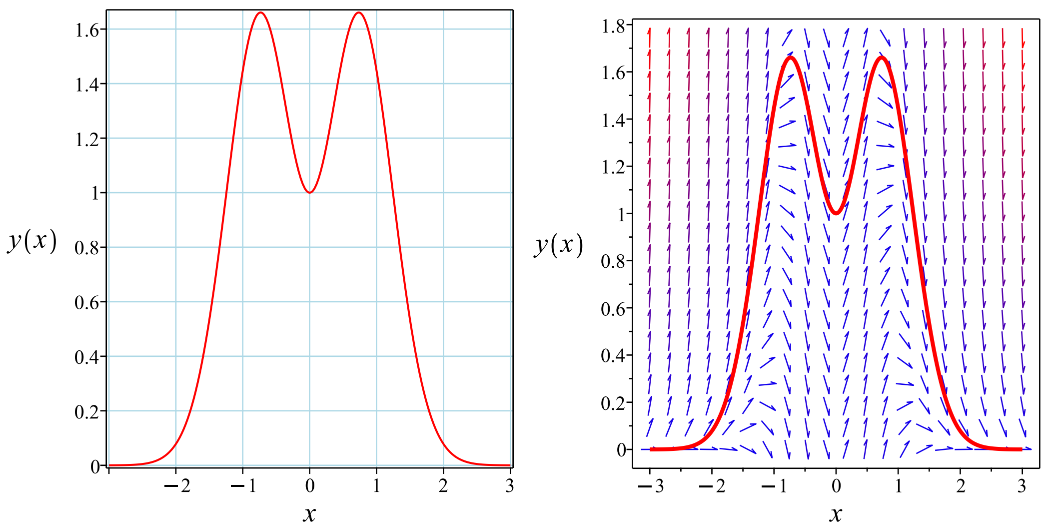
Substituting c_1 found above in the general solution gives

$$y = 3(x^2 + 1)^{\frac{3}{2}} e^{-\frac{3x^2}{2}} - 2\,e^{-\frac{3x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = 3(x^2 + 1)^{\frac{3}{2}} e^{-\frac{3x^2}{2}} - 2\,e^{-\frac{3x^2}{2}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = 3(x^2 + 1)^{\frac{3}{2}} e^{-\frac{3x^2}{2}} - 2\,e^{-\frac{3x^2}{2}}$$

Verified OK.

4.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(x^{2}+1\right)\mathrm{d}y&=\left(-3y\,x^{3}+6x\,\mathrm{e}^{-\frac{3x^{2}}{2}}\right)\mathrm{d}x\\ \left(3y\,x^{3}-6x\,\mathrm{e}^{-\frac{3x^{2}}{2}}\right)\mathrm{d}x+\left(x^{2}+1\right)\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3y\,x^{3}-6x\,\mathrm{e}^{-\frac{3x^{2}}{2}}\\ N(x,y)&=x^{2}+1\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y\,x^{3}-6x\,\mathrm{e}^{-\frac{3x^{2}}{2}}\right)\\ &= 3x^{3}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^{2}+1\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x^{2}+1}\left(\left(3x^{3}\right)-\left(2x\right)\right)\\ &= \frac{3x^{3}-2x}{x^{2}+1}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{3x^{3}-2x}{x^{2}+1}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{\frac{3x^{2}}{2}-\frac{5\ln\left(x^{2}+1\right)}{2}}\\ &= \frac{\mathrm{e}^{\frac{3x^{2}}{2}}}{\left(x^{2}+1\right)^{\frac{5}{2}}}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{e^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{5}{2}}}\left(3y\,x^3 - 6x\,e^{-\frac{3x^2}{2}}\right) \\
&= \frac{3x\left(y\,e^{\frac{3x^2}{2}}x^2 - 2\right)}{(x^2+1)^{\frac{5}{2}}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{e^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{5}{2}}}(x^2+1) \\
&= \frac{e^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\overline{M} + \overline{N}\frac{dy}{dx} = 0
$$

$$
\left(\frac{3x\left(y\,e^{\frac{3x^2}{2}}x^2 - 2\right)}{(x^2+1)^{\frac{5}{2}}}\right) + \left(\frac{e^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}}\right)\frac{dy}{dx} = 0
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\int \frac{\partial \phi}{\partial x}\, dx = \int \overline{M}\, dx
$$

$$
\int \frac{\partial \phi}{\partial x}\, dx = \int \frac{3x\left(y\,e^{\frac{3x^2}{2}}x^2 - 2\right)}{(x^2+1)^{\frac{5}{2}}}\, dx
$$

$$
\phi = \int_0^x \frac{3\_a\left(y\,e^{\frac{3\_a^2}{2}}\_a^2 - 2\right)}{(\_a^2+1)^{\frac{5}{2}}} d\_a + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \int_0^x \frac{3\_a^3 \mathrm{e}^{\frac{3\_a^2}{2}}}{(\_a^2+1)^{\frac{5}{2}}} d\_a + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}} = 3\left(\int_0^x \frac{\_a^3 \mathrm{e}^{\frac{3\_a^2}{2}}}{(\_a^2+1)^{\frac{5}{2}}} d\_a\right) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$\begin{aligned} f'(y) &= -\frac{3\left(\int_0^x \frac{\_a^3 \mathrm{e}^{\frac{3\_a^2}{2}}}{(\_a^2+1)^{\frac{5}{2}}} d\_a\right)(x^2+1)^{\frac{3}{2}} - \mathrm{e}^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}} \\ &= -\frac{3\left(\left(\int_0^x \frac{\_a^3 \mathrm{e}^{\frac{3\_a^2}{2}}}{(\_a^2+1)^{\frac{5}{2}}} d\_a\right)(x^2+1)^{\frac{3}{2}} - \frac{\mathrm{e}^{\frac{3x^2}{2}}}{3}\right)}{(x^2+1)^{\frac{3}{2}}} \end{aligned}$$

Integrating the above w.r.t y results in

$$\begin{aligned} \int f'(y)\,\mathrm{d}y &= \int \left(-\frac{3\left(\left(\int_0^x \frac{\_a^3 \mathrm{e}^{\frac{3\_a^2}{2}}}{(\_a^2+1)^{\frac{5}{2}}} d\_a\right)(x^2+1)^{\frac{3}{2}} - \frac{\mathrm{e}^{\frac{3x^2}{2}}}{3}\right)}{(x^2+1)^{\frac{3}{2}}}\right) \mathrm{d}y \\ f(y) &= -\frac{3\left(\left(\int_0^x \frac{\_a^3 \mathrm{e}^{\frac{3\_a^2}{2}}}{(\_a^2+1)^{\frac{5}{2}}} d\_a\right)(x^2+1)^{\frac{3}{2}} - \frac{\mathrm{e}^{\frac{3x^2}{2}}}{3}\right) y}{(x^2+1)^{\frac{3}{2}}} + c_1 \end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \int_0^x \frac{3\_a\left(y\,\mathrm{e}^{\frac{3\_a^2}{2}}\_a^2 - 2\right)}{(\_a^2+1)^{\frac{5}{2}}} d\_a - \frac{3\left(\left(\int_0^x \frac{\_a^3 \mathrm{e}^{\frac{3\_a^2}{2}}}{(\_a^2+1)^{\frac{5}{2}}} d\_a\right)(x^2+1)^{\frac{3}{2}} - \frac{\mathrm{e}^{\frac{3x^2}{2}}}{3}\right) y}{(x^2+1)^{\frac{3}{2}}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \int_0^x \frac{3\_a\left(y\,\mathrm{e}^{\frac{3\_a^2}{2}}\_a^2 - 2\right)}{\left(\_a^2+1\right)^{\frac{5}{2}}} d\_a - \frac{3\left(\left(\int_0^x \frac{\_a^3\,\mathrm{e}^{\frac{3\_a^2}{2}}}{\left(\_a^2+1\right)^{\frac{5}{2}}} d\_a\right)\left(x^2+1\right)^{\frac{3}{2}} - \frac{\mathrm{e}^{\frac{3x^2}{2}}}{3}\right) y}{\left(x^2+1\right)^{\frac{3}{2}}}$$

The solution becomes

$$y = \left(x^2+1\right)^{\frac{3}{2}} \left(c_1 + 6\left(\int_0^x \frac{\_a}{\left(\_a^2+1\right)^{\frac{5}{2}}} d\_a\right)\right) \mathrm{e}^{-\frac{3x^2}{2}}$$

Initial conditions are used to solve for c_1. Substituting $x = 0$ and $y = 1$ in the above solution gives an equation to solve for the constant of integration.

$$1 = c_1$$

$$c_1 = 1$$

Substituting c_1 found above in the general solution gives

$$y = 6\left(x^2+1\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{3x^2}{2}} \left(\int_0^x \frac{\_a}{\left(\_a^2+1\right)^{\frac{5}{2}}} d\_a\right) + \left(x^2+1\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{3x^2}{2}}$$

Summary

The solution(s) found are the following

$$y = 6\left(x^2+1\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{3x^2}{2}} \left(\int_0^x \frac{\_a}{\left(\_a^2+1\right)^{\frac{5}{2}}} d\_a\right) + \left(x^2+1\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{3x^2}{2}} \tag{1}$$

Verification of solutions

$$y = 6\left(x^2+1\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{3x^2}{2}} \left(\int_0^x \frac{\_a}{\left(\_a^2+1\right)^{\frac{5}{2}}} d\_a\right) + \left(x^2+1\right)^{\frac{3}{2}} \mathrm{e}^{-\frac{3x^2}{2}}$$

Verified OK.

4.25.5 Maple step by step solution

Let's solve

$$\left[3yx^3+\left(x^2+1\right)y'=\frac{6x}{\mathrm{e}^{\frac{3x^2}{2}}}, y(0)=1\right]$$

- Highest derivative means the order of the ODE is 1

$$y'$$

- Isolate the derivative

$$y'=-\frac{3x^3y}{x^2+1}+\frac{6x}{(x^2+1)\,\mathrm{e}^{\frac{3x^2}{2}}}$$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

$$y'+\frac{3x^3y}{x^2+1}=\frac{6x}{(x^2+1)\,\mathrm{e}^{\frac{3x^2}{2}}}$$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

$$\mu(x)\left(y'+\frac{3x^3y}{x^2+1}\right)=\frac{6\mu(x)x}{(x^2+1)\,\mathrm{e}^{\frac{3x^2}{2}}}$$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

$$\mu(x)\left(y'+\frac{3x^3y}{x^2+1}\right)=\mu'(x)\,y+\mu(x)\,y'$$

- Isolate $\mu'(x)$

$$\mu'(x)=\frac{3\mu(x)x^3}{x^2+1}$$

- Solve to find the integrating factor

$$\mu(x)=\frac{\left(\mathrm{e}^{\frac{3x^2}{2}}\right)^2\mathrm{e}^{-\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}}$$

- Integrate both sides with respect to x

$$\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{6\mu(x)x}{(x^2+1)\,\mathrm{e}^{\frac{3x^2}{2}}}dx+c_1$$

- Evaluate the integral on the lhs

$$\mu(x)\,y=\int\frac{6\mu(x)x}{(x^2+1)\,\mathrm{e}^{\frac{3x^2}{2}}}dx+c_1$$

- Solve for y

$$y=\frac{\int\frac{6\mu(x)x}{\left(x^2+1\right)\mathrm{e}^{\frac{3x^2}{2}}}dx+c_1}{\mu(x)}$$

- Substitute $\mu(x) = \frac{\left(e^{\frac{3x^2}{2}}\right)^2 e^{-\frac{3x^2}{2}}}{(x^2+1)^{\frac{3}{2}}}$

  $$y = \frac{(x^2+1)^{\frac{3}{2}} \left(\int \frac{6x\,e^{-\frac{3x^2}{2}} e^{\frac{3x^2}{2}}}{(x^2+1)^{\frac{5}{2}}} dx + c_1\right)}{\left(e^{\frac{3x^2}{2}}\right)^2 e^{-\frac{3x^2}{2}}}$$

- Evaluate the integrals on the rhs

  $$y = \frac{(x^2+1)^{\frac{3}{2}} \left(-\frac{2}{(x^2+1)^{\frac{3}{2}}} + c_1\right)}{\left(e^{\frac{3x^2}{2}}\right)^2 e^{-\frac{3x^2}{2}}}$$

- Simplify

  $$y = \left((x^2+1)^{\frac{3}{2}} c_1 - 2\right) e^{-\frac{3x^2}{2}}$$

- Use initial condition $y(0) = 1$

  $$1 = c_1 - 2$$

- Solve for c_1

  $$c_1 = 3$$

- Substitute $c_1 = 3$ into general solution and simplify

  $$y = \left(3x^2\sqrt{x^2+1} + 3\sqrt{x^2+1} - 2\right) e^{-\frac{3x^2}{2}}$$

- Solution to the IVP

  $$y = \left(3x^2\sqrt{x^2+1} + 3\sqrt{x^2+1} - 2\right) e^{-\frac{3x^2}{2}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 29

```
dsolve([3*x^3*y(x)+(x^2+1)*diff(y(x),x) = 6*x/exp(3/2*x^2),y(0) = 1],y(x), singsol=all)
```

$$y(x) = \left(3x^2\sqrt{x^2+1} + 3\sqrt{x^2+1} - 2\right) e^{-\frac{3x^2}{2}}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 28

```
DSolve[{3*x^3*y[x]+(x^2+1)*y'[x] == 6*x/Exp[3/2*x^2],y[0]==1},y[x],x,IncludeSingularSolution
```

$$y(x) \to e^{-\frac{3x^2}{2}}\left(3\left(x^2+1\right)^{3/2} - 2\right)$$

5 Section 1.6, Substitution methods and exact equations. Page 74

5.1 problem 1

Internal problem ID [79]
Internal file name [`OUTPUT/79_Sunday_June_05_2022_01_34_22_AM_72549356/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{(x+y)\,y'+y=x}$$

5.1.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x+u(x)\,x)\,(u'(x)\,x+u(x))+u(x)\,x=x$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^2+2u-1}{x\,(u+1)}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+2u-1}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+2u-1}{u+1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+2u-1}{u+1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln\left(u^2+2u-1\right)}{2} = -\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2+2u-1} = \mathrm{e}^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2+2u-1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u\left(x\right)^2+2u\left(x\right)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

The solution is

$$\sqrt{u\left(x\right)^2+2u\left(x\right)-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}+\frac{2y}{x}-1} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

$$\sqrt{\frac{y^2+2yx-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y^2+2yx-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x} \tag{1}$$

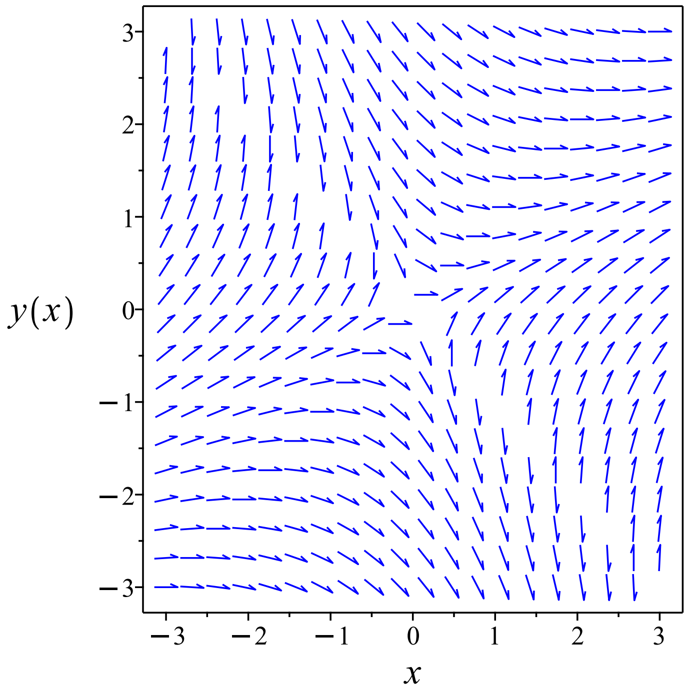


Figure 237: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2+2yx-x^2}{x^2}} = \frac{c_3 \mathrm{e}^{c_2}}{x}$$

Verified OK.

5.1.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x-y}{x+y} \tag{1}$$

Which becomes

$$(y)\,dy = (-x)\,dy + (x-y)\,dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\,dy + (x-y)\,dx = d\left(\frac{1}{2}x^2 - yx\right)$$

Hence (2) becomes

$$(y)\, dy = d\left(\frac{1}{2}x^2 - yx\right)$$

Integrating both sides gives gives these solutions

$$y = -x + \sqrt{2x^2 + 2c_1} + c_1$$
$$y = -x - \sqrt{2x^2 + 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = -x + \sqrt{2x^2 + 2c_1} + c_1 \tag{1}$$
$$y = -x - \sqrt{2x^2 + 2c_1} + c_1 \tag{2}$$

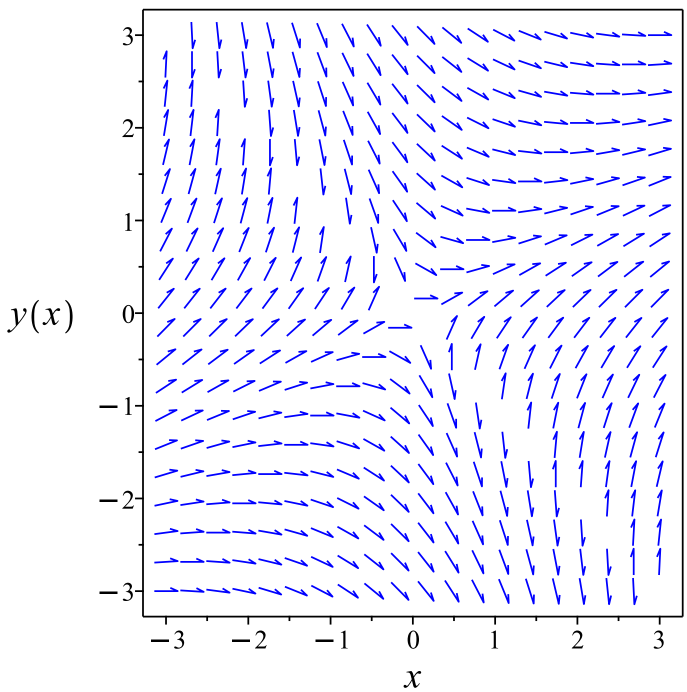


Figure 238: Slope field plot

Verification of solutions

$$y = -x + \sqrt{2x^2 + 2c_1} + c_1$$

Verified OK.

$$y = -x - \sqrt{2x^2 + 2c_1} + c_1$$

Verified OK.

5.1.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{-x+y}{x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(-x+y)(b_3-a_2)}{x+y} - \frac{(-x+y)^2 a_3}{(x+y)^2} - \left(\frac{1}{x+y} + \frac{-x+y}{(x+y)^2}\right)(x a_2 + y a_3 + a_1) - \left(-\frac{1}{x+y} + \frac{-x+y}{(x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^2 a_2 + x^2 a_3 - 3x^2 b_2 - x^2 b_3 + 2xy a_2 - 2xy a_3 - 2xy b_2 - 2xy b_3 - y^2 a_2 + 3y^2 a_3 - y^2 b_2 + y^2 b_3 - 2x b_1 + 2}{(x+y)^2} = 0$$

Setting the numerator to zero gives

$$-x^2 a_2 - x^2 a_3 + 3x^2 b_2 + x^2 b_3 - 2xy a_2 + 2xy a_3 + 2xy b_2 + 2xy b_3 + y^2 a_2 - 3y^2 a_3 + y^2 b_2 - y^2 b_3 + 2x b_1 - 2y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-a_2 v_1^2 - 2a_2 v_1 v_2 + a_2 v_2^2 - a_3 v_1^2 + 2a_3 v_1 v_2 - 3a_3 v_2^2 + 3b_2 v_1^2 + 2b_2 v_1 v_2 + b_2 v_2^2 + b_3 v_1^2 + 2b_3 v_1 v_2 - b_3 v_2^2 - 2a_1 v_2 + 2b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 + 3b_2 + b_3) v_1^2 + (-2a_2 + 2a_3 + 2b_2 + 2b_3) v_1 v_2 + 2b_1 v_1 + (a_2 - 3a_3 + b_2 - b_3) v_2^2 - 2a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-2a_1 &= 0 \\
2b_1 &= 0 \\
-2a_2 + 2a_3 + 2b_2 + 2b_3 &= 0 \\
-a_2 - a_3 + 3b_2 + b_3 &= 0 \\
a_2 - 3a_3 + b_2 - b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= 2b_2 + b_3 \\
a_3 &= b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(-\frac{-x+y}{x+y}\right)(x) \\
&= \frac{-x^2 + 2yx + y^2}{x+y} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{-x^2+2yx+y^2}{x+y}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln(-x^2 + 2yx + y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{-x+y}{x+y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x-y}{x^2-2yx-y^2} \\
S_y &= \frac{-x-y}{x^2-2yx-y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2+2yx-x^2)}{2} = c_1$$

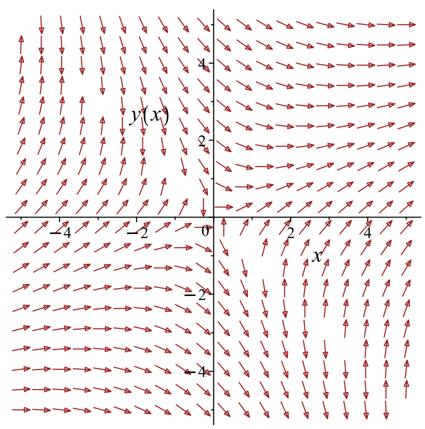
Which simplifies to

$$\frac{\ln(y^2+2yx-x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{-x+y}{x+y}$



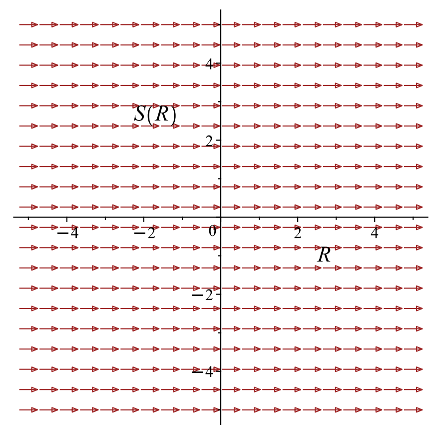
</td><td>

$R = x$

$S = \frac{\ln(-x^2 + 2yx + y^2)}{2}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 + 2yx - x^2)}{2} = c_1 \tag{1}$$

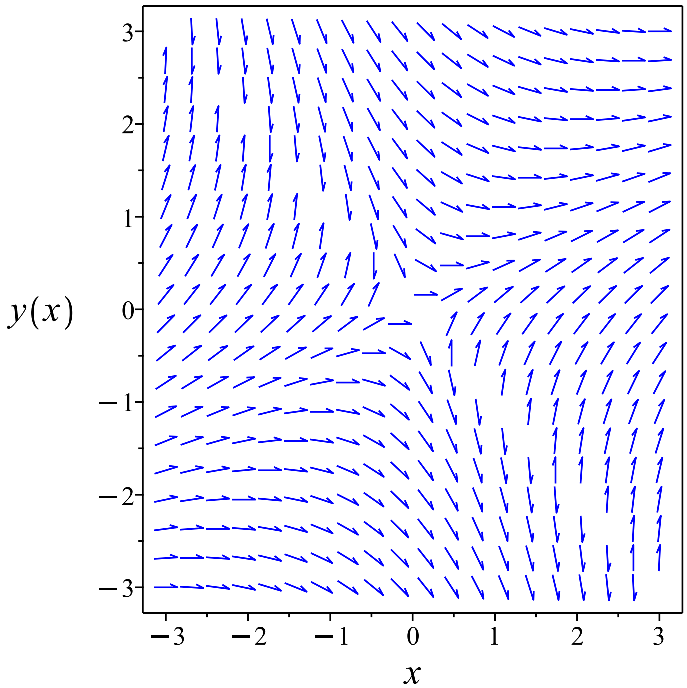


Figure 239: Slope field plot

Verification of solutions

$$\frac{\ln \left(y^2+2yx-x^2\right)}{2}=c_1$$

Verified OK.

5.1.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x + y)\,\mathrm{d}y = (x - y)\,\mathrm{d}x$$
$$(-x + y)\,\mathrm{d}x + (x + y)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x + y$$
$$N(x, y) = x + y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(-x + y) = 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x + y) = 1$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int -x+y\,\mathrm{d}x$$
$$\phi=-\frac{x(x-2y)}{2}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=x+y$. Therefore equation (4) becomes

$$x+y=x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (y)\,\mathrm{d}y$$
$$f(y)=\frac{y^2}{2}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(x-2y)}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(x-2y)}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x(x-2y)}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

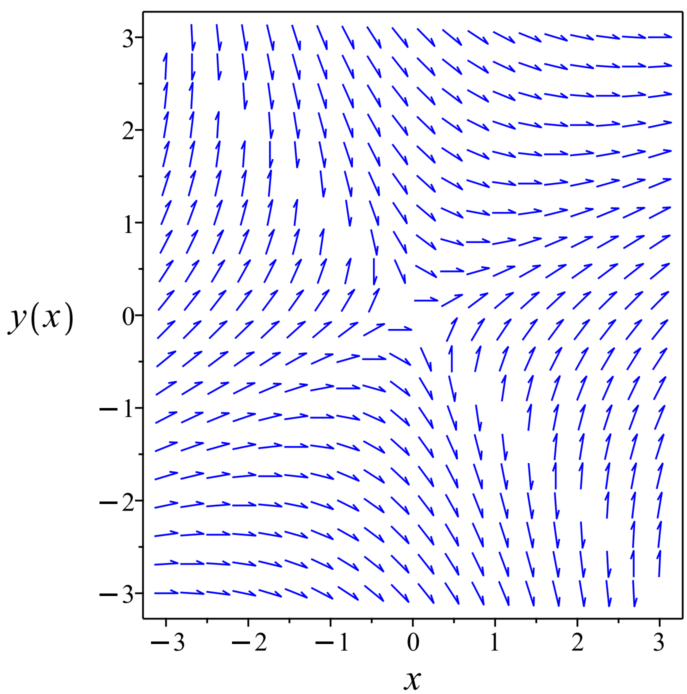


Figure 240: Slope field plot

Verification of solutions

$$-\frac{x(x-2y)}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

5.1.5 Maple step by step solution

Let's solve

$(x+y)\,y'+y=x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $1=1$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (-x+y)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=-\frac{x^2}{2}+yx+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $x+y=x+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=y$

- Solve for $f_1(y)$

  $f_1(y)=\frac{y^2}{2}$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x, y) = -\frac{1}{2}x^2 + yx + \frac{1}{2}y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

$$-\frac{1}{2}x^2 + yx + \frac{1}{2}y^2 = c_1$$

- Solve for y

$$\left\{y = -x - \sqrt{2x^2 + 2c_1}, y = -x + \sqrt{2x^2 + 2c_1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve((x+y(x))*diff(y(x),x) = x-y(x),y(x), singsol=all)
```

$$y(x) = \frac{-c_1 x - \sqrt{2c_1^2 x^2 + 1}}{c_1}$$
$$y(x) = \frac{-c_1 x + \sqrt{2c_1^2 x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.465 (sec). Leaf size: 94

```
DSolve[(x+y[x])*y'[x]== x-y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$
\begin{aligned}
y(x) &\to -x - \sqrt{2x^2 + e^{2c_1}} \\
y(x) &\to -x + \sqrt{2x^2 + e^{2c_1}} \\
y(x) &\to -\sqrt{2}\sqrt{x^2} - x \\
y(x) &\to \sqrt{2}\sqrt{x^2} - x
\end{aligned}
$$

5.2 problem 2

Internal problem ID [80]
Internal file name [`OUTPUT/80_Sunday_June_05_2022_01_34_22_AM_8417198/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{2xyy' - y^2 = x^2}$$

5.2.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^2u(x)\,(u'(x)\,x + u(x)) - u(x)^2\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2-1}{2ux} \end{aligned}$$

Where $f(x)=-\frac{1}{2x}$ and $g(u)=\frac{u^2-1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-1}{u}}\,du=-\frac{1}{2x}\,dx$$
$$\int\frac{1}{\frac{u^2-1}{u}}\,du=\int-\frac{1}{2x}\,dx$$
$$\frac{\ln(u-1)}{2}+\frac{\ln(u+1)}{2}=-\frac{\ln(x)}{2}+c_2$$

The above can be written as

$$\left(\frac{1}{2}\right)(\ln(u-1)+\ln(u+1))=-\frac{\ln(x)}{2}+2c_2$$
$$\ln(u-1)+\ln(u+1)=(2)\left(-\frac{\ln(x)}{2}+2c_2\right)$$
$$=-\ln(x)+4c_2$$

Raising both side to exponential gives

$$\mathrm{e}^{\ln(u-1)+\ln(u+1)}=\mathrm{e}^{-\ln(x)+2c_2}$$

Which simplifies to

$$u^2-1=\frac{2c_2}{x}$$
$$=\frac{c_3}{x}$$

The solution is

$$u(x)^2-1=\frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2}-1=\frac{c_3}{x}$$
$$\frac{y^2}{x^2}-1=\frac{c_3}{x}$$

Summary

The solution(s) found are the following

$$\frac{y^2}{x^2}-1=\frac{c_3}{x} \tag{1}$$

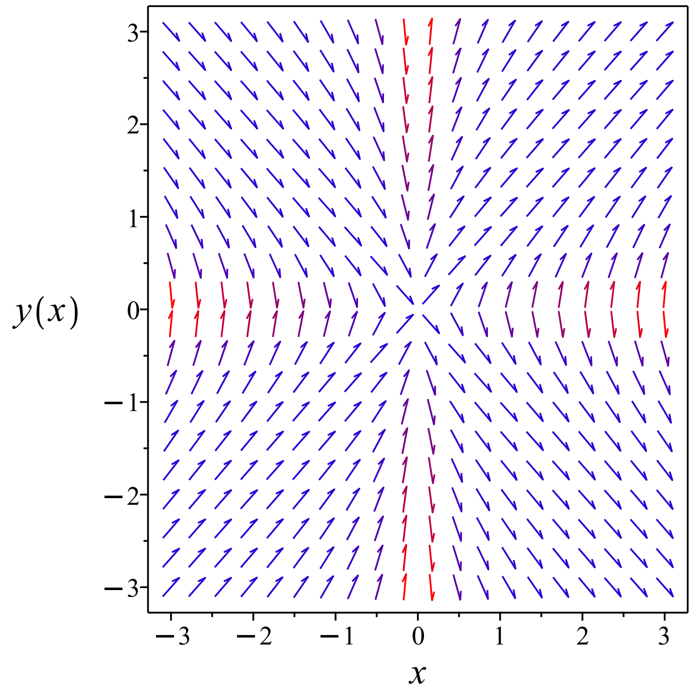


Figure 241: Slope field plot

Verification of solutions

$$\frac{y^2}{x^2} - 1 = \frac{c_3}{x}$$

Verified OK.

5.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2 + y^2}{2xy}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 196: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^2 + y^2}{2xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^2}{2x^2} \\ S_y &= \frac{y}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{2} + c_1 \tag{4}$$

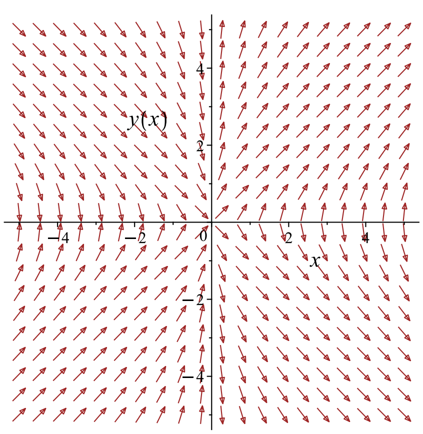
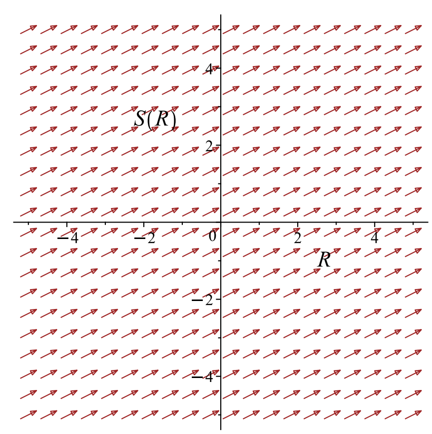
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Which simplifies to

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ 	$R = x$ $S = \frac{y^2}{2x}$	$\frac{dS}{dR} = \frac{1}{2}$ 

Summary
The solution(s) found are the following

$$\frac{y^2}{2x} = \frac{x}{2} + c_1 \tag{1}$$

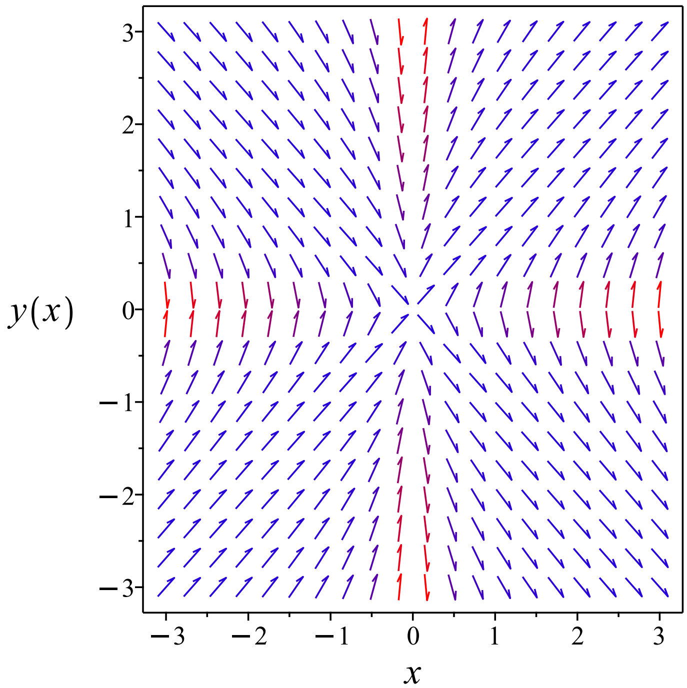


Figure 242: Slope field plot

Verification of solutions

$$\frac{y^2}{2x} = \frac{x}{2} + c_1$$

Verified OK.

5.2.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + y^2}{2xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{2x}y + \frac{x}{2}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{2x} \\
f_1(x) &= \frac{x}{2} \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{y^2}{2x} + \frac{x}{2} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{w(x)}{2x} + \frac{x}{2} \\
w' &= \frac{w}{x} + x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(x) \\ \frac{d}{dx}\left(\frac{w}{x}\right) &= \left(\frac{1}{x}\right)(x) \\ d\left(\frac{w}{x}\right) &= dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x} &= \int dx \\ \frac{w}{x} &= x + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = c_1 x + x^2$$

which simplifies to

$$w(x) = x(x + c_1)$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = x(x + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \sqrt{x(x + c_1)} \\ y(x) &= -\sqrt{x(x + c_1)}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \sqrt{x (x + c_1)} \tag{1}$$
$$y = -\sqrt{x (x + c_1)} \tag{2}$$

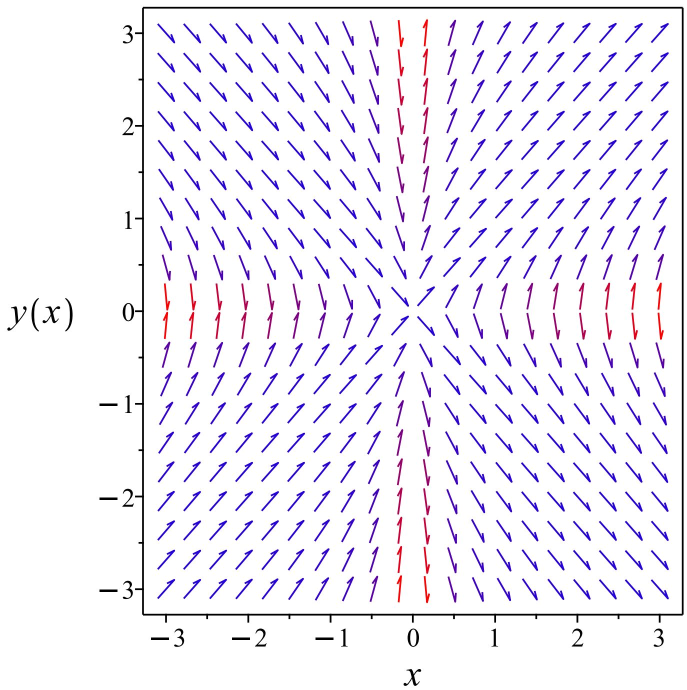


Figure 243: Slope field plot

Verification of solutions

$$y = \sqrt{x (x + c_1)}$$

Verified OK.

$$y = -\sqrt{x (x + c_1)}$$

Verified OK.

5.2.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(2yx)\,\mathrm{d}y&=\left(x^2+y^2\right)\mathrm{d}x\\ \left(-x^2-y^2\right)\mathrm{d}x+(2yx)\,\mathrm{d}y&=0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x^2-y^2\\ N(x,y)&=2yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^2 - y^2)\\ &= -2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2yx)\\ &= 2y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{2xy}((-2y) - (2y))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{2}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)}\\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^2}(-x^2 - y^2)\\ &= \frac{-x^2 - y^2}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(2yx) \\
&= \frac{2y}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^2 - y^2}{x^2}\right) + \left(\frac{2y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-x^2 - y^2}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -x + \frac{y^2}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{2y}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2y}{x}$. Therefore equation (4) becomes

$$
\frac{2y}{x} = \frac{2y}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + \frac{y^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + \frac{y^2}{x}$$

Summary

The solution(s) found are the following

$$-x + \frac{y^2}{x} = c_1 \tag{1}$$

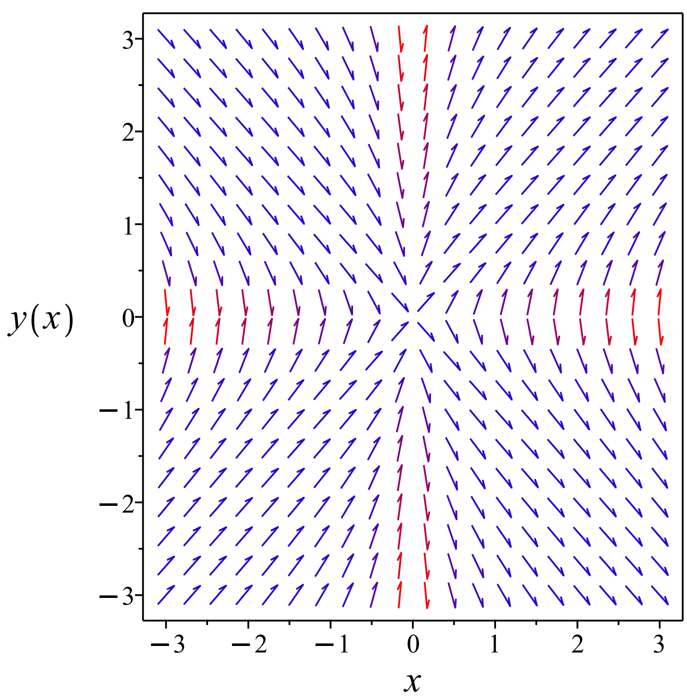


Figure 244: Slope field plot

Verification of solutions

$$-x+\frac{y^2}{x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(2*x*y(x)*diff(y(x),x) = x^2+y(x)^2,y(x), singsol=all)
```

$$y(x)=\sqrt{(c_1+x)\,x}$$
$$y(x)=-\sqrt{(c_1+x)\,x}$$

✓ Solution by Mathematica

Time used: 0.17 (sec). Leaf size: 38

```
DSolve[2*x*y[x]*y'[x] == x^2+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\sqrt{x}\sqrt{x+c_1}$$
$$y(x)\to \sqrt{x}\sqrt{x+c_1}$$

5.3 problem 3

Internal problem ID [81]
Internal file name [`OUTPUT/81_Sunday_June_05_2022_01_34_23_AM_42870616/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x - y - 2\sqrt{yx} = 0}$$

5.3.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y + 2\sqrt{yx}}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
& b_2+\frac{\left(y+2 \sqrt{y x}\right)\left(b_3-a_2\right)}{x}-\frac{\left(y+2 \sqrt{y x}\right)^2 a_3}{x^2} \\
& -\left(\frac{y}{\sqrt{y x}\, x}-\frac{y+2 \sqrt{y x}}{x^2}\right)\left(x a_2+y a_3+a_1\right)-\frac{\left(1+\frac{x}{\sqrt{y x}}\right)\left(x b_2+y b_3+b_1\right)}{x}=0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{4(y x)^{\frac{3}{2}} a_3-x^2 y b_3+3 x\, y^2 a_3+x^3 b_2+x^2 y a_2-x y a_1+\sqrt{y x}\, x b_1-\sqrt{y x}\, y a_1+x^2 b_1}{\sqrt{y x}\, x^2}=0
$$

Setting the numerator to zero gives

$$
-4(y x)^{\frac{3}{2}} a_3-x^3 b_2-x^2 y a_2+x^2 y b_3-3 x\, y^2 a_3-\sqrt{y x}\, x b_1+\sqrt{y x}\, y a_1-x^2 b_1+x y a_1=0 \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
-x^3 b_2-x^2 y a_2+x^2 y b_3-4 y x \sqrt{y x}\, a_3-3 x\, y^2 a_3-x^2 b_1-\sqrt{y x}\, x b_1+x y a_1+\sqrt{y x}\, y a_1=0
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\{x, y, \sqrt{y x}\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$
\{x=v_1, y=v_2, \sqrt{y x}=v_3\}
$$

The above PDE (6E) now becomes

$$
-v_1^2 v_2 a_2-3 v_1 v_2^2 a_3-4 v_2 v_1 v_3 a_3-v_1^3 b_2+v_1^2 v_2 b_3+v_1 v_2 a_1+v_3 v_2 a_1-v_1^2 b_1-v_3 v_1 b_1=0 \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3\}
$$

Equation (7E) now becomes

$$-v_1^3b_2 + (b_3 - a_2)\, v_1^2v_2 - v_1^2b_1 - 3v_1v_2^2a_3 - 4v_2v_1v_3a_3 + v_1v_2a_1 - v_3v_1b_1 + v_3v_2a_1 = 0 \quad (8E)$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -4a_3 &= 0 \\ -3a_3 &= 0 \\ -b_1 &= 0 \\ -b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(\frac{y + 2\sqrt{yx}}{x}\right)(x) \\ &= -2\sqrt{yx} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-2\sqrt{yx}} dy \end{aligned}$$

Which results in

$$S = -\frac{y}{\sqrt{yx}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y + 2\sqrt{yx}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{\sqrt{y}}{2x^{\frac{3}{2}}} \\ S_y &= -\frac{1}{2\sqrt{y}\,\sqrt{x}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sqrt{yx}}{\sqrt{y}\, x^{\frac{3}{2}}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\sqrt{y}}{\sqrt{x}} = -\ln(x) + c_1$$

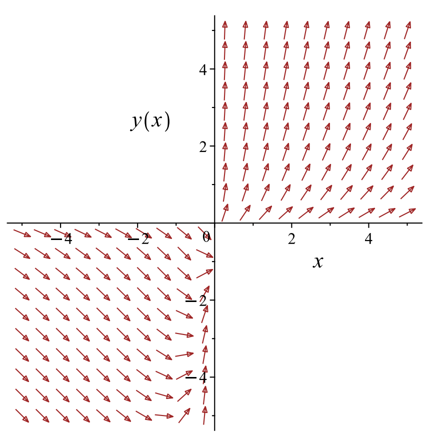
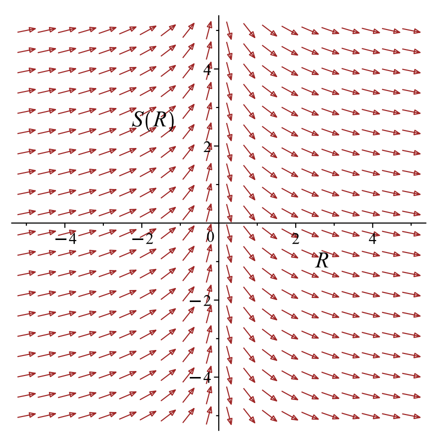
Which simplifies to

$$-\frac{\sqrt{y}}{\sqrt{x}} = -\ln(x) + c_1$$

Which gives

$$y = x\ln(x)^2 - 2x\ln(x)c_1 + c_1^2 x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+2\sqrt{yx}}{x}$ 	$R = x$ $S = -\frac{\sqrt{y}}{\sqrt{x}}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = x\ln(x)^2 - 2x\ln(x)\,c_1 + c_1^2 x \tag{1}$$

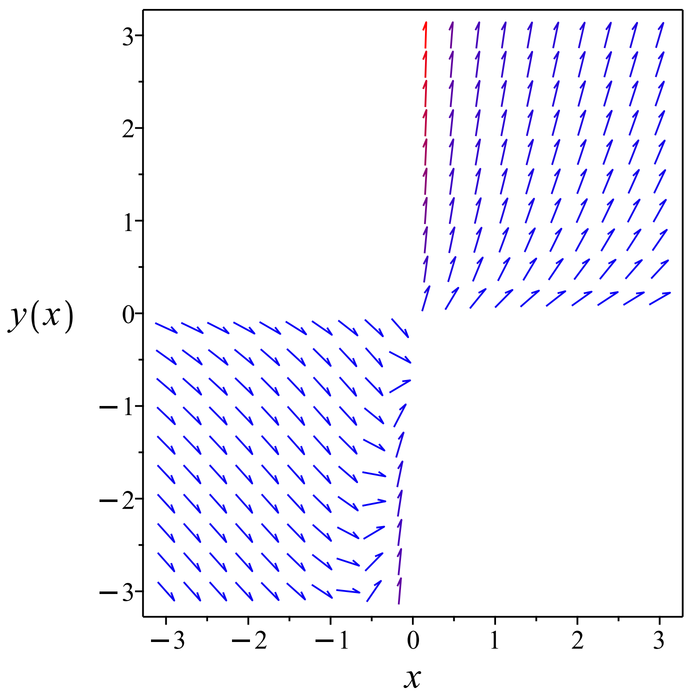


Figure 245: Slope field plot

Verification of solutions

$$y = x\ln(x)^2 - 2x\ln(x)\,c_1 + c_1^2 x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x) = y(x)+2*(x*y(x))^(1/2),y(x), singsol=all)
```

$$-\frac{y(x)}{\sqrt{xy(x)}}+\ln(x)-c_1=0$$

✓ Solution by Mathematica

Time used: 0.159 (sec). Leaf size: 19

```
DSolve[x*y'[x] == y[x]+2*(x*y[x])^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{1}{4}x(2\log(x)+c_1)^2$$

5.4 problem 4

Internal problem ID [82]
Internal file name [`OUTPUT/82_Sunday_June_05_2022_01_34_24_AM_98227792/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x-y)\,y'-y=x}$$

5.4.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x-u(x)\,x)\,(u'(x)\,x+u(x))-u(x)\,x=x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2+1}{(u-1)\,x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2+1}{u-1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+1}{u-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u^2+1}{u-1}}\, du = \int -\frac{1}{x}\, dx$$

$$\frac{\ln (u^2+1)}{2} - \arctan (u) = -\ln (x) + c_2$$

The solution is

$$\frac{\ln \left(u(x)^2+1\right)}{2} - \arctan (u(x)) + \ln (x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{\ln \left(\frac{y^2}{x^2}+1\right)}{2} - \arctan \left(\frac{y}{x}\right) + \ln (x) - c_2 = 0$$

$$\frac{\ln \left(\frac{y^2}{x^2}+1\right)}{2} - \arctan \left(\frac{y}{x}\right) + \ln (x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{\ln \left(\frac{y^2}{x^2}+1\right)}{2} - \arctan \left(\frac{y}{x}\right) + \ln (x) - c_2 = 0 \tag{1}$$

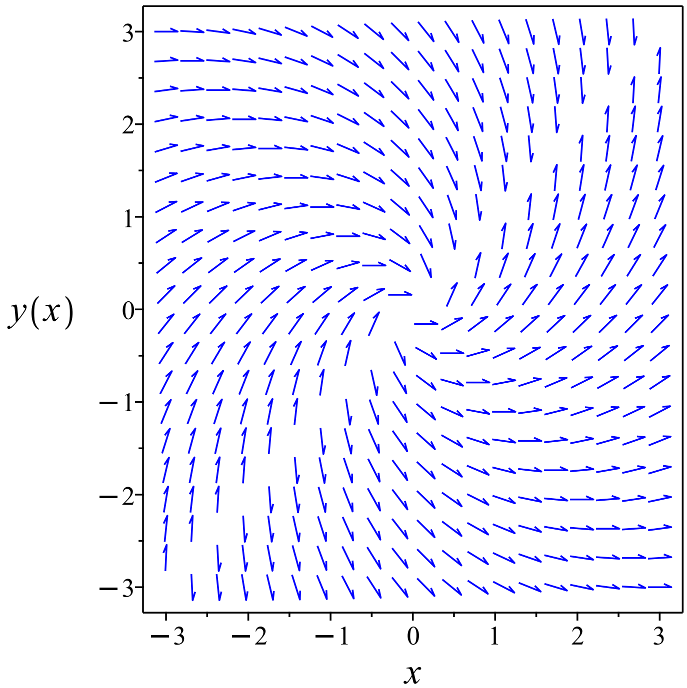


Figure 246: Slope field plot

Verification of solutions

$$\frac{\ln\left(\frac{y^2}{x^2}+1\right)}{2}-\arctan\left(\frac{y}{x}\right)+\ln(x)-c_2=0$$

Verified OK.

5.4.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{x+y}{-x+y}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 - \frac{(x+y)(b_3-a_2)}{-x+y} - \frac{(x+y)^2 a_3}{(-x+y)^2} \\
&- \left(-\frac{1}{-x+y} - \frac{x+y}{(-x+y)^2}\right)(xa_2 + ya_3 + a_1) \\
&- \left(-\frac{1}{-x+y} + \frac{x+y}{(-x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0
\end{aligned}
\tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{x^2a_2 + x^2a_3 + x^2b_2 - x^2b_3 - 2xya_2 + 2xya_3 + 2xyb_2 + 2xyb_3 - y^2a_2 - y^2a_3 - y^2b_2 + y^2b_3 + 2xb_1 - 2ya}{(x-y)^2}
$$
$$= 0$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-x^2a_2 - x^2a_3 - x^2b_2 + x^2b_3 + 2xya_2 - 2xya_3 - 2xyb_2 \\
&- 2xyb_3 + y^2a_2 + y^2a_3 + y^2b_2 - y^2b_3 - 2xb_1 + 2ya_1 = 0
\end{aligned}
\tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-a_2v_1^2 + 2a_2v_1v_2 + a_2v_2^2 - a_3v_1^2 - 2a_3v_1v_2 + a_3v_2^2 - b_2v_1^2 \\
&- 2b_2v_1v_2 + b_2v_2^2 + b_3v_1^2 - 2b_3v_1v_2 - b_3v_2^2 + 2a_1v_2 - 2b_1v_1 = 0
\end{aligned}
\tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(-a_2 - a_3 - b_2 + b_3)\, v_1^2 + (2a_2 - 2a_3 - 2b_2 - 2b_3)\, v_1 v_2 \\ - 2b_1 v_1 + (a_2 + a_3 + b_2 - b_3)\, v_2^2 + 2a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -2b_1 &= 0 \\ -a_2 - a_3 - b_2 + b_3 &= 0 \\ a_2 + a_3 + b_2 - b_3 &= 0 \\ 2a_2 - 2a_3 - 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= -b_2 \\ b_1 &= 0 \\ b_2 &= b_2 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{x + y}{-x + y}\right)(x) \\ &= \frac{-x^2 - y^2}{x - y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-x^2 - y^2}{x - y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln\left(x^2 + y^2\right)}{2} - \arctan\left(\frac{y}{x}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{x + y}{-x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{x + y}{x^2 + y^2} \\ S_y &= \frac{-x + y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

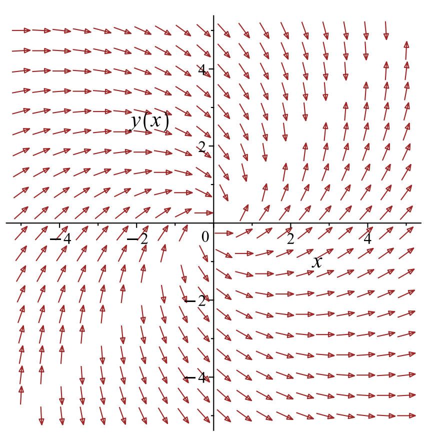
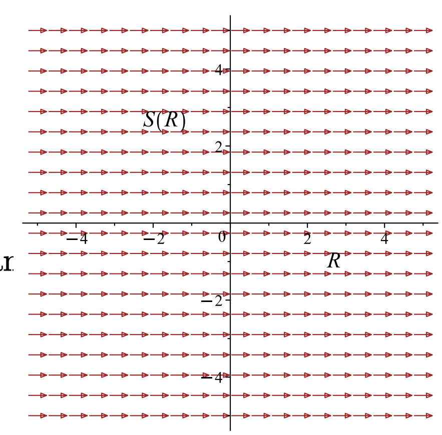
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Which simplifies to

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{x+y}{-x+y}$ 	$R = x$ $S = \frac{\ln(x^2+y^2)}{2} - \text{arctar}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1 \tag{1}$$

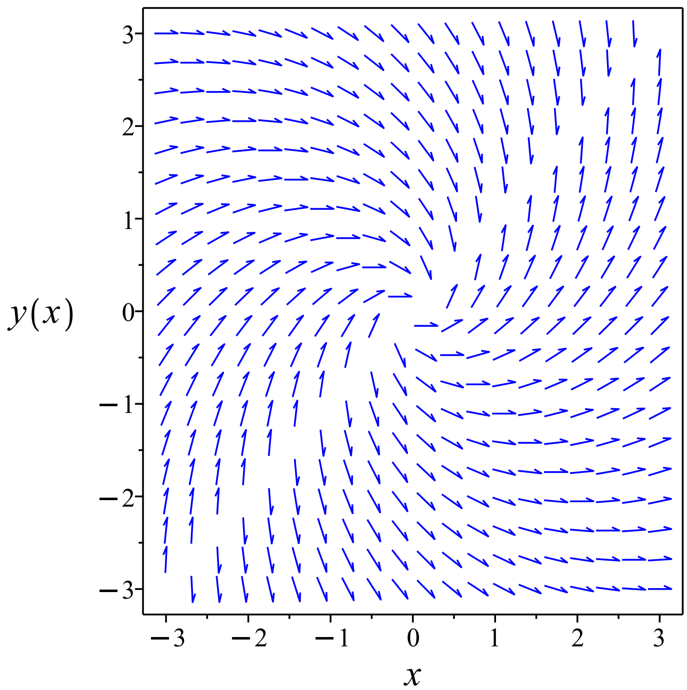


Figure 247: Slope field plot

Verification of solutions

$$\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{y}{x}\right) = c_1$$

Verified OK.

5.4.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x - y) \, \mathrm{d}y &= (x + y) \, \mathrm{d}x \\ (-x - y) \, \mathrm{d}x + (x - y) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x - y \\ N(x, y) &= x - y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x - y) \\ &= -1\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x-y)\\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x^2+y^2}$ is an integrating factor. Therefore by multiplying $M = -y - x$ and $N = x - y$ by this integrating factor the ode becomes exact. The new M, N are

$$
\begin{aligned}
M &= \frac{-y-x}{x^2+y^2}\\
N &= \frac{x-y}{x^2+y^2}
\end{aligned}
$$

To solve an ode of the form

$$
M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}
$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$
\frac{d}{dx}\phi(x,y) = 0
$$

Hence

$$
\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}
$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M\\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x \partial y} = \frac{\partial^2\phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x-y}{x^2+y^2}\right)\mathrm{d}y = \left(-\frac{-x-y}{x^2+y^2}\right)\mathrm{d}x$$
$$\left(\frac{-x-y}{x^2+y^2}\right)\mathrm{d}x + \left(\frac{x-y}{x^2+y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = \frac{-x-y}{x^2+y^2}$$
$$N(x,y) = \frac{x-y}{x^2+y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-x-y}{x^2+y^2}\right)\\ &= \frac{-x^2+2yx+y^2}{(x^2+y^2)^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x-y}{x^2+y^2}\right)\\ &= \frac{-x^2+2yx+y^2}{(x^2+y^2)^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{-x-y}{x^2+y^2} \mathrm{d}x$$
$$\phi = -\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{y}{x^2+y^2} + \frac{x}{y^2\left(\frac{x^2}{y^2}+1\right)} + f'(y) \tag{4}$$
$$= \frac{x-y}{x^2+y^2} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x-y}{x^2+y^2}$. Therefore equation (4) becomes

$$\frac{x-y}{x^2+y^2} = \frac{x-y}{x^2+y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln(x^2+y^2)}{2} - \arctan\left(\frac{x}{y}\right)$$

Summary

The solution(s) found are the following

$$-\frac{\ln(x^2+y^2)}{2}-\arctan\left(\frac{x}{y}\right)=c_1 \tag{1}$$

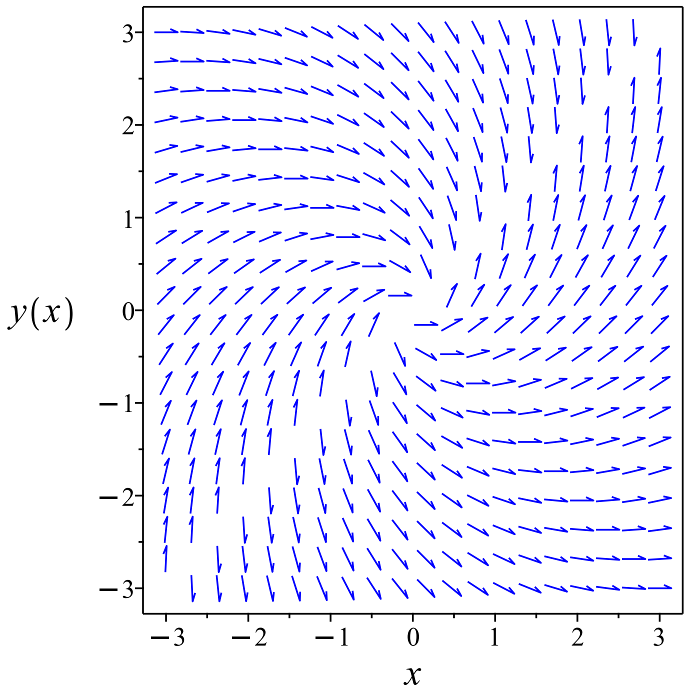


Figure 248: Slope field plot

Verification of solutions

$$-\frac{\ln(x^2+y^2)}{2}-\arctan\left(\frac{x}{y}\right)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 24

```
dsolve((x-y(x))*diff(y(x),x) = x+y(x),y(x), singsol=all)
```

$$y(x) = \tan\left(\operatorname{RootOf}\left(-2\_Z + \ln\left(\sec\left(\_Z\right)^2\right) + 2\ln\left(x\right) + 2c_1\right)\right) x$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 36

```
DSolve[(x-y[x])*y'[x] == x+y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$\text{Solve}\left[\frac{1}{2}\log\left(\frac{y(x)^2}{x^2}+1\right) - \arctan\left(\frac{y(x)}{x}\right) = -\log(x) + c_1, y(x)\right]$$

5.5 problem 5

Internal problem ID [83]
Internal file name [`OUTPUT/83_Sunday_June_05_2022_01_34_25_AM_29678176/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{x(x+y)\,y'-y(x-y)=0}$$

5.5.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x(x+u(x)\,x)\,(u'(x)\,x+u(x))-u(x)\,x(x-u(x)\,x)=0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u)\\
&= f(x)g(u)\\
&= -\frac{2u^2}{x\,(u+1)}
\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{u+1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{u^2}{u+1}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln(u) - \frac{1}{u} = -2\ln(x) + c_2$$

The solution is

$$\ln(u(x)) - \frac{1}{u(x)} + 2\ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\ln\left(\frac{y}{x}\right) - \frac{x}{y} + 2\ln(x) - c_2 = 0$$
$$\ln\left(\frac{y}{x}\right) - \frac{x}{y} + 2\ln(x) - c_2 = 0$$

<u>Summary</u>

The solution(s) found are the following

$$\ln\left(\frac{y}{x}\right) - \frac{x}{y} + 2\ln(x) - c_2 = 0 \tag{1}$$

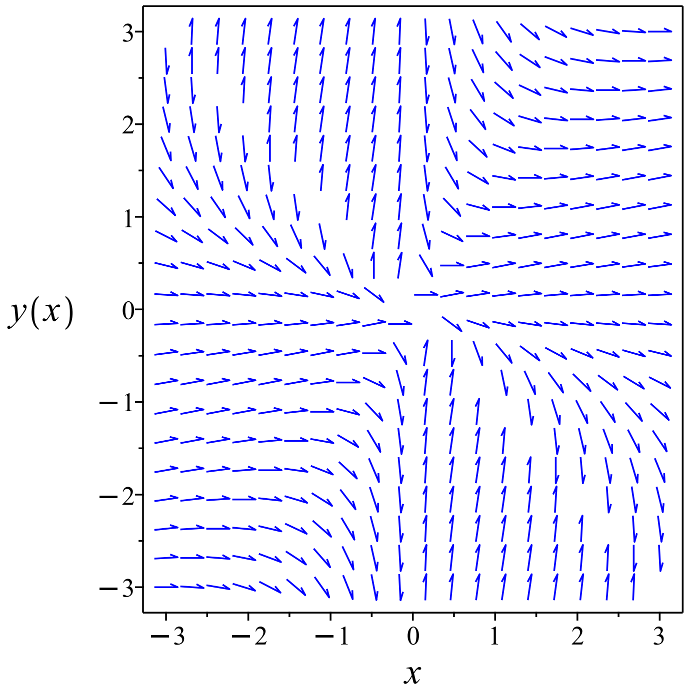


Figure 249: Slope field plot

Verification of solutions

$$\ln\left(\frac{y}{x}\right)-\frac{x}{y}+2\ln\left(x\right)-c_2=0$$

Verified OK.

5.5.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y(-x+y)}{x\left(x+y\right)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{y(-x+y)(b_3 - a_2)}{x(x+y)} - \frac{y^2(-x+y)^2 a_3}{x^2(x+y)^2} \\ &- \left(\frac{y}{x(x+y)} + \frac{y(-x+y)}{x^2(x+y)} + \frac{y(-x+y)}{x(x+y)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{-x+y}{x(x+y)} - \frac{y}{x(x+y)} + \frac{y(-x+y)}{x(x+y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^3yb_2 - 2x^2y^2a_2 + 2x^2y^2b_2 + 2x^2y^2b_3 - 2y^4a_3 - x^3b_1 + x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + x\,y^2b_1 - y^3a_1}{x^2(x+y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &4x^3yb_2 - 2x^2y^2a_2 + 2x^2y^2b_2 + 2x^2y^2b_3 - 2y^4a_3 - x^3b_1 \\ &+ x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + x\,y^2b_1 - y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2a_2v_1^2v_2^2 - 2a_3v_2^4 + 4b_2v_1^3v_2 + 2b_2v_1^2v_2^2 + 2b_3v_1^2v_2^2 + a_1v_1^2v_2 \\ &- 2a_1v_1v_2^2 - a_1v_2^3 - b_1v_1^3 + 2b_1v_1^2v_2 + b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2v_1^3v_2 - b_1v_1^3 + (-2a_2 + 2b_2 + 2b_3)\, v_1^2v_2^2 + (a_1 + 2b_1)\, v_1^2v_2 + (-2a_1 + b_1)\, v_1v_2^2 - 2a_3v_2^4 - a_1v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-a_1 &= 0\\
-2a_3 &= 0\\
-b_1 &= 0\\
4b_2 &= 0\\
-2a_1 + b_1 &= 0\\
a_1 + 2b_1 &= 0\\
-2a_2 + 2b_2 + 2b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0\\
a_2 &= b_3\\
a_3 &= 0\\
b_1 &= 0\\
b_2 &= 0\\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi\\
&= y - \left(-\frac{y(-x + y)}{x\,(x + y)}\right)(x)\\
&= \frac{2y^2}{x + y}\\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y^2}{x+y}} dy \end{aligned}$$

Which results in

$$S = \frac{\ln (y)}{2} - \frac{x}{2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-x + y)}{x (x + y)}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{2y} \\ S_y &= \frac{x + y}{2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y)\, y - x}{2y} = -\frac{\ln(x)}{2} + c_1$$

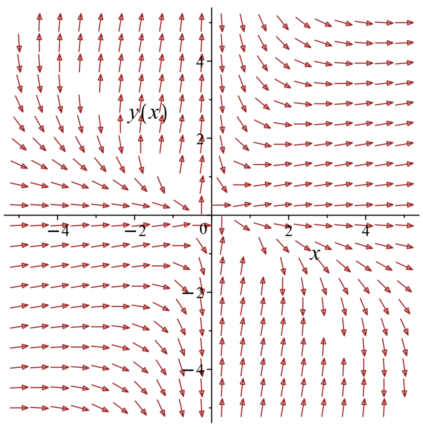
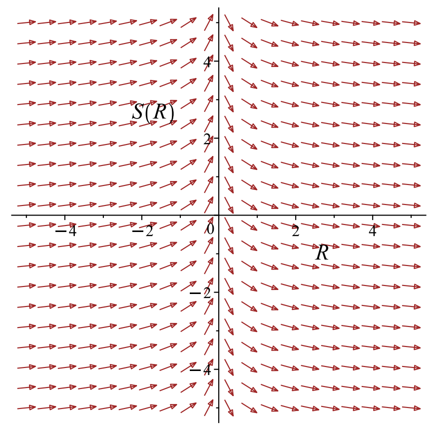
Which simplifies to

$$\frac{\ln(y)\, y - x}{2y} = -\frac{\ln(x)}{2} + c_1$$

Which gives

$$y = \frac{x}{\operatorname{LambertW}\left(x^2 \mathrm{e}^{-2c_1}\right)}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x+y)}{x(x+y)}$ 	$R = x$ $S = \frac{\ln(y)\, y - x}{2y}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{\text{LambertW}\left(x^2 e^{-2c_1}\right)} \tag{1}$$

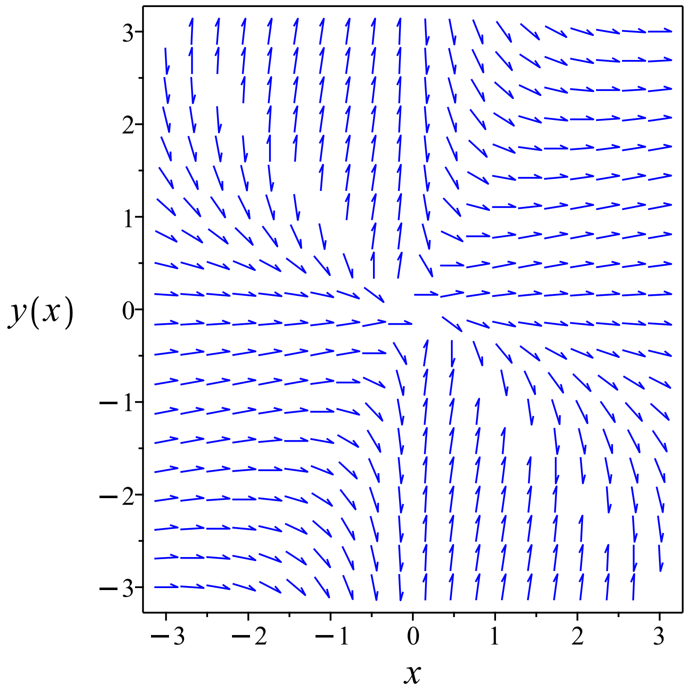


Figure 250: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x}{\mathrm{LambertW}\left(x^2 \mathrm{e}^{-2c_1}\right)}$$

Verified OK.

5.5.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x(x+y))\,\mathrm{d}y &= (y(x-y))\,\mathrm{d}x\\ (-y(x-y))\,\mathrm{d}x + (x(x+y))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -y(x-y)$$
$$N(x, y) = x(x+y)$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y(x-y))\\ &= -x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(x+y))\\ &= 2x + y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = -y(x-y)$ and $N = x(x+y)$ by this integrating factor the ode becomes exact. The new M, N are

$$M = -\frac{x-y}{xy}$$
$$N = \frac{x+y}{y^2}$$

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{x+y}{y^2}\right) \mathrm{d}y = \left(\frac{x-y}{yx}\right) \mathrm{d}x$$

$$\left(-\frac{x-y}{yx}\right) \mathrm{d}x + \left(\frac{x+y}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{x-y}{yx}$$
$$N(x,y) = \frac{x+y}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x-y}{yx}\right)$$
$$= \frac{1}{y^2}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x+y}{y^2}\right)$$
$$= \frac{1}{y^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{x-y}{yx} \mathrm{d}x$$
$$\phi = \ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x+y}{y^2}$. Therefore equation (4) becomes

$$\frac{x+y}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln(x) - \frac{x}{y} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) - \frac{x}{y} + \ln(y)$$

The solution becomes

$$y = \frac{x}{\text{LambertW}(x^2 \mathrm{e}^{-c_1})}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\text{LambertW}(x^2 \mathrm{e}^{-c_1})} \tag{1}$$

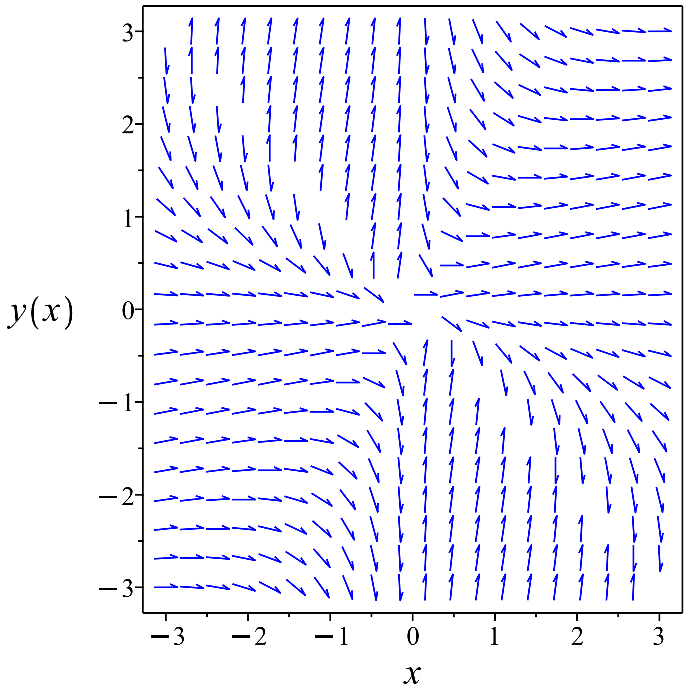


Figure 251: Slope field plot

Verification of solutions

$$y = \frac{x}{\operatorname{LambertW}\left(x^2 e^{-c_1}\right)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve(x*(x+y(x))*diff(y(x),x) = y(x)*(x-y(x)),y(x), singsol=all)
```

$$y(x) = \frac{x}{\operatorname{LambertW}(c_1 x^2)}$$

✓ Solution by Mathematica

Time used: 4.218 (sec). Leaf size: 25

```
DSolve[x*(x+y[x])*y'[x] == y[x]*(x-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{W(e^{-c_1} x^2)}$$
$$y(x) \to 0$$

5.6 problem 6

Internal problem ID [84]
Internal file name [`OUTPUT/84_Sunday_June_05_2022_01_34_26_AM_1049457/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{(x+2y)\,y'-y=0}$$

5.6.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(x+2u(x)\,x)\,(u'(x)\,x+u(x))-u(x)\,x=0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u^2}{x\,(2u+1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2}{2u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u^2}{2u+1}}\, du = -\frac{2}{x}\, dx$$

$$\int \frac{1}{\frac{u^2}{2u+1}}\, du = \int -\frac{2}{x}\, dx$$

$$2\ln\left(u\right) - \frac{1}{u} = -2\ln\left(x\right) + c_2$$

The solution is

$$2\ln\left(u(x)\right) - \frac{1}{u\left(x\right)} + 2\ln\left(x\right) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$2\ln\left(\frac{y}{x}\right) - \frac{x}{y} + 2\ln\left(x\right) - c_2 = 0$$
$$2\ln\left(\frac{y}{x}\right) - \frac{x}{y} + 2\ln\left(x\right) - c_2 = 0$$

Summary

The solution(s) found are the following

$$2\ln\left(\frac{y}{x}\right) - \frac{x}{y} + 2\ln\left(x\right) - c_2 = 0 \tag{1}$$

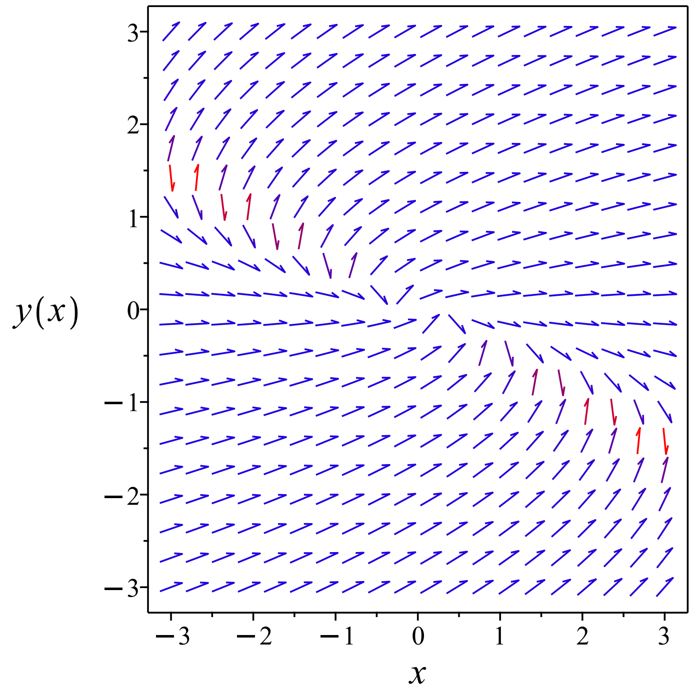


Figure 252: Slope field plot

Verification of solutions

$$2\ln\left(\frac{y}{x}\right)-\frac{x}{y}+2\ln\left(x\right)-c_2=0$$

Verified OK.

5.6.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=\frac{y}{x+2y}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 + \frac{y(b_3 - a_2)}{x + 2y} - \frac{y^2 a_3}{(x + 2y)^2} + \frac{y(x a_2 + y a_3 + a_1)}{(x + 2y)^2} - \left(\frac{1}{x + 2y} - \frac{2y}{(x + 2y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{4xyb_2 - 2y^2 a_2 + 4y^2 b_2 + 2y^2 b_3 - x b_1 + y a_1}{(x + 2y)^2} = 0$$

Setting the numerator to zero gives

$$4xyb_2 - 2y^2 a_2 + 4y^2 b_2 + 2y^2 b_3 - x b_1 + y a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$-2a_2 v_2^2 + 4b_2 v_1 v_2 + 4b_2 v_2^2 + 2b_3 v_2^2 + a_1 v_2 - b_1 v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$4b_2 v_1 v_2 - b_1 v_1 + (-2a_2 + 4b_2 + 2b_3) v_2^2 + a_1 v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -b_1 &= 0 \\ 4b_2 &= 0 \\ -2a_2 + 4b_2 + 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= a_3 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{y}{x+2y}\right)(x) \\ &= \frac{2y^2}{x+2y} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{2y^2}{x+2y}} dy \end{aligned}$$

Which results in

$$S = \ln(y) - \frac{x}{2y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y}{x + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{2y} \\ S_y &= \frac{x + 2y}{2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{2\ln(y)\,y - x}{2y} = c_1$$

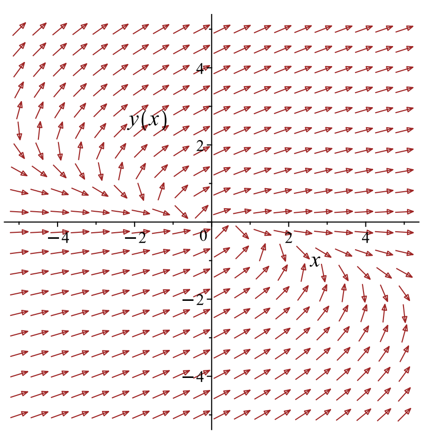
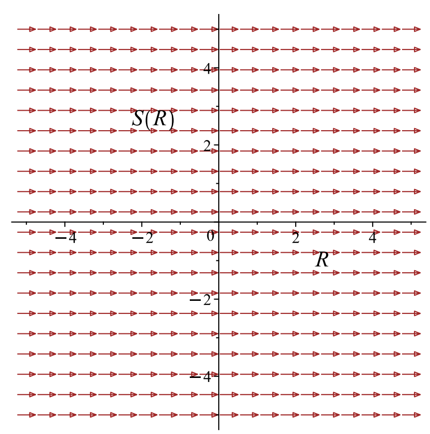
Which simplifies to

$$\frac{2\ln(y)\,y - x}{2y} = c_1$$

Which gives

$$y = \mathrm{e}^{\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{-c_1}}{2}\right)+c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y}{x+2y}$ 	$R = x$ $S = \frac{2\ln(y)\,y - x}{2y}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$y = e^{\operatorname{LambertW}\left(\frac{x\,e^{-c_1}}{2}\right)+c_1} \tag{1}$$

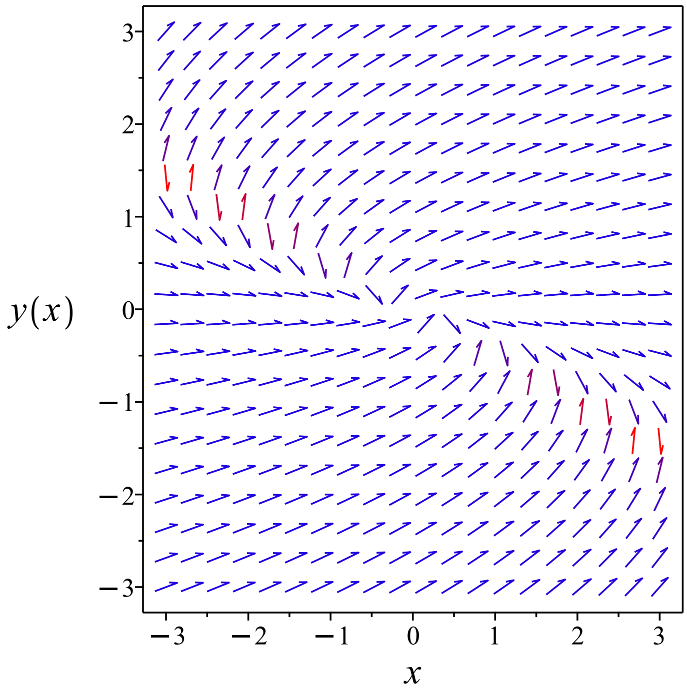


Figure 253: Slope field plot

Verification of solutions

$$y = e^{\operatorname{LambertW}\left(\frac{x\,e^{-c_1}}{2}\right)+c_1}$$

Verified OK.

5.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y} \frac{d y}{d x}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \mathrm{d} x+N(x, y) \mathrm{d} y=0 \tag{1A}$$

Therefore

$$\begin{aligned}
(x+2 y) \mathrm{d} y &= (y) \mathrm{d} x \\
(-y) \mathrm{d} x+(x+2 y) \mathrm{d} y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= -y \\
N(x, y) &= x+2 y
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-y) \\
&= -1
\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x+2y)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x+2y}((-1)-(1))\\ &= -\frac{2}{x+2y}\end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned}B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\ &= -\frac{1}{y}((1)-(-1))\\ &= -\frac{2}{y}\end{aligned}$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$\begin{aligned}\mu &= e^{\int B\,\mathrm{d}y}\\ &= e^{\int -\frac{2}{y}\,\mathrm{d}y}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(y)}\\ &= \frac{1}{y^2}\end{aligned}$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{y^2}(-y)\\ &= -\frac{1}{y}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(x+2y) \\
&= \frac{x+2y}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{1}{y}\right) + \left(\frac{x+2y}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{y}\,\mathrm{d}x \\
\phi &= -\frac{x}{y} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x+2y}{y^2}$. Therefore equation (4) becomes

$$
\frac{x+2y}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{2}{y}\right) \mathrm{d}y$$
$$f(y) = 2\ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x}{y} + 2\ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x}{y} + 2\ln(y)$$

The solution becomes

$$y = \mathrm{e}^{\operatorname{LambertW}\left(\frac{x\,\mathrm{e}^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{\operatorname{LambertW}\left(\frac{x\,\mathrm{e}^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}} \tag{1}$$

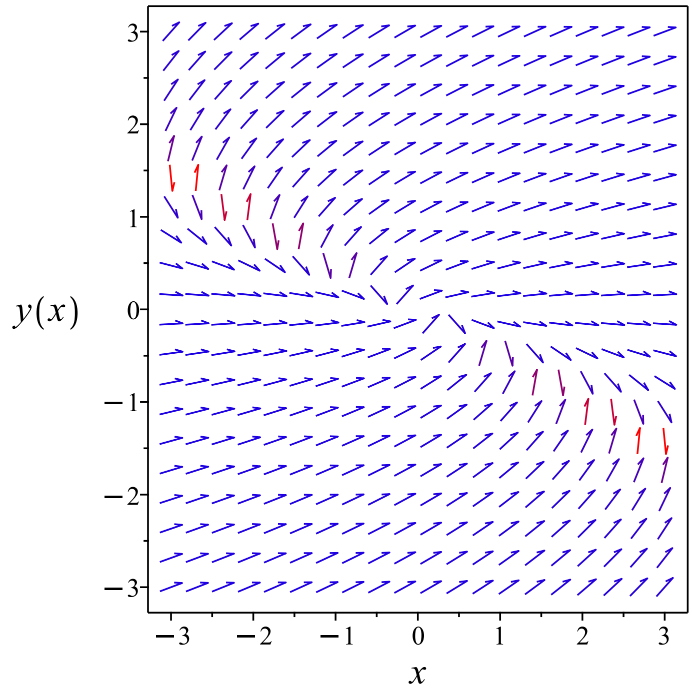


Figure 254: Slope field plot

Verification of solutions

$$y = e^{\mathrm{LambertW}\left(\frac{x\,e^{-\frac{c_1}{2}}}{2}\right)+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
<- 1st order linear successful
<- inverse linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve((x+2*y(x))*diff(y(x),x) = y(x),y(x), singsol=all)
```

$$y(x) = \frac{x}{2\,\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{\frac{c_1}{2}}}{2}\right)}$$

✓ Solution by Mathematica

Time used: 4.677 (sec). Leaf size: 31

```
DSolve[(x+2*y[x])*y'[x] == y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{2W\left(\frac{1}{2}e^{-\frac{c_1}{2}}x\right)}$$

$$y(x) \to 0$$

5.7 problem 7

Internal problem ID [85]
Internal file name [`OUTPUT/85_Sunday_June_05_2022_01_34_27_AM_34787952/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{xy^2y' - y^3 = x^3}$$

5.7.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^3u(x)^2\,(u'(x)\,x + u(x)) - u(x)^3\,x^3 = x^3$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{1}{u^2x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{1}{u^2}$. Integrating both sides gives

$$\frac{1}{\frac{1}{u^2}}\, du = \frac{1}{x}\, dx$$
$$\int \frac{1}{\frac{1}{u^2}}\, du = \int \frac{1}{x}\, dx$$
$$\frac{u^3}{3} = \ln(x) + c_2$$

The solution is

$$\frac{u(x)^3}{3} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^3}{3x^3} - \ln(x) - c_2 = 0$$
$$\frac{y^3}{3x^3} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$\frac{y^3}{3x^3} - \ln(x) - c_2 = 0 \tag{1}$$

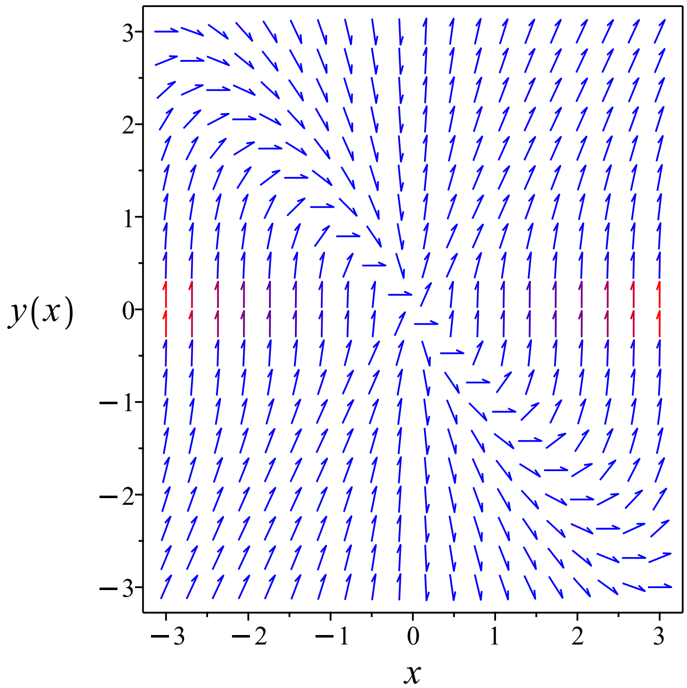


Figure 255: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^3} - \ln(x) - c_2 = 0$$

Verified OK.

5.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + y^3}{x\,y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 198: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{x^3}{y^2} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^3}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^3 + y^3}{x\, y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^3}{x^4} \\ S_y &= \frac{y^2}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

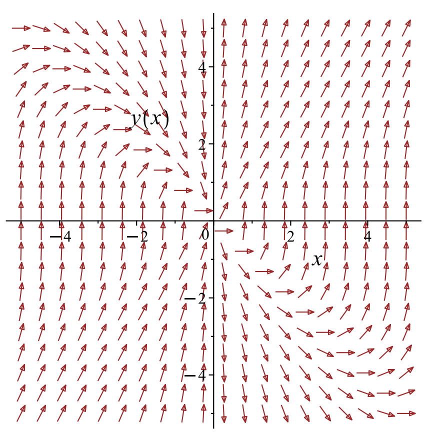
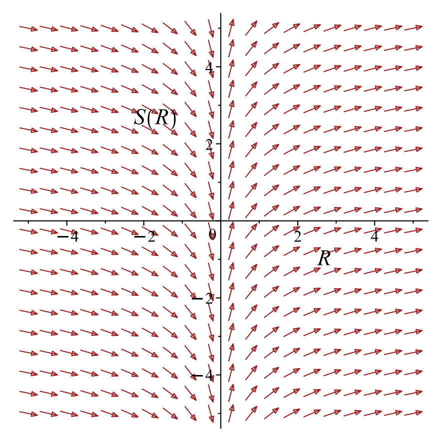
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x^3} = \ln(x) + c_1$$

Which simplifies to

$$\frac{y^3}{3x^3} = \ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+y^3}{x\,y^2}$ 	$R = x$ $S = \frac{y^3}{3x^3}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$\frac{y^3}{3x^3} = \ln(x) + c_1 \tag{1}$$

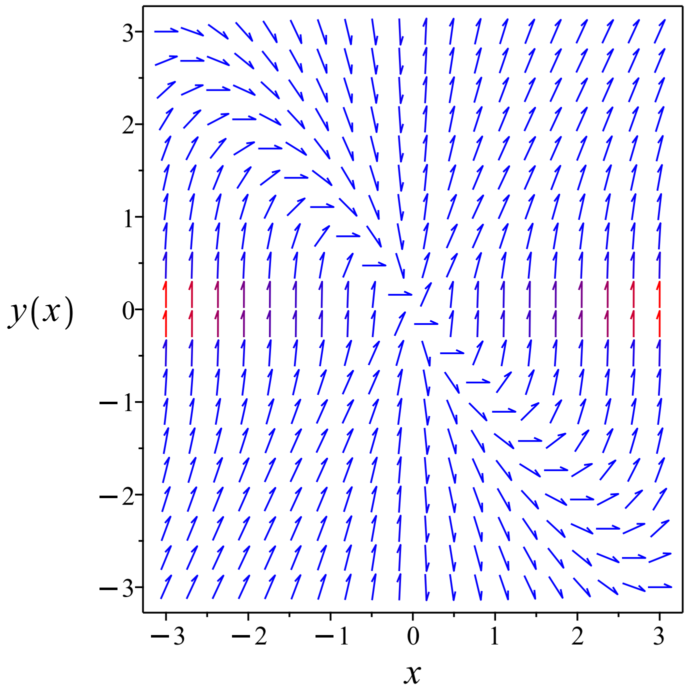


Figure 256: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^3} = \ln(x) + c_1$$

Verified OK.

5.7.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^3 + y^3}{x\, y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + x^2 \frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{1}{x} \\
f_1(x) &= x^2 \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = \frac{y^3}{x} + x^2 \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= \frac{w(x)}{x} + x^2 \\
w' &= \frac{3w}{x} + 3x^2
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{3}{x} \\
q(x) &= 3x^2
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = 3x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(3x^2\right) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(3x^2\right) \\ d\left(\frac{w}{x^3}\right) &= \left(\frac{3}{x}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int \frac{3}{x}\, dx \\ \frac{w}{x^3} &= 3\ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = 3\ln(x)\, x^3 + c_1 x^3$$

which simplifies to

$$w(x) = x^3(3\ln(x) + c_1)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x^3(3\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= (3\ln(x) + c_1)^{\frac{1}{3}}\, x \\ y(x) &= \frac{(3\ln(x) + c_1)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)x}{2} \\ y(x) &= -\frac{(3\ln(x) + c_1)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)x}{2}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = (3\ln(x) + c_1)^{\frac{1}{3}} x \tag{1}$$

$$y = \frac{(3\ln(x) + c_1)^{\frac{1}{3}} (i\sqrt{3} - 1) x}{2} \tag{2}$$

$$y = -\frac{(3\ln(x) + c_1)^{\frac{1}{3}} (1 + i\sqrt{3}) x}{2} \tag{3}$$

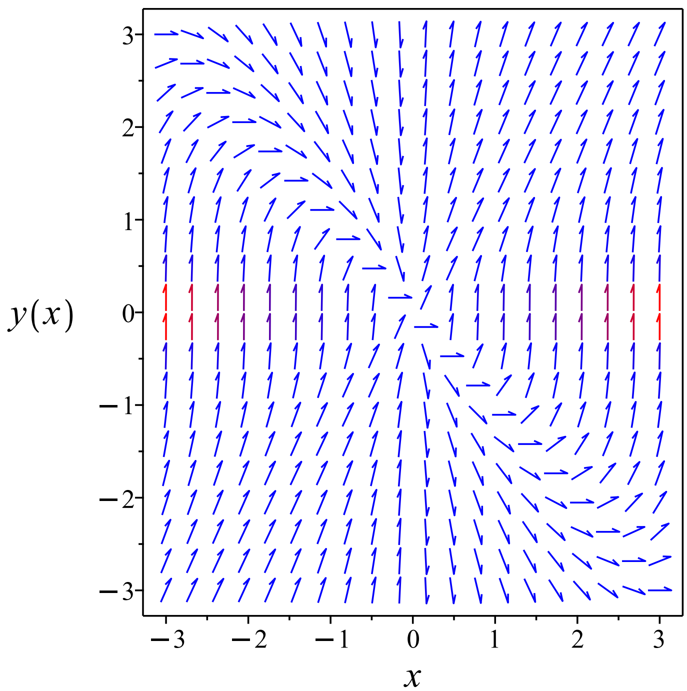


Figure 257: Slope field plot

Verification of solutions

$$y = (3\ln(x) + c_1)^{\frac{1}{3}} x$$

Verified OK.

$$y = \frac{(3\ln(x) + c_1)^{\frac{1}{3}} (i\sqrt{3} - 1) x}{2}$$

Verified OK.

$$y = -\frac{(3\ln(x) + c_1)^{\frac{1}{3}} (1 + i\sqrt{3}) x}{2}$$

Verified OK.

5.7.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x\,y^2)\,\mathrm{d}y &= (x^3+y^3)\,\mathrm{d}x\\ (-x^3-y^3)\,\mathrm{d}x + (x\,y^2)\,\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x^3-y^3\\ N(x,y) &= x\,y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x^3-y^3)\\ &= -3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x\,y^2)\\ &= y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x\,y^2}\left((-3y^2)-(y^2)\right)\\ &= -\frac{4}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\frac{4}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-4\ln(x)} \\ &= \frac{1}{x^4}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^4}\left(-x^3 - y^3\right) \\ &= \frac{-x^3 - y^3}{x^4}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^4}\left(x\,y^2\right) \\ &= \frac{y^2}{x^3}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{-x^3 - y^3}{x^4}\right) + \left(\frac{y^2}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-x^3 - y^3}{x^4} \, \mathrm{d}x$$

$$\phi = \frac{y^3}{3x^3} - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{y^2}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y^2}{x^3}$. Therefore equation (4) becomes

$$\frac{y^2}{x^3} = \frac{y^2}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^3}{3x^3} - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^3}{3x^3} - \ln(x)$$

Summary

The solution(s) found are the following

$$\frac{y^3}{3x^3} - \ln(x) = c_1 \tag{1}$$

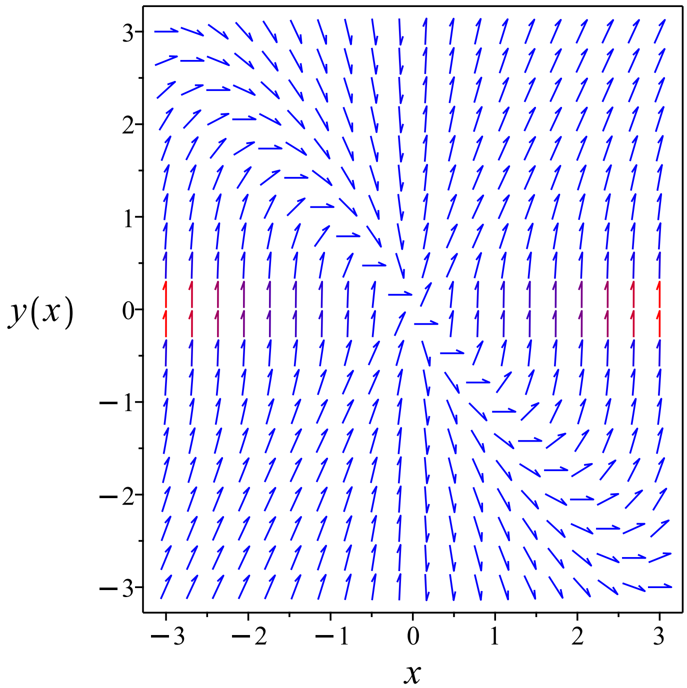


Figure 258: Slope field plot

Verification of solutions

$$\frac{y^3}{3x^3} - \ln(x) = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 58

```
dsolve(x*y(x)^2*diff(y(x),x) = x^3+y(x)^3,y(x), singsol=all)
```

$$y(x) = (3\ln(x) + c_1)^{\frac{1}{3}} x$$
$$y(x) = -\frac{(3\ln(x) + c_1)^{\frac{1}{3}} (1 + i\sqrt{3}) x}{2}$$
$$y(x) = \frac{(3\ln(x) + c_1)^{\frac{1}{3}} (i\sqrt{3} - 1) x}{2}$$

✓ Solution by Mathematica

Time used: 0.195 (sec). Leaf size: 63

```
DSolve[x*y[x]^2*y'[x] == x^3+y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x\sqrt[3]{3\log(x) + c_1}$$
$$y(x) \to -\sqrt[3]{-1}x\sqrt[3]{3\log(x) + c_1}$$
$$y(x) \to (-1)^{2/3}x\sqrt[3]{3\log(x) + c_1}$$

5.8 problem 8

Internal problem ID [86]
Internal file name [`OUTPUT/86_Sunday_June_05_2022_01_34_28_AM_64359794/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{y'x^2 - \mathrm{e}^{\frac{y}{x}}x^2 - yx = 0}$$

5.8.1 Solving as homogeneousTypeD ode

Writing the ode as

$$y' = \mathrm{e}^{\frac{y}{x}} + \frac{y}{x} \tag{A}$$

The given ode has the form

$$y' = \frac{y}{x} + g(x)\, f\left(b\frac{y}{x}\right)^{\frac{n}{m}} \tag{1}$$

Where b is scalar and $g(x)$ is function of x and n, m are integers. The solution is given in Kamke page 20. Using the substitution $y(x) = u(x)\, x$ then

$$\frac{dy}{dx} = \frac{du}{dx}x + u$$

Hence the given ode becomes

$$\begin{aligned} \frac{du}{dx}x + u &= u + g(x)\, f(bu)^{\frac{n}{m}} \\ u' &= \frac{1}{x}g(x)\, f(bu)^{\frac{n}{m}} \end{aligned} \tag{2}$$

The above ode is always separable. This is easily solved for u assuming the integration can be resolved, and then the solution to the original ode becomes $y = ux$. Comapring the given ode (A) with the form (1) shows that

$$
\begin{aligned}
g(x) &= 1 \\
b &= 1 \\
f\left(\frac{bx}{y}\right) &= e^{\frac{y}{x}}
\end{aligned}
$$

Substituting the above in (2) results in the $u(x)$ ode as

$$
u'(x) = \frac{e^{u(x)}}{x}
$$

Which is now solved as separable In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= \frac{e^{u}}{x}
\end{aligned}
$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{e^{u}}\, du &= \frac{1}{x}\, dx \\
\int \frac{1}{e^{u}}\, du &= \int \frac{1}{x}\, dx \\
-e^{-u} &= \ln(x) + c_1
\end{aligned}
$$

The solution is

$$
-e^{-u(x)} - \ln(x) - c_1 = 0
$$

Therefore the solution is found using $y = ux$. Hence

$$
-e^{-\frac{y}{x}} - \ln(x) - c_1 = 0
$$

Summary

The solution(s) found are the following

$$
-e^{-\frac{y}{x}} - \ln(x) - c_1 = 0 \tag{1}
$$

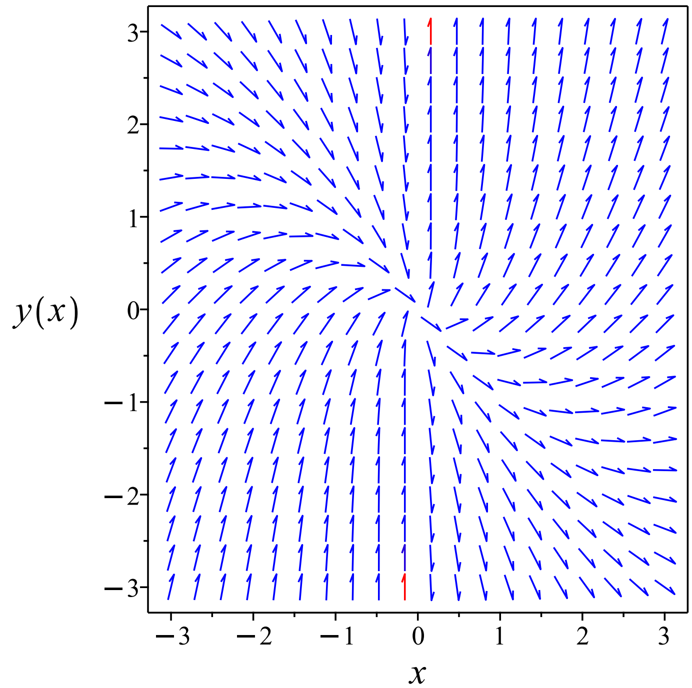


Figure 259: Slope field plot

Verification of solutions

$$-e^{-\frac{y}{x}} - \ln(x) - c_1 = 0$$

Verified OK.

5.8.2 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - e^{u(x)}x^2 - u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{e^u}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = e^u$. Integrating both sides gives

$$\frac{1}{e^u}\,du = \frac{1}{x}\,dx$$

$$\int \frac{1}{e^u} du = \int \frac{1}{x} dx$$
$$-e^{-u} = \ln(x) + c_2$$

The solution is

$$-e^{-u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$
$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0 \quad (1)$$

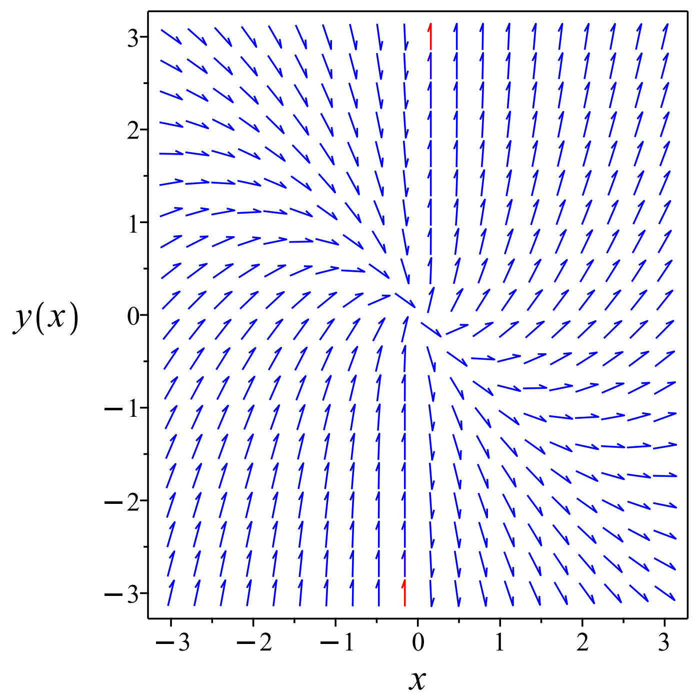


Figure 260: Slope field plot

Verification of solutions

$$-e^{-\frac{y}{x}} - \ln(x) - c_2 = 0$$

Verified OK.

5.8.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{e^{\frac{y}{x}} x + y}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type D`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 200: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= x^2 \\ \eta(x,y) &= yx \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi} \\ &= \frac{yx}{x^2} \\ &= \frac{y}{x}\end{aligned}$$

This is easily solved to give

$$y = c_1 x$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = \frac{y}{x}$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi} \\ &= \frac{dx}{x^2}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T} \\ &= -\frac{1}{x}\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{e^{\frac{y}{x}}x + y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= -\frac{y}{x^2} \\ R_y &= \frac{1}{x} \\ S_x &= \frac{1}{x^2} \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{e^{-\frac{y}{x}}}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -S(R)\,e^{-R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 e^{e^{-R}} \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x} = c_1 e^{e^{-\frac{y}{x}}}$$

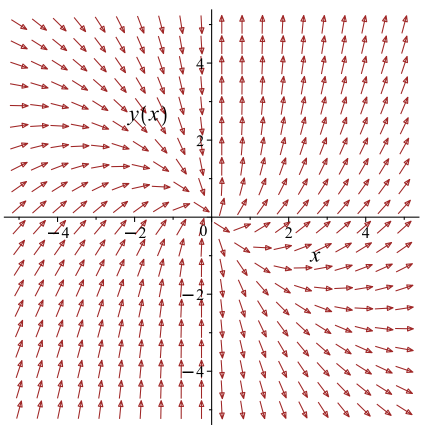
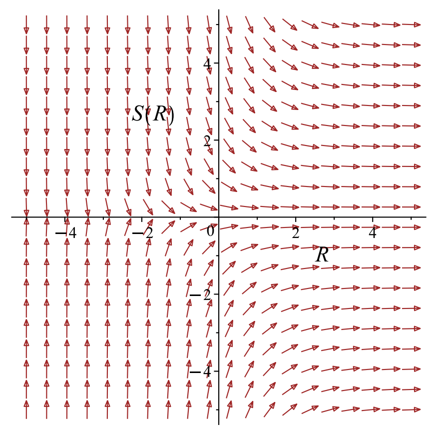
Which simplifies to

$$-\frac{1}{x} = c_1 \mathrm{e}^{\mathrm{e}^{-\frac{y}{x}}}$$

Which gives

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{\mathrm{e}^{\frac{y}{x}} x + y}{x}$ 	$R = \frac{y}{x}$ $S = -\frac{1}{x}$	$\frac{dS}{dR} = -S(R)\,\mathrm{e}^{-R}$ 

Summary

The solution(s) found are the following

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x \tag{1}$$

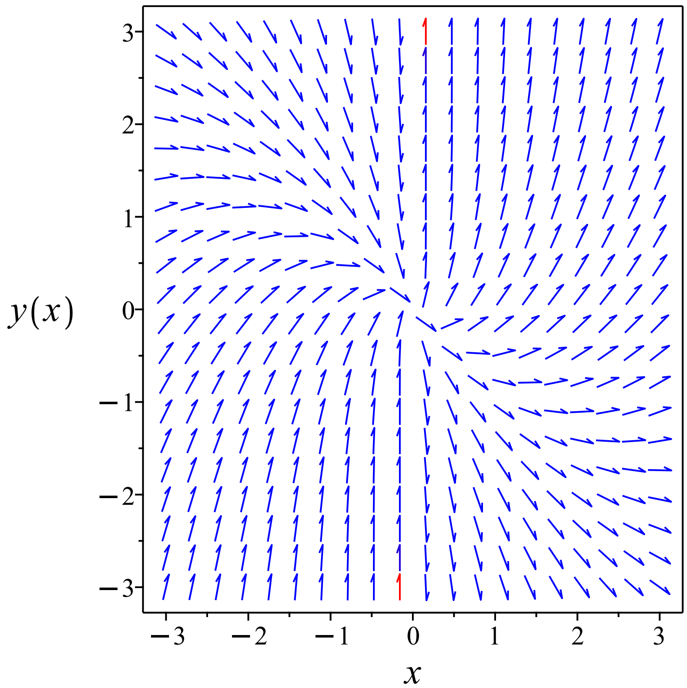


Figure 261: Slope field plot

Verification of solutions

$$y = -\ln\left(\ln\left(-\frac{1}{c_1 x}\right)\right) x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x) = exp(y(x)/x)*x^2+x*y(x),y(x), singsol=all)
```

$$y(x) = \ln\left(-\frac{1}{\ln(x) + c_1}\right)x$$

✓ Solution by Mathematica

Time used: 0.303 (sec). Leaf size: 18

```
DSolve[x^2*y'[x] == Exp[y[x]/x]*x^2+x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x\log(-\log(x) - c_1)$$

5.9 problem 9

Internal problem ID [87]
Internal file name [`OUTPUT/87_Sunday_June_05_2022_01_34_29_AM_6495535/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y'x^2 - yx - y^2 = 0}$$

5.9.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - u(x)\,x^2 - u(x)^2\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u^2}\,du = \int \frac{1}{x}\,dx$$
$$-\frac{1}{u} = \ln(x) + c_2$$

The solution is

$$-\frac{1}{u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} - \ln(x) - c_2 = 0$$
$$-\frac{x}{y} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} - \ln(x) - c_2 = 0 \tag{1}$$

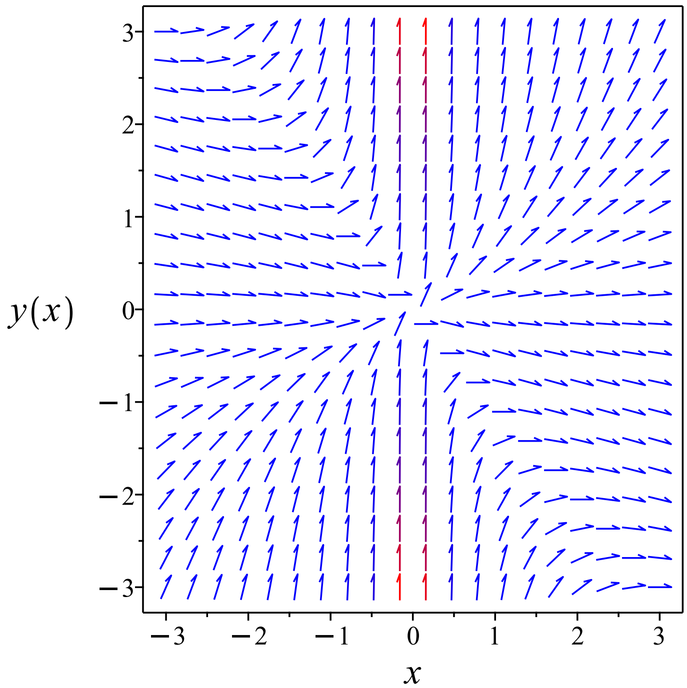


Figure 262: Slope field plot

Verification of solutions

$$-\frac{x}{y} - \ln(x) - c_2 = 0$$

Verified OK.

5.9.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x+y)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 202: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(x+y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = \ln(x) + c_1$$

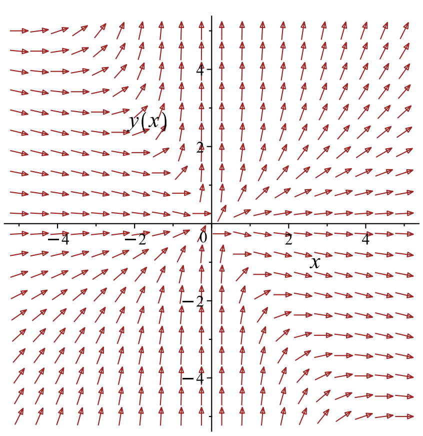
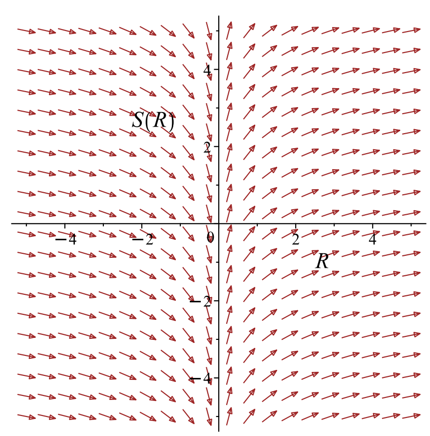
Which simplifies to

$$-\frac{x}{y} = \ln(x) + c_1$$

Which gives

$$y = -\frac{x}{\ln(x) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x+y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{x}{\ln(x) + c_1} \tag{1}$$

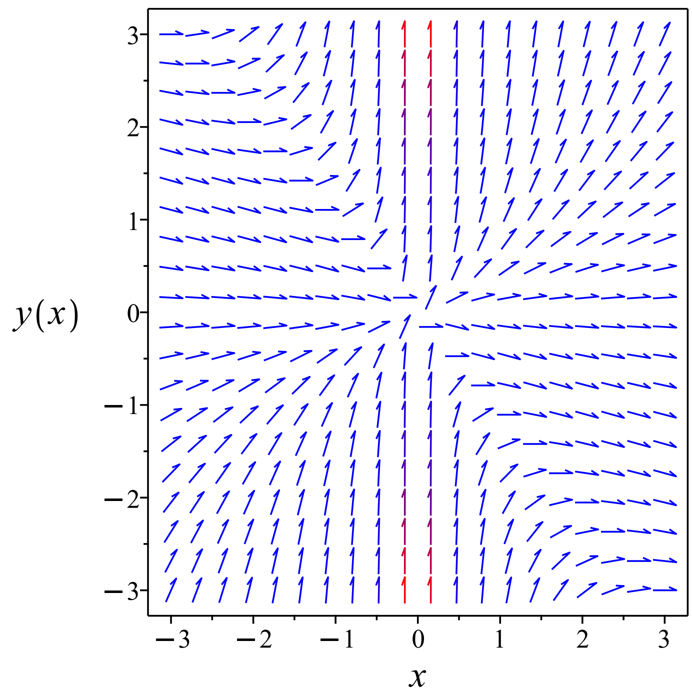


Figure 263: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) + c_1}$$

Verified OK.

5.9.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= \frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{xy} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} + \frac{1}{x^2} \\ w' &= -\frac{w}{x} - \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -\frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x^2}\right) \\ \frac{d}{dx}(wx) &= (x)\left(-\frac{1}{x^2}\right) \\ d(wx) &= \left(-\frac{1}{x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} wx &= \int -\frac{1}{x}\, dx \\ wx &= -\ln(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{-\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{-\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{-\ln(x) + c_1}$$

Summary
The solution(s) found are the following

$$y = \frac{x}{-\ln(x) + c_1} \tag{1}$$

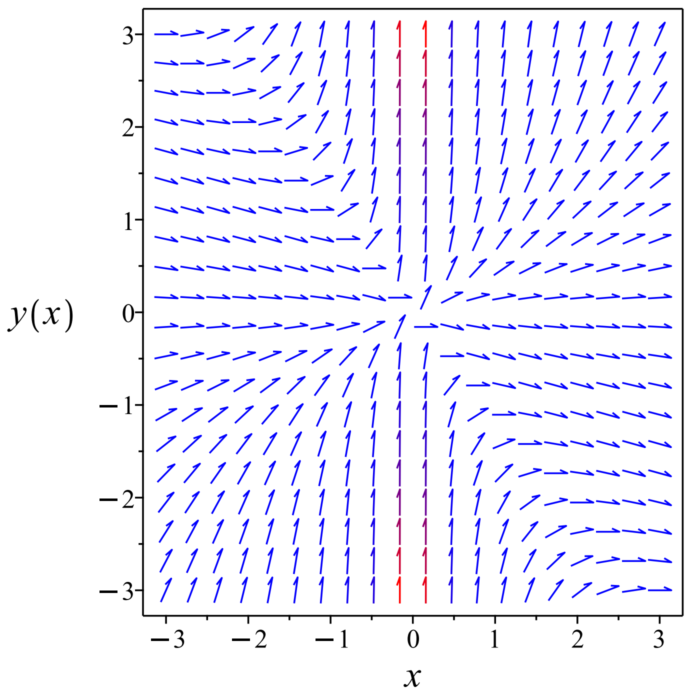


Figure 264: Slope field plot

Verification of solutions

$$y = \frac{x}{-\ln(x) + c_1}$$

Verified OK.

5.9.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y &= (yx + y^2)\,\mathrm{d}x\\ (-yx - y^2)\,\mathrm{d}x + (x^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -yx - y^2\\ N(x,y) &= x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-yx - y^2\right)\\ &= -x - 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2\right)\\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = -y^2 - yx$ and $N = x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{-y^2 - yx}{xy^2}\\ N &= \frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x}{y^2}\right)\mathrm{d}y &= \left(-\frac{-yx - y^2}{x\,y^2}\right)\mathrm{d}x\\ \left(\frac{-yx - y^2}{x\,y^2}\right)\mathrm{d}x + \left(\frac{x}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{-yx - y^2}{x\,y^2}\\ N(x, y) &= \frac{x}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-yx - y^2}{x\,y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right)\\ &= \frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-yx - y^2}{x\, y^2} \, \mathrm{d}x$$

$$\phi = -\ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) - \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) - \frac{x}{y}$$

The solution becomes

$$y = -\frac{x}{\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\ln(x) + c_1} \tag{1}$$

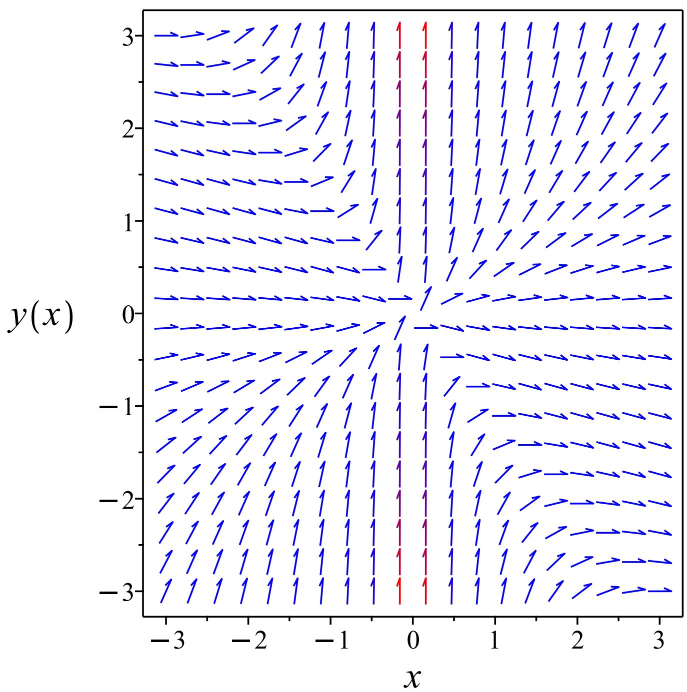


Figure 265: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) + c_1}$$

Verified OK.

5.9.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(x+y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{1}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x) + c_1$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 x}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x}{\ln(x) + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\ln(x) + c_3} \tag{1}$$

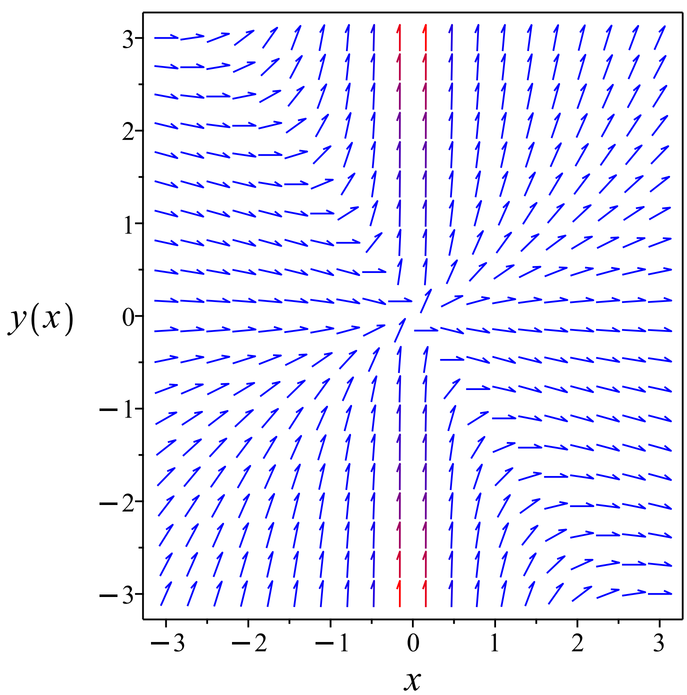


Figure 266: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x) = x*y(x)+y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1 - \ln(x)}$$

✓ Solution by Mathematica

Time used: 0.119 (sec). Leaf size: 21

```
DSolve[x^2*y'[x] == x*y[x]+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{-\log(x) + c_1}$$
$$y(x) \to 0$$

5.10 problem 10

Internal problem ID [88]
Internal file name [`OUTPUT/88_Sunday_June_05_2022_01_34_30_AM_21734797/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{xyy' - 3y^2 = x^2}$$

5.10.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$x^2 u(x)\,(u'(x)\,x + u(x)) - 3u(x)^2\,x^2 = x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{2u^2+1}{ux} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{\frac{2u^2+1}{u}}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{\frac{2u^2+1}{u}}\,du &= \int \frac{1}{x}\,dx \\
\frac{\ln(2u^2+1)}{4} &= \ln(x) + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
(2u^2+1)^{\frac{1}{4}} = e^{\ln(x)+c_2}
$$

Which simplifies to

$$
(2u^2+1)^{\frac{1}{4}} = c_3 x
$$

Which simplifies to

$$
(2u(x)^2+1)^{\frac{1}{4}} = c_3 e^{c_2} x
$$

The solution is

$$
(2u(x)^2+1)^{\frac{1}{4}} = c_3 e^{c_2} x
$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$
\begin{aligned}
\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}} &= c_3 e^{c_2} x \\
\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} &= c_3 e^{c_2} x
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = c_3 e^{c_2} x \tag{1}
$$

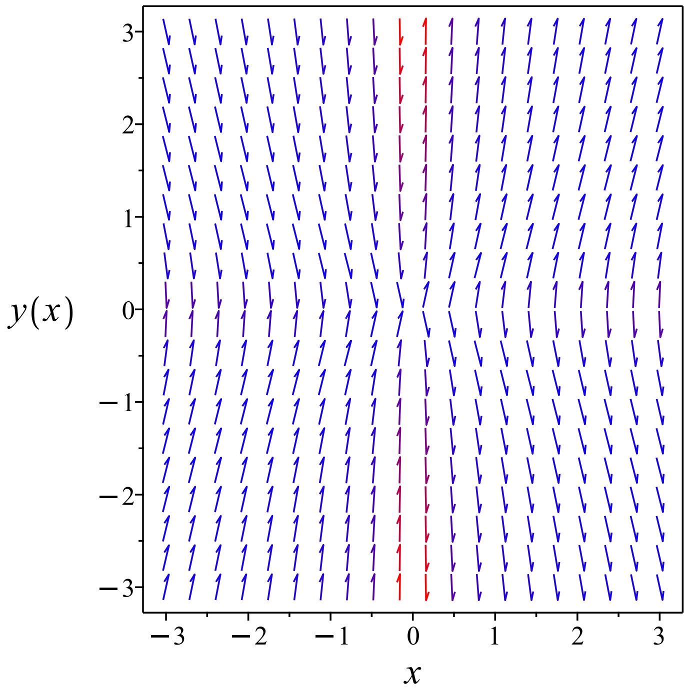


Figure 267: Slope field plot

Verification of solutions

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = c_3 \mathrm{e}^{c_2} x$$

Verified OK.

5.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^2+3y^2}{xy}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 204: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x^6}{y}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^6}{y}} dy \end{aligned}$$

Which results in

$$S = \frac{y^2}{2x^6}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x^2 + 3y^2}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y^2}{x^7} \\ S_y &= \frac{y}{x^6} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^5} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^5}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{4R^4} + c_1 \tag{4}$$

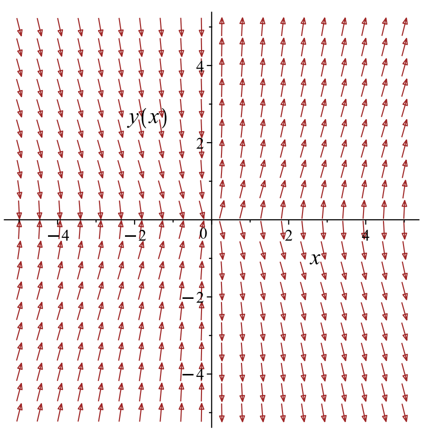
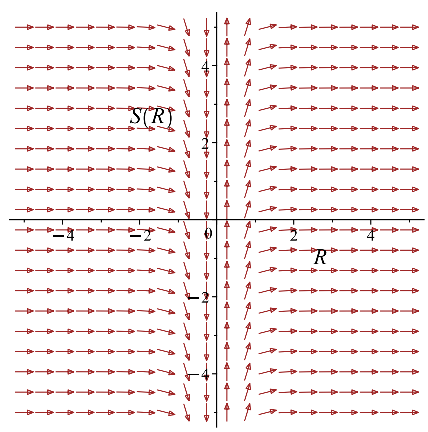
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^2}{2x^6} = -\frac{1}{4x^4} + c_1$$

Which simplifies to

$$\frac{y^2}{2x^6} = -\frac{1}{4x^4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^2+3y^2}{xy}$ 	$R = x$ $S = \frac{y^2}{2x^6}$	$\frac{dS}{dR} = \frac{1}{R^5}$ 

Summary

The solution(s) found are the following

$$\frac{y^2}{2x^6} = -\frac{1}{4x^4} + c_1 \tag{1}$$

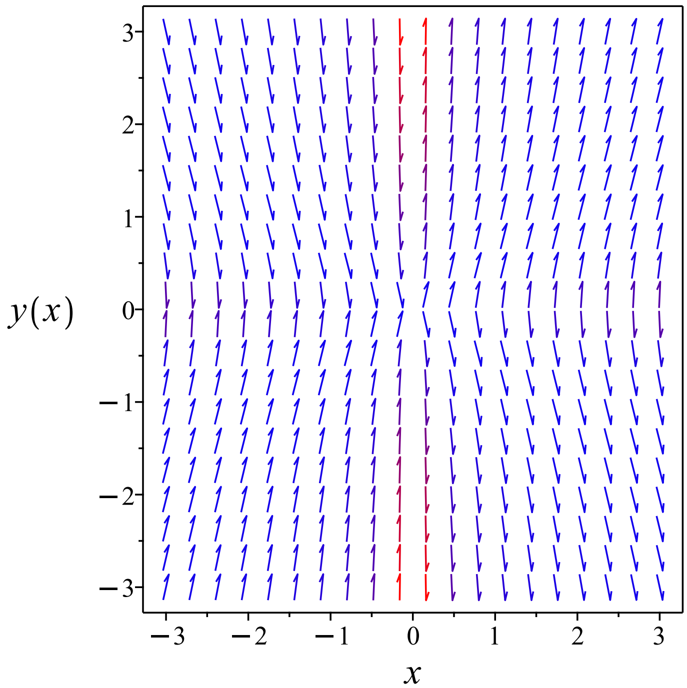


Figure 268: Slope field plot

<u>Verification of solutions</u>

$$\frac{y^2}{2x^6} = -\frac{1}{4x^4} + c_1$$

Verified OK.

5.10.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{x^2 + 3y^2}{xy} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{3}{x}y + x\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= \frac{3}{x} \\
f_1(x) &= x \\
n &= -1
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$
y'y = \frac{3y^2}{x} + x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^2
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 2yy' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{2} &= \frac{3w(x)}{x} + x \\
w' &= \frac{6w}{x} + 2x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{6}{x} \\
q(x) &= 2x
\end{aligned}
$$

Hence the ode is

$$w'(x) - \frac{6w(x)}{x} = 2x$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{6}{x}dx} \\ &= \frac{1}{x^6}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(2x) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^6}\right) &= \left(\frac{1}{x^6}\right)(2x) \\ \mathrm{d}\left(\frac{w}{x^6}\right) &= \left(\frac{2}{x^5}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^6} &= \int \frac{2}{x^5}\,\mathrm{d}x \\ \frac{w}{x^6} &= -\frac{1}{2x^4} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^6}$ results in

$$w(x) = -\frac{1}{2}x^2 + c_1x^6$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{1}{2}x^2 + c_1x^6$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{4c_1x^4 - 2}\,x}{2} \\ y(x) &= -\frac{\sqrt{4c_1x^4 - 2}\,x}{2}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\sqrt{4c_1x^4 - 2}\,x}{2} \tag{1}$$

$$y = -\frac{\sqrt{4c_1x^4 - 2}\,x}{2} \tag{2}$$

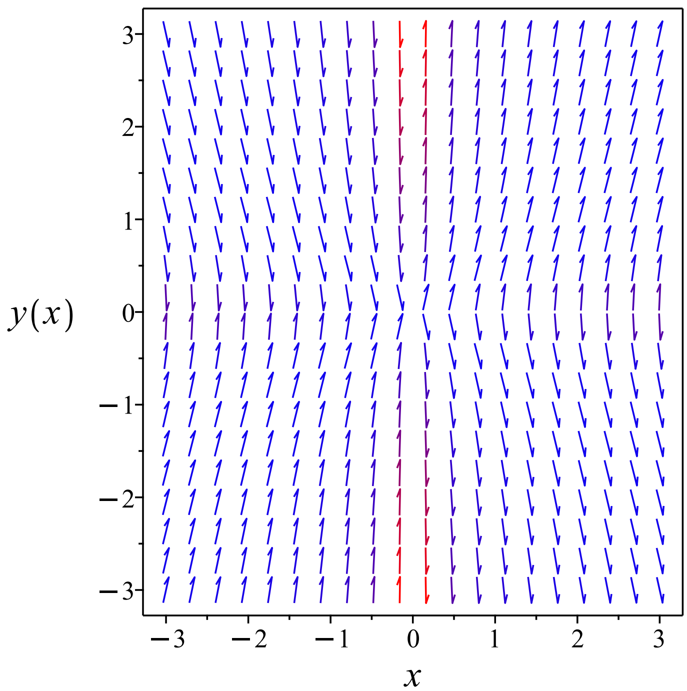


Figure 269: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{4c_1x^4 - 2}\,x}{2}$$

Verified OK.

$$y = -\frac{\sqrt{4c_1x^4 - 2}\,x}{2}$$

Verified OK.

5.10.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(yx)\,\mathrm{d}y &= \left(x^2 + 3y^2\right)\mathrm{d}x\\ \left(-x^2 - 3y^2\right)\mathrm{d}x + (yx)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -x^2 - 3y^2\\ N(x, y) &= yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 3y^2\right)\\ &= -6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(yx)\\ &= y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{xy}((-6y) - (y))\\ &= -\frac{7}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, \mathrm{d}x}\\ &= e^{\int -\frac{7}{x} \, \mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-7\ln(x)}\\ &= \frac{1}{x^7}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \frac{1}{x^7}\left(-x^2 - 3y^2\right)\\ &= \frac{-x^2 - 3y^2}{x^7}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^7}(yx) \\
&= \frac{y}{x^6}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-x^2 - 3y^2}{x^7}\right) + \left(\frac{y}{x^6}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \frac{-x^2 - 3y^2}{x^7} \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{x^2 + 2y^2}{4x^6} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{y}{x^6} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{y}{x^6}$. Therefore equation (4) becomes

$$
\frac{y}{x^6} = \frac{y}{x^6} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^2 + 2y^2}{4x^6} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^2 + 2y^2}{4x^6}$$

Summary

The solution(s) found are the following

$$\frac{2y^2 + x^2}{4x^6} = c_1 \tag{1}$$

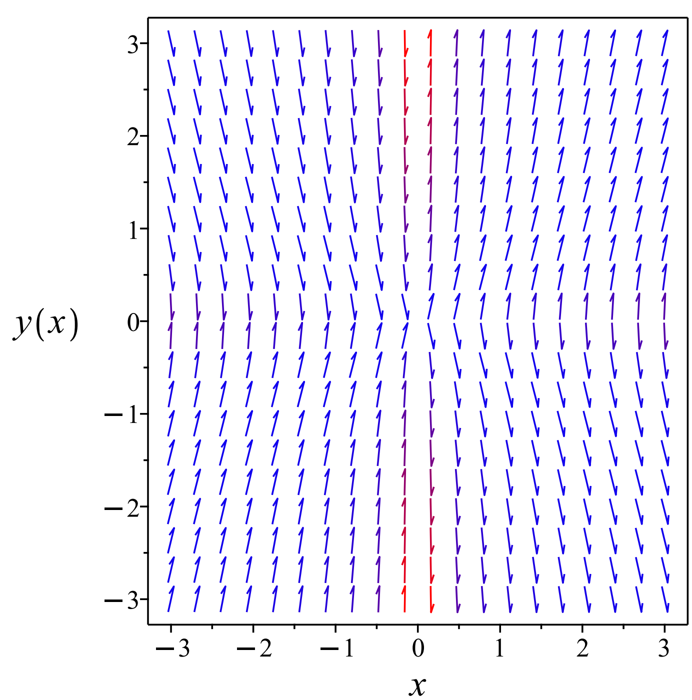


Figure 270: Slope field plot

Verification of solutions

$$\frac{2y^2+x^2}{4x^6}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(x*y(x)*diff(y(x),x) = x^2+3*y(x)^2,y(x), singsol=all)
```

$$y(x)=-\frac{\sqrt{4c_1x^4-2}\,x}{2}$$
$$y(x)=\frac{\sqrt{4c_1x^4-2}\,x}{2}$$

✓ Solution by Mathematica

Time used: 0.6 (sec). Leaf size: 42

```
DSolve[x*y[x]*y'[x] == x^2+3*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -x\sqrt{-\frac{1}{2}+c_1x^4}$$
$$y(x)\to x\sqrt{-\frac{1}{2}+c_1x^4}$$

5.11 problem 11

Internal problem ID [89]
Internal file name [OUTPUT/89_Sunday_June_05_2022_01_34_31_AM_60925206/index.tex]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{(x^2 - y^2)\, y' - 2yx = 0}$$

5.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$(x^2 - u(x)^2\, x^2)\, (u'(x)\, x + u(x)) - 2u(x)\, x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(u^2+1)}{(u^2-1)\, x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u(u^2+1)}{u^2-1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u^2+1)}{u^2-1}}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{\frac{u(u^2+1)}{u^2-1}}\, du = \int -\frac{1}{x}\, dx$$

$$-\ln(u) + \ln\left(u^2+1\right) = -\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{-\ln(u)+\ln\left(u^2+1\right)} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{u^2+1}{u} = \frac{c_3}{x}$$

The solution is

$$\frac{u(x)^2+1}{u(x)} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{x\left(\frac{y^2}{x^2}+1\right)}{y} = \frac{c_3}{x}$$

$$\frac{x^2+y^2}{xy} = \frac{c_3}{x}$$

Which simplifies to

$$\frac{x^2+y^2}{y} = c_3$$

Summary

The solution(s) found are the following

$$\frac{x^2+y^2}{y} = c_3 \tag{1}$$

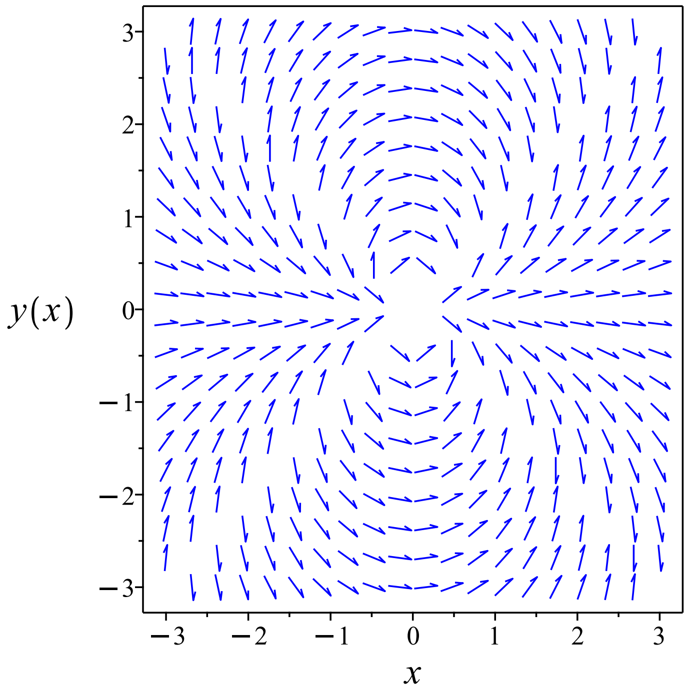


Figure 271: Slope field plot

Verification of solutions

$$\frac{x^2 + y^2}{y} = c_3$$

Verified OK.

5.11.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2yx}{-x^2 + y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} &b_2 - \frac{2yx(b_3 - a_2)}{-x^2 + y^2} - \frac{4y^2x^2a_3}{(-x^2 + y^2)^2} \\ &- \left(-\frac{2y}{-x^2 + y^2} - \frac{4y\,x^2}{(-x^2 + y^2)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(-\frac{2x}{-x^2 + y^2} + \frac{4y^2x}{(-x^2 + y^2)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{x^4b_2 + 2y^2x^2a_3 + 4x^2y^2b_2 - 4x\,y^3a_2 + 4x\,y^3b_3 - 2y^4a_3 - y^4b_2 + 2x^3b_1 - 2x^2ya_1 + 2x\,y^2b_1 - 2y^3a_1}{(x^2 - y^2)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned} &-x^4b_2 - 2y^2x^2a_3 - 4x^2y^2b_2 + 4x\,y^3a_2 - 4x\,y^3b_3 + 2y^4a_3 \\ &+ y^4b_2 - 2x^3b_1 + 2x^2ya_1 - 2x\,y^2b_1 + 2y^3a_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &4a_2v_1v_2^3 - 2a_3v_1^2v_2^2 + 2a_3v_2^4 - b_2v_1^4 - 4b_2v_1^2v_2^2 + b_2v_2^4 \\ &- 4b_3v_1v_2^3 + 2a_1v_1^2v_2 + 2a_1v_2^3 - 2b_1v_1^3 - 2b_1v_1v_2^2 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$-b_2 v_1^4 - 2b_1 v_1^3 + (-2a_3 - 4b_2) v_1^2 v_2^2 + 2a_1 v_1^2 v_2 + (4a_2 - 4b_3) v_1 v_2^3 - 2b_1 v_1 v_2^2 + (2a_3 + b_2) v_2^4 + 2a_1 v_2^3 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} 2a_1 &= 0 \\ -2b_1 &= 0 \\ -b_2 &= 0 \\ 4a_2 - 4b_3 &= 0 \\ -2a_3 - 4b_2 &= 0 \\ 2a_3 + b_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned} \eta &= \eta - \omega(x, y)\, \xi \\ &= y - \left(-\frac{2yx}{-x^2 + y^2}\right)(x) \\ &= \frac{-y\, x^2 - y^3}{x^2 - y^2} \\ \xi &= 0 \end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{-y\,x^2 - y^3}{x^2 - y^2}} dy \end{aligned}$$

Which results in

$$S = -\ln(y) + \ln\left(x^2 + y^2\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2yx}{-x^2 + y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x}{x^2 + y^2} \\ S_y &= -\frac{1}{y} + \frac{2y}{x^2 + y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

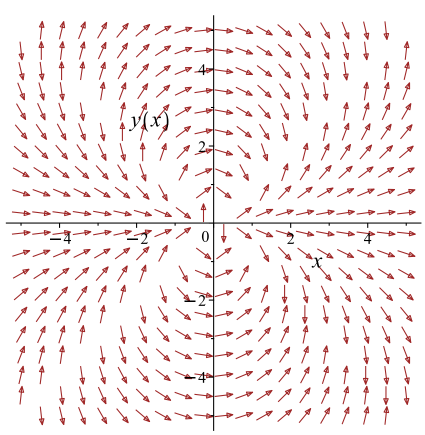
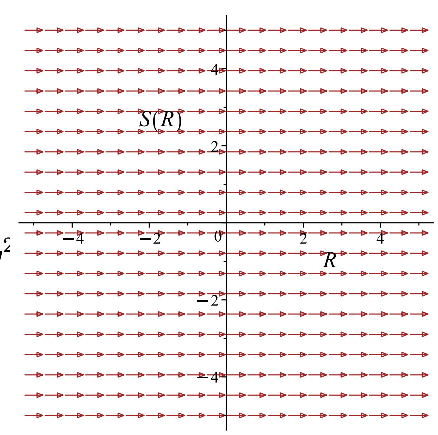
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln(y) + \ln\left(x^2 + y^2\right) = c_1$$

Which simplifies to

$$-\ln(y) + \ln\left(x^2 + y^2\right) = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2yx}{-x^2+y^2}$ 	$R = x$ $S = -\ln(y) + \ln\left(x^2 + y^2\right)$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$-\ln(y)+\ln\left(x^2+y^2\right)=c_1 \tag{1}$$

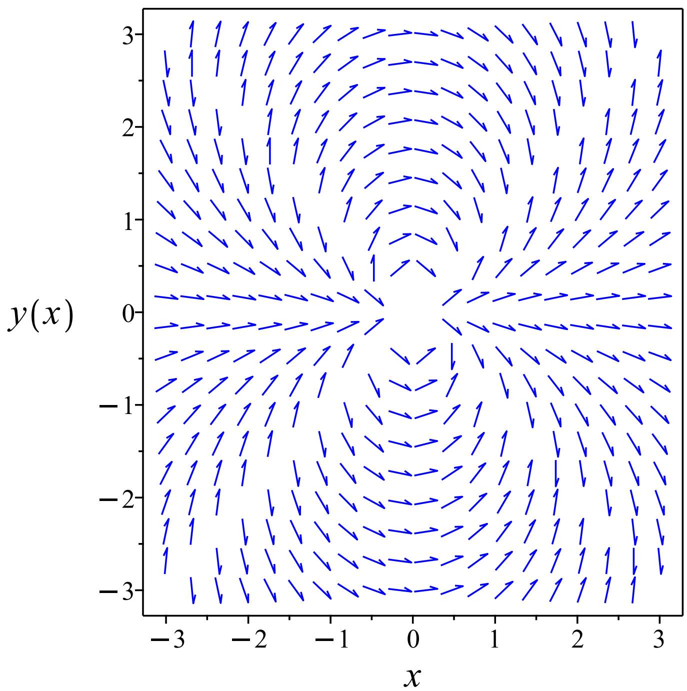


Figure 272: Slope field plot

Verification of solutions

$$-\ln(y)+\ln\left(x^2+y^2\right)=c_1$$

Verified OK.

5.11.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 - y^2)\,\mathrm{d}y &= (2yx)\,\mathrm{d}x \\ (-2yx)\,\mathrm{d}x + (x^2 - y^2)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2yx \\ N(x,y) &= x^2 - y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-2yx) \\ &= -2x\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(x^2 - y^2\right)\\
&= 2x
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\
&= \frac{1}{x^2 - y^2}((-2x) - (2x))\\
&= -\frac{4x}{x^2 - y^2}
\end{aligned}
$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$
\begin{aligned}
B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\\
&= -\frac{1}{2xy}((2x) - (-2x))\\
&= -\frac{2}{y}
\end{aligned}
$$

Since B does not depend on x, it can be used to obtain an integrating factor. Let the integrating factor be μ. Then

$$
\begin{aligned}
\mu &= e^{\int B \, \mathrm{d}y}\\
&= e^{\int -\frac{2}{y} \, \mathrm{d}y}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-2\ln(y)}\\
&= \frac{1}{y^2}
\end{aligned}
$$

M and N are now multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M\\
&= \frac{1}{y^2}(-2yx)\\
&= -\frac{2x}{y}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{y^2}(x^2 - y^2) \\
&= \frac{x^2 - y^2}{y^2}
\end{aligned}
$$

So now a modified ODE is obtained from the original ODE which will be exact and can be solved using the standard method. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{2x}{y}\right) + \left(\frac{x^2 - y^2}{y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \overline{M} \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int -\frac{2x}{y} \mathrm{d}x \\
\phi &= -\frac{x^2}{y} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{x^2}{y^2} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x^2 - y^2}{y^2}$. Therefore equation (4) becomes

$$
\frac{x^2 - y^2}{y^2} = \frac{x^2}{y^2} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = -1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int (-1)\,\mathrm{d}y$$
$$f(y) = -y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^2}{y} - y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{y} - y$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{y} - y = c_1 \tag{1}$$

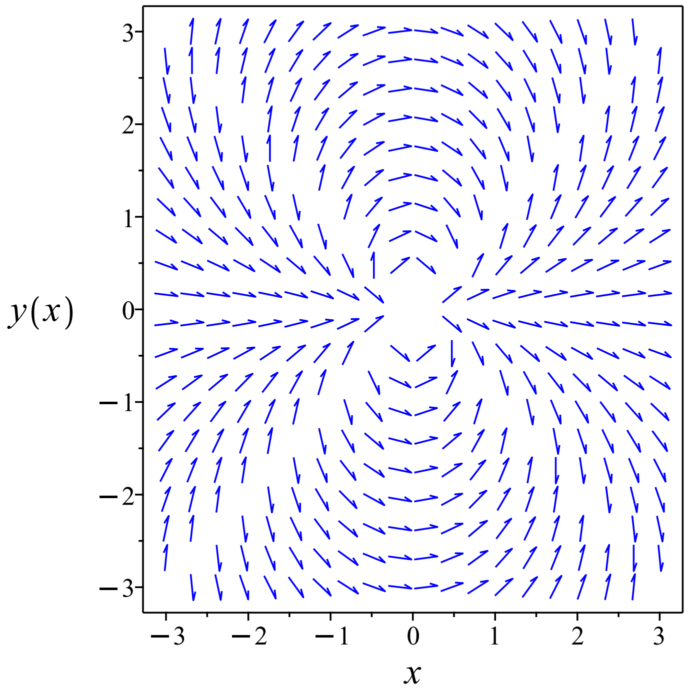


Figure 273: Slope field plot

Verification of solutions

$$-\frac{x^2}{y} - y = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 47

```
dsolve((x^2-y(x)^2)*diff(y(x),x) = 2*x*y(x),y(x), singsol=all)
```

$$y(x) = \frac{1-\sqrt{-4c_1^2x^2+1}}{2c_1}$$
$$y(x) = \frac{1+\sqrt{-4c_1^2x^2+1}}{2c_1}$$

✓ Solution by Mathematica

Time used: 1.035 (sec). Leaf size: 66

```
DSolve[(x^2-y[x]^2)*y'[x]== 2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(e^{c_1} - \sqrt{-4x^2+e^{2c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{-4x^2+e^{2c_1}} + e^{c_1}\right)$$
$$y(x) \to 0$$

5.12 problem 12

Internal problem ID [90]
Internal file name [`OUTPUT/90_Sunday_June_05_2022_01_34_33_AM_27424058/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _dAlembert]
```

$$\boxed{xyy' - y^2 - x\sqrt{4x^2 + y^2} = 0}$$

5.12.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y^2 + x\sqrt{4x^2 + y^2}}{xy} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2+\frac{\left(y^2+x\sqrt{4x^2+y^2}\right)(b_3-a_2)}{xy}-\frac{\left(y^2+x\sqrt{4x^2+y^2}\right)^2 a_3}{x^2y^2} \\ -\left(\frac{\sqrt{4x^2+y^2}+\frac{4x^2}{\sqrt{4x^2+y^2}}}{xy}-\frac{y^2+x\sqrt{4x^2+y^2}}{x^2y}\right)(xa_2+ya_3+a_1) \\ -\left(\frac{2y+\frac{xy}{\sqrt{4x^2+y^2}}}{xy}-\frac{y^2+x\sqrt{4x^2+y^2}}{x\,y^2}\right)(xb_2+yb_3+b_1)=0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{\left(4x^2+y^2\right)^{\frac{3}{2}}x^2a_3-4x^5b_2+8x^4ya_2-8x^4yb_3+12x^3y^2a_3+x^2y^3a_2-x^2y^3b_3+2x\,y^4a_3+\sqrt{4x^2+y^2}\,x\,y^2b}{x^2y^2\sqrt{4x^2+y^2}} \\ =0$$

Setting the numerator to zero gives

$$-\left(4x^2+y^2\right)^{\frac{3}{2}}x^2a_3+4x^5b_2-8x^4ya_2+8x^4yb_3-12x^3y^2a_3-x^2y^3a_2+x^2y^3b_3 \\ -2x\,y^4a_3-\sqrt{4x^2+y^2}\,x\,y^2b_1+\sqrt{4x^2+y^2}\,y^3a_1+4x^4b_1-4x^3ya_1=0 \tag{6E}$$

Simplifying the above gives

$$-\left(4x^2+y^2\right)^{\frac{3}{2}}x^2a_3+\left(4x^2+y^2\right)x^3b_2-\left(4x^2+y^2\right)x^2ya_2+2\left(4x^2+y^2\right)x^2yb_3 \\ -2\left(4x^2+y^2\right)x\,y^2a_3-4x^4ya_2-4x^3y^2a_3-x^3y^2b_2-x^2y^3b_3+\left(4x^2+y^2\right)x^2b_1 \\ -\sqrt{4x^2+y^2}\,x\,y^2b_1+\sqrt{4x^2+y^2}\,y^3a_1-4x^3ya_1-x^2y^2b_1=0 \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$4x^5b_2-4x^4\sqrt{4x^2+y^2}\,a_3-8x^4ya_2+8x^4yb_3-12x^3y^2a_3-x^2\sqrt{4x^2+y^2}\,y^2a_3-x^2y^3a_2 \\ +x^2y^3b_3-2x\,y^4a_3+4x^4b_1-4x^3ya_1-\sqrt{4x^2+y^2}\,x\,y^2b_1+\sqrt{4x^2+y^2}\,y^3a_1=0$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\left\{x,y,\sqrt{4x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{4x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$-8v_1^4 v_2 a_2 - v_1^2 v_2^3 a_2 - 4v_1^4 v_3 a_3 - 12v_1^3 v_2^2 a_3 - v_1^2 v_3 v_2^2 a_3 - 2v_1 v_2^4 a_3 + 4v_1^5 b_2 + 8v_1^4 v_2 b_3 + v_1^2 v_2^3 b_3 - 4v_1^3 v_2 a_1 + v_3 v_2^3 a_1 + 4v_1^4 b_1 - v_3 v_1 v_2^2 b_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$4v_1^5 b_2 + (-8a_2 + 8b_3) v_1^4 v_2 - 4v_1^4 v_3 a_3 + 4v_1^4 b_1 - 12v_1^3 v_2^2 a_3 - 4v_1^3 v_2 a_1 + (b_3 - a_2) v_1^2 v_2^3 - v_1^2 v_3 v_2^2 a_3 - 2v_1 v_2^4 a_3 - v_3 v_1 v_2^2 b_1 + v_3 v_2^3 a_1 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
a_1 &= 0 \\
-4a_1 &= 0 \\
-12a_3 &= 0 \\
-4a_3 &= 0 \\
-2a_3 &= 0 \\
-a_3 &= 0 \\
-b_1 &= 0 \\
4b_1 &= 0 \\
4b_2 &= 0 \\
-8a_2 + 8b_3 &= 0 \\
b_3 - a_2 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(\frac{y^2 + x\sqrt{4x^2+y^2}}{xy}\right)(x) \\
&= -\frac{x\sqrt{4x^2+y^2}}{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\frac{x\sqrt{4x^2+y^2}}{y}} dy
\end{aligned}
$$

Which results in

$$S = -\frac{\sqrt{4x^2 + y^2}}{x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y^2 + x\sqrt{4x^2 + y^2}}{xy}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{y^2}{x^2\sqrt{4x^2 + y^2}} \\
S_y &= -\frac{y}{x\sqrt{4x^2 + y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

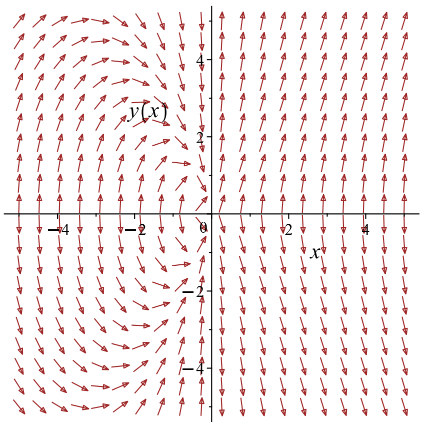
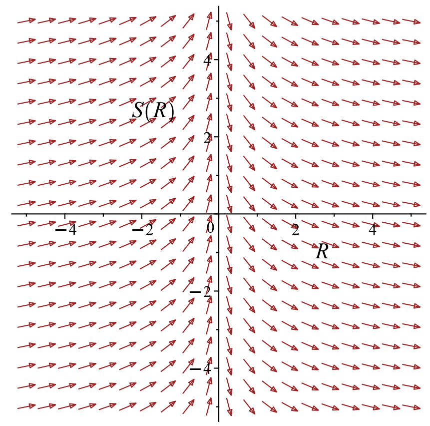
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\sqrt{4x^2 + y^2}}{x} = -\ln(x) + c_1$$

Which simplifies to

$$
-\frac{\sqrt{4x^2+y^2}}{x} = -\ln(x) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2+x\sqrt{4x^2+y^2}}{xy}$ 	$R = x$ $S = -\frac{\sqrt{4x^2+y^2}}{x}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary
The solution(s) found are the following

$$
-\frac{\sqrt{4x^2+y^2}}{x} = -\ln(x) + c_1 \tag{1}
$$

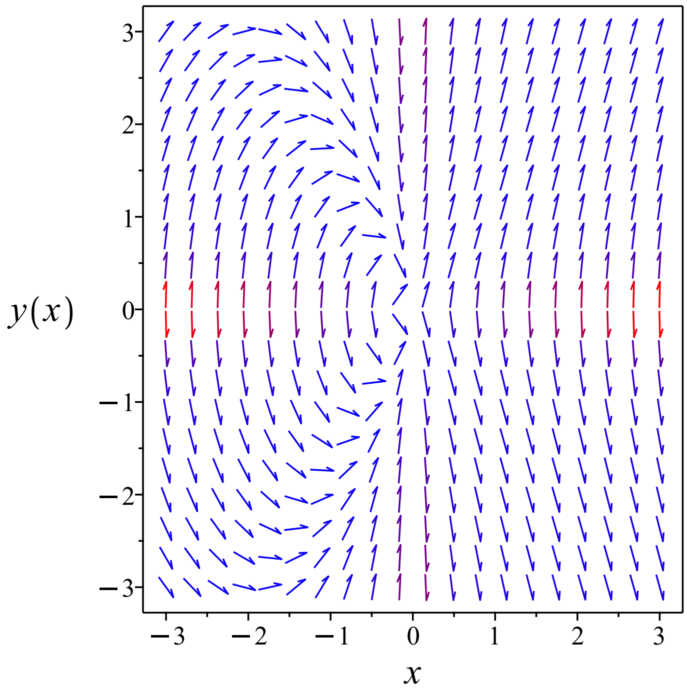


Figure 274: Slope field plot

Verification of solutions

$$-\frac{\sqrt{4x^2+y^2}}{x} = -\ln(x) + c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(x*y(x)*diff(y(x),x) = y(x)^2+x*(4*x^2+y(x)^2)^(1/2),y(x), singsol=all)
```

$$\frac{x \ln (x)-c_1 x-\sqrt{4 x^2+y(x)^2}}{x}=0$$

✓ Solution by Mathematica

Time used: 0.256 (sec). Leaf size: 54

```
DSolve[x*y[x]*y'[x] == y[x]^2+x*(4*x^2+y[x]^2)^(1/2),y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \rightarrow -x \sqrt{\log^2(x)+2 c_1 \log(x)-4+c_1{}^2}$$
$$y(x) \rightarrow x \sqrt{\log^2(x)+2 c_1 \log(x)-4+c_1{}^2}$$

5.13 problem 13

Internal problem ID [91]
Internal file name [`OUTPUT/91_Sunday_June_05_2022_01_34_34_AM_26106079/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{y'x - y - \sqrt{x^2 + y^2} = 0}$$

5.13.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{y + \sqrt{x^2 + y^2}}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{\left(y+\sqrt{x^2+y^2}\right)(b_3-a_2)}{x} - \frac{\left(y+\sqrt{x^2+y^2}\right)^2 a_3}{x^2} \\
&- \left(\frac{1}{\sqrt{x^2+y^2}} - \frac{y+\sqrt{x^2+y^2}}{x^2}\right)(xa_2+ya_3+a_1) \\
&- \frac{\left(1+\frac{y}{\sqrt{x^2+y^2}}\right)(xb_2+yb_3+b_1)}{x} = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
-\frac{(x^2+y^2)^{\frac{3}{2}} a_3 + x^3a_2 - x^3b_3 + 2x^2ya_3 + x^2yb_2 + y^3a_3 + \sqrt{x^2+y^2}\, xb_1 - \sqrt{x^2+y^2}\, ya_1 + xyb_1 - y^2a_1}{\sqrt{x^2+y^2}\, x^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 - x^3a_2 + x^3b_3 - 2x^2ya_3 - x^2yb_2 - y^3a_3 \\
&- \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - xyb_1 + y^2a_1 = 0
\end{aligned} \tag{6E}
$$

Simplifying the above gives

$$
\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3 + \left(x^2+y^2\right) xb_3 - \left(x^2+y^2\right) ya_3 - x^3a_2 - x^2ya_3 - x^2yb_2 \\
&- x\,y^2b_3 + \left(x^2+y^2\right) a_1 - \sqrt{x^2+y^2}\, xb_1 + \sqrt{x^2+y^2}\, ya_1 - x^2a_1 - xyb_1 = 0
\end{aligned} \tag{6E}
$$

Since the PDE has radicals, simplifying gives

$$
\begin{aligned}
&-x^3a_2 + x^3b_3 - x^2\sqrt{x^2+y^2}\, a_3 - 2x^2ya_3 - x^2yb_2 - \sqrt{x^2+y^2}\, y^2a_3 \\
&- y^3a_3 - \sqrt{x^2+y^2}\, xb_1 - xyb_1 + \sqrt{x^2+y^2}\, ya_1 + y^2a_1 = 0
\end{aligned}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$
\left\{x, y, \sqrt{x^2+y^2}\right\}
$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2 + y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&-v_1^3 a_2 - 2v_1^2 v_2 a_3 - v_1^2 v_3 a_3 - v_2^3 a_3 - v_3 v_2^2 a_3 - v_1^2 v_2 b_2 \\ &+ v_1^3 b_3 + v_2^2 a_1 + v_3 v_2 a_1 - v_1 v_2 b_1 - v_3 v_1 b_1 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned}&(b_3 - a_2) v_1^3 + (-2a_3 - b_2) v_1^2 v_2 - v_1^2 v_3 a_3 - v_1 v_2 b_1 \\ &- v_3 v_1 b_1 - v_2^3 a_3 - v_3 v_2^2 a_3 + v_2^2 a_1 + v_3 v_2 a_1 = 0\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ -a_3 &= 0 \\ -b_1 &= 0 \\ -2a_3 - b_2 &= 0 \\ b_3 - a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= y - \left(\frac{y + \sqrt{x^2 + y^2}}{x}\right)(x) \\
&= -\sqrt{x^2 + y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{-\sqrt{x^2 + y^2}} dy
\end{aligned}$$

Which results in

$$S = -\ln\left(y + \sqrt{x^2 + y^2}\right)$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y + \sqrt{x^2 + y^2}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{x}{\sqrt{x^2 + y^2}\,\left(y + \sqrt{x^2 + y^2}\right)} \\
S_y &= -\frac{1}{\sqrt{x^2 + y^2}}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{2\left(\sqrt{x^2 + y^2}\, y + x^2 + y^2\right)}{x\sqrt{x^2 + y^2}\,\left(y + \sqrt{x^2 + y^2}\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{2}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -2\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\ln\left(y + \sqrt{x^2 + y^2}\right) = -2\ln(x) + c_1$$

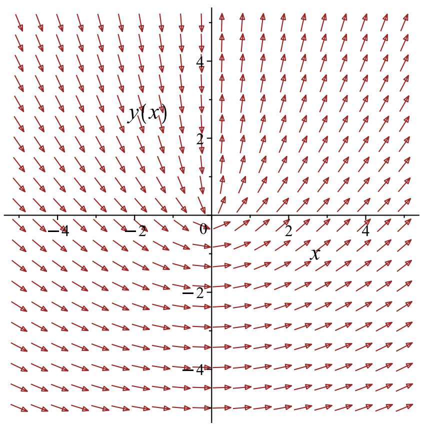
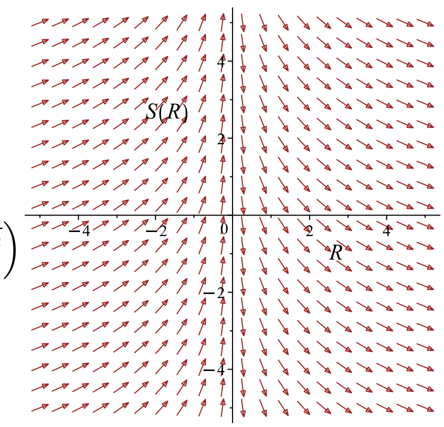
Which simplifies to

$$-\ln\left(y + \sqrt{x^2 + y^2}\right) = -2\ln(x) + c_1$$

Which gives

$$y = -\frac{e^{-c_1}\left(e^{2c_1} - x^2\right)}{2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x}$ 	$R = x$ $S = -\ln\left(y + \sqrt{x^2 + y^2}\right)$	$\frac{dS}{dR} = -\frac{2}{R}$ 

Summary

The solution(s) found are the following

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{2} \tag{1}$$

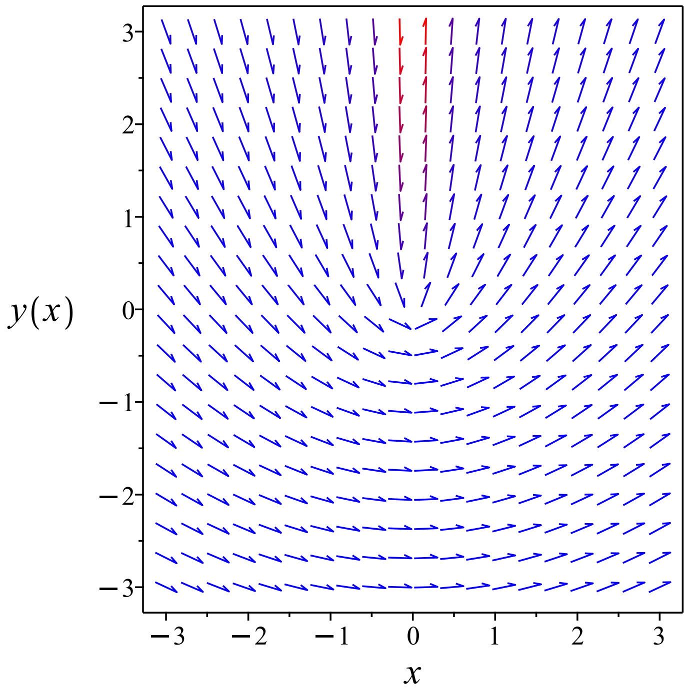


Figure 275: Slope field plot

Verification of solutions

$$y = -\frac{e^{-c_1}(e^{2c_1} - x^2)}{2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 26

```
dsolve(x*diff(y(x),x) = y(x)+(x^2+y(x)^2)^(1/2),y(x), singsol=all)
```

$$\frac{-c_1 x^2 + \sqrt{x^2 + y(x)^2} + y(x)}{x^2} = 0$$

✓ Solution by Mathematica

Time used: 0.341 (sec). Leaf size: 27

```
DSolve[x*y'[x] == y[x]+(x^2+y[x]^2)^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2} e^{-c_1} \left(-1 + e^{2c_1} x^2\right)$$

5.14 problem 14

Internal problem ID [92]
Internal file name [`OUTPUT/92_Sunday_June_05_2022_01_34_36_AM_9049263/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _dAlembert]
```

$$\boxed{yy' - \sqrt{x^2 + y^2} = -x}$$

5.14.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= \frac{-x + \sqrt{x^2 + y^2}}{y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}
&b_2+\frac{\left(-x+\sqrt{x^2+y^2}\right)(b_3-a_2)}{y}-\frac{\left(-x+\sqrt{x^2+y^2}\right)^2 a_3}{y^2} \\
&-\frac{\left(-1+\frac{x}{\sqrt{x^2+y^2}}\right)(xa_2+ya_3+a_1)}{y} \\
&-\left(\frac{1}{\sqrt{x^2+y^2}}-\frac{-x+\sqrt{x^2+y^2}}{y^2}\right)(xb_2+yb_3+b_1)=0
\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$-\frac{(x^2+y^2)^{\frac{3}{2}} a_3+\sqrt{x^2+y^2}\, x^2a_3+\sqrt{x^2+y^2}\, x^2b_2-2\sqrt{x^2+y^2}\, xya_2+2\sqrt{x^2+y^2}\, xyb_3-\sqrt{x^2+y^2}\, y^2a_3-}{}$$

$$=0$$

Setting the numerator to zero gives

$$\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3-\sqrt{x^2+y^2}\, x^2a_3-\sqrt{x^2+y^2}\, x^2b_2+2\sqrt{x^2+y^2}\, xya_2 \\
&-2\sqrt{x^2+y^2}\, xyb_3+\sqrt{x^2+y^2}\, y^2a_3+b_2\sqrt{x^2+y^2}\, y^2 \\
&+2x^3a_3+x^3b_2-2x^2ya_2+2x^2yb_3+x\,y^2a_3-y^3a_2+y^3b_3 \\
&-\sqrt{x^2+y^2}\, xb_1+\sqrt{x^2+y^2}\, ya_1+x^2b_1-xya_1=0
\end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned}
&-\left(x^2+y^2\right)^{\frac{3}{2}} a_3+2\left(x^2+y^2\right) xa_3+\left(x^2+y^2\right) xb_2-\left(x^2+y^2\right) ya_2 \\
&+2\left(x^2+y^2\right) yb_3-\sqrt{x^2+y^2}\, x^2a_3-\sqrt{x^2+y^2}\, x^2b_2 \\
&+2\sqrt{x^2+y^2}\, xya_2-2\sqrt{x^2+y^2}\, xyb_3+\sqrt{x^2+y^2}\, y^2a_3 \\
&+b_2\sqrt{x^2+y^2}\, y^2-x^2ya_2-x\,y^2a_3-x\,y^2b_2-y^3b_3+\left(x^2+y^2\right) b_1 \\
&-\sqrt{x^2+y^2}\, xb_1+\sqrt{x^2+y^2}\, ya_1-xya_1-y^2b_1=0
\end{aligned} \tag{6E}$$

Since the PDE has radicals, simplifying gives

$$\begin{aligned}
&2x^3a_3+x^3b_2-2\sqrt{x^2+y^2}\, x^2a_3-\sqrt{x^2+y^2}\, x^2b_2-2x^2ya_2+2x^2yb_3 \\
&+2\sqrt{x^2+y^2}\, xya_2-2\sqrt{x^2+y^2}\, xyb_3+x\,y^2a_3+b_2\sqrt{x^2+y^2}\, y^2 \\
&-y^3a_2+y^3b_3+x^2b_1-\sqrt{x^2+y^2}\, xb_1-xya_1+\sqrt{x^2+y^2}\, ya_1=0
\end{aligned}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\left\{x, y, \sqrt{x^2+y^2}\right\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\left\{x = v_1, y = v_2, \sqrt{x^2+y^2} = v_3\right\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &-2v_1^2v_2a_2 + 2v_3v_1v_2a_2 - v_2^3a_2 + 2v_1^3a_3 - 2v_3v_1^2a_3 + v_1v_2^2a_3 + v_1^3b_2 - v_3v_1^2b_2 \\ &+ b_2v_3v_2^2 + 2v_1^2v_2b_3 - 2v_3v_1v_2b_3 + v_2^3b_3 - v_1v_2a_1 + v_3v_2a_1 + v_1^2b_1 - v_3v_1b_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2, v_3\}$$

Equation (7E) now becomes

$$\begin{aligned} &(2a_3+b_2)\, v_1^3 + (-2a_2+2b_3)\, v_1^2v_2 + (-2a_3-b_2)\, v_1^2v_3 + v_1^2b_1 + v_1v_2^2a_3 \\ &+ (2a_2-2b_3)\, v_1v_2v_3 - v_1v_2a_1 - v_3v_1b_1 + (b_3-a_2)\, v_2^3 + b_2v_3v_2^2 + v_3v_2a_1 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} a_1 &= 0 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ -a_1 &= 0 \\ -b_1 &= 0 \\ -2a_2+2b_3 &= 0 \\ 2a_2-2b_3 &= 0 \\ -2a_3-b_2 &= 0 \\ 2a_3+b_2 &= 0 \\ b_3-a_2 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{-x + \sqrt{x^2 + y^2}}{y}\right)(x) \\
&= \frac{x^2 - \sqrt{x^2 + y^2}\, x + y^2}{y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{x^2 - \sqrt{x^2 + y^2}\, x + y^2}{y}} dy
\end{aligned}
$$

Which results in

$$S = \ln(y) - \frac{x \ln\left(\frac{2x^2 + 2\sqrt{x^2}\,\sqrt{x^2+y^2}}{y}\right)}{\sqrt{x^2}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-x + \sqrt{x^2+y^2}}{y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{\sqrt{x^2+y^2} + x}{x\sqrt{x^2+y^2}} \\
S_y &= \frac{2x^2 + y^2 + 2\sqrt{x^2+y^2}\, x}{y\sqrt{x^2+y^2}\, \left(\sqrt{x^2+y^2} + x\right)}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{\sqrt{x^2+y^2}\, x + x^2 + y^2}{x\sqrt{x^2+y^2}\, \left(\sqrt{x^2+y^2} + x\right)} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

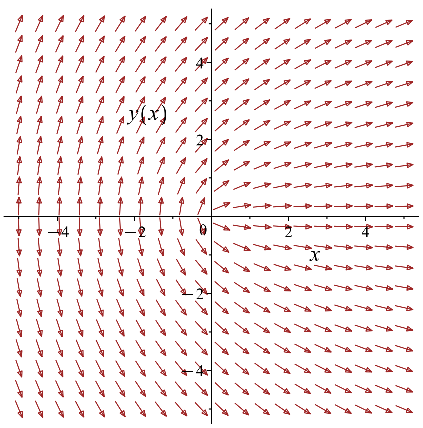
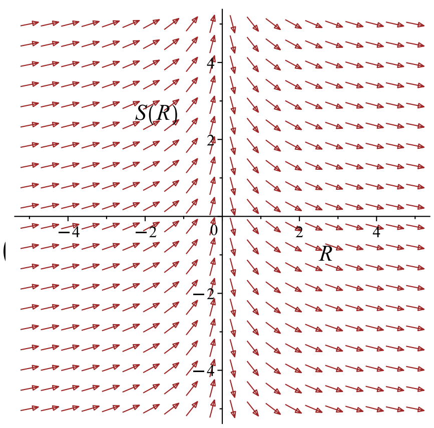
The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

Which gives

$$y = \mathrm{e}^{\frac{\ln(2)}{2}+\frac{\ln(2\,\mathrm{e}^{c_1}+2x)}{2}+\frac{c_1}{2}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-x+\sqrt{x^2+y^2}}{y}$ 	$R = x$ $S = 2\ln(y) - \ln(2) - \ln($	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = \mathrm{e}^{\frac{\ln(2)}{2}+\frac{\ln(2\,\mathrm{e}^{c_1}+2x)}{2}+\frac{c_1}{2}} \tag{1}$$

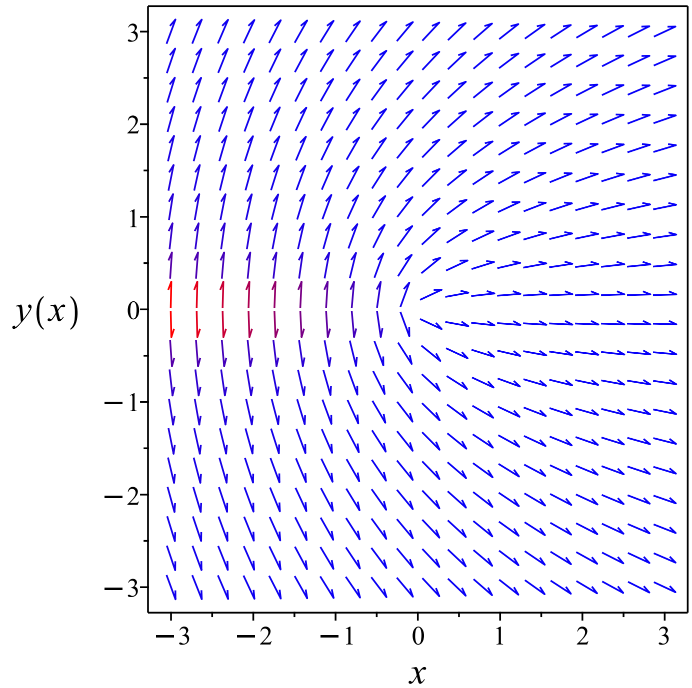


Figure 276: Slope field plot

Verification of solutions

$$y = e^{\frac{\ln(2)}{2}+\frac{\ln(2\,e^{c_1}+2x)}{2}+\frac{c_1}{2}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = y(x)/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 27

```
dsolve(x+y(x)*diff(y(x),x) = (x^2+y(x)^2)^(1/2),y(x), singsol=all)
```

$$\frac{-y(x)^2 c_1 + \sqrt{x^2 + y(x)^2} + x}{y(x)^2} = 0$$

✓ Solution by Mathematica

Time used: 0.402 (sec). Leaf size: 57

```
DSolve[x+y[x]*y'[x] == (x^2+y[x]^2)^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$\begin{aligned} y(x) &\to -e^{\frac{c_1}{2}}\sqrt{2x + e^{c_1}} \\ y(x) &\to e^{\frac{c_1}{2}}\sqrt{2x + e^{c_1}} \\ y(x) &\to 0 \end{aligned}$$

5.15 problem 15

Internal problem ID [93]
Internal file name [`OUTPUT/93_Sunday_June_05_2022_01_34_37_AM_70160291/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeD2", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class B`]]
```

$$\boxed{y(3x+y)+x(x+y)\,y'=0}$$

5.15.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y=u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x(3x+u(x)\,x)+x(x+u(x)\,x)\,(u'(x)\,x+u(x))=0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{2u(u+2)}{x\,(u+1)}\end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u(u+2)}{u+1}$. Integrating both sides gives

$$\frac{1}{\frac{u(u+2)}{u+1}}\, du = -\frac{2}{x}\, dx$$
$$\int \frac{1}{\frac{u(u+2)}{u+1}}\, du = \int -\frac{2}{x}\, dx$$
$$\frac{\ln\left(u(u+2)\right)}{2} = -2\ln\left(x\right) + c_2$$

Raising both side to exponential gives

$$\sqrt{u\left(u+2\right)} = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u\left(u+2\right)} = \frac{c_3}{x^2}$$

Which simplifies to

$$\sqrt{u\left(x\right)\left(u\left(x\right)+2\right)} = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$

The solution is

$$\sqrt{u\left(x\right)\left(u\left(x\right)+2\right)} = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y\left(\frac{y}{x}+2\right)}{x}} = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$
$$\sqrt{\frac{y\left(y+2x\right)}{x^2}} = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y\left(y+2x\right)}{x^2}} = \frac{c_3\,\mathrm{e}^{c_2}}{x^2} \tag{1}$$

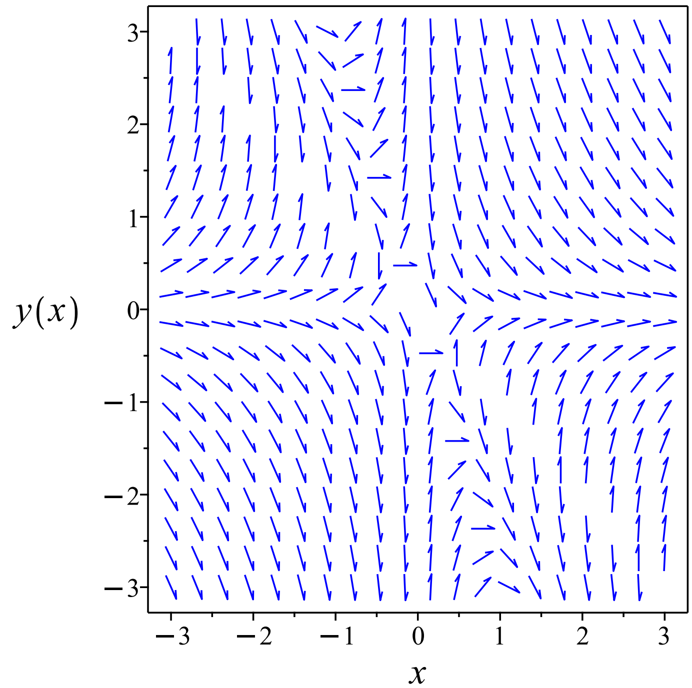


Figure 277: Slope field plot

Verification of solutions

$$\sqrt{\frac{y\left(y+2x\right)}{x^{2}}}=\frac{c_{3}\mathrm{e}^{c_{2}}}{x^{2}}$$

Verified OK.

5.15.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y'=-\frac{y(3x+y)}{x\left(x+y\right)}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi=xa_2+ya_3+a_1 \tag{1E}$$
$$\eta=xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{y(3x+y)(b_3-a_2)}{x(x+y)} - \frac{y^2(3x+y)^2 a_3}{x^2(x+y)^2} - \left(-\frac{3y}{x(x+y)} + \frac{y(3x+y)}{x^2(x+y)} + \frac{y(3x+y)}{x(x+y)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{3x+y}{x(x+y)} - \frac{y}{x(x+y)} + \frac{y(3x+y)}{x(x+y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$\frac{4x^4b_2 + 4x^3yb_2 + 2x^2y^2a_2 - 12x^2y^2a_3 + 2x^2y^2b_2 - 2x^2y^2b_3 - 8x\,y^3a_3 - 2y^4a_3 + 3x^3b_1 - 3x^2ya_1 + 2x^2yb_1}{x^2(x+y)^2} = 0$$

Setting the numerator to zero gives

$$4x^4b_2 + 4x^3yb_2 + 2x^2y^2a_2 - 12x^2y^2a_3 + 2x^2y^2b_2 - 2x^2y^2b_3 - 8x\,y^3a_3 - 2y^4a_3 + 3x^3b_1 - 3x^2ya_1 + 2x^2yb_1 - 2x\,y^2a_1 + x\,y^2b_1 - y^3a_1 = 0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$2a_2v_1^2v_2^2 - 12a_3v_1^2v_2^2 - 8a_3v_1v_2^3 - 2a_3v_2^4 + 4b_2v_1^4 + 4b_2v_1^3v_2 + 2b_2v_1^2v_2^2 - 2b_3v_1^2v_2^2 - 3a_1v_1^2v_2 - 2a_1v_1v_2^2 - a_1v_2^3 + 3b_1v_1^3 + 2b_1v_1^2v_2 + b_1v_1v_2^2 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &4b_2v_1^4 + 4b_2v_1^3v_2 + 3b_1v_1^3 + (2a_2 - 12a_3 + 2b_2 - 2b_3)\,v_1^2v_2^2 \\ &+ (-3a_1 + 2b_1)\,v_1^2v_2 - 8a_3v_1v_2^3 + (-2a_1 + b_1)\,v_1v_2^2 - 2a_3v_2^4 - a_1v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -a_1 &= 0 \\ -8a_3 &= 0 \\ -2a_3 &= 0 \\ 3b_1 &= 0 \\ 4b_2 &= 0 \\ -3a_1 + 2b_1 &= 0 \\ -2a_1 + b_1 &= 0 \\ 2a_2 - 12a_3 + 2b_2 - 2b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x,y)\,\xi \\ &= y - \left(-\frac{y(3x+y)}{x\,(x+y)}\right)(x) \\ &= \frac{4yx+2y^2}{x+y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{4yx+2y^2}{x+y}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln\left(y(2x+y)\right)}{4}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(3x+y)}{x\,(x+y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{1}{4x+2y} \\
S_y &= \frac{x+y}{2y\,(2x+y)}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{2R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}
$$

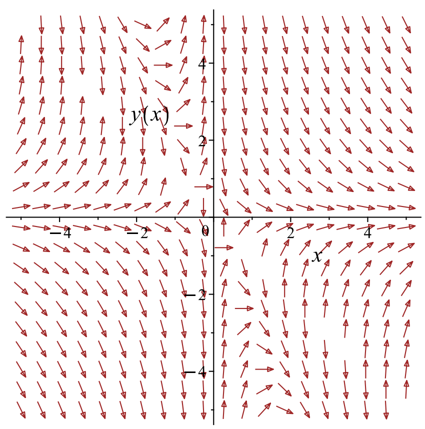
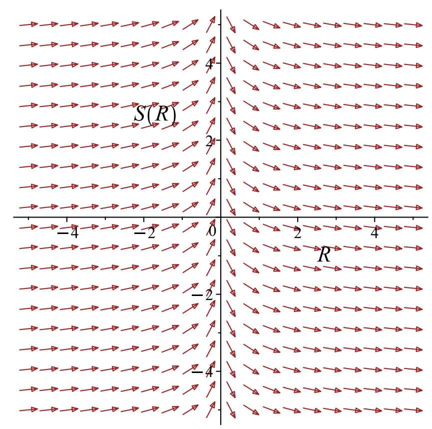
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln(y)}{4} + \frac{\ln(y+2x)}{4} = -\frac{\ln(x)}{2} + c_1
$$

Which simplifies to

$$
\frac{\ln(y)}{4} + \frac{\ln(y+2x)}{4} = -\frac{\ln(x)}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = -\frac{y(3x+y)}{x(x+y)}$

</td><td>$R = x$
$S = \frac{\ln(y)}{4} + \frac{\ln(2x+y)}{4}$</td><td>$\frac{dS}{dR} = -\frac{1}{2R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y)}{4} + \frac{\ln(y+2x)}{4} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

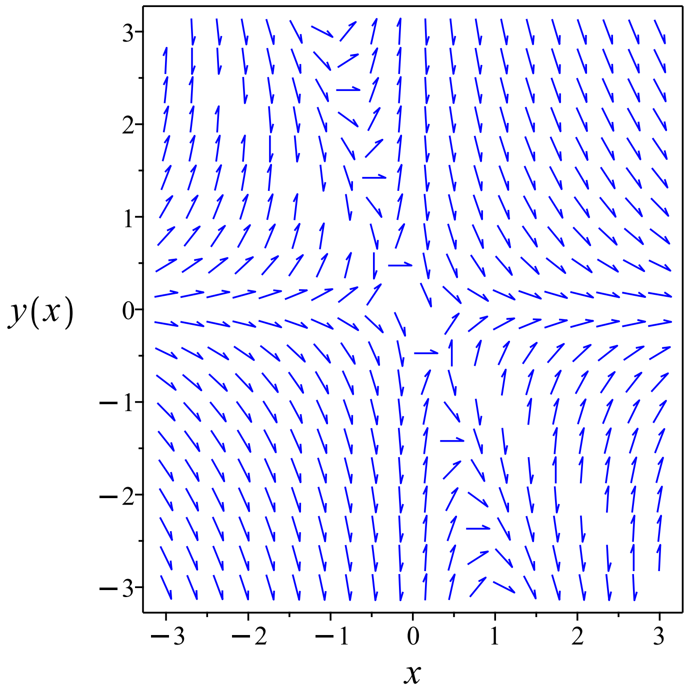


Figure 278: Slope field plot

Verification of solutions

$$\frac{\ln(y)}{4} + \frac{\ln(y+2x)}{4} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

5.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x(x + y))\, \mathrm{d}y &= (-y(3x + y))\, \mathrm{d}x \\
(y(3x + y))\, \mathrm{d}x + (x(x + y))\, \mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= y(3x + y) \\
N(x, y) &= x(x + y)
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(y(3x + y)) \\
&= 3x + 2y
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x(x + y)) \\
&= 2x + y
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x\,(x+y)}((3x+2y) - (2x+y)) \\
&= \frac{1}{x}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{1}{x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{\ln(x)} \\
&= x
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= x(y(3x+y)) \\
&= y(3x+y)\,x
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x(x(x+y)) \\
&= x^2(x+y)
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
(y(3x+y)\,x) + \left(x^2(x+y)\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y(3x + y) \, x \, \mathrm{d}x$$

$$\phi = \frac{y \, x^2 (2x + y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x^2 (2x + y)}{2} + \frac{y \, x^2}{2} + f'(y) \tag{4}$$

$$= x^2 (x + y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2(x + y)$. Therefore equation (4) becomes

$$x^2 (x + y) = x^2 (x + y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y \, x^2 (2x + y)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y\, x^2(2x + y)}{2}$$

Summary

The solution(s) found are the following

$$\frac{yx^2(y + 2x)}{2} = c_1 \tag{1}$$

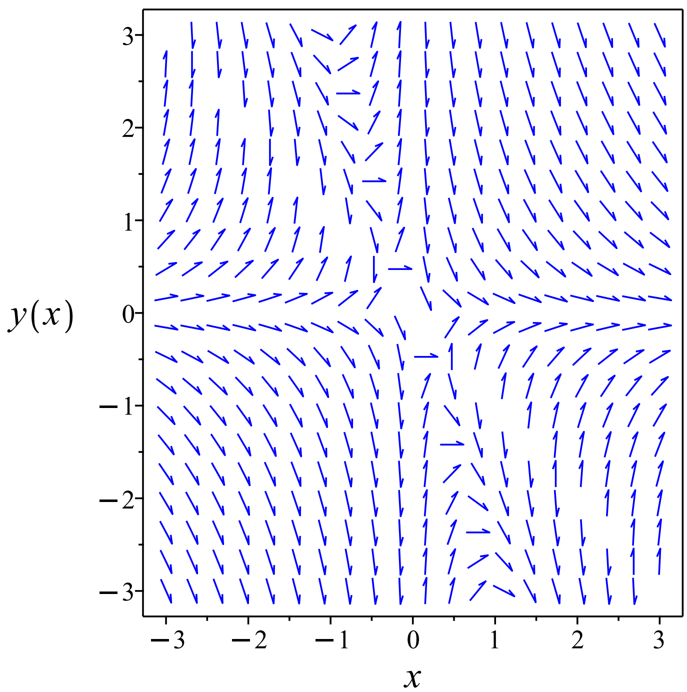


Figure 279: Slope field plot

Verification of solutions

$$\frac{yx^2(y + 2x)}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 59

```
dsolve(y(x)*(3*x+y(x))+x*(x+y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{-c_1 x^2 - \sqrt{c_1^2 x^4 + 1}}{c_1 x}$$
$$y(x) = \frac{-c_1 x^2 + \sqrt{c_1^2 x^4 + 1}}{c_1 x}$$

✓ Solution by Mathematica

Time used: 0.607 (sec). Leaf size: 93

```
DSolve[y[x]*(3*x+y[x])+x*(x+y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2 + \sqrt{x^4 + e^{2c_1}}}{x}$$
$$y(x) \to -x + \frac{\sqrt{x^4 + e^{2c_1}}}{x}$$
$$y(x) \to -\frac{\sqrt{x^4} + x^2}{x}$$
$$y(x) \to \frac{\sqrt{x^4}}{x} - x$$

5.16 problem 16

Internal problem ID [94]
Internal file name [`OUTPUT/94_Sunday_June_05_2022_01_34_38_AM_14516920/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _dAlembert]
```

$$\boxed{y' - \sqrt{1 + x + y} = 0}$$

5.16.1 Solving as homogeneousTypeC ode

Let

$$z = 1 + x + y \tag{1}$$

Then

$$z'(x) = 1 + y'$$

Therefore

$$y' = z'(x) - 1$$

Hence the given ode can now be written as

$$z'(x) - 1 = \sqrt{z}$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{\sqrt{z}+1} dz$$
$$x + c_1 = 2\sqrt{z} + \ln\left(-1+\sqrt{z}\right) - \ln\left(\sqrt{z}+1\right) - \ln\left(z-1\right)$$

Replacing z back by its value from (1) then the above gives the solution as

$$2\sqrt{1+x+y} + \ln\left(-1+\sqrt{1+x+y}\right) - \ln\left(\sqrt{1+x+y}+1\right) - \ln\left(x+y\right) = x + c_1$$

Summary

The solution(s) found are the following

$$2\sqrt{1+x+y} + \ln\left(-1+\sqrt{1+x+y}\right) - \ln\left(\sqrt{1+x+y}+1\right) - \ln\left(x+y\right) = x + c_1 \tag{1}$$

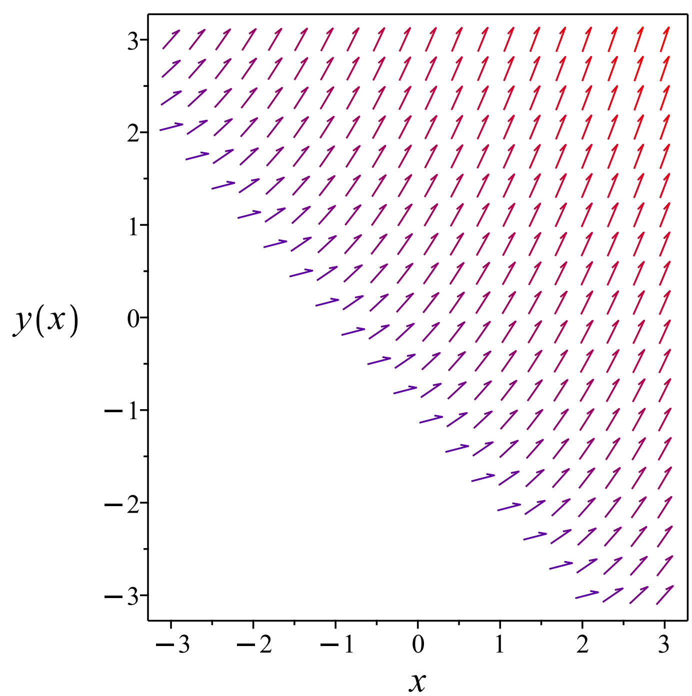


Figure 280: Slope field plot

Verification of solutions

$$2\sqrt{1+x+y} + \ln\left(-1+\sqrt{1+x+y}\right) - \ln\left(\sqrt{1+x+y}+1\right) - \ln\left(x+y\right) = x + c_1$$

Verified OK.

5.16.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \sqrt{x+y+1} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 206: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= 1\\ \eta(x,y) &= -1\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{-1}{1}\\ &= -1\end{aligned}$$

This is easily solved to give

$$y = -x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \sqrt{x+y+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1\\ R_y &= 1\\ S_x &= 1\\ S_y &= 0\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{\sqrt{x+y+1}+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{\sqrt{R+1}+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = 2\sqrt{R+1} + \ln\left(-1+\sqrt{R+1}\right) - \ln\left(\sqrt{R+1}+1\right) - \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = 2\sqrt{1+x+y} + \ln\left(-1+\sqrt{1+x+y}\right) - \ln\left(\sqrt{1+x+y}+1\right) - \ln(x+y) + c_1$$

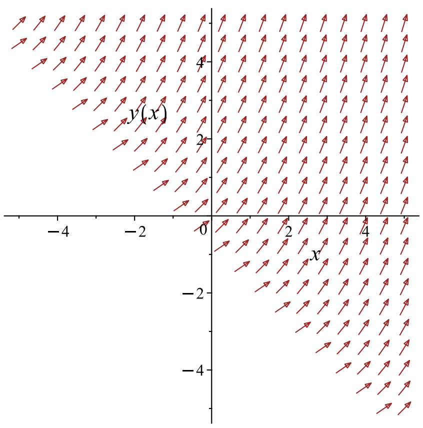
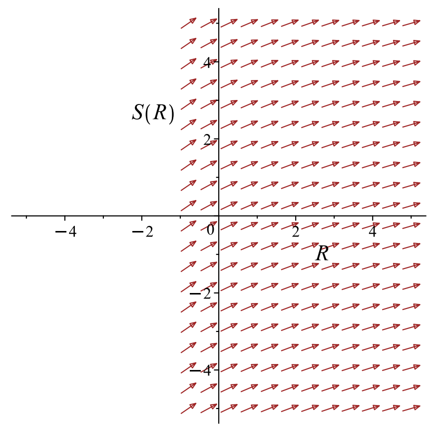
Which simplifies to

$$x = 2\sqrt{1+x+y} + \ln\left(-1+\sqrt{1+x+y}\right) - \ln\left(\sqrt{1+x+y}+1\right) - \ln(x+y) + c_1$$

Which gives

$$y = e^{-2\,\mathrm{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-x-2+c_1} - 2\,e^{-\mathrm{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-\frac{x}{2}-1+\frac{c_1}{2}} - x$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \sqrt{x+y+1}$ 	$R = x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{\sqrt{R+1}+1}$ 

Summary

The solution(s) found are the following

$$y = e^{-2\,\mathrm{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-x-2+c_1} - 2\,e^{-\mathrm{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-\frac{x}{2}-1+\frac{c_1}{2}} - x \tag{1}$$

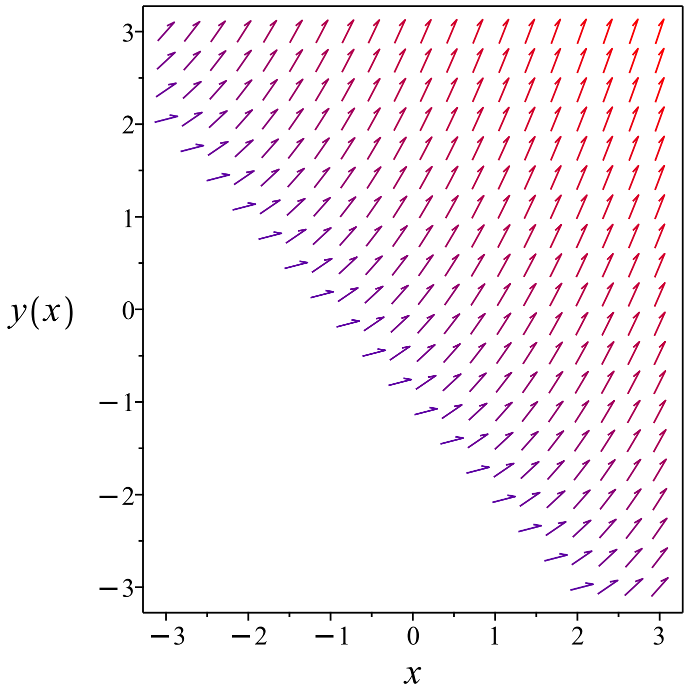


Figure 281: Slope field plot

Verification of solutions

$$y = e^{-2\,\mathrm{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-x-2+c_1} - 2\,e^{-\mathrm{LambertW}\left(-e^{-\frac{x}{2}-1+\frac{c_1}{2}}\right)-\frac{x}{2}-1+\frac{c_1}{2}} - x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 44

```
dsolve(diff(y(x),x) = (1+x+y(x))^(1/2),y(x), singsol=all)
```

$$x-2\sqrt{1+x+y(x)}-\ln\left(-1+\sqrt{1+x+y(x)}\right)+\ln\left(1+\sqrt{1+x+y(x)}\right)+\ln\left(x+y(x)\right)-c_1=0$$

✓ Solution by Mathematica

Time used: 9.342 (sec). Leaf size: 56

```
DSolve[y'[x] == (1+x+y[x])^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to W\left(-e^{\frac{1}{2}(-x-3-c_1)}\right)^2+2W\left(-e^{\frac{1}{2}(-x-3-c_1)}\right)-x$$
$$y(x)\to -x$$

5.17 problem 17

Internal problem ID [95]
Internal file name [`OUTPUT/95_Sunday_June_05_2022_01_34_41_AM_1305094/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "homogeneousTypeC", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{y' - (4x + y)^2 = 0}$$

5.17.1 Solving as homogeneousTypeC ode

Let

$$z = 4x + y \tag{1}$$

Then

$$z'(x) = 4 + y'$$

Therefore

$$y' = z'(x) - 4$$

Hence the given ode can now be written as

$$z'(x) - 4 = z^2$$

This is separable first order ode. Integrating

$$\int dx = \int \frac{1}{z^2 + 4} dz$$

$$x + c_1 = \frac{\arctan\left(\frac{z}{2}\right)}{2}$$

Replacing z back by its value from (1) then the above gives the solution as

$$y = -4x + 2\tan(2x + 2c_1)$$

$$y = -4x + 2\tan(2x + 2c_1)$$

Summary

The solution(s) found are the following

$$y = -4x + 2\tan(2x + 2c_1) \tag{1}$$

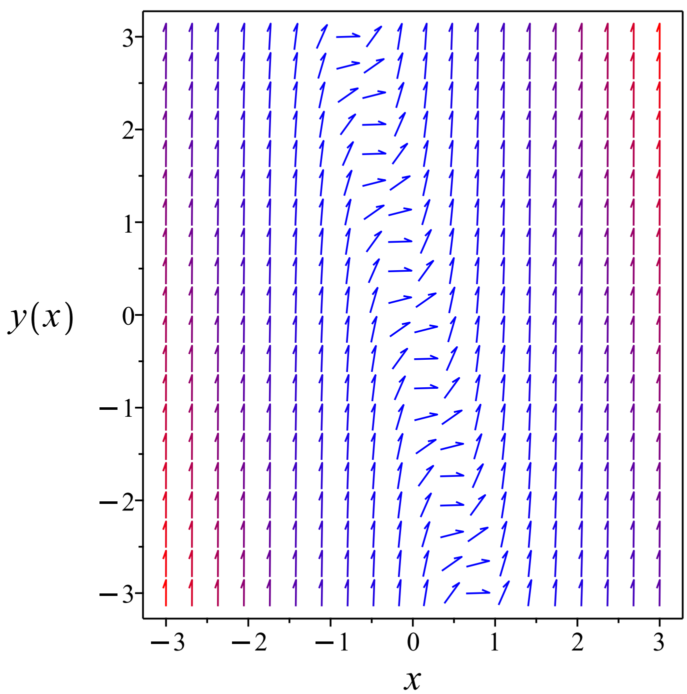


Figure 282: Slope field plot

Verification of solutions

$$y = -4x + 2\tan(2x + 2c_1)$$

Verified OK.

5.17.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = (4x + y)^2$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `homogeneous Type C`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 208: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x,y) = 1$$
$$\eta(x,y) = -4 \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= \frac{\eta}{\xi}\\ &= \frac{-4}{1}\\ &= -4\end{aligned}$$

This is easily solved to give

$$y = -4x + c_1$$

Where now the coordinate R is taken as the constant of integration. Hence

$$R = 4x + y$$

And S is found from

$$\begin{aligned}dS &= \frac{dx}{\xi}\\ &= \frac{dx}{1}\end{aligned}$$

Integrating gives

$$\begin{aligned}S &= \int \frac{dx}{T}\\ &= x\end{aligned}$$

Where the constant of integration is set to zero as we just need one solution. Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y)S_y}{R_x + \omega(x, y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = (4x + y)^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 4 \\ R_y &= 1 \\ S_x &= 1 \\ S_y &= 0 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{4 + (4x + y)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2 + 4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{\arctan\left(\frac{R}{2}\right)}{2} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x = \frac{\arctan\left(2x + \frac{y}{2}\right)}{2} + c_1$$

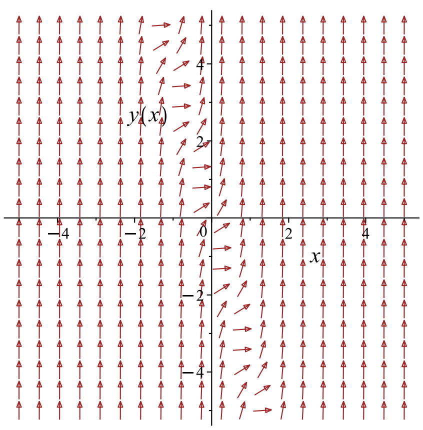
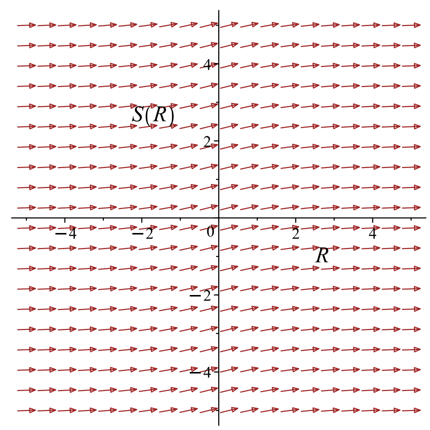
Which simplifies to

$$x = \frac{\arctan\left(2x + \frac{y}{2}\right)}{2} + c_1$$

Which gives

$$y = -4x - 2\tan\left(-2x + 2c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = (4x + y)^2$ 	$R = 4x + y$ $S = x$	$\frac{dS}{dR} = \frac{1}{R^2+4}$ 

Summary
The solution(s) found are the following

$$y = -4x - 2\tan\left(-2x + 2c_1\right) \tag{1}$$

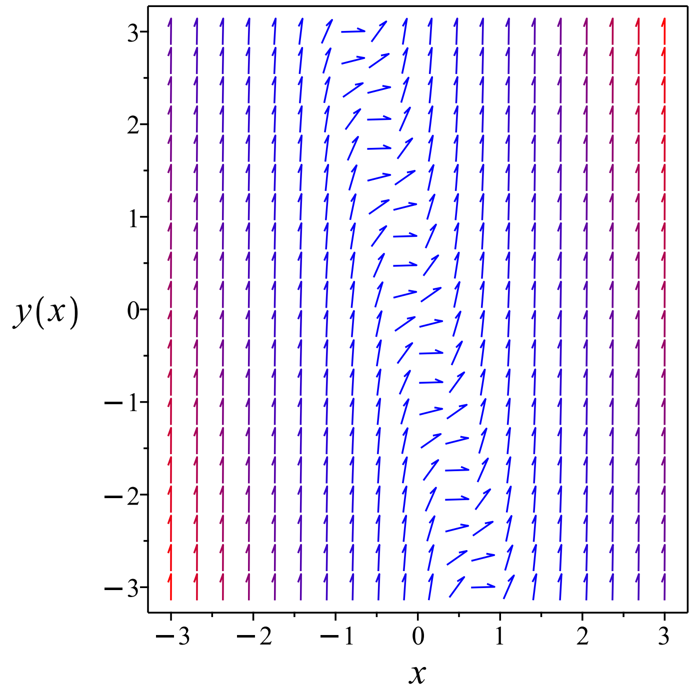


Figure 283: Slope field plot

Verification of solutions

$$y = -4x - 2\tan(-2x + 2c_1)$$

Verified OK.

5.17.3 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= (4x + y)^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = 16x^2 + 8yx + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 16x^2$, $f_1(x) = 8x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= 8x \\ f_2^2 f_0 &= 16x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) - 8xu'(x) + 16x^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = \mathrm{e}^{2x^2} (c_1 \cos(2x) + c_2 \sin(2x))$$

The above shows that

$$u'(x) = 2\,\mathrm{e}^{2x^2} (2\cos(2x)\, c_1 x + 2\sin(2x)\, c_2 x + c_2 \cos(2x) - c_1 \sin(2x))$$

Using the above in (1) gives the solution

$$y = -\frac{2(2\cos(2x)\, c_1 x + 2\sin(2x)\, c_2 x + c_2 \cos(2x) - c_1 \sin(2x))}{c_1 \cos(2x) + c_2 \sin(2x)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{(-4c_3 x - 2)\cos(2x) - 4\sin(2x)\left(x - \frac{c_3}{2}\right)}{c_3 \cos(2x) + \sin(2x)}$$

Summary

The solution(s) found are the following

$$y = \frac{(-4c_3x - 2)\cos(2x) - 4\sin(2x)\left(x - \frac{c_3}{2}\right)}{c_3\cos(2x) + \sin(2x)} \tag{1}$$

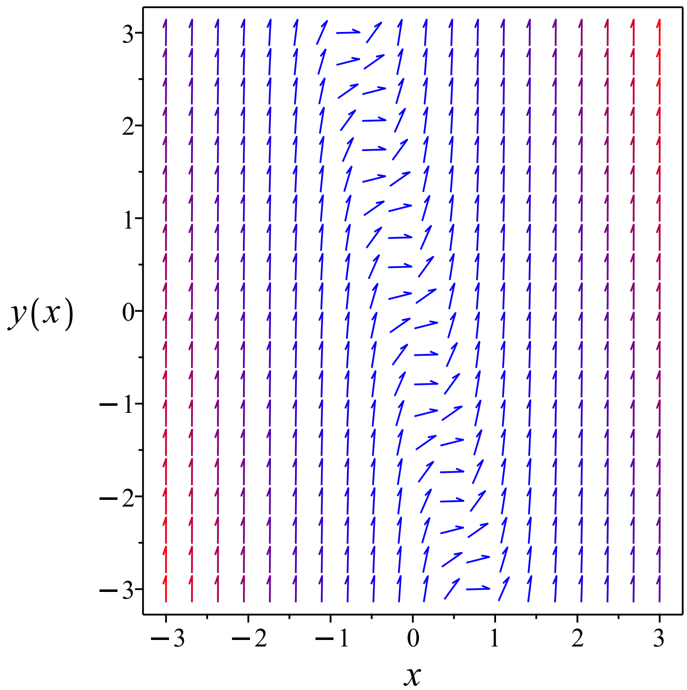


Figure 284: Slope field plot

Verification of solutions

$$y = \frac{(-4c_3x - 2)\cos(2x) - 4\sin(2x)\left(x - \frac{c_3}{2}\right)}{c_3\cos(2x) + \sin(2x)}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = -4, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple
Time used: 0.015 (sec). Leaf size: 18

```
dsolve(diff(y(x),x) = (4*x+y(x))^2,y(x), singsol=all)
```

$$y(x) = -4x - 2\tan(-2x + 2c_1)$$

✓ Solution by Mathematica
Time used: 0.121 (sec). Leaf size: 41

```
DSolve[y'[x] == (4*x+y[x])^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -4x + \frac{1}{c_1 e^{4ix} - \frac{i}{4}} - 2i$$
$$y(x) \to -4x - 2i$$

5.18 problem 18

Internal problem ID [96]
Internal file name [`OUTPUT/96_Sunday_June_05_2022_01_34_42_AM_93737581/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{(x+y)\,y'=0}$$

5.18.1 Solving as quadrature ode

Integrating both sides gives

$$\begin{aligned} y &= \int 0 \, \mathrm{d}x \\ &= c_1 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \tag{1}$$

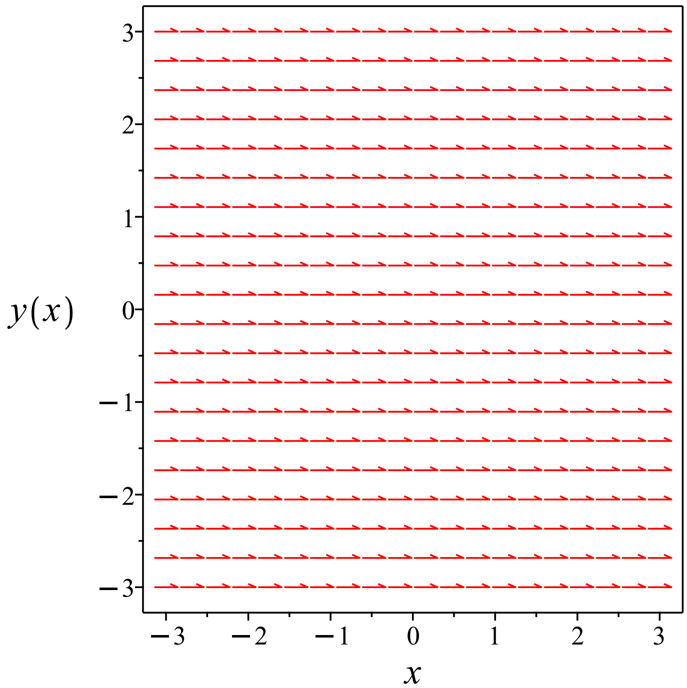


Figure 285: Slope field plot

Verification of solutions

$$y = c_1$$

Verified OK.

5.18.2 Maple step by step solution

Let's solve

$(x + y)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Integrate both sides with respect to x

  $\int (x + y)\, y' dx = \int 0 dx + c_1$

- Cannot compute integral

  $\int (x + y)\, y' dx = c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 11

```
dsolve((x+y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = -x$$
$$y(x) = c_1$$

✓ Solution by Mathematica

Time used: 0.002 (sec). Leaf size: 14

```
DSolve[(x+y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -x$$
$$y(x) \to c_1$$

5.19 problem 19

Internal problem ID [97]
Internal file name [`OUTPUT/97_Sunday_June_05_2022_01_34_42_AM_28173696/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2yx + y'x^2 - 5y^3 = 0}$$

5.19.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= \frac{y(5y^2 - 2x)}{x^2} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 211: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= y^3 x^4
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 x^4} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{2x^4 y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(5y^2 - 2x)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{x^5 y^2} \\ S_y &= \frac{1}{y^3 x^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{5}{x^6} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{5}{R^6}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R^5} + c_1 \tag{4}$$

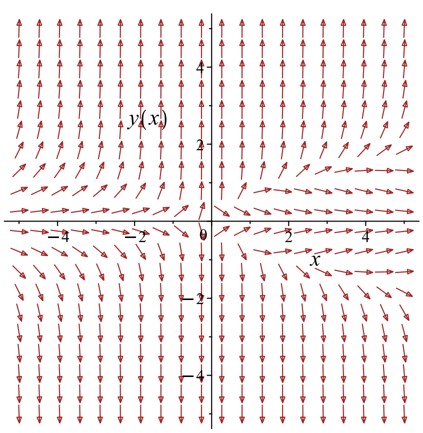
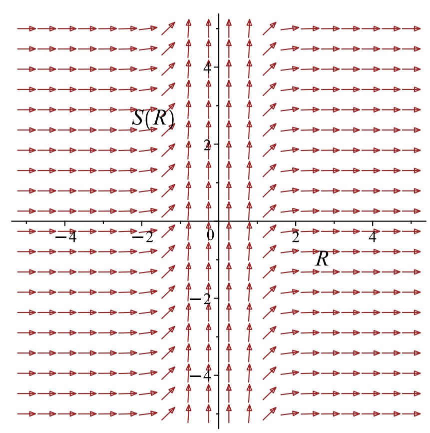
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{2x^4y^2} = -\frac{1}{x^5} + c_1$$

Which simplifies to

$$-\frac{1}{2x^4y^2} = -\frac{1}{x^5} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \frac{y(5y^2-2x)}{x^2}$
</td><td>$R = x$
$S = -\frac{1}{2x^4y^2}$</td><td>$\frac{dS}{dR} = \frac{5}{R^6}$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$-\frac{1}{2x^4y^2} = -\frac{1}{x^5} + c_1 \tag{1}$$

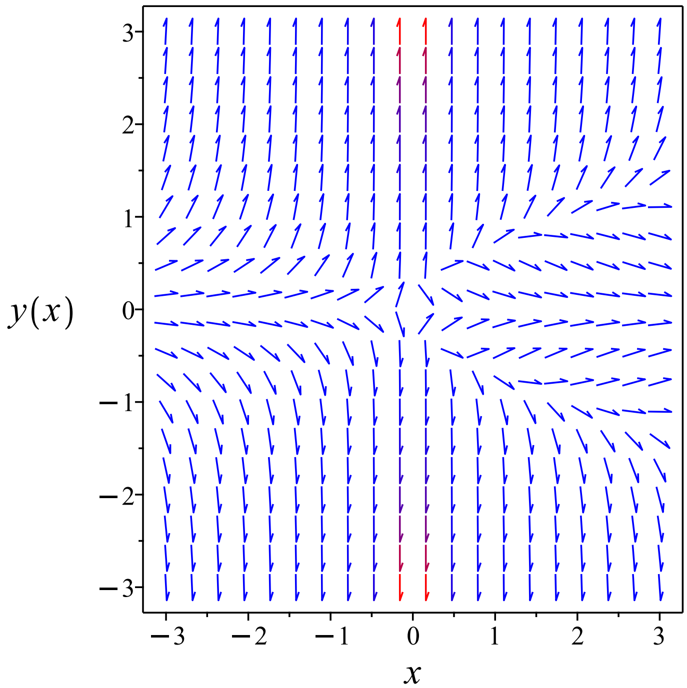


Figure 286: Slope field plot

Verification of solutions

$$-\frac{1}{2x^4y^2} = -\frac{1}{x^5} + c_1$$

Verified OK.

5.19.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(5y^2-2x)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x}y + \frac{5}{x^2}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= \frac{5}{x^2} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y'\frac{1}{y^3} = -\frac{2}{x\,y^2} + \frac{5}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -\frac{2w(x)}{x} + \frac{5}{x^2} \\ w' &= \frac{4w}{x} - \frac{10}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= -\frac{10}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{4w(x)}{x} = -\frac{10}{x^2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int -\frac{4}{x}dx} \\
&= \frac{1}{x^4}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{10}{x^2}\right) \\
\frac{d}{dx}\left(\frac{w}{x^4}\right) &= \left(\frac{1}{x^4}\right)\left(-\frac{10}{x^2}\right) \\
d\left(\frac{w}{x^4}\right) &= \left(-\frac{10}{x^6}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\frac{w}{x^4} &= \int -\frac{10}{x^6}\, dx \\
\frac{w}{x^4} &= \frac{2}{x^5} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^4}$ results in

$$w(x) = \frac{2}{x} + c_1 x^4$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{2}{x} + c_1 x^4$$

Solving for y gives

$$\begin{aligned}
y(x) &= \frac{\sqrt{(c_1 x^5 + 2)\, x}}{c_1 x^5 + 2} \\
y(x) &= -\frac{\sqrt{(c_1 x^5 + 2)\, x}}{c_1 x^5 + 2}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{(c_1 x^5 + 2) x}}{c_1 x^5 + 2} \tag{1}$$

$$y = -\frac{\sqrt{(c_1 x^5 + 2) x}}{c_1 x^5 + 2} \tag{2}$$

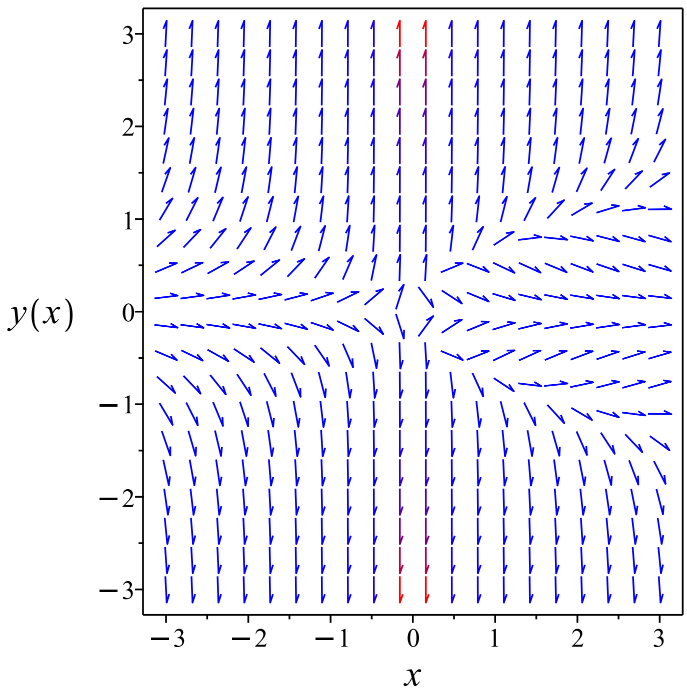


Figure 287: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{(c_1 x^5 + 2) x}}{c_1 x^5 + 2}$$

Verified OK.

$$y = -\frac{\sqrt{(c_1 x^5 + 2) x}}{c_1 x^5 + 2}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 50

```
dsolve(2*x*y(x)+x^2*diff(y(x),x) = 5*y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(c_1 x^5 + 2)\, x}}{c_1 x^5 + 2}$$
$$y(x) = -\frac{\sqrt{(c_1 x^5 + 2)\, x}}{c_1 x^5 + 2}$$

✓ Solution by Mathematica

Time used: 0.399 (sec). Leaf size: 51

```
DSolve[2*x*y[x]+x^2*y'[x] == 5*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{x}}{\sqrt{2 + c_1 x^5}}$$
$$y(x) \to \frac{\sqrt{x}}{\sqrt{2 + c_1 x^5}}$$
$$y(x) \to 0$$

5.20 problem 20

Internal problem ID [98]
Internal file name [`OUTPUT/98_Sunday_June_05_2022_01_34_43_AM_69592259/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{2xy^3 + y^2y' = 6x}$$

5.20.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{2x(y^3-3)}{y^2} \end{aligned}$$

Where $f(x) = -2x$ and $g(y) = \frac{y^3-3}{y^2}$. Integrating both sides gives

$$\frac{1}{\frac{y^3-3}{y^2}}\,dy = -2x\,dx$$

$$\int \frac{1}{\frac{y^3-3}{y^2}}\,dy = \int -2x\,dx$$

$$\frac{\ln(y^3-3)}{3} = -x^2 + c_1$$

Raising both side to exponential gives

$$(y^3-3)^{\frac{1}{3}} = e^{-x^2+c_1}$$

Which simplifies to

$$(y^3-3)^{\frac{1}{3}} = c_2 e^{-x^2}$$

The solution is

$$(y^3-3)^{\frac{1}{3}} = c_2 e^{-x^2+c_1}$$

Summary

The solution(s) found are the following

$$(y^3-3)^{\frac{1}{3}} = c_2 e^{-x^2+c_1} \tag{1}$$

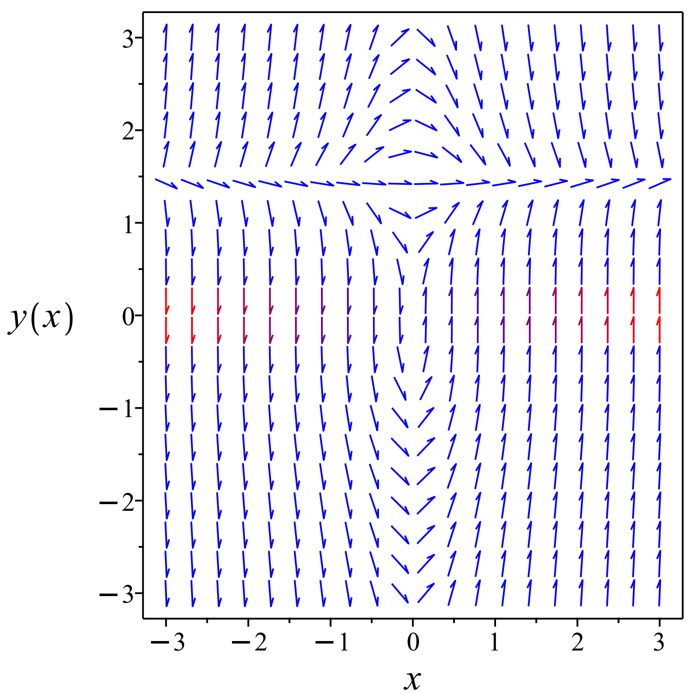


Figure 288: Slope field plot

Verification of solutions

$$(y^3-3)^{\frac{1}{3}} = c_2 e^{-x^2+c_1}$$

Verified OK.

5.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{2x(y^3-3)}{y^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 213: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{1}{2x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{1}{2x}} dx
\end{aligned}
$$

Which results in

$$
S = -x^2
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2x(y^3-3)}{y^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= -2x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{y^2}{y^3 - 3} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{R^2}{R^3 - 3}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = \frac{\ln (R^3 - 3)}{3} + c_1 \tag{4}
$$

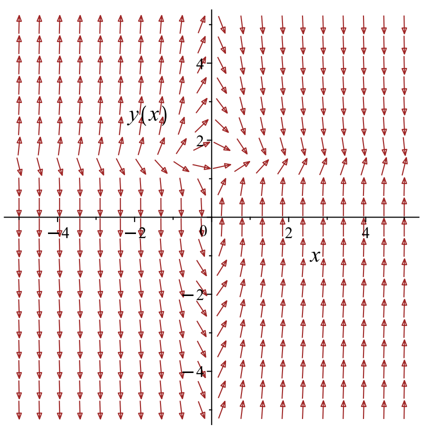
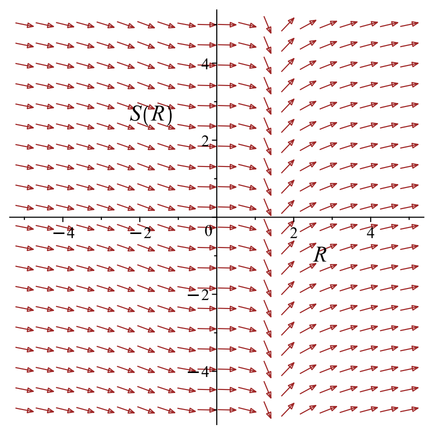
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
-x^2 = \frac{\ln (y^3 - 3)}{3} + c_1
$$

Which simplifies to

$$
-x^2 = \frac{\ln (y^3 - 3)}{3} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2x(y^3-3)}{y^2}$ 	$R = y$ $S = -x^2$	$\frac{dS}{dR} = \frac{R^2}{R^3-3}$ 

Summary

The solution(s) found are the following

$$-x^2 = \frac{\ln(y^3 - 3)}{3} + c_1 \tag{1}$$

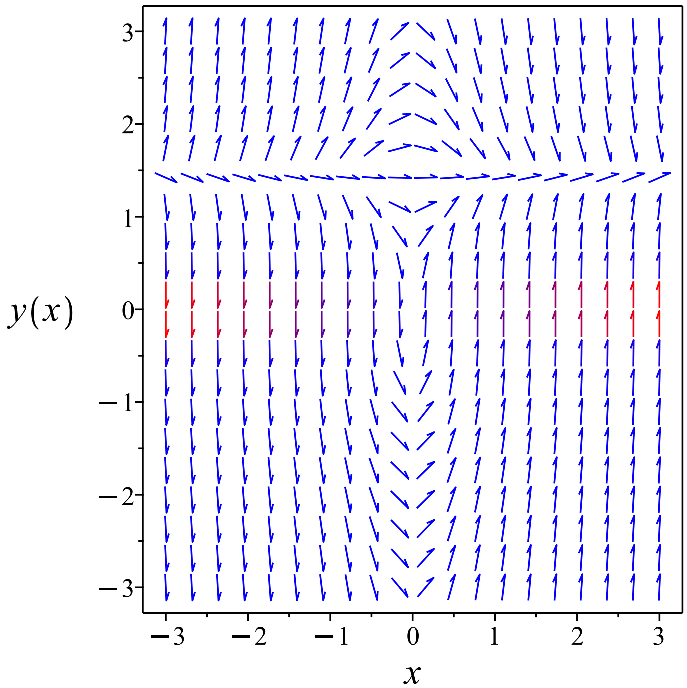


Figure 289: Slope field plot

Verification of solutions

$$-x^2 = \frac{\ln(y^3 - 3)}{3} + c_1$$

Verified OK.

5.20.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{2x(y^3 - 3)}{y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -2xy + 6x\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -2x \\
f_1(x) &= 6x \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y' y^2 = -2x\, y^3 + 6x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2 y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= -2w(x)\, x + 6x \\
w' &= -6wx + 18x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 6x \\
q(x) &= 18x
\end{aligned}
$$

Hence the ode is

$$
w'(x) + 6w(x)\, x = 18x
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int 6x dx} \\
&= e^{3x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(18x) \\
\frac{d}{dx}\left(e^{3x^2} w\right) &= \left(e^{3x^2}\right)(18x) \\
d\left(e^{3x^2} w\right) &= \left(18x\, e^{3x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{3x^2} w &= \int 18x\, e^{3x^2}\, dx \\
e^{3x^2} w &= 3\, e^{3x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{3x^2}$ results in

$$
w(x) = 3\, e^{-3x^2} e^{3x^2} + c_1 e^{-3x^2}
$$

which simplifies to

$$
w(x) = 3 + c_1 e^{-3x^2}
$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$
y^3 = 3 + c_1 e^{-3x^2}
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \left(3 + c_1 e^{-3x^2}\right)^{\frac{1}{3}} \\
y(x) &= \frac{\left(3 + c_1 e^{-3x^2}\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2} \\
y(x) &= -\frac{\left(3 + c_1 e^{-3x^2}\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$y = \left(3 + c_1 \mathrm{e}^{-3x^2}\right)^{\frac{1}{3}} \tag{1}$$

$$y = \frac{\left(3 + c_1 \mathrm{e}^{-3x^2}\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2} \tag{2}$$

$$y = -\frac{\left(3 + c_1 \mathrm{e}^{-3x^2}\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2} \tag{3}$$

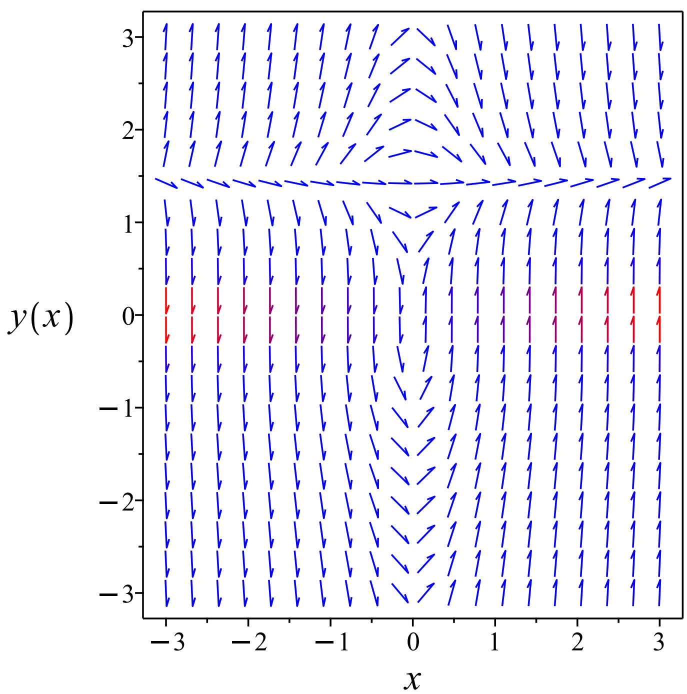


Figure 290: Slope field plot

Verification of solutions

$$y = \left(3 + c_1 \mathrm{e}^{-3x^2}\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(3 + c_1 \mathrm{e}^{-3x^2}\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

Verified OK.

$$y = -\frac{\left(3 + c_1 \mathrm{e}^{-3x^2}\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$

Verified OK.

5.20.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned} \frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N \end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{y^2}{2\left(y^3-3\right)}\right)\mathrm{d}y &= (x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x + \left(-\frac{y^2}{2\left(y^3-3\right)}\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -x\\ N(x,y) &= -\frac{y^2}{2\left(y^3-3\right)}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{y^2}{2\left(y^3-3\right)}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -x \,\mathrm{d}x \\
\phi &= -\frac{x^2}{2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{y^2}{2(y^3-3)}$. Therefore equation (4) becomes

$$
-\frac{y^2}{2\,(y^3-3)} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{y^2}{2\,(y^3-3)}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int \left(-\frac{y^2}{2y^3-6}\right) \mathrm{d}y \\
f(y) &= -\frac{\ln\left(y^3-3\right)}{6} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{x^2}{2} - \frac{\ln\left(y^3-3\right)}{6} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^2}{2} - \frac{\ln(y^3 - 3)}{6}$$

Summary

The solution(s) found are the following

$$-\frac{x^2}{2} - \frac{\ln(y^3 - 3)}{6} = c_1 \tag{1}$$

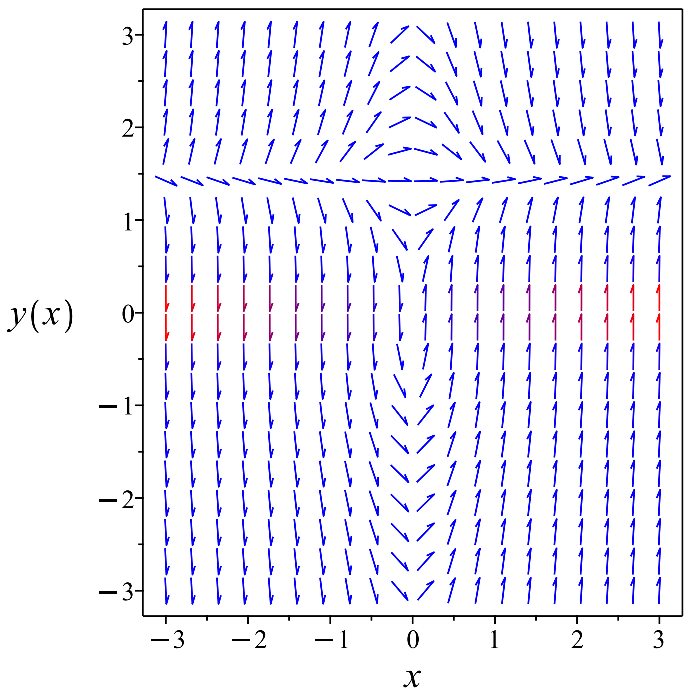


Figure 291: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \frac{\ln(y^3 - 3)}{6} = c_1$$

Verified OK.

5.20.5 Maple step by step solution

Let's solve

$2xy^3 + y^2y' = 6x$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y^2y'}{y^3-3} = -2x$

- Integrate both sides with respect to x

  $\int \frac{y^2y'}{y^3-3}dx = \int -2xdx + c_1$

- Evaluate integral

  $\frac{\ln(y^3-3)}{3} = -x^2 + c_1$

- Solve for y

  $y = \left(3 + e^{-3x^2+3c_1}\right)^{\frac{1}{3}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 66

```
dsolve(2*x*y(x)^3+y(x)^2*diff(y(x),x) = 6*x,y(x), singsol=all)
```

$$y(x) = \left(e^{-3x^2}c_1 + 3\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{\left(e^{-3x^2}c_1 + 3\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{\left(e^{-3x^2}c_1 + 3\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}$$

✓ Solution by Mathematica

Time used: 1.937 (sec). Leaf size: 115

```
DSolve[2*x*y[x]^3+y[x]^2*y'[x] == 6*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{3 + e^{-3x^2+3c_1}}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{3 + e^{-3x^2+3c_1}}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{3 + e^{-3x^2+3c_1}}$$
$$y(x) \to -\sqrt[3]{-3}$$
$$y(x) \to \sqrt[3]{3}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{3}$$

5.21 problem 21

Internal problem ID [99]
Internal file name [`OUTPUT/99_Sunday_June_05_2022_01_34_44_AM_997980/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"quadrature"**

Maple gives the following as the ode type

`[_quadrature]`

$$\boxed{y' - y^3 - y = 0}$$

5.21.1 Solving as quadrature ode

Integrating both sides gives

$$\int \frac{1}{y^3 + y} dy = \int dx$$
$$\ln(y) - \frac{\ln(y^2+1)}{2} = x + c_1$$

Raising both side to exponential gives

$$e^{\ln(y) - \frac{\ln(y^2+1)}{2}} = e^{x+c_1}$$

Which simplifies to

$$\frac{y}{\sqrt{y^2+1}} = c_2 e^x$$

Summary
The solution(s) found are the following

$$y = c_2 e^x \sqrt{-\frac{1}{c_2^2 e^{2x} - 1}} \tag{1}$$

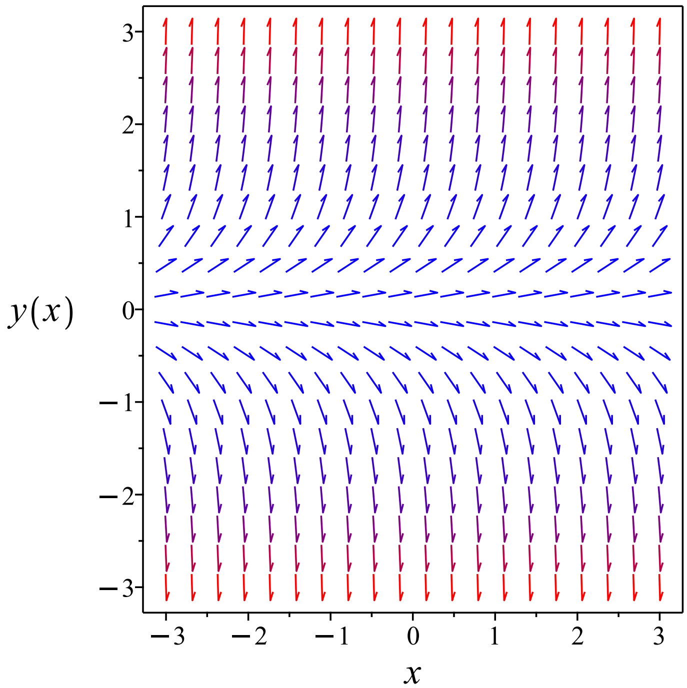


Figure 292: Slope field plot

Verification of solutions

$$y = c_2 e^{x} \sqrt{-\frac{1}{c_2^2 e^{2x} - 1}}$$

Verified OK.

5.21.2 Maple step by step solution

Let's solve

$y' - y^3 - y = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^3+y} = 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^3+y} dx = \int 1 dx + c_1$

- Evaluate integral

$$\ln (y) - \frac{\ln(1+y^2)}{2} = x + c_1$$

- Solve for y

$$\left\{y = \frac{\sqrt{-(\mathrm{e}^{2x+2c_1}-1)\,\mathrm{e}^{2x+2c_1}}}{\mathrm{e}^{2x+2c_1}-1}, y = -\frac{\sqrt{-(\mathrm{e}^{2x+2c_1}-1)\,\mathrm{e}^{2x+2c_1}}}{\mathrm{e}^{2x+2c_1}-1}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x) = y(x)+y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{1}{\sqrt{\mathrm{e}^{-2x}c_1 - 1}}$$
$$y(x) = -\frac{1}{\sqrt{\mathrm{e}^{-2x}c_1 - 1}}$$

✓ Solution by Mathematica

Time used: 60.06 (sec). Leaf size: 57

```
DSolve[y'[x] == y[x]+y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{ie^{x+c_1}}{\sqrt{-1+e^{2(x+c_1)}}}$$
$$y(x) \to \frac{ie^{x+c_1}}{\sqrt{-1+e^{2(x+c_1)}}}$$

5.22 problem 22

Internal problem ID [100]
Internal file name [`OUTPUT/100_Sunday_June_05_2022_01_34_48_AM_80663676/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2yx + y'x^2 - 5y^4 = 0}$$

5.22.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(5y^3 - 2x)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 217: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= y^4 x^6 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^4 x^6} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{3x^6 y^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(5y^3 - 2x)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{x^7 y^3} \\ S_y &= \frac{1}{y^4 x^6} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{5}{x^8} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{5}{R^8}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{5}{7R^7} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{3x^6y^3} = -\frac{5}{7x^7} + c_1$$

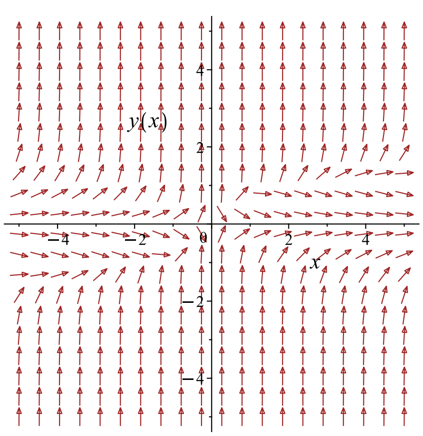
Which simplifies to

$$-\frac{1}{3x^6y^3} = -\frac{5}{7x^7} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y(5y^3-2x)}{x^2}$



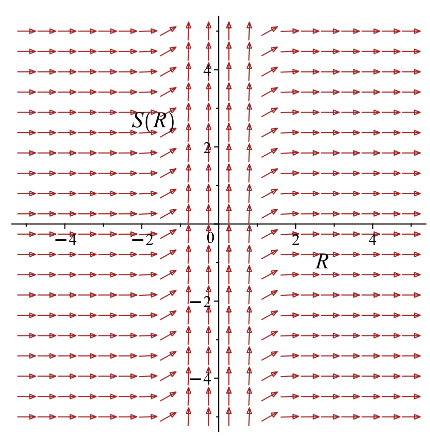
</td><td>

$R = x$

$S = -\frac{1}{3x^6y^3}$

</td><td>

$\frac{dS}{dR} = \frac{5}{R^8}$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$-\frac{1}{3x^6y^3} = -\frac{5}{7x^7} + c_1 \tag{1}$$

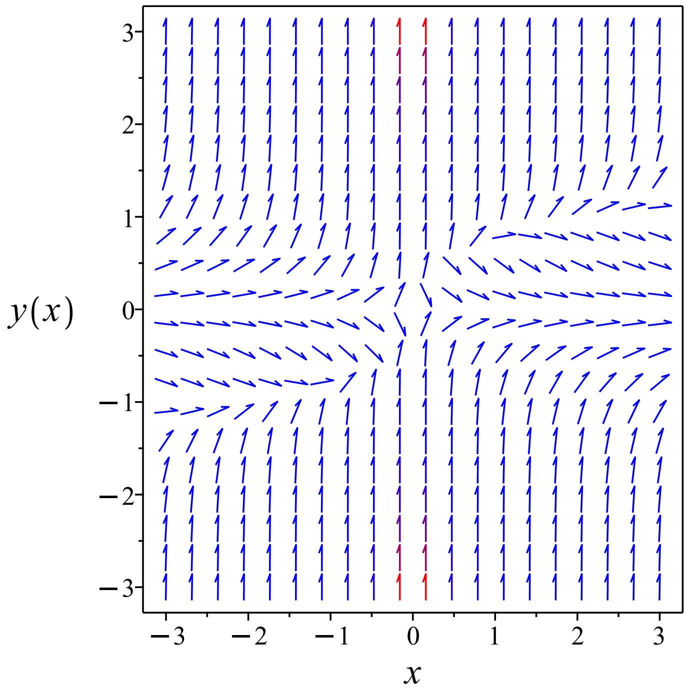


Figure 293: Slope field plot

Verification of solutions

$$-\frac{1}{3x^6y^3} = -\frac{5}{7x^7} + c_1$$

Verified OK.

5.22.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(5y^3 - 2x)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x}y + \frac{5}{x^2}y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= \frac{5}{x^2} \\ n &= 4 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$y' \frac{1}{y^4} = -\frac{2}{x\,y^3} + \frac{5}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{3}{y^4} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{3} &= -\frac{2w(x)}{x} + \frac{5}{x^2} \\ w' &= \frac{6w}{x} - \frac{15}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{6}{x} \\ q(x) &= -\frac{15}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{6w(x)}{x} = -\frac{15}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{6}{x}dx}\\ &= \frac{1}{x^6}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{15}{x^2}\right)\\ \frac{d}{dx}\left(\frac{w}{x^6}\right) &= \left(\frac{1}{x^6}\right)\left(-\frac{15}{x^2}\right)\\ d\left(\frac{w}{x^6}\right) &= \left(-\frac{15}{x^8}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^6} &= \int -\frac{15}{x^8}\, dx\\ \frac{w}{x^6} &= \frac{15}{7x^7} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^6}$ results in

$$w(x) = \frac{15}{7x} + c_1 x^6$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = \frac{15}{7x} + c_1 x^6$$

Solving for y gives

$$y(x) = \frac{7^{\frac{1}{3}}\left(x(7c_1x^7+15)^2\right)^{\frac{1}{3}}}{7c_1x^7+15}$$

$$y(x) = \frac{7^{\frac{1}{3}}\left(x(7c_1x^7+15)^2\right)^{\frac{1}{3}}(i\sqrt{3}-1)}{14c_1x^7+30}$$

$$y(x) = -\frac{7^{\frac{1}{3}}\left(x(7c_1x^7+15)^2\right)^{\frac{1}{3}}(1+i\sqrt{3})}{14c_1x^7+30}$$

Summary

The solution(s) found are the following

$$y = \frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}}}{7 c_1 x^7 + 15} \tag{1}$$

$$y = \frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}} (i\sqrt{3} - 1)}{14 c_1 x^7 + 30} \tag{2}$$

$$y = -\frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}} (1 + i\sqrt{3})}{14 c_1 x^7 + 30} \tag{3}$$

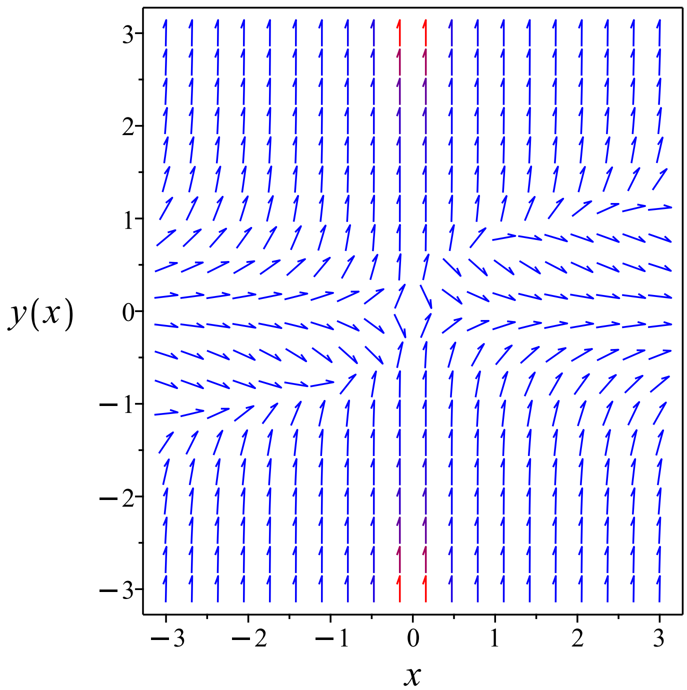


Figure 294: Slope field plot

Verification of solutions

$$y = \frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}}}{7 c_1 x^7 + 15}$$

Verified OK.

$$y = \frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}} (i\sqrt{3} - 1)}{14 c_1 x^7 + 30}$$

Verified OK.

$$y = -\frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}} (1 + i\sqrt{3})}{14 c_1 x^7 + 30}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 111

```
dsolve(2*x*y(x)+x^2*diff(y(x),x) = 5*y(x)^4,y(x), singsol=all)
```

$$y(x) = \frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}}}{7 c_1 x^7 + 15}$$
$$y(x) = -\frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}} (1 + i\sqrt{3})}{14 c_1 x^7 + 30}$$
$$y(x) = \frac{7^{\frac{1}{3}} \left(x (7 c_1 x^7 + 15)^2\right)^{\frac{1}{3}} (i\sqrt{3} - 1)}{14 c_1 x^7 + 30}$$

✓ Solution by Mathematica

Time used: 0.454 (sec). Leaf size: 96

```
DSolve[2*x*y[x]+x^2*y'[x] == 5*y[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt[3]{-7}\sqrt[3]{x}}{\sqrt[3]{15+7c_1x^7}}$$
$$y(x) \to \frac{\sqrt[3]{7}\sqrt[3]{x}}{\sqrt[3]{15+7c_1x^7}}$$
$$y(x) \to \frac{(-1)^{2/3}\sqrt[3]{7}\sqrt[3]{x}}{\sqrt[3]{15+7c_1x^7}}$$
$$y(x) \to 0$$

5.23 problem 23

Internal problem ID [101]
Internal file name [`OUTPUT/101_Sunday_June_05_2022_01_34_49_AM_42528764/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{6y + y'x - 3xy^{\frac{4}{3}} = 0}$$

5.23.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3\left(-x\,y^{\frac{4}{3}} + 2y\right)}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 219: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= y^{\frac{4}{3}} x^2 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^{\frac{4}{3}} x^2} dy \end{aligned}$$

Which results in

$$S = -\frac{3}{y^{\frac{1}{3}} x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{3\left(-x\, y^{\frac{4}{3}} + 2y\right)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{6}{y^{\frac{1}{3}} x^3} \\ S_y &= \frac{1}{y^{\frac{4}{3}} x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{x^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{3}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{3}{y^{\frac{1}{3}} x^2} = -\frac{3}{x} + c_1$$

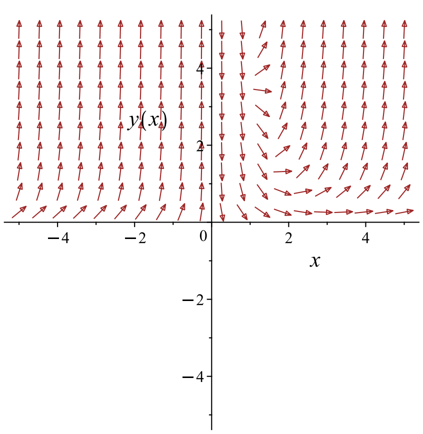
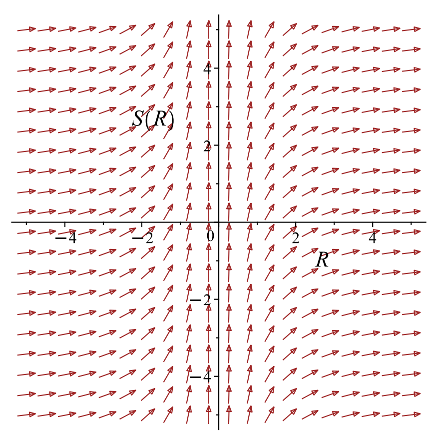
Which simplifies to

$$-\frac{3}{y^{\frac{1}{3}} x^2} = -\frac{3}{x} + c_1$$

Which gives

$$y = -\frac{27}{x^3 (c_1 x - 3)^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3\left(-x y^{\frac{4}{3}} + 2y\right)}{x}$ 	$R = x$ $S = -\frac{3}{y^{\frac{1}{3}} x^2}$	$\frac{dS}{dR} = \frac{3}{R^2}$ 

Summary
The solution(s) found are the following

$$y = -\frac{27}{x^3 (c_1 x - 3)^3} \tag{1}$$

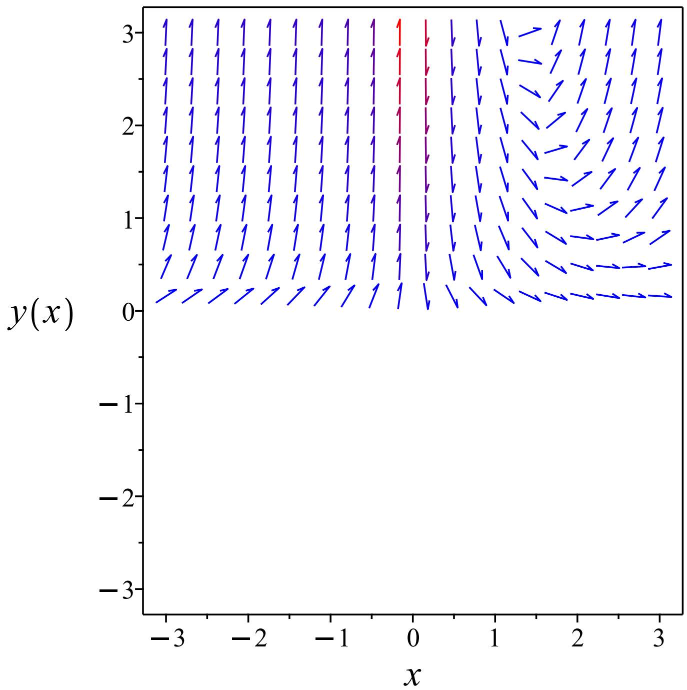


Figure 295: Slope field plot

Verification of solutions

$$y = -\frac{27}{x^3 (c_1 x - 3)^3}$$

Verified OK.

5.23.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{3\left(-x\, y^{\frac{4}{3}} + 2y\right)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{6}{x} y + 3y^{\frac{4}{3}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned}
f_0(x) &= -\frac{6}{x} \\
f_1(x) &= 3 \\
n &= \frac{4}{3}
\end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{4}{3}}$ gives

$$y' \frac{1}{y^{\frac{4}{3}}} = -\frac{6}{x\, y^{\frac{1}{3}}} + 3 \tag{4}$$

Let

$$\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^{\frac{1}{3}}}
\end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{3y^{\frac{4}{3}}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned}
-3w'(x) &= -\frac{6w(x)}{x} + 3 \\
w' &= \frac{2w}{x} - 1
\end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{2}{x}\\ q(x) &= -1\end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -1$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{2}{x}dx}\\ &= \frac{1}{x^2}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)(-1)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)(-1)\\ \mathrm{d}\left(\frac{w}{x^2}\right) &= \left(-\frac{1}{x^2}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^2} &= \int -\frac{1}{x^2}\,\mathrm{d}x\\ \frac{w}{x^2} &= \frac{1}{x} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = c_1x^2 + x$$

Replacing w in the above by $\frac{1}{y^{\frac{1}{3}}}$ using equation (5) gives the final solution.

$$\frac{1}{y^{\frac{1}{3}}} = c_1x^2 + x$$

Summary

The solution(s) found are the following

$$\frac{1}{y^{\frac{1}{3}}} = c_1 x^2 + x \tag{1}$$

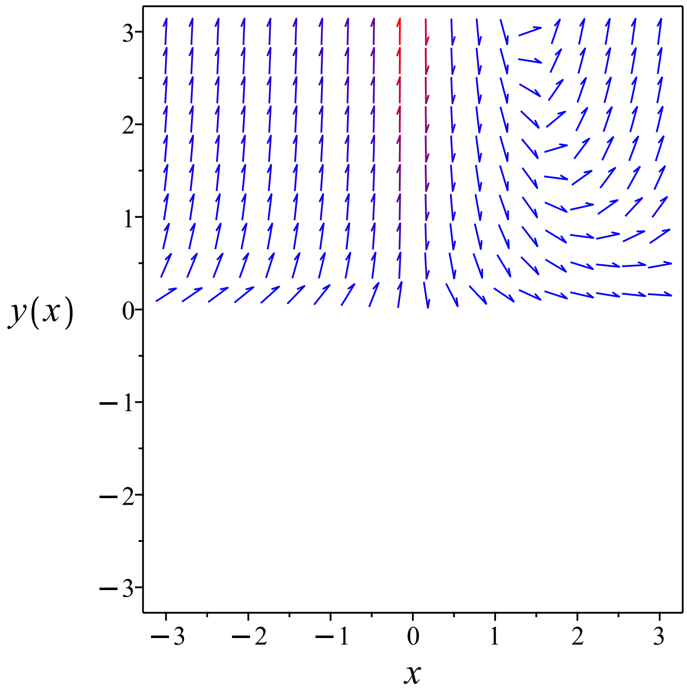


Figure 296: Slope field plot

Verification of solutions

$$\frac{1}{y^{\frac{1}{3}}} = c_1 x^2 + x$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(6*y(x)+x*diff(y(x),x) = 3*x*y(x)^(4/3),y(x), singsol=all)
```

$$\frac{1}{y(x)^{\frac{1}{3}}} - x - c_1 x^2 = 0$$

✓ Solution by Mathematica

Time used: 0.146 (sec). Leaf size: 22

```
DSolve[6*y[x]+x*y'[x] == 3*x*y[x]^(4/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{x^3(1+c_1 x)^3}$$
$$y(x) \to 0$$

5.24 problem 24

Internal problem ID [102]
Internal file name [`OUTPUT/102_Sunday_June_05_2022_01_34_50_AM_75999386/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_Bernoulli]
```

$$\boxed{y^3 e^{-2x} + 2y'x - 2yx = 0}$$

5.24.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(y^2 - 2x\,e^{2x})\,e^{-2x}}{2x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 221: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= y^3 \mathrm{e}^{-2x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{y^3 e^{-2x}} dy \end{aligned}$$

Which results in

$$S = -\frac{e^{2x}}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(y^2 - 2x\, e^{2x})\, e^{-2x}}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{e^{2x}}{y^2} \\ S_y &= \frac{e^{2x}}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{2} + c_1 \tag{4}$$

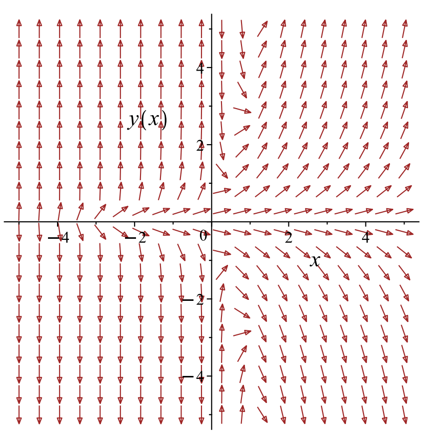
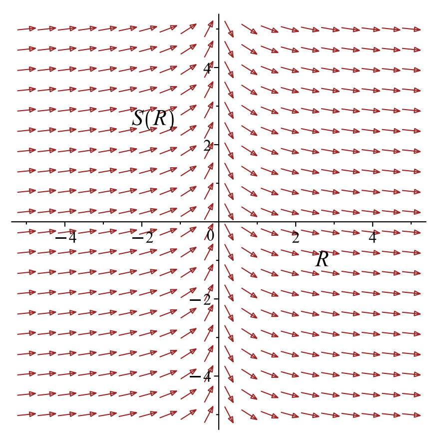
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{e^{2x}}{2y^2} = -\frac{\ln(x)}{2} + c_1$$

Which simplifies to

$$-\frac{e^{2x}}{2y^2} = -\frac{\ln(x)}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y^2-2x\,e^{2x})\,e^{-2x}}{2x}$ 	$R = x$ $S = -\frac{e^{2x}}{2y^2}$	$\frac{dS}{dR} = -\frac{1}{2R}$ 

Summary

The solution(s) found are the following

$$-\frac{e^{2x}}{2y^2} = -\frac{\ln(x)}{2} + c_1 \tag{1}$$

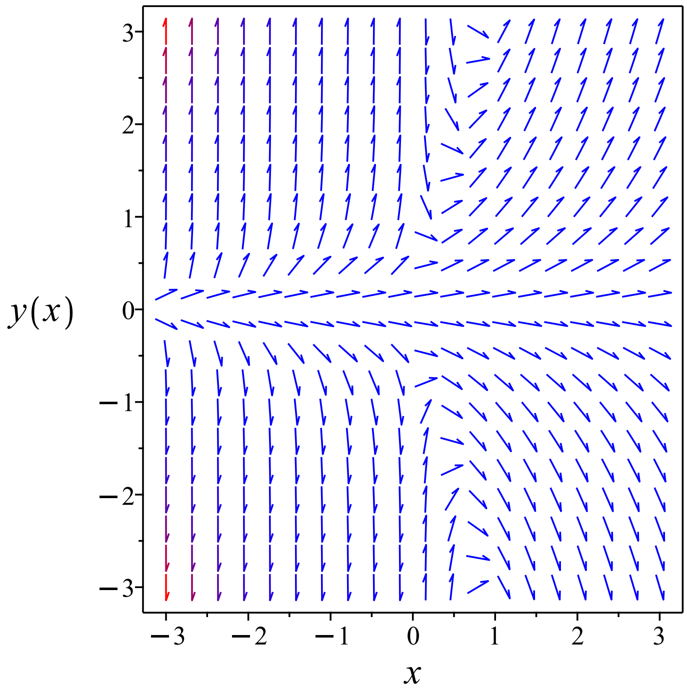


Figure 297: Slope field plot

Verification of solutions

$$-\frac{e^{2x}}{2y^2} = -\frac{\ln(x)}{2} + c_1$$

Verified OK.

5.24.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y(y^2 - 2x\,e^{2x})\,e^{-2x}}{2x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = y - \frac{e^{-2x}}{2x}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= 1 \\
f_1(x) &= -\frac{e^{-2x}}{2x} \\
n &= 3
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$
y'\frac{1}{y^3} = \frac{1}{y^2} - \frac{e^{-2x}}{2x} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \frac{1}{y^2}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = -\frac{2}{y^3}y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
-\frac{w'(x)}{2} &= w(x) - \frac{e^{-2x}}{2x} \\
w' &= -2w + \frac{e^{-2x}}{x}
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= \frac{e^{-2x}}{x}
\end{aligned}
$$

Hence the ode is

$$w'(x) + 2w(x) = \frac{e^{-2x}}{x}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int 2dx} \\ &= e^{2x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{e^{-2x}}{x}\right) \\ \frac{d}{dx}\left(e^{2x}w\right) &= \left(e^{2x}\right)\left(\frac{e^{-2x}}{x}\right) \\ d\left(e^{2x}w\right) &= \frac{1}{x}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}e^{2x}w &= \int \frac{1}{x}\,dx \\ e^{2x}w &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{2x}$ results in

$$w(x) = e^{-2x}\ln(x) + c_1 e^{-2x}$$

which simplifies to

$$w(x) = e^{-2x}(\ln(x) + c_1)$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = e^{-2x}(\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{\sqrt{e^{-2x}(\ln(x) + c_1)}} \\ y(x) &= -\frac{1}{\sqrt{e^{-2x}(\ln(x) + c_1)}}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\sqrt{e^{-2x} (\ln (x) + c_1)}} \tag{1}$$

$$y = -\frac{1}{\sqrt{e^{-2x} (\ln (x) + c_1)}} \tag{2}$$

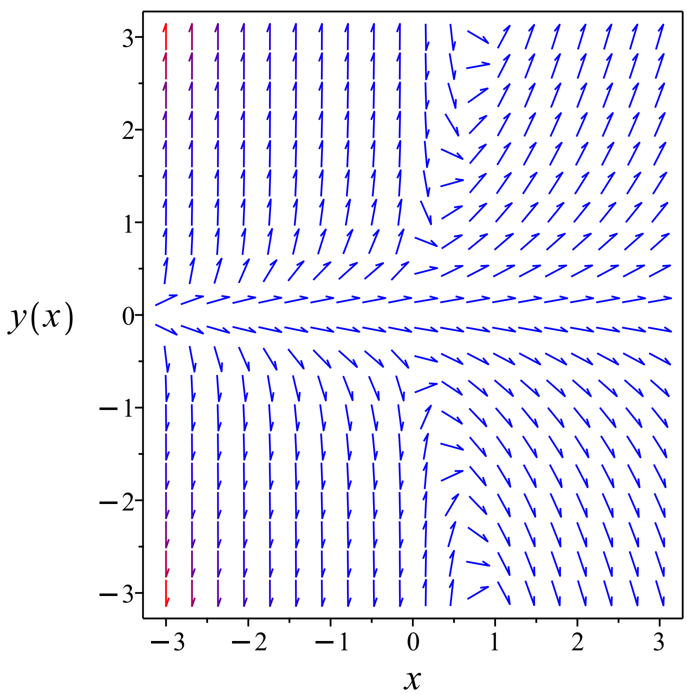


Figure 298: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{e^{-2x} (\ln (x) + c_1)}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{e^{-2x} (\ln (x) + c_1)}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 47

```
dsolve(y(x)^3/exp(2*x)+2*x*diff(y(x),x) = 2*x*y(x),y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{(\ln (x) + c_1)\, e^{2x}}}{\ln (x) + c_1}$$
$$y(x) = \frac{\sqrt{(\ln (x) + c_1)\, e^{2x}}}{-\ln (x) - c_1}$$

✓ Solution by Mathematica

Time used: 0.342 (sec). Leaf size: 41

```
DSolve[y[x]^3/Exp[2*x]+2*x*y'[x] == 2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{e^x}{\sqrt{\log(x) + c_1}}$$
$$y(x) \to \frac{e^x}{\sqrt{\log(x) + c_1}}$$
$$y(x) \to 0$$

5.25 problem 25

Internal problem ID [103]
Internal file name [`OUTPUT/103_Sunday_June_05_2022_01_34_52_AM_240953/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_Bernoulli]`

$$\boxed{\sqrt{x^4+1}\, y^2(y+y'x)=x}$$

5.25.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{\sqrt{x^4+1}\, y^3-x}{\sqrt{x^4+1}\, y^2 x} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 223: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x, y) &= 0 \\ \eta(x, y) &= \frac{1}{y^2 x^3}\end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{y^2 x^3}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3 x^3}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{\sqrt{x^4+1}\, y^3 - x}{\sqrt{x^4+1}\, y^2 x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y^3 x^2 \\ S_y &= x^3 y^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{x^3}{\sqrt{x^4+1}} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{R^3}{\sqrt{R^4+1}}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \int \frac{R^3}{\sqrt{R^4+1}} dR + c_1 \tag{4}$$

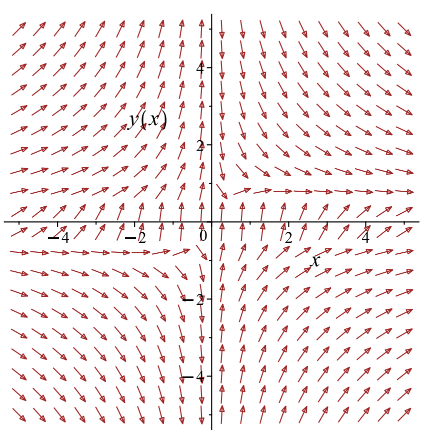
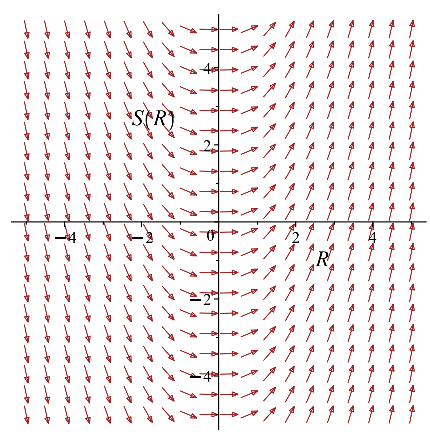
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 x^3}{3} = \int \frac{x^3}{\sqrt{x^4+1}} dx + c_1$$

Which simplifies to

$$\frac{y^3 x^3}{3} = \int \frac{x^3}{\sqrt{x^4+1}} dx + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{\sqrt{x^4+1}\, y^3 - x}{\sqrt{x^4+1}\, y^2 x}$ 	$R = x$ $S = \frac{y^3 x^3}{3}$	$\frac{dS}{dR} = \frac{R^3}{\sqrt{R^4+1}}$ 

Summary
The solution(s) found are the following

$$\frac{y^3 x^3}{3} = \int \frac{x^3}{\sqrt{x^4+1}} dx + c_1 \tag{1}$$

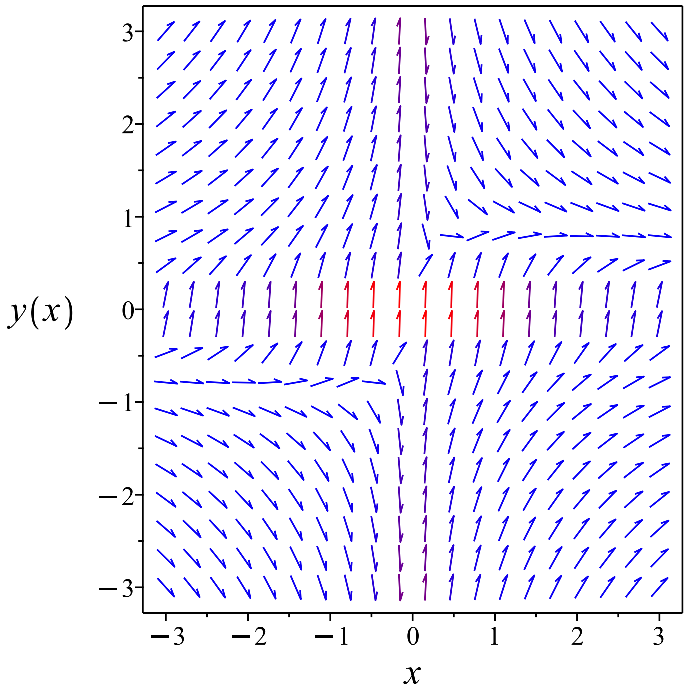


Figure 299: Slope field plot

Verification of solutions

$$\frac{y^3 x^3}{3} = \int \frac{x^3}{\sqrt{x^4+1}}dx + c_1$$

Verified OK.

5.25.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{\sqrt{x^4+1}\,y^3 - x}{\sqrt{x^4+1}\,y^2 x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y + \frac{1}{\sqrt{x^4+1}}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= \frac{1}{\sqrt{x^4+1}} \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = -\frac{y^3}{x} + \frac{1}{\sqrt{x^4+1}} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= -\frac{w(x)}{x} + \frac{1}{\sqrt{x^4+1}} \\ w' &= -\frac{3w}{x} + \frac{3}{\sqrt{x^4+1}} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$
\begin{aligned}
p(x) &= \frac{3}{x} \\
q(x) &= \frac{3}{\sqrt{x^4+1}}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + \frac{3w(x)}{x} = \frac{3}{\sqrt{x^4+1}}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{x}dx} \\
&= x^3
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{3}{\sqrt{x^4+1}}\right) \\
\frac{d}{dx}(x^3 w) &= (x^3)\left(\frac{3}{\sqrt{x^4+1}}\right) \\
d(x^3 w) &= \left(\frac{3x^3}{\sqrt{x^4+1}}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
x^3 w &= \int \frac{3x^3}{\sqrt{x^4+1}}\, dx \\
x^3 w &= \frac{3\sqrt{x^4+1}}{2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
w(x) = \frac{3\sqrt{x^4+1}}{2x^3} + \frac{c_1}{x^3}
$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$
y^3 = \frac{3\sqrt{x^4+1}}{2x^3} + \frac{c_1}{x^3}
$$

Solving for y gives

$$y(x) = \frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}}{2x}$$
$$y(x) = \frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4x}$$
$$y(x) = -\frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}}{2x} \tag{1}$$
$$y = \frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4x} \tag{2}$$
$$y = -\frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4x} \tag{3}$$

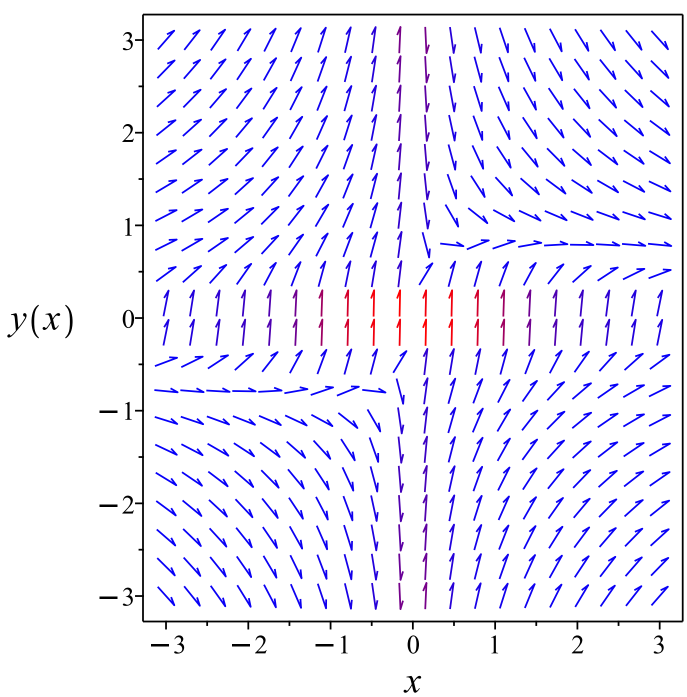


Figure 300: Slope field plot

Verification of solutions

$$y = \frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}}{2x}$$

Verified OK.

$$y = \frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{4x}$$

Verified OK.

$$y = -\frac{\left(12\sqrt{x^4+1}+8c_1\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{4x}$$

Verified OK.

5.25.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sqrt{x^4+1}\,y^2x\right)\mathrm{d}y &= \left(-\sqrt{x^4+1}\,y^3+x\right)\mathrm{d}x\\ \left(\sqrt{x^4+1}\,y^3-x\right)\mathrm{d}x + \left(\sqrt{x^4+1}\,y^2x\right)\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= \sqrt{x^4+1}\,y^3-x\\ N(x,y) &= \sqrt{x^4+1}\,y^2x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\sqrt{x^4+1}\,y^3-x\right)\\ &= 3\sqrt{x^4+1}\,y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sqrt{x^4+1}\,y^2x\right)\\ &= \frac{y^2(3x^4+1)}{\sqrt{x^4+1}}\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{\sqrt{x^4+1}\,y^2x}\left(\left(3\sqrt{x^4+1}\,y^2\right) - \left(\frac{2y^2x^4}{\sqrt{x^4+1}} + \sqrt{x^4+1}\,y^2\right)\right)\\ &= \frac{2}{(x^4+1)\,x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int \frac{2}{\left(x^4+1\right)x} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\frac{\ln\left(x^4+1\right)}{2}+2\ln(x)} \\
&= \frac{x^2}{\sqrt{x^4+1}}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{x^2}{\sqrt{x^4+1}}\left(\sqrt{x^4+1}\, y^3 - x\right) \\
&= -\frac{x^2\left(-\sqrt{x^4+1}\, y^3 + x\right)}{\sqrt{x^4+1}}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{x^2}{\sqrt{x^4+1}}\left(\sqrt{x^4+1}\, y^2 x\right) \\
&= x^3 y^2
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(-\frac{x^2\left(-\sqrt{x^4+1}\, y^3 + x\right)}{\sqrt{x^4+1}}\right) + \left(x^3 y^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \overline{M} \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{x^2\left(-\sqrt{x^4+1}\, y^3 + x\right)}{\sqrt{x^4+1}} \mathrm{d}x$$

$$\phi = \frac{y^3 x^3}{3} - \frac{\sqrt{x^4+1}}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^3 y^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3 y^2$. Therefore equation (4) becomes

$$x^3 y^2 = x^3 y^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{y^3 x^3}{3} - \frac{\sqrt{x^4+1}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{y^3 x^3}{3} - \frac{\sqrt{x^4+1}}{2}$$

Summary

The solution(s) found are the following

$$\frac{y^3 x^3}{3} - \frac{\sqrt{x^4+1}}{2} = c_1 \tag{1}$$

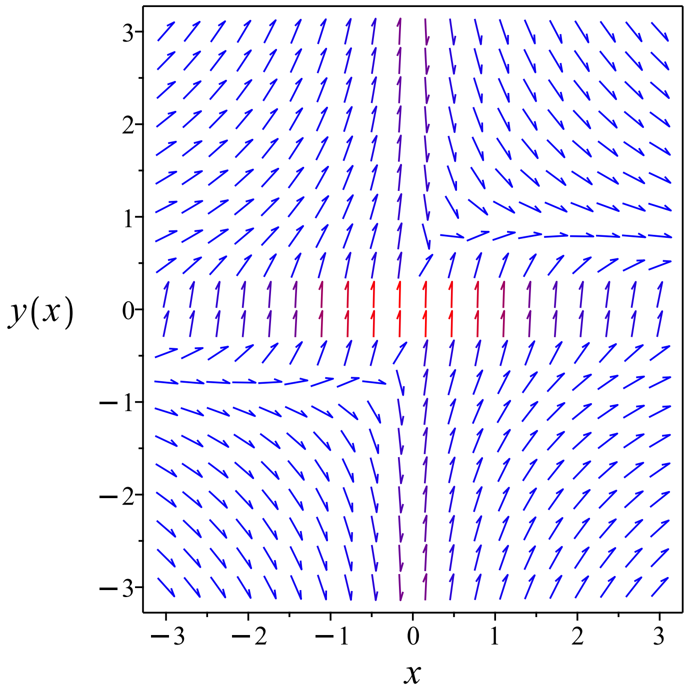


Figure 301: Slope field plot

Verification of solutions

$$\frac{y^3x^3}{3} - \frac{\sqrt{x^4+1}}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 97

```
dsolve((x^4+1)^(1/2)*y(x)^2*(y(x)+x*diff(y(x),x)) = x,y(x), singsol=all)
```

$$y(x) = \frac{\left(3\left(\int \frac{x^3}{\sqrt{x^4+1}}dx\right)+c_1\right)^{\frac{1}{3}}}{x}$$
$$y(x) = -\frac{\left(3\left(\int \frac{x^3}{\sqrt{x^4+1}}dx\right)+c_1\right)^{\frac{1}{3}}\left(1+i\sqrt{3}\right)}{2x}$$
$$y(x) = \frac{\left(3\left(\int \frac{x^3}{\sqrt{x^4+1}}dx\right)+c_1\right)^{\frac{1}{3}}\left(i\sqrt{3}-1\right)}{2x}$$

✓ Solution by Mathematica

Time used: 3.932 (sec). Leaf size: 106

```
DSolve[(x^4+1)^(1/2)*y[x]^2*(y[x]+x*y'[x]) ==x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{\frac{3\sqrt{x^4+1}}{2x^3}+\frac{c_1}{x^3}}$$
$$y(x) \to -\sqrt[3]{-\frac{1}{2}}\sqrt[3]{\frac{3\sqrt{x^4+1}+2c_1}{x^3}}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{\frac{3\sqrt{x^4+1}}{2x^3}+\frac{c_1}{x^3}}$$

5.26 problem 26

Internal problem ID [104]
Internal file name [`OUTPUT/104_Sunday_June_05_2022_01_34_54_AM_49783570/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries], _Bernoulli]
```

$$\boxed{y^3 + 3y^2y' = \mathrm{e}^{-x}}$$

5.26.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^3 - \mathrm{e}^{-x}}{3y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 225: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x, y) &= 0 \\ \eta(x, y) &= \frac{e^{-x}}{y^2} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{e^{-x}}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3 e^x}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y^3 - e^{-x}}{3y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^3 e^x}{3} \\ S_y &= y^2 e^x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \frac{R}{3} + c_1 \tag{4}$$

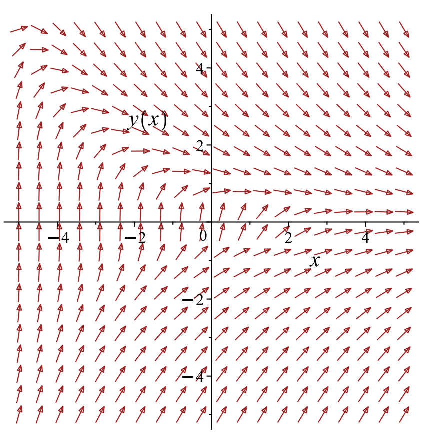
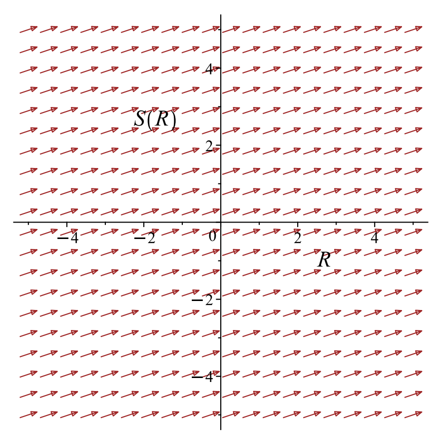
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3 e^x}{3} = \frac{x}{3} + c_1$$

Which simplifies to

$$\frac{y^3 e^x}{3} = \frac{x}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^3 - e^{-x}}{3y^2}$ 	$R = x$ $S = \frac{y^3 e^x}{3}$	$\frac{dS}{dR} = \frac{1}{3}$ 

Summary

The solution(s) found are the following

$$\frac{y^3 e^x}{3} = \frac{x}{3} + c_1 \tag{1}$$

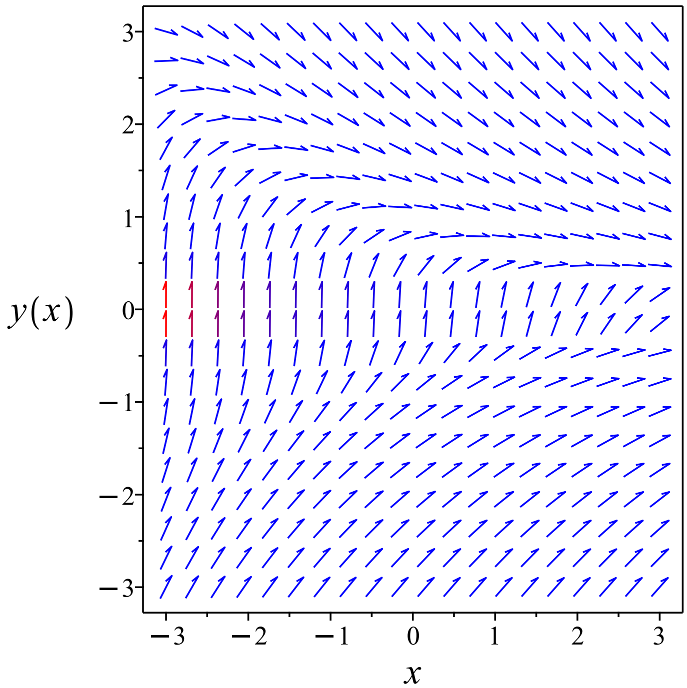


Figure 302: Slope field plot

Verification of solutions

$$\frac{y^3 e^x}{3} = \frac{x}{3} + c_1$$

Verified OK.

5.26.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^3 - e^{-x}}{3y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{3}y + \frac{e^{-x}}{3}\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{1}{3} \\
f_1(x) &= \frac{e^{-x}}{3} \\
n &= -2
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$
y'y^2 = -\frac{y^3}{3} + \frac{e^{-x}}{3} \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= y^3 \qquad (5)
\end{aligned}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = 3y^2y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
\frac{w'(x)}{3} &= -\frac{w(x)}{3} + \frac{e^{-x}}{3} \\
w' &= -w + e^{-x} \qquad (7)
\end{aligned}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= e^{-x}
\end{aligned}
$$

Hence the ode is

$$
w'(x) + w(x) = e^{-x}
$$

The integrating factor μ is

$$\mu = e^{\int 1dx}$$
$$= e^{x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(e^{-x}\right)$$
$$\frac{d}{dx}(w\,e^{x}) = (e^{x})\left(e^{-x}\right)$$
$$d(w\,e^{x}) = dx$$

Integrating gives

$$w\,e^{x} = \int dx$$
$$w\,e^{x} = x + c_1$$

Dividing both sides by the integrating factor $\mu = e^{x}$ results in

$$w(x) = x\,e^{-x} + c_1 e^{-x}$$

which simplifies to

$$w(x) = e^{-x}(x + c_1)$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = e^{-x}(x + c_1)$$

Solving for y gives

$$y(x) = \left(e^{-x}(x + c_1)\right)^{\frac{1}{3}}$$
$$y(x) = \frac{\left(e^{-x}(x + c_1)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}$$
$$y(x) = -\frac{\left(e^{-x}(x + c_1)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \left(e^{-x}(x + c_1)\right)^{\frac{1}{3}} \tag{1}$$
$$y = \frac{\left(e^{-x}(x + c_1)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2} \tag{2}$$
$$y = -\frac{\left(e^{-x}(x + c_1)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2} \tag{3}$$

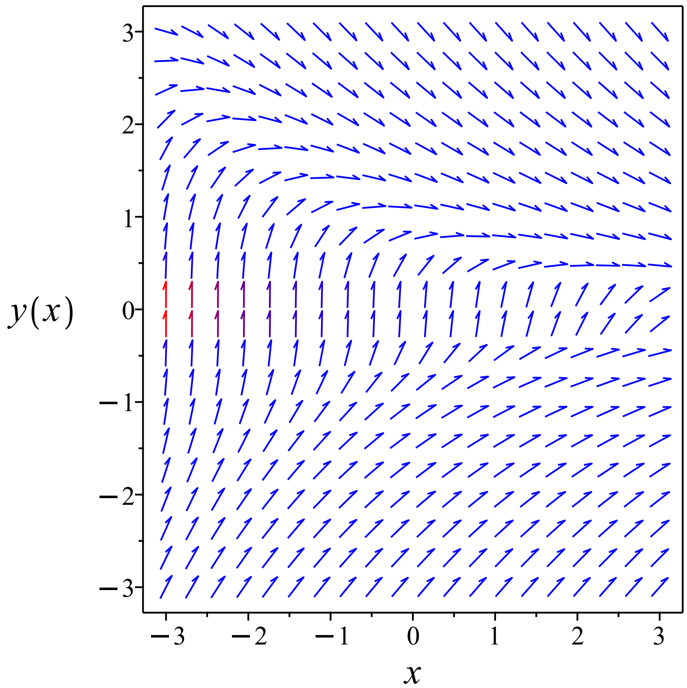


Figure 303: Slope field plot

Verification of solutions

$$y = \left(e^{-x}(x + c_1)\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(e^{-x}(x + c_1)\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

Verified OK.

$$y = -\frac{\left(e^{-x}(x + c_1)\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$

Verified OK.

5.26.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(3y^2)\,\mathrm{d}y&=(-y^3+\mathrm{e}^{-x})\,\mathrm{d}x\\ (y^3-\mathrm{e}^{-x})\,\mathrm{d}x+(3y^2)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=y^3-\mathrm{e}^{-x}\\ N(x,y)&=3y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(y^3 - \mathrm{e}^{-x}\right)\\ &= 3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3y^2\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{3y^2}\left(\left(3y^2\right) - (0)\right)\\ &= 1\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int 1\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{x}\\ &= \mathrm{e}^{x}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= \mathrm{e}^{x}\left(y^3 - \mathrm{e}^{-x}\right)\\ &= y^3\mathrm{e}^{x} - 1\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= e^x (3y^2) \\
&= 3y^2 e^x
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\
(y^3 e^x - 1) + (3y^2 e^x)\frac{dy}{dx} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} dx &= \int \overline{M} dx \\
\int \frac{\partial \phi}{\partial x} dx &= \int y^3 e^x - 1 dx
\end{aligned}
$$

$$
\phi = -x + y^3 e^x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 3y^2 e^x + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3y^2 e^x$. Therefore equation (4) becomes

$$
3y^2 e^x = 3y^2 e^x + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x + y^3 e^x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x + y^3 e^x$$

Summary

The solution(s) found are the following

$$y^3 e^x - x = c_1 \tag{1}$$

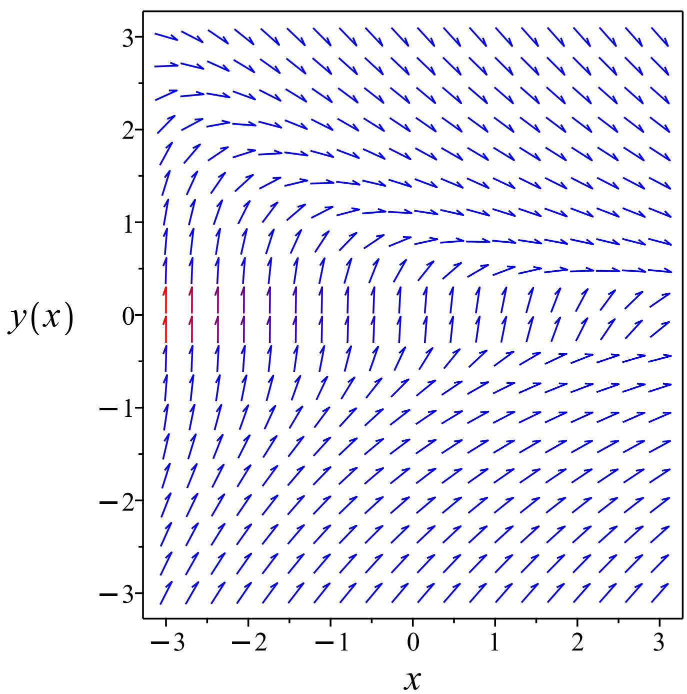


Figure 304: Slope field plot

Verification of solutions

$$y^3 e^x - x = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 73

```
dsolve(y(x)^3+3*y(x)^2*diff(y(x),x) = exp(-x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x}\left(\left(c_1 + x\right)\mathrm{e}^{2x}\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{\left(\left(c_1 + x\right)\mathrm{e}^{2x}\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)\mathrm{e}^{-x}}{2}$$
$$y(x) = \frac{\left(\left(c_1 + x\right)\mathrm{e}^{2x}\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)\mathrm{e}^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.307 (sec). Leaf size: 72

```
DSolve[y[x]^3+3*y[x]^2*y'[x] == Exp[-x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/3}\sqrt[3]{x + c_1}$$
$$y(x) \to -\sqrt[3]{-1}e^{-x/3}\sqrt[3]{x + c_1}$$
$$y(x) \to (-1)^{2/3}e^{-x/3}\sqrt[3]{x + c_1}$$

5.27 problem 27

Internal problem ID [105]
Internal file name [`OUTPUT/105_Sunday_June_05_2022_01_34_55_AM_33710285/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{3xy^2y' - y^3 = 3x^4}$$

5.27.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3x^4 + y^3}{3x\,y^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 227: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{x}{y^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x}{y^2}} dy \end{aligned}$$

Which results in

$$S = \frac{y^3}{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3x^4 + y^3}{3x\, y^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{y^3}{3x^2} \\ S_y &= \frac{y^2}{x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \frac{R^3}{3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y^3}{3x} = \frac{x^3}{3} + c_1$$

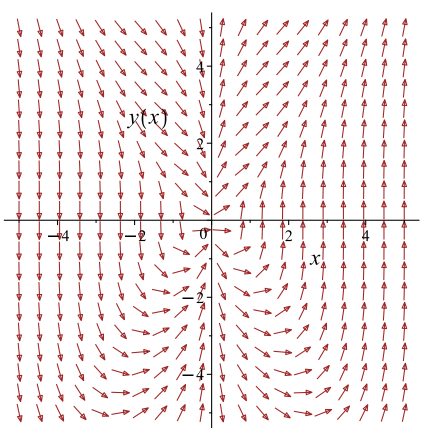
Which simplifies to

$$\frac{y^3}{3x} = \frac{x^3}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>

$\frac{dy}{dx} = \frac{3x^4+y^3}{3x\,y^2}$



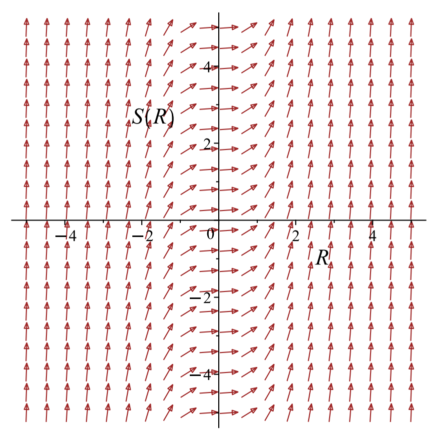
</td><td>

$R = x$

$S = \frac{y^3}{3x}$

</td><td>

$\frac{dS}{dR} = R^2$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{y^3}{3x} = \frac{x^3}{3} + c_1 \tag{1}$$

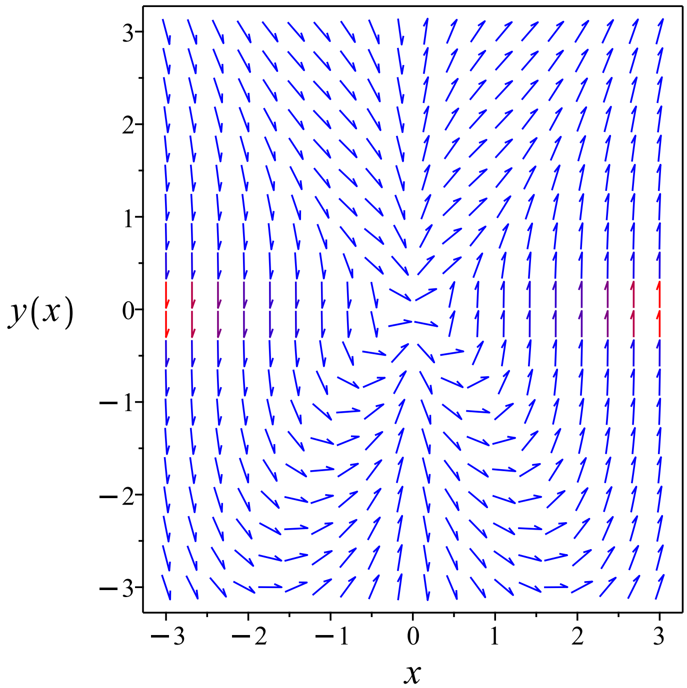


Figure 305: Slope field plot

Verification of solutions

$$\frac{y^3}{3x} = \frac{x^3}{3} + c_1$$

Verified OK.

5.27.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{3x^4 + y^3}{3x\, y^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{3x}y + x^3\frac{1}{y^2} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{3x} \\ f_1(x) &= x^3 \\ n &= -2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y^2}$ gives

$$y'y^2 = \frac{y^3}{3x} + x^3 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^3 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 3y^2y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{3} &= \frac{w(x)}{3x} + x^3 \\ w' &= \frac{w}{x} + 3x^3 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= 3x^3 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{w(x)}{x} = 3x^3$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x}dx} = \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(3x^3\right)$$
$$\frac{d}{dx}\left(\frac{w}{x}\right) = \left(\frac{1}{x}\right)\left(3x^3\right)$$
$$d\left(\frac{w}{x}\right) = \left(3x^2\right) dx$$

Integrating gives

$$\frac{w}{x} = \int 3x^2\, dx$$
$$\frac{w}{x} = x^3 + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$w(x) = x^4 + c_1 x$$

Replacing w in the above by y^3 using equation (5) gives the final solution.

$$y^3 = x^4 + c_1 x$$

Solving for y gives

$$y(x) = \left(x\left(x^3 + c_1\right)\right)^{\frac{1}{3}}$$
$$y(x) = \frac{\left(x(x^3 + c_1)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2}$$
$$y(x) = -\frac{\left(x(x^3 + c_1)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2}$$

Summary

The solution(s) found are the following

$$y = \left(x\left(x^3 + c_1\right)\right)^{\frac{1}{3}} \tag{1}$$
$$y = \frac{\left(x(x^3 + c_1)\right)^{\frac{1}{3}}\left(i\sqrt{3} - 1\right)}{2} \tag{2}$$
$$y = -\frac{\left(x(x^3 + c_1)\right)^{\frac{1}{3}}\left(1 + i\sqrt{3}\right)}{2} \tag{3}$$

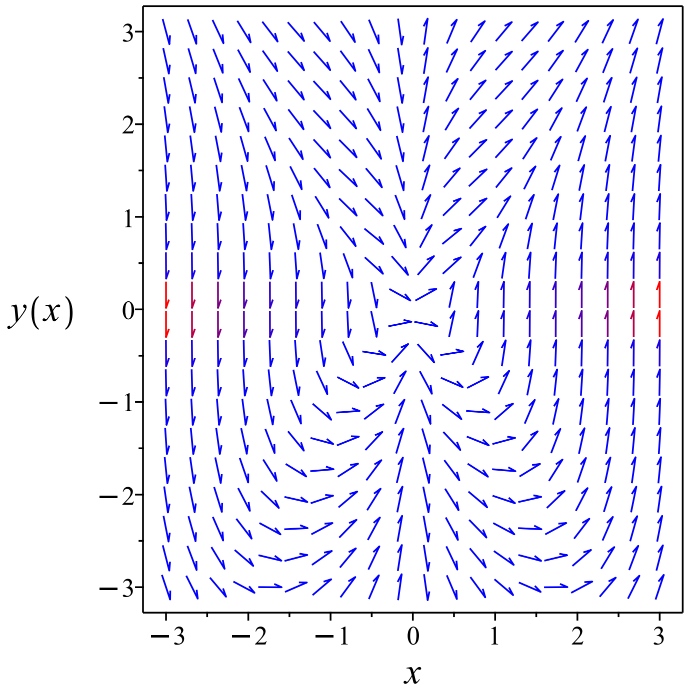


Figure 306: Slope field plot

Verification of solutions

$$y = \left(x\left(x^3 + c_1\right)\right)^{\frac{1}{3}}$$

Verified OK.

$$y = \frac{\left(x(x^3 + c_1)\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

Verified OK.

$$y = -\frac{\left(x(x^3 + c_1)\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$

Verified OK.

5.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(3x\,y^2\right)\mathrm{d}y&=\left(3x^4+y^3\right)\mathrm{d}x\\ \left(-3x^4-y^3\right)\mathrm{d}x+\left(3x\,y^2\right)\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-3x^4-y^3\\ N(x,y)&=3x\,y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-3x^4 - y^3\right) \\ &= -3y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(3x\,y^2\right) \\ &= 3y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{3x\,y^2}\left(\left(-3y^2\right) - \left(3y^2\right)\right) \\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x} \\ &= e^{\int -\frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}\left(-3x^4 - y^3\right) \\ &= \frac{-3x^4 - y^3}{x^2}\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^2}(3x\,y^2) \\
&= \frac{3y^2}{x}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-3x^4 - y^3}{x^2}\right) + \left(\frac{3y^2}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{-3x^4 - y^3}{x^2}\,\mathrm{d}x \\
\phi &= -x^3 + \frac{y^3}{x} + f(y)
\end{aligned} \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = \frac{3y^2}{x} + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3y^2}{x}$. Therefore equation (4) becomes

$$
\frac{3y^2}{x} = \frac{3y^2}{x} + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -x^3 + \frac{y^3}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -x^3 + \frac{y^3}{x}$$

Summary

The solution(s) found are the following

$$-x^3 + \frac{y^3}{x} = c_1 \tag{1}$$

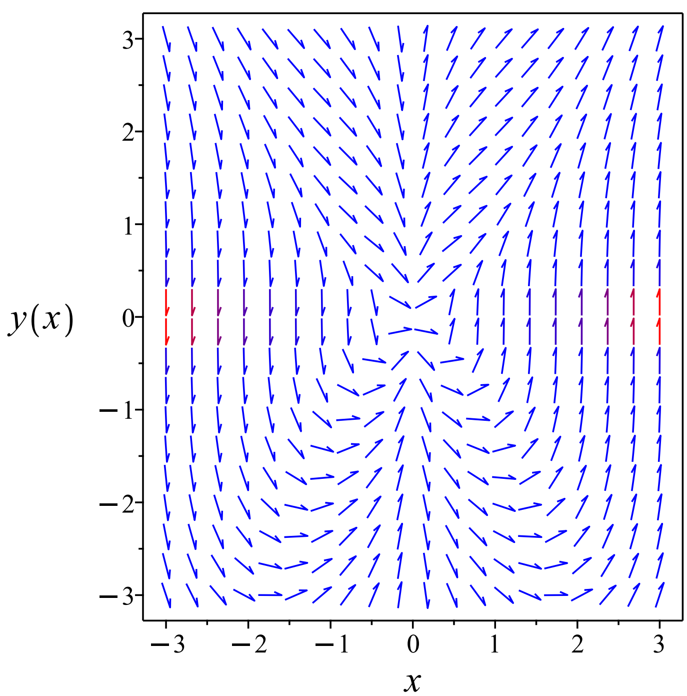


Figure 307: Slope field plot

Verification of solutions

$$-x^3 + \frac{y^3}{x} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 57

```
dsolve(3*x*y(x)^2*diff(y(x),x) = 3*x^4+y(x)^3,y(x), singsol=all)
```

$$y(x) = \left(\left(x^3 + c_1\right) x\right)^{\frac{1}{3}}$$
$$y(x) = -\frac{\left(\left(x^3 + c_1\right) x\right)^{\frac{1}{3}} \left(1 + i\sqrt{3}\right)}{2}$$
$$y(x) = \frac{\left(\left(x^3 + c_1\right) x\right)^{\frac{1}{3}} \left(i\sqrt{3} - 1\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.217 (sec). Leaf size: 72

```
DSolve[3*x*y[x]^2*y'[x] == 3*x^4+y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \sqrt[3]{x}\sqrt[3]{x^3 + c_1}$$
$$y(x) \to -\sqrt[3]{-1}\sqrt[3]{x}\sqrt[3]{x^3 + c_1}$$
$$y(x) \to (-1)^{2/3}\sqrt[3]{x}\sqrt[3]{x^3 + c_1}$$

5.28 problem 28

Internal problem ID [106]
Internal file name [OUTPUT/106_Sunday_June_05_2022_01_34_56_AM_89190887/index.tex]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

```
[[_1st_order, `_with_symmetry_[F(x),G(x)]`]]
```

$$\boxed{\mathrm{e}^{y} x y' - 2\,\mathrm{e}^{y} = 2\,\mathrm{e}^{2x} x^{3}}$$

5.28.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^y x)\,\mathrm{d}y &= \left(2\,\mathrm{e}^y + 2\,\mathrm{e}^{2x}x^3\right)\mathrm{d}x\\ \left(-2\,\mathrm{e}^y - 2\,\mathrm{e}^{2x}x^3\right)\mathrm{d}x + (\mathrm{e}^y x)\,\mathrm{d}y &= 0 \qquad (2\mathrm{A})\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= -2\,\mathrm{e}^y - 2\,\mathrm{e}^{2x}x^3\\ N(x,y) &= \mathrm{e}^y x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-2\,\mathrm{e}^y - 2\,\mathrm{e}^{2x}x^3\right)\\ &= -2\,\mathrm{e}^y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^y x)\\ &= \mathrm{e}^y\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{\mathrm{e}^{-y}}{x}\left((-2\,\mathrm{e}^y) - (\mathrm{e}^y)\right)\\ &= -\frac{3}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A\,\mathrm{d}x} \\
&= e^{\int -\frac{3}{x}\,\mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-3\ln(x)} \\
&= \frac{1}{x^3}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^3}\left(-2\,\mathrm{e}^{y} - 2\,\mathrm{e}^{2x}x^3\right) \\
&= \frac{-2\,\mathrm{e}^{y} - 2\,\mathrm{e}^{2x}x^3}{x^3}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^3}(\mathrm{e}^{y}x) \\
&= \frac{\mathrm{e}^{y}}{x^2}
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-2\,\mathrm{e}^{y} - 2\,\mathrm{e}^{2x}x^3}{x^3}\right) + \left(\frac{\mathrm{e}^{y}}{x^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \overline{M}\, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x = \int \frac{-2\,\mathrm{e}^{y} - 2\,\mathrm{e}^{2x} x^{3}}{x^{3}}\, \mathrm{d}x$$

$$\phi = \frac{-x^{2}\mathrm{e}^{2x} + \mathrm{e}^{y}}{x^{2}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{y}}{x^{2}} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\mathrm{e}^{y}}{x^{2}}$. Therefore equation (4) becomes

$$\frac{\mathrm{e}^{y}}{x^{2}} = \frac{\mathrm{e}^{y}}{x^{2}} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^{2}\mathrm{e}^{2x} + \mathrm{e}^{y}}{x^{2}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^{2}\mathrm{e}^{2x} + \mathrm{e}^{y}}{x^{2}}$$

The solution becomes

$$y = \ln\left(x^{2}\mathrm{e}^{2x} + c_1 x^{2}\right)$$

Summary
The solution(s) found are the following

$$y = \ln\left(x^2 e^{2x} + c_1 x^2\right) \tag{1}$$

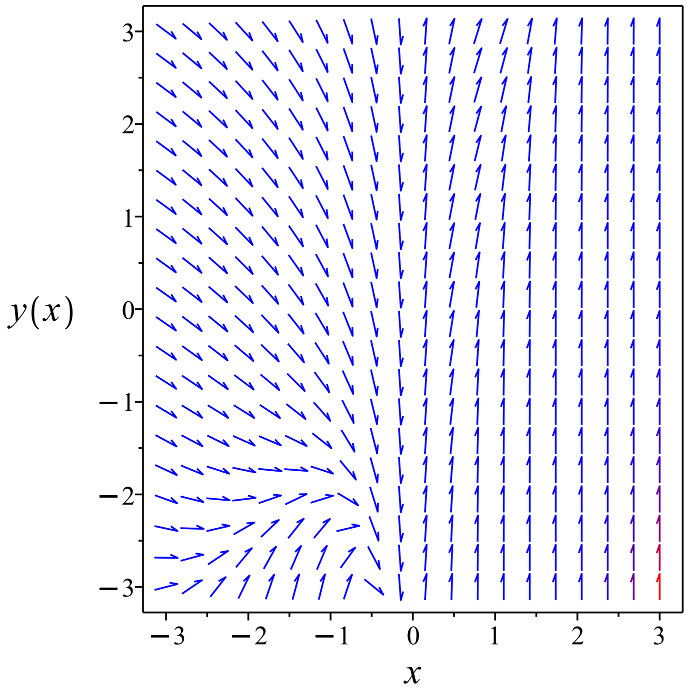


Figure 308: Slope field plot

Verification of solutions

$$y = \ln\left(x^2 e^{2x} + c_1 x^2\right)$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve(exp(y(x))*x*diff(y(x),x) = 2*exp(y(x))+2*exp(2*x)*x^3,y(x), singsol=all)
```

$$y(x) = \ln\left(x^2\left(\mathrm{e}^{2x} - c_1\right)\right)$$

✓ Solution by Mathematica

Time used: 4.305 (sec). Leaf size: 18

```
DSolve[Exp[y[x]]*x*y'[x] == 2*Exp[y[x]]+2*Exp[2*x]*x^3,y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \log\left(x^2\left(e^{2x} + c_1\right)\right)$$

5.29 problem 29

Internal problem ID [107]
Internal file name [`OUTPUT/107_Sunday_June_05_2022_01_34_57_AM_21095011/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exactWithIntegrationFactor"**

Maple gives the following as the ode type

`` [`y=_G(x,y')`] ``

$$\boxed{2x\cos(y)\sin(y)\,y' - \sin(y)^2 = 4x^2}$$

5.29.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(2\cos(y)\,x\sin(y))\,\mathrm{d}y &= \left(4x^2 + \sin(y)^2\right)\mathrm{d}x\\ \left(-4x^2 - \sin(y)^2\right)\mathrm{d}x + (2\cos(y)\,x\sin(y))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -4x^2 - \sin(y)^2\\ N(x, y) &= 2\cos(y)\,x\sin(y)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-4x^2 - \sin(y)^2\right)\\ &= -\sin(2y)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(2\cos(y)\,x\sin(y))\\ &= \sin(2y)\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{\sec(y)\csc(y)}{2x}((-2\cos(y)\sin(y)) - (2\cos(y)\sin(y)))\\ &= -\frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A \, dx} \\ &= e^{\int -\frac{2}{x} \, dx}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{-2\ln(x)} \\ &= \frac{1}{x^2}\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{x^2}\left(-4x^2 - \sin(y)^2\right) \\ &= \frac{-4x^2 - \sin(y)^2}{x^2}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{x^2}(2\cos(y)\, x \sin(y)) \\ &= \frac{\sin(2y)}{x}\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{dy}{dx} &= 0 \\ \left(\frac{-4x^2 - \sin(y)^2}{x^2}\right) + \left(\frac{\sin(2y)}{x}\right)\frac{dy}{dx} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int \overline{M} \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int \frac{-4x^2 - \sin(y)^2}{x^2} \, dx$$
$$\phi = -4x + \frac{\sin(y)^2}{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{2\cos(y)\sin(y)}{x} + f'(y) \tag{4}$$
$$= \frac{\sin(2y)}{x} + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{\sin(2y)}{x}$. Therefore equation (4) becomes

$$\frac{\sin(2y)}{x} = \frac{\sin(2y)}{x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -4x + \frac{\sin(y)^2}{x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -4x + \frac{\sin(y)^2}{x}$$

Summary

The solution(s) found are the following

$$-4x+\frac{\sin(y)^2}{x}=c_1 \tag{1}$$

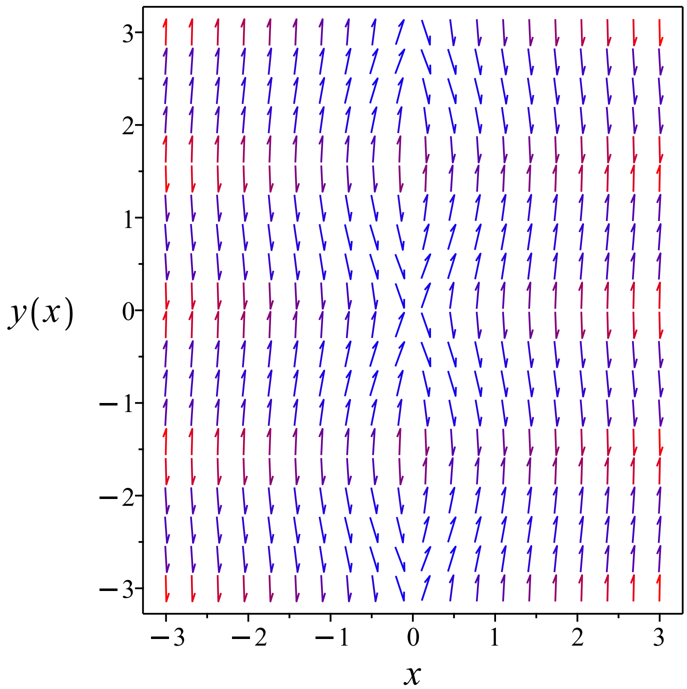


Figure 309: Slope field plot

Verification of solutions

$$-4x+\frac{\sin(y)^2}{x}=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 31

```
dsolve(2*x*cos(y(x))*sin(y(x))*diff(y(x),x) = 4*x^2+sin(y(x))^2,y(x), singsol=all)
```

$$y(x) = \arcsin\left(\sqrt{-x\left(c_1 - 4x\right)}\right)$$
$$y(x) = -\arcsin\left(\sqrt{-x\left(c_1 - 4x\right)}\right)$$

✓ Solution by Mathematica

Time used: 6.406 (sec). Leaf size: 41

```
DSolve[2*x*Cos[y[x]]*Sin[y[x]]*y'[x] == 4*x^2+Sin[y[x]]^2,y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to -\arcsin\left(2\sqrt{x(x + 2c_1)}\right)$$
$$y(x) \to \arcsin\left(2\sqrt{x(x + 2c_1)}\right)$$

5.30 problem 30

Internal problem ID [108]
Internal file name [`OUTPUT/108_Sunday_June_05_2022_01_35_00_AM_34058302/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 30.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_1st_order, _with_linear_symmetries]]
```

$$\boxed{(e^y + x)\, y' - x\, e^{-y} = -1}$$

5.30.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{(e^y - x)\, e^{-y}}{e^y + x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$
$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(e^y - x)\,e^{-y}(b_3 - a_2)}{e^y + x} - \frac{(e^y - x)^2\,e^{-2y} a_3}{(e^y + x)^2} \\ & - \left(\frac{e^{-y}}{e^y + x} + \frac{(e^y - x)\,e^{-y}}{(e^y + x)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(-\frac{1}{e^y + x} + \frac{(e^y - x)\,e^{-y}}{e^y + x} + \frac{e^y - x}{(e^y + x)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{(e^{4y} b_2 + e^{3y} x b_2 - e^{3y} y b_3 + 3\,e^{2y} x^2 b_2 + 2\,e^{2y} x y b_3 + e^y x^3 b_2 + e^y x^2 y b_3 + e^{3y} a_2 - e^{3y} b_1 - e^{3y} b_3 - 2\,e^{2y} x a_2 + 2}{(e^y + x)^2}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned} & e^{4y} b_2 + e^{3y} x b_2 - e^{3y} y b_3 + 3\,e^{2y} x^2 b_2 + 2\,e^{2y} x y b_3 + e^y x^3 b_2 + e^y x^2 y b_3 \\ & + e^{3y} a_2 - e^{3y} b_1 - e^{3y} b_3 - 2\,e^{2y} x a_2 + 2\,e^{2y} x b_1 - 2\,e^{2y} y a_3 - e^y x^2 a_2 \\ & + e^y x^2 b_1 + e^y x^2 b_3 - 2\,e^{2y} a_1 - e^{2y} a_3 + 2\,e^y x a_3 - x^2 a_3 = 0 \end{aligned} \tag{6E}$$

Simplifying the above gives

$$\begin{aligned} & e^{4y} b_2 + e^{3y} x b_2 - e^{3y} y b_3 + 3\,e^{2y} x^2 b_2 + 2\,e^{2y} x y b_3 + e^y x^3 b_2 + e^y x^2 y b_3 \\ & + e^{3y} a_2 - e^{3y} b_1 - e^{3y} b_3 - 2\,e^{2y} x a_2 + 2\,e^{2y} x b_1 - 2\,e^{2y} y a_3 - e^y x^2 a_2 \\ & + e^y x^2 b_1 + e^y x^2 b_3 - 2\,e^{2y} a_1 - e^{2y} a_3 + 2\,e^y x a_3 - x^2 a_3 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y, e^y, e^{2y}, e^{3y}, e^{4y}\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2, e^y = v_3, e^{2y} = v_4, e^{3y} = v_5, e^{4y} = v_6\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&v_3 v_1^3 b_2 + v_3 v_1^2 v_2 b_3 - v_3 v_1^2 a_2 + v_3 v_1^2 b_1 + 3 v_4 v_1^2 b_2 + v_3 v_1^2 b_3 + 2 v_4 v_1 v_2 b_3 \\
&- 2 v_4 v_1 a_2 - v_1^2 a_3 + 2 v_3 v_1 a_3 - 2 v_4 v_2 a_3 + 2 v_4 v_1 b_1 + v_5 v_1 b_2 \\
&- v_5 v_2 b_3 - 2 v_4 a_1 + v_5 a_2 - v_4 a_3 - v_5 b_1 + v_6 b_2 - v_5 b_3 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$
\{v_1, v_2, v_3, v_4, v_5, v_6\}
$$

Equation (7E) now becomes

$$
\begin{aligned}
&v_3 v_1^3 b_2 + v_3 v_1^2 v_2 b_3 + (-a_2 + b_1 + b_3) v_1^2 v_3 + 3 v_4 v_1^2 b_2 - v_1^2 a_3 \\
&+ 2 v_4 v_1 v_2 b_3 + 2 v_3 v_1 a_3 + (-2 a_2 + 2 b_1) v_1 v_4 + v_5 v_1 b_2 - 2 v_4 v_2 a_3 \\
&- v_5 v_2 b_3 + (-2 a_1 - a_3) v_4 + (a_2 - b_1 - b_3) v_5 + v_6 b_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
b_3 &= 0 \\
-2 a_3 &= 0 \\
-a_3 &= 0 \\
2 a_3 &= 0 \\
3 b_2 &= 0 \\
-b_3 &= 0 \\
2 b_3 &= 0 \\
-2 a_1 - a_3 &= 0 \\
-2 a_2 + 2 b_1 &= 0 \\
-a_2 + b_1 + b_3 &= 0 \\
a_2 - b_1 - b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= b_1 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = 1$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(-\frac{(\mathrm{e}^{y} - x)\,\mathrm{e}^{-y}}{\mathrm{e}^{y} + x}\right)(x) \\
&= \frac{\mathrm{e}^{2y} + 2\,\mathrm{e}^{y} x - x^{2}}{\mathrm{e}^{2y} + \mathrm{e}^{y} x} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{\mathrm{e}^{2y} + 2\,\mathrm{e}^{y} x - x^{2}}{\mathrm{e}^{2y} + \mathrm{e}^{y} x}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(\mathrm{e}^{2y} + 2\,\mathrm{e}^{y} x - x^{2}\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{(e^y - x)\, e^{-y}}{e^y + x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{e^y - x}{e^{2y} + 2\, e^y x - x^2} \\ S_y &= \frac{e^y (e^y + x)}{e^{2y} + 2\, e^y x - x^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln (e^{2y} + 2\, e^y x - x^2)}{2} = c_1$$

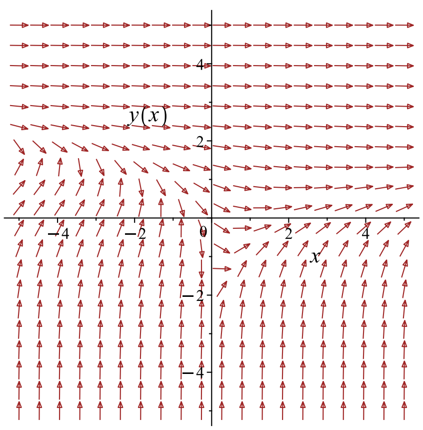
Which simplifies to

$$\frac{\ln (e^{2y} + 2\, e^y x - x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{(e^y - x)e^{-y}}{e^y + x}$



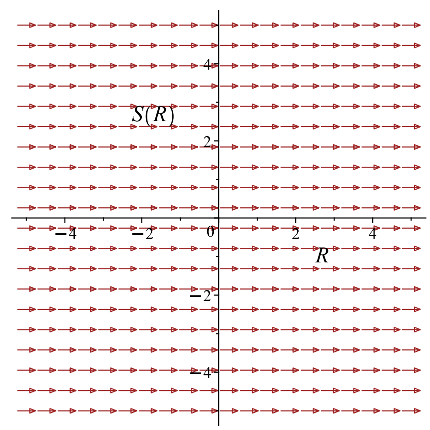
</td><td>

$R = x$

$S = \frac{\ln(e^{2y} + 2e^y x - x^2)}{2}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(e^{2y} + 2e^y x - x^2)}{2} = c_1 \tag{1}$$

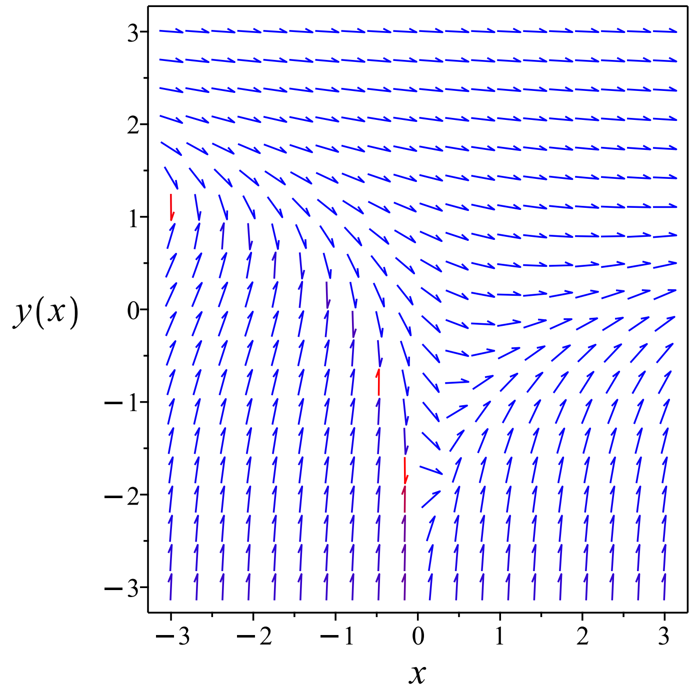


Figure 310: Slope field plot

Verification of solutions

$$\frac{\ln\left(e^{2y}+2\,e^{y}x-x^{2}\right)}{2}=c_1$$

Verified OK.

5.30.2 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (\mathrm{e}^y(\mathrm{e}^y + x)) \, \mathrm{d}y &= (-\mathrm{e}^y + x) \, \mathrm{d}x \\ (\mathrm{e}^y - x) \, \mathrm{d}x + (\mathrm{e}^y(\mathrm{e}^y + x)) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= \mathrm{e}^y - x \\ N(x, y) &= \mathrm{e}^y(\mathrm{e}^y + x) \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^y - x) \\ &= \mathrm{e}^y \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^y(\mathrm{e}^y + x)) \\ &= \mathrm{e}^y \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{y} - x \, \mathrm{d}x$$

$$\phi = -\frac{x(-2\,\mathrm{e}^{y} + x)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{y} x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{y}(\mathrm{e}^{y} + x)$. Therefore equation (4) becomes

$$\mathrm{e}^{y}(\mathrm{e}^{y} + x) = \mathrm{e}^{y} x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{2y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\mathrm{e}^{2y}) \, \mathrm{d}y$$

$$f(y) = \frac{\mathrm{e}^{2y}}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(-2\,\mathrm{e}^{y} + x)}{2} + \frac{\mathrm{e}^{2y}}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(-2\,\mathrm{e}^{y} + x)}{2} + \frac{\mathrm{e}^{2y}}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x(-2\,\mathrm{e}^{y} + x)}{2} + \frac{\mathrm{e}^{2y}}{2} = c_1 \tag{1}$$

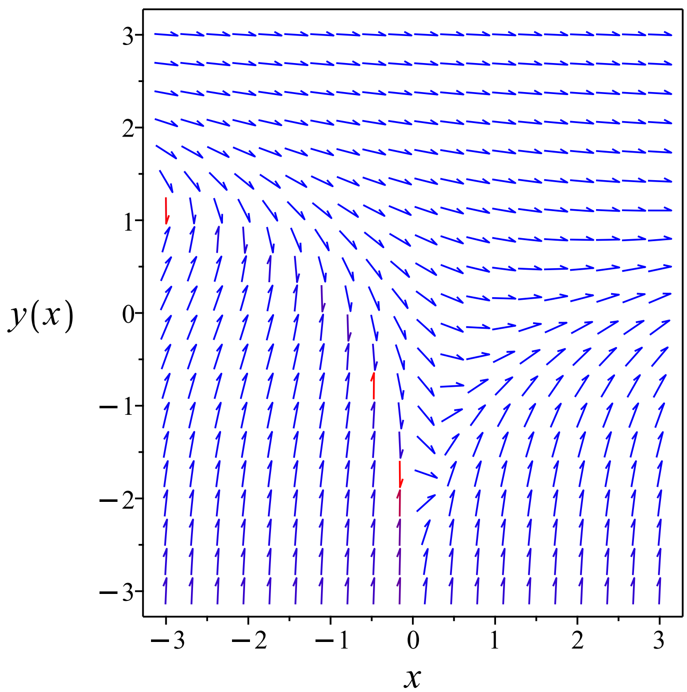


Figure 311: Slope field plot

Verification of solutions

$$-\frac{x(-2\,\mathrm{e}^{y} + x)}{2} + \frac{\mathrm{e}^{2y}}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
differential order: 1; found: 1 linear symmetries. Trying reduction of order
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1/x, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      <- quadrature successful
<- 1st order, canonical coordinates successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 37

```
dsolve((exp(y(x))+x)*diff(y(x),x) = -1+x/exp(y(x)),y(x), singsol=all)
```

$$y(x) = \ln\left(-x - \sqrt{2x^2 + c_1}\right)$$
$$y(x) = \ln\left(-x + \sqrt{2x^2 + c_1}\right)$$

✓ Solution by Mathematica

Time used: 2.698 (sec). Leaf size: 52

```
DSolve[(Exp[y[x]]+x)*y'[x]== -1+x/Exp[y[x]],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log\left(-x - \sqrt{2}\sqrt{x^2 + c_1}\right)$$
$$y(x) \to \log\left(-x + \sqrt{2}\sqrt{x^2 + c_1}\right)$$

5.31 problem 31

Internal problem ID [109]
Internal file name [`OUTPUT/109_Sunday_June_05_2022_01_35_01_AM_80422406/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 31.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{3y + (3x + 2y)\, y' = -2x}$$

5.31.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$3u(x)\, x + (3x + 2u(x)\, x)\, (u'(x)\, x + u(x)) = -2x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2(u^2 + 3u + 1)}{x\,(2u + 3)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{u^2+3u+1}{2u+3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2+3u+1}{2u+3}}\, du = -\frac{2}{x}\, dx$$
$$\int \frac{1}{\frac{u^2+3u+1}{2u+3}}\, du = \int -\frac{2}{x}\, dx$$
$$\ln\left(u^2+3u+1\right) = -2\ln(x) + c_2$$

Raising both side to exponential gives

$$u^2+3u+1 = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$u^2+3u+1 = \frac{c_3}{x^2}$$

Which simplifies to

$$u(x)^2 + 3u(x) + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

The solution is

$$u(x)^2 + 3u(x) + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{y^2}{x^2} + \frac{3y}{x} + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$
$$\frac{y^2}{x^2} + \frac{3y}{x} + 1 = \frac{c_3 \mathrm{e}^{c_2}}{x^2}$$

Which simplifies to

$$y^2 + 3yx + x^2 = c_3 \mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$y^2 + 3yx + x^2 = c_3 \mathrm{e}^{c_2} \tag{1}$$

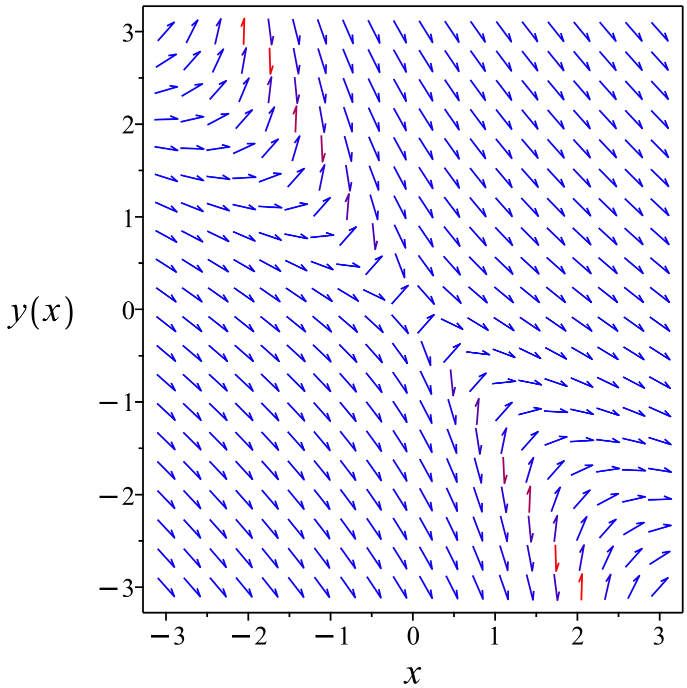


Figure 312: Slope field plot

Verification of solutions

$$y^2 + 3yx + x^2 = c_3 e^{c_2}$$

Verified OK.

5.31.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-2x - 3y}{3x + 2y} \tag{1}$$

Which becomes

$$(2y)\, dy = (-3x)\, dy + (-2x - 3y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-3x)\, dy + (-2x - 3y)\, dx = d\left(-x^2 - 3yx\right)$$

Hence (2) becomes

$$(2y)\, dy = d\left(-x^2 - 3yx\right)$$

Integrating both sides gives gives these solutions

$$y = -\frac{3x}{2} + \frac{\sqrt{5x^2 + 4c_1}}{2} + c_1$$
$$y = -\frac{3x}{2} - \frac{\sqrt{5x^2 + 4c_1}}{2} + c_1$$

Summary

The solution(s) found are the following

$$y = -\frac{3x}{2} + \frac{\sqrt{5x^2 + 4c_1}}{2} + c_1 \tag{1}$$
$$y = -\frac{3x}{2} - \frac{\sqrt{5x^2 + 4c_1}}{2} + c_1 \tag{2}$$

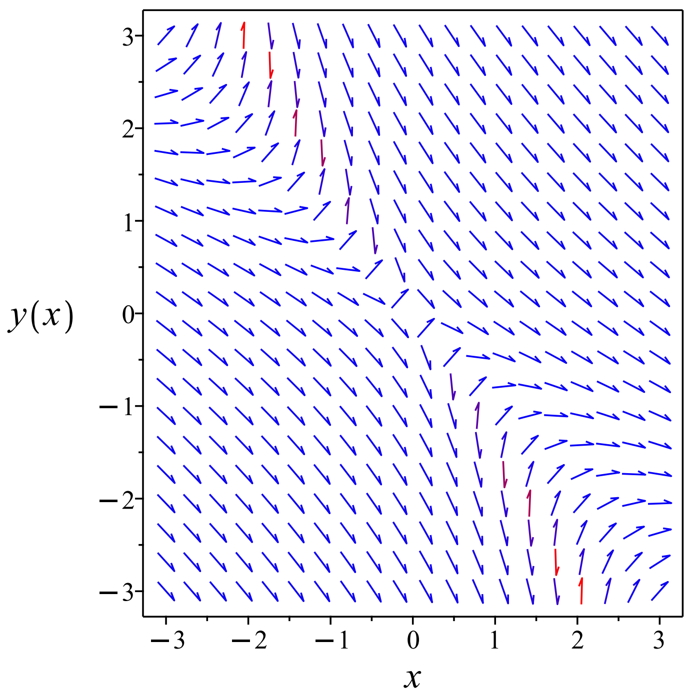


Figure 313: Slope field plot

Verification of solutions

$$y = -\frac{3x}{2} + \frac{\sqrt{5x^2 + 4c_1}}{2} + c_1$$

Verified OK.

$$y = -\frac{3x}{2} - \frac{\sqrt{5x^2 + 4c_1}}{2} + c_1$$

Verified OK.

5.31.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2x + 3y}{3x + 2y} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = x a_2 + y a_3 + a_1 \tag{1E}$$

$$\eta = x b_2 + y b_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2 - \frac{(2x + 3y)(b_3 - a_2)}{3x + 2y} - \frac{(2x + 3y)^2 a_3}{(3x + 2y)^2} \\ & - \left(-\frac{2}{3x + 2y} + \frac{6x + 9y}{(3x + 2y)^2}\right)(x a_2 + y a_3 + a_1) \\ & - \left(-\frac{3}{3x + 2y} + \frac{4x + 6y}{(3x + 2y)^2}\right)(x b_2 + y b_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{6x^2a_2 - 4x^2a_3 + 14x^2b_2 - 6x^2b_3 + 8xya_2 - 12xya_3 + 12xyb_2 - 8xyb_3 + 6y^2a_2 - 14y^2a_3 + 4y^2b_2 - 6y^2b_3 +}{(3x+2y)^2}$$

$= 0$

Setting the numerator to zero gives

$$\begin{aligned} &6x^2a_2 - 4x^2a_3 + 14x^2b_2 - 6x^2b_3 + 8xya_2 - 12xya_3 + 12xyb_2 \\ &- 8xyb_3 + 6y^2a_2 - 14y^2a_3 + 4y^2b_2 - 6y^2b_3 + 5xb_1 - 5ya_1 = 0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} &6a_2v_1^2 + 8a_2v_1v_2 + 6a_2v_2^2 - 4a_3v_1^2 - 12a_3v_1v_2 - 14a_3v_2^2 + 14b_2v_1^2 \\ &+ 12b_2v_1v_2 + 4b_2v_2^2 - 6b_3v_1^2 - 8b_3v_1v_2 - 6b_3v_2^2 - 5a_1v_2 + 5b_1v_1 = 0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(6a_2 - 4a_3 + 14b_2 - 6b_3)\, v_1^2 + (8a_2 - 12a_3 + 12b_2 - 8b_3)\, v_1v_2 \\ &+ 5b_1v_1 + (6a_2 - 14a_3 + 4b_2 - 6b_3)\, v_2^2 - 5a_1v_2 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -5a_1 &= 0 \\ 5b_1 &= 0 \\ 6a_2 - 14a_3 + 4b_2 - 6b_3 &= 0 \\ 6a_2 - 4a_3 + 14b_2 - 6b_3 &= 0 \\ 8a_2 - 12a_3 + 12b_2 - 8b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0\\
a_2 &= -3b_2 + b_3\\
a_3 &= -b_2\\
b_1 &= 0\\
b_2 &= b_2\\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x\\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi\\
&= y - \left(-\frac{2x+3y}{3x+2y}\right)(x)\\
&= \frac{2x^2+6yx+2y^2}{3x+2y}\\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy\\
&= \int \frac{1}{\frac{2x^2+6yx+2y^2}{3x+2y}} dy
\end{aligned}
$$

Which results in

$$S = \frac{\ln(x^2 + 3yx + y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2x + 3y}{3x + 2y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2x + 3y}{2x^2 + 6yx + 2y^2} \\ S_y &= \frac{3x + 2y}{2x^2 + 6yx + 2y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

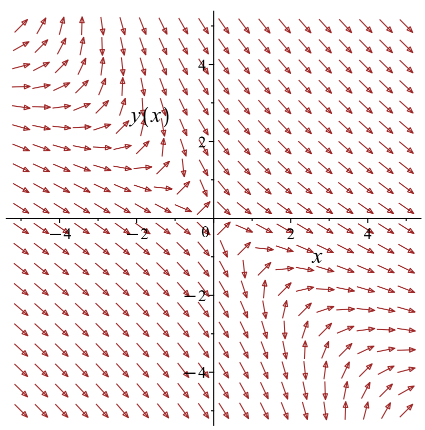
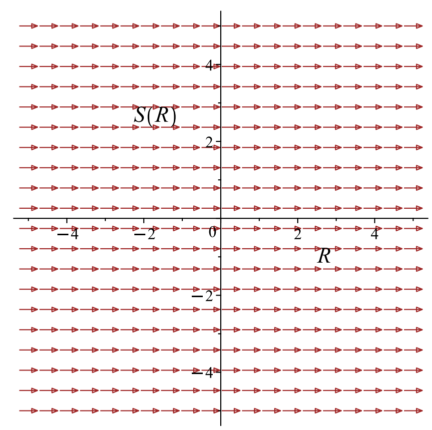
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 + 3yx + x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln\left(y^2+3yx+x^2\right)}{2}=c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx}=-\frac{2x+3y}{3x+2y}$ 	$R=x$ $S=\frac{\ln\left(x^2+3yx+y^2\right)}{2}$	$\frac{dS}{dR}=0$ 

Summary

The solution(s) found are the following

$$\frac{\ln\left(y^2+3yx+x^2\right)}{2}=c_1 \tag{1}$$

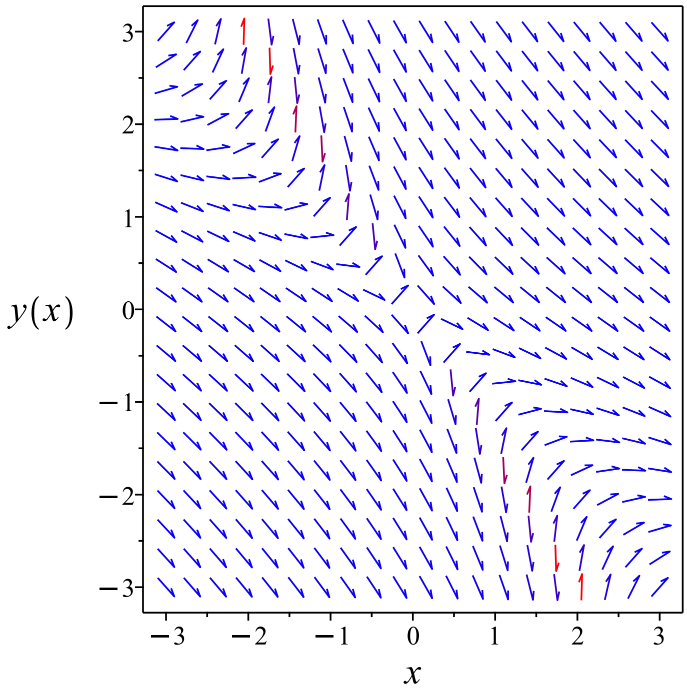


Figure 314: Slope field plot

Verification of solutions

$$\frac{\ln(y^2 + 3yx + x^2)}{2} = c_1$$

Verified OK.

5.31.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (3x + 2y)\,\mathrm{d}y &= (-2x - 3y)\,\mathrm{d}x \\ (2x + 3y)\,\mathrm{d}x + (3x + 2y)\,\mathrm{d}y &= 0 \qquad (2A) \end{aligned}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 2x + 3y$$
$$N(x, y) = 3x + 2y$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2x + 3y) \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x + 2y) \\ &= 3 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int 2x + 3y \, dx$$
$$\phi = x(x + 3y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x + 2y$. Therefore equation (4) becomes

$$3x + 2y = 3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int (2y) \, dy$$
$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x(x + 3y) + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(x + 3y) + y^2$$

Summary

The solution(s) found are the following

$$x(x + 3y) + y^2 = c_1 \tag{1}$$

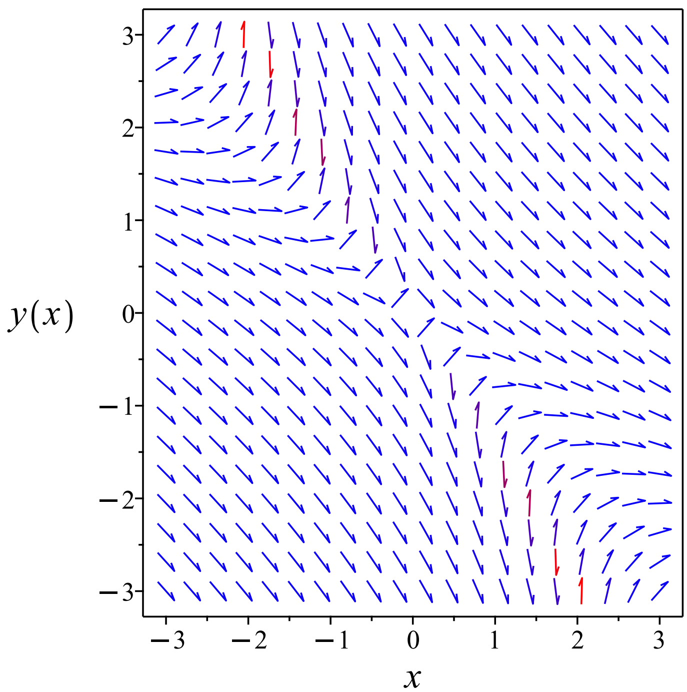


Figure 315: Slope field plot

Verification of solutions

$$x(x + 3y) + y^2 = c_1$$

Verified OK.

5.31.5 Maple step by step solution

Let's solve

$3y + (3x + 2y)\, y' = -2x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $3 = 3$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2x + 3y)\, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = x^2 + 3yx + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$
- Compute derivative

  $3x + 2y = 3x + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 2y$
- Solve for $f_1(y)$

  $f_1(y) = y^2$
- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = x^2 + 3yx + y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

$$x^2 + 3yx + y^2 = c_1$$

- Solve for y

$$\left\{y = -\frac{3x}{2} - \frac{\sqrt{5x^2+4c_1}}{2}, y = -\frac{3x}{2} + \frac{\sqrt{5x^2+4c_1}}{2}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 53

```
dsolve(2*x+3*y(x)+(3*x+2*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{-3c_1x - \sqrt{5c_1^2x^2 + 4}}{2c_1}$$
$$y(x) = \frac{-3c_1x + \sqrt{5c_1^2x^2 + 4}}{2c_1}$$

✓ Solution by Mathematica

Time used: 0.445 (sec). Leaf size: 110

```
DSolve[2*x+3*y[x]+(3*x+2*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}\left(-3x - \sqrt{5x^2 + 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(-3x + \sqrt{5x^2 + 4e^{c_1}}\right)$$
$$y(x) \to \frac{1}{2}\left(-\sqrt{5}\sqrt{x^2} - 3x\right)$$
$$y(x) \to \frac{1}{2}\left(\sqrt{5}\sqrt{x^2} - 3x\right)$$

5.32 problem 32

Internal problem ID [110]
Internal file name [`OUTPUT/110_Sunday_June_05_2022_01_35_02_AM_31607378/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 32.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class A`]]
```

$$\boxed{-y + (-x + 6y)\, y' = -4x}$$

5.32.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$-u(x)\, x + (-x + 6u(x)\, x)\, (u'(x)\, x + u(x)) = -4x$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2(3u^2 - u + 2)}{x\, (6u - 1)} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = \frac{3u^2-u+2}{6u-1}$. Integrating both sides gives

$$\frac{1}{\frac{3u^2-u+2}{6u-1}}\,du = -\frac{2}{x}\,dx$$
$$\int \frac{1}{\frac{3u^2-u+2}{6u-1}}\,du = \int -\frac{2}{x}\,dx$$
$$\ln\left(3u^2-u+2\right) = -2\ln(x) + c_2$$

Raising both side to exponential gives

$$3u^2 - u + 2 = \mathrm{e}^{-2\ln(x)+c_2}$$

Which simplifies to

$$3u^2 - u + 2 = \frac{c_3}{x^2}$$

Which simplifies to

$$3u(x)^2 - u(x) + 2 = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$

The solution is

$$3u(x)^2 - u(x) + 2 = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{3y^2}{x^2} - \frac{y}{x} + 2 = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$
$$\frac{3y^2}{x^2} - \frac{y}{x} + 2 = \frac{c_3\,\mathrm{e}^{c_2}}{x^2}$$

Which simplifies to

$$3y^2 - yx + 2x^2 = c_3\,\mathrm{e}^{c_2}$$

Summary

The solution(s) found are the following

$$3y^2 - yx + 2x^2 = c_3\,\mathrm{e}^{c_2} \tag{1}$$

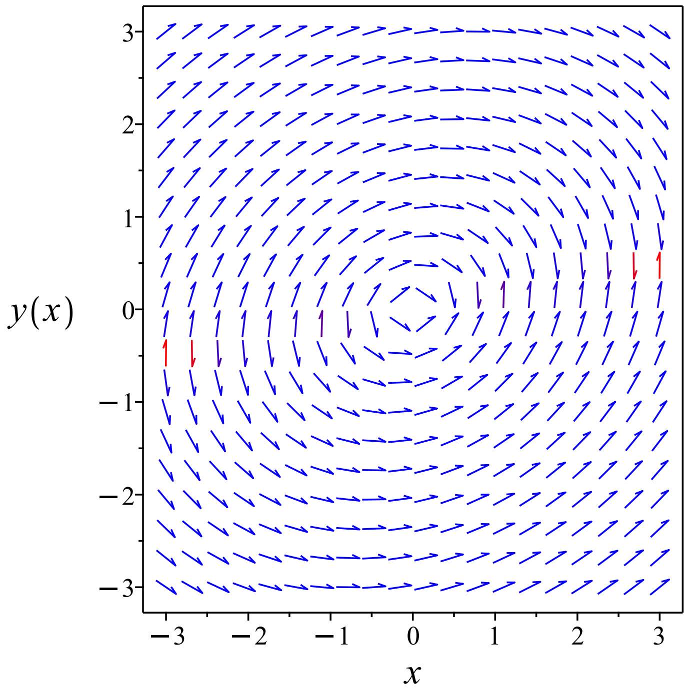


Figure 316: Slope field plot

Verification of solutions

$$3y^2 - yx + 2x^2 = c_3 e^{c_2}$$

Verified OK.

5.32.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{-4x + y}{-x + 6y} \tag{1}$$

Which becomes

$$(-6y)\, dy = (-x)\, dy + (4x - y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-x)\, dy + (4x - y)\, dx = d(2x^2 - yx)$$

Hence (2) becomes

$$(-6y)\, dy = d(2x^2 - yx)$$

Integrating both sides gives gives these solutions

$$y = \frac{x}{6} + \frac{\sqrt{-23x^2 - 12c_1}}{6} + c_1$$
$$y = \frac{x}{6} - \frac{\sqrt{-23x^2 - 12c_1}}{6} + c_1$$

Summary

The solution(s) found are the following

$$y = \frac{x}{6} + \frac{\sqrt{-23x^2 - 12c_1}}{6} + c_1 \tag{1}$$
$$y = \frac{x}{6} - \frac{\sqrt{-23x^2 - 12c_1}}{6} + c_1 \tag{2}$$

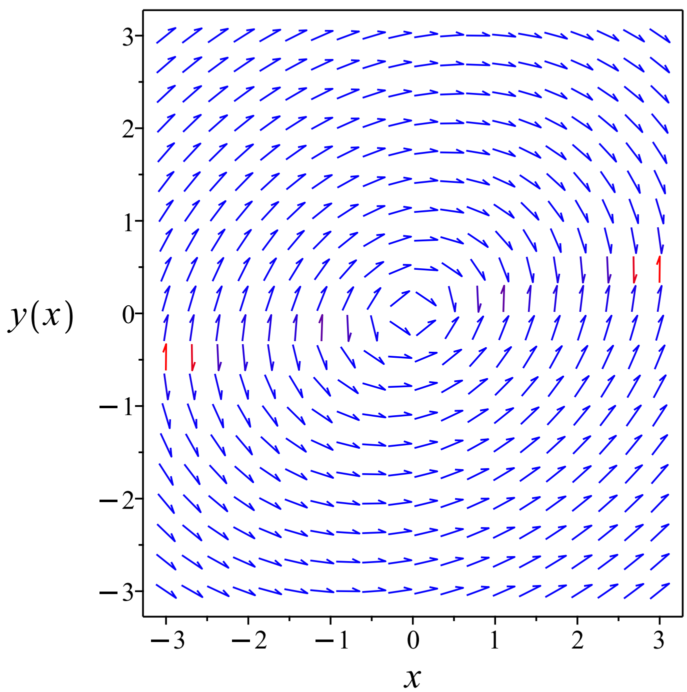


Figure 317: Slope field plot

Verification of solutions

$$y = \frac{x}{6} + \frac{\sqrt{-23x^2 - 12c_1}}{6} + c_1$$

Verified OK.

$$y = \frac{x}{6} - \frac{\sqrt{-23x^2 - 12c_1}}{6} + c_1$$

Verified OK.

5.32.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = \frac{-4x + y}{-x + 6y}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} b_2 &+ \frac{(-4x + y)(b_3 - a_2)}{-x + 6y} - \frac{(-4x + y)^2 a_3}{(-x + 6y)^2} \\ &- \left(-\frac{4}{-x + 6y} + \frac{-4x + y}{(-x + 6y)^2}\right)(xa_2 + ya_3 + a_1) \\ &- \left(\frac{1}{-x + 6y} - \frac{6(-4x + y)}{(-x + 6y)^2}\right)(xb_2 + yb_3 + b_1) = 0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$
-\frac{4x^2a_2 + 16x^2a_3 + 22x^2b_2 - 4x^2b_3 - 48xya_2 - 8xya_3 + 12xyb_2 + 48xyb_3 + 6y^2a_2 - 22y^2a_3 - 36y^2b_2 - 6}{(x-6y)^2}
$$

$= 0$

Setting the numerator to zero gives

$$
\begin{aligned}
&-4x^2a_2 - 16x^2a_3 - 22x^2b_2 + 4x^2b_3 + 48xya_2 + 8xya_3 - 12xyb_2 \\
&- 48xyb_3 - 6y^2a_2 + 22y^2a_3 + 36y^2b_2 + 6y^2b_3 - 23xb_1 + 23ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$
\begin{aligned}
&-4a_2v_1^2 + 48a_2v_1v_2 - 6a_2v_2^2 - 16a_3v_1^2 + 8a_3v_1v_2 + 22a_3v_2^2 - 22b_2v_1^2 \\
&- 12b_2v_1v_2 + 36b_2v_2^2 + 4b_3v_1^2 - 48b_3v_1v_2 + 6b_3v_2^2 + 23a_1v_2 - 23b_1v_1 = 0
\end{aligned} \tag{7E}
$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$
\begin{aligned}
&(-4a_2 - 16a_3 - 22b_2 + 4b_3)\, v_1^2 + (48a_2 + 8a_3 - 12b_2 - 48b_3)\, v_1v_2 \\
&- 23b_1v_1 + (-6a_2 + 22a_3 + 36b_2 + 6b_3)\, v_2^2 + 23a_1v_2 = 0
\end{aligned} \tag{8E}
$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
23a_1 &= 0 \\
-23b_1 &= 0 \\
-6a_2 + 22a_3 + 36b_2 + 6b_3 &= 0 \\
-4a_2 - 16a_3 - 22b_2 + 4b_3 &= 0 \\
48a_2 + 8a_3 - 12b_2 - 48b_3 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= 0 \\
a_2 &= \frac{b_2}{2} + b_3 \\
a_3 &= -\frac{3b_2}{2} \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\, \xi \\
&= y - \left(\frac{-4x + y}{-x + 6y}\right)(x) \\
&= \frac{-4x^2 + 2yx - 6y^2}{x - 6y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$S = \int \frac{1}{\eta} dy$$
$$= \int \frac{1}{\frac{-4x^2+2yx-6y^2}{x-6y}} dy$$

Which results in

$$S = \frac{\ln(2x^2 - yx + 3y^2)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-4x + y}{-x + 6y}$$

Evaluating all the partial derivatives gives

$$R_x = 1$$
$$R_y = 0$$
$$S_x = \frac{4x - y}{4x^2 - 2yx + 6y^2}$$
$$S_y = \frac{-x + 6y}{4x^2 - 2yx + 6y^2}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

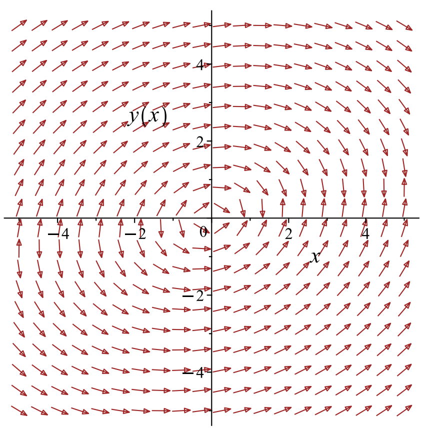
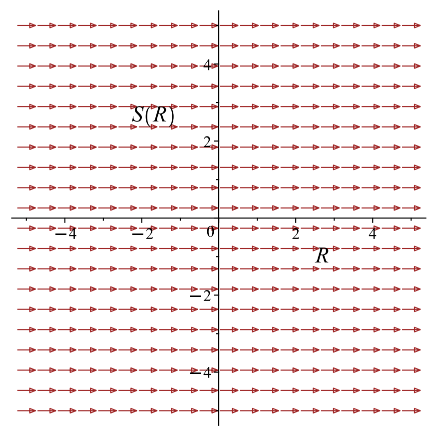
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(3y^2 - yx + 2x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(3y^2 - yx + 2x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-4x+y}{-x+6y}$ 	$R = x$ $S = \frac{\ln(2x^2 - yx + 3y^2)}{2}$	$\frac{dS}{dR} = 0$ 

Summary
The solution(s) found are the following

$$\frac{\ln(3y^2 - yx + 2x^2)}{2} = c_1 \tag{1}$$

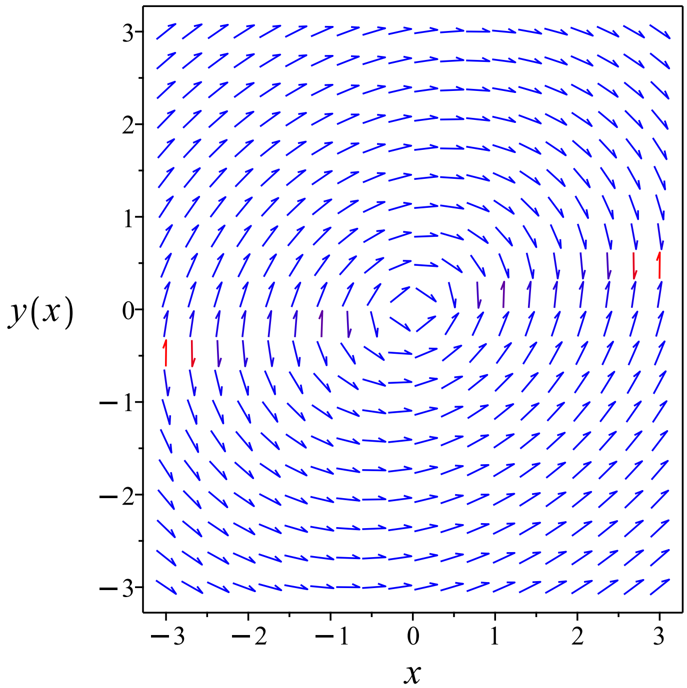


Figure 318: Slope field plot

Verification of solutions

$$\frac{\ln\left(3y^2 - yx + 2x^2\right)}{2} = c_1$$

Verified OK.

5.32.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(-x + 6y)\,\mathrm{d}y &= (-4x + y)\,\mathrm{d}x \\
(4x - y)\,\mathrm{d}x + (-x + 6y)\,\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 4x - y \\
N(x, y) &= -x + 6y
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(4x - y) \\
&= -1
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x + 6y) \\
&= -1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$
$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int 4x-y\,\mathrm{d}x$$
$$\phi=x(2x-y)+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=-x+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-x+6y$. Therefore equation (4) becomes

$$-x+6y=-x+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=6y$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int (6y)\,\mathrm{d}y$$
$$f(y)=3y^2+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=x(2x-y)+3y^2+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x(2x - y) + 3y^2$$

Summary

The solution(s) found are the following

$$x(2x - y) + 3y^2 = c_1 \tag{1}$$

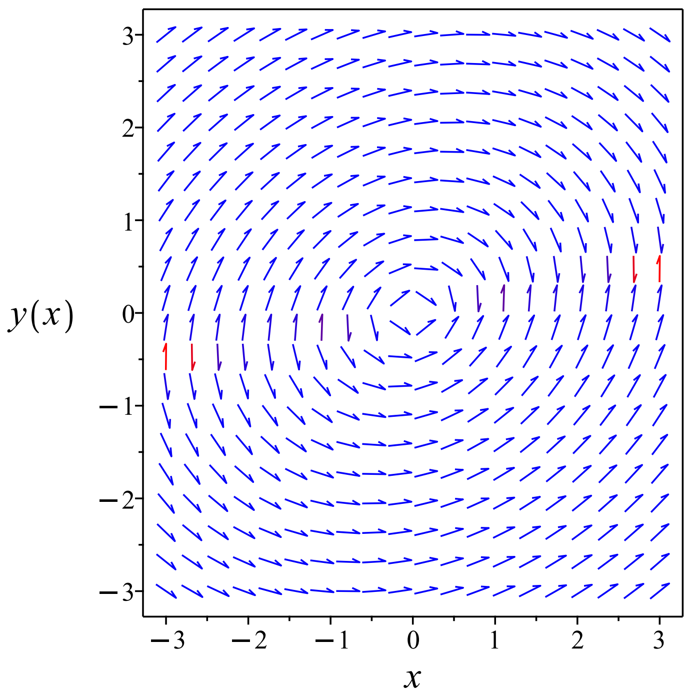


Figure 319: Slope field plot

Verification of solutions

$$x(2x - y) + 3y^2 = c_1$$

Verified OK.

5.32.5 Maple step by step solution

Let's solve

$-y+(-x+6y)\, y' = -4x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$

  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

  - Evaluate derivatives

    $-1 = -1$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (4x-y)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = 2x^2 - yx + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-x+6y = -x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 6y$

- Solve for $f_1(y)$

  $f_1(y) = 3y^2$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x, y) = 2x^2 - yx + 3y^2$$

- Substitute $F(x, y)$ into the solution of the ODE

  $$2x^2 - yx + 3y^2 = c_1$$

- Solve for y

  $$\left\{y = \frac{x}{6} - \frac{\sqrt{-23x^2 + 12c_1}}{6}, y = \frac{x}{6} + \frac{\sqrt{-23x^2 + 12c_1}}{6}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 51

```
dsolve(4*x-y(x)+(-x+6*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x - \sqrt{-23c_1^2 x^2 + 12}}{6c_1}$$
$$y(x) = \frac{c_1 x + \sqrt{-23c_1^2 x^2 + 12}}{6c_1}$$

✓ Solution by Mathematica

Time used: 0.446 (sec). Leaf size: 106

```
DSolve[4*x-y[x]+(-x+6*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}\left(x - \sqrt{-23x^2 + 12e^{c_1}}\right)$$
$$y(x) \to \frac{1}{6}\left(x + \sqrt{-23x^2 + 12e^{c_1}}\right)$$
$$y(x) \to \frac{1}{6}\left(x - \sqrt{23}\sqrt{-x^2}\right)$$
$$y(x) \to \frac{1}{6}\left(\sqrt{23}\sqrt{-x^2} + x\right)$$

5.33 problem 33

Internal problem ID [111]
Internal file name [`OUTPUT/111_Sunday_June_05_2022_01_35_03_AM_2506565/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 33.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, _dAlembert]
```

$$\boxed{2y^2 + (4yx + 6y^2)\, y' = -3x^2}$$

5.33.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$2u(x)^2\, x^2 + \left(4u(x)\, x^2 + 6u(x)^2\, x^2\right) \left(u'(x)\, x + u(x)\right) = -3x^2$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3(2u^3 + 2u^2 + 1)}{2ux\,(3u + 2)} \end{aligned}$$

Where $f(x) = -\frac{3}{2x}$ and $g(u) = \frac{2u^3+2u^2+1}{u(3u+2)}$. Integrating both sides gives

$$\frac{1}{\frac{2u^3+2u^2+1}{u(3u+2)}}\,du = -\frac{3}{2x}\,dx$$
$$\int \frac{1}{\frac{2u^3+2u^2+1}{u(3u+2)}}\,du = \int -\frac{3}{2x}\,dx$$
$$\frac{\ln(2u^3+2u^2+1)}{2} = -\frac{3\ln(x)}{2} + c_2$$

Raising both side to exponential gives

$$\sqrt{2u^3+2u^2+1} = \mathrm{e}^{-\frac{3\ln(x)}{2}+c_2}$$

Which simplifies to

$$\sqrt{2u^3+2u^2+1} = \frac{c_3}{x^{\frac{3}{2}}}$$

Which simplifies to

$$\sqrt{2u(x)^3+2u(x)^2+1} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{3}{2}}}$$

The solution is

$$\sqrt{2u(x)^3+2u(x)^2+1} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{3}{2}}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{2y^3}{x^3}+\frac{2y^2}{x^2}+1} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{3}{2}}}$$
$$\sqrt{\frac{2y^3+2xy^2+x^3}{x^3}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{3}{2}}}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{2y^3+2xy^2+x^3}{x^3}} = \frac{c_3\mathrm{e}^{c_2}}{x^{\frac{3}{2}}} \tag{1}$$

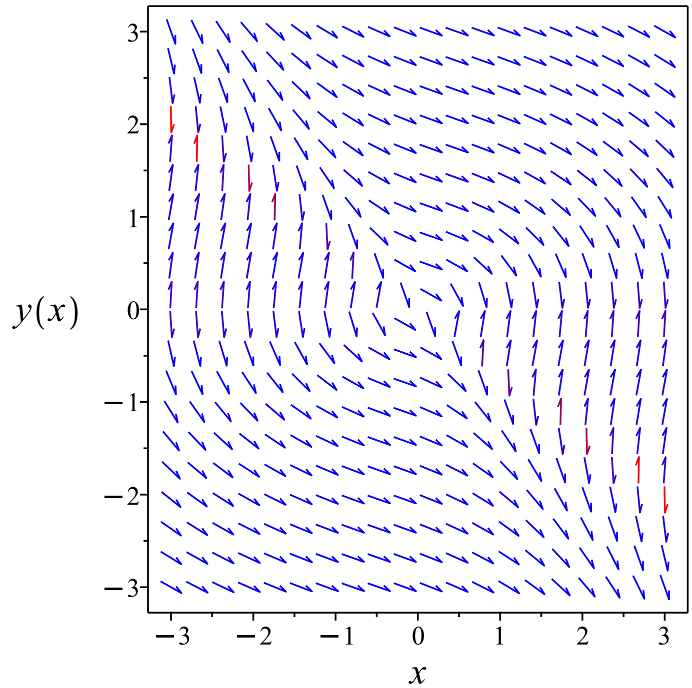


Figure 320: Slope field plot

Verification of solutions

$$\sqrt{\frac{2y^3+2xy^2+x^3}{x^3}}=\frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{3}{2}}}$$

Verified OK.

5.33.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{3x^2+2y^2}{2y(2x+3y)} \\ y' &= \omega(x,y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ,η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2+ya_3+a_1 \tag{1E}$$

$$\eta = xb_2+yb_3+b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$b_2 - \frac{(3x^2+2y^2)(b_3-a_2)}{2y(2x+3y)} - \frac{(3x^2+2y^2)^2 a_3}{4y^2(2x+3y)^2} - \left(-\frac{3x}{y(2x+3y)} + \frac{3x^2+2y^2}{y(2x+3y)^2}\right)(xa_2+ya_3+a_1) - \left(-\frac{2}{2x+3y} + \frac{3x^2+2y^2}{2y^2(2x+3y)} + \frac{\frac{9x^2}{2}+3y^2}{y(2x+3y)^2}\right)(xb_2+yb_3+b_1) = 0 \tag{5E}$$

Putting the above in normal form gives

$$-\frac{9x^4a_3+12x^4b_2-24x^3ya_2+36x^3yb_2+24x^3yb_3-54x^2y^2a_2-24x^2y^2b_2+54x^2y^2b_3-36x\,y^3a_3-48x\,y^3}{4y^2\,(2x-}$$
$$=0$$

Setting the numerator to zero gives

$$-9x^4a_3-12x^4b_2+24x^3ya_2-36x^3yb_2-24x^3yb_3+54x^2y^2a_2+24x^2y^2b_2-54x^2y^2b_3+36x\,y^3a_3+48x\,y^3b_2+12y^4a_2-12y^4a_3+36y^4b_2-12y^4b_3-12x^3b_1+12x^2ya_1-36x^2yb_1+36x\,y^2a_1+8x\,y^2b_1-8y^3a_1=0 \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$24a_2v_1^3v_2+54a_2v_1^2v_2^2+12a_2v_2^4-9a_3v_1^4+36a_3v_1v_2^3-12a_3v_2^4-12b_2v_1^4-36b_2v_1^3v_2+24b_2v_1^2v_2^2+48b_2v_1v_2^3+36b_2v_2^4-24b_3v_1^3v_2-54b_3v_1^2v_2^2-12b_3v_2^4+12a_1v_1^2v_2+36a_1v_1v_2^2-8a_1v_2^3-12b_1v_1^3-36b_1v_1^2v_2+8b_1v_1v_2^2=0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} &(-9a_3 - 12b_2)\, v_1^4 + (24a_2 - 36b_2 - 24b_3)\, v_1^3 v_2 - 12b_1 v_1^3 \\ &+ (54a_2 + 24b_2 - 54b_3)\, v_1^2 v_2^2 + (12a_1 - 36b_1)\, v_1^2 v_2 + (36a_3 + 48b_2)\, v_1 v_2^3 \\ &+ (36a_1 + 8b_1)\, v_1 v_2^2 + (12a_2 - 12a_3 + 36b_2 - 12b_3)\, v_2^4 - 8a_1 v_2^3 = 0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned} -8a_1 &= 0 \\ -12b_1 &= 0 \\ 12a_1 - 36b_1 &= 0 \\ 36a_1 + 8b_1 &= 0 \\ -9a_3 - 12b_2 &= 0 \\ 36a_3 + 48b_2 &= 0 \\ 24a_2 - 36b_2 - 24b_3 &= 0 \\ 54a_2 + 24b_2 - 54b_3 &= 0 \\ 12a_2 - 12a_3 + 36b_2 - 12b_3 &= 0 \end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned} a_1 &= 0 \\ a_2 &= b_3 \\ a_3 &= 0 \\ b_1 &= 0 \\ b_2 &= 0 \\ b_3 &= b_3 \end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned} \xi &= x \\ \eta &= y \end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{3x^2+2y^2}{2y\,(2x+3y)}\right)(x) \\
&= \frac{3x^3+6x\,y^2+6y^3}{4yx+6y^2} \\
\xi &= 0
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{3x^3+6x\,y^2+6y^3}{4yx+6y^2}} dy
\end{aligned}$$

Which results in

$$S = \frac{\ln\left(x^3+2x\,y^2+2y^3\right)}{3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x^2+2y^2}{2y\,(2x+3y)}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3x^2 + 2y^2}{3x^3 + 6x\,y^2 + 6y^3} \\
S_y &= \frac{4yx + 6y^2}{3x^3 + 6x\,y^2 + 6y^3}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = 0 \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = 0
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = c_1 \tag{4}
$$

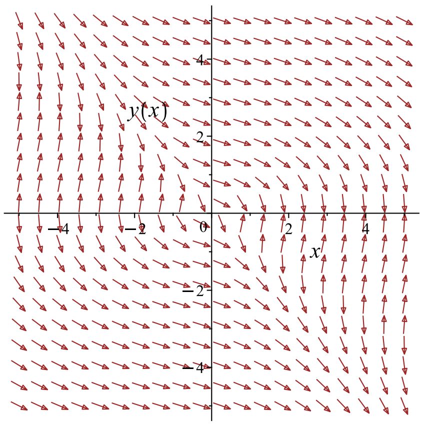
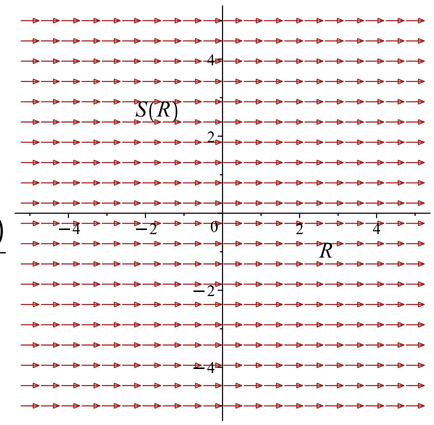
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{\ln (2y^3 + 2xy^2 + x^3)}{3} = c_1
$$

Which simplifies to

$$
\frac{\ln (2y^3 + 2xy^2 + x^3)}{3} = c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^2+2y^2}{2y(2x+3y)}$ 	$R = x$ $S = \frac{\ln(x^3 + 2x\,y^2 + 2y^3)}{3}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$\frac{\ln(2y^3 + 2xy^2 + x^3)}{3} = c_1 \tag{1}$$

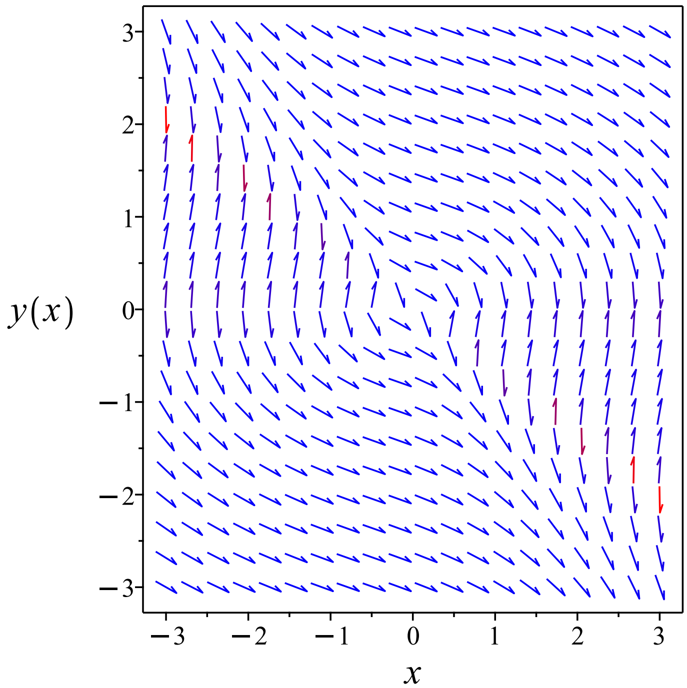


Figure 321: Slope field plot

<u>Verification of solutions</u>

$$\frac{\ln(2y^3 + 2xy^2 + x^3)}{3} = c_1$$

Verified OK.

5.33.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(4yx + 6y^2\right) \mathrm{d}y = \left(-3x^2 - 2y^2\right) \mathrm{d}x$$
$$\left(3x^2 + 2y^2\right) \mathrm{d}x + \left(4yx + 6y^2\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 3x^2 + 2y^2$$
$$N(x, y) = 4yx + 6y^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3x^2 + 2y^2\right) \\ &= 4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(4yx + 6y^2\right) \\ &= 4y\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M && (1) \\
\frac{\partial \phi}{\partial y} &= N && (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int 3x^2 + 2y^2 \, \mathrm{d}x \\
\phi &= x^3 + 2x\, y^2 + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 4yx + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 4yx + 6y^2$. Therefore equation (4) becomes

$$
4yx + 6y^2 = 4yx + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 6y^2
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int (6y^2) \, \mathrm{d}y \\
f(y) &= 2y^3 + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = x^3 + 2x\, y^2 + 2y^3 + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3 + 2x\,y^2 + 2y^3$$

Summary

The solution(s) found are the following

$$2y^3 + 2xy^2 + x^3 = c_1 \tag{1}$$

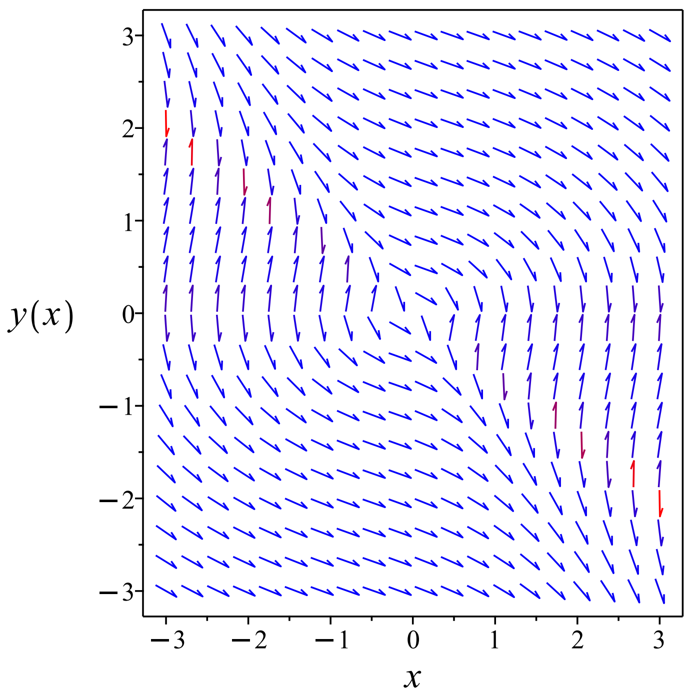


Figure 322: Slope field plot

Verification of solutions

$$2y^3 + 2xy^2 + x^3 = c_1$$

Verified OK.

5.33.4 Maple step by step solution

Let's solve

$2y^2 + (4yx + 6y^2)\, y' = -3x^2$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $4y = 4y$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (3x^2 + 2y^2)\, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = x^3 + 2x\, y^2 + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$
- Compute derivative

  $4yx + 6y^2 = 4yx + \frac{d}{dy} f_1(y)$
- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 6y^2$
- Solve for $f_1(y)$

  $f_1(y) = 2y^3$
- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = x^3 + 2x\,y^2 + 2y^3$$

- Substitute $F(x, y)$ into the solution of the ODE

$$x^3 + 2x\,y^2 + 2y^3 = c_1$$

- Solve for y

$$\left\{y = \frac{\left(-62x^3+54c_1+6\sqrt{105x^6-186c_1x^3+81c_1^2}\right)^{\frac{1}{3}}}{6} + \frac{2x^2}{3\left(-62x^3+54c_1+6\sqrt{105x^6-186c_1x^3+81c_1^2}\right)^{\frac{1}{3}}} - \frac{x}{3}, y = -\frac{\left(-62x^3+\right.}{}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 405

```
dsolve(3*x^2+2*y(x)^2+(4*x*y(x)+6*y(x)^2)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{\frac{\left(54-62x^3c_1^3+6\sqrt{105c_1^6x^6-186x^3c_1^3+81}\right)^{\frac{1}{3}}}{2}+\frac{2x^2c_1^2}{\left(54-62x^3c_1^3+6\sqrt{105c_1^6x^6-186x^3c_1^3+81}\right)^{\frac{1}{3}}}-c_1x}{3c_1}$$

$$y(x) = \frac{4i\sqrt{3}\,c_1^2x^2-i\sqrt{3}\left(54-62x^3c_1^3+6\sqrt{105c_1^6x^6-186x^3c_1^3+81}\right)^{\frac{2}{3}}-4c_1^2x^2-4c_1x\left(54-62x^3c_1^3+6\sqrt{105c\right.}{12\left(54-62x^3c_1^3+6\sqrt{105c_1^6x^6-186x^3c_1^3\right.}}$$

$$y(x) = \frac{\left(i\sqrt{3}-1\right)\left(54-62x^3c_1^3+6\sqrt{105c_1^6x^6-186x^3c_1^3+81}\right)^{\frac{2}{3}}-4x\left(ixc_1\sqrt{3}+c_1x+\left(54-62x^3c_1^3+6\sqrt{10\right.\right.}{12\left(54-62x^3c_1^3+6\sqrt{105c_1^6x^6-186x^3c_1^3+81}\right)^{\frac{1}{3}}c_1}$$

✓ Solution by Mathematica

Time used: 39.668 (sec). Leaf size: 679

```
DSolve[3*x^2+2*y[x]^2+(4*x*y[x]+6*y[x]^2)*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to \frac{\sqrt[3]{-124x^3+\sqrt{-256x^6+(-124x^3+108e^{2c_1})^2}+108e^{2c_1}}}{6\sqrt[3]{2}} + \frac{2\sqrt[3]{2}x^2}{3\sqrt[3]{-124x^3+\sqrt{-256x^6+(-124x^3+108e^{2c_1})^2}+108e^{2c_1}}} - \frac{x}{3}$$

$$y(x) \to \frac{1}{12}i\left(\sqrt{3}+i\right)\sqrt[3]{-62x^3+6\sqrt{3}\sqrt{35x^6-62e^{2c_1}x^3+27e^{4c_1}}+54e^{2c_1}} - \frac{i\left(\sqrt{3}-i\right)x^2}{3\sqrt[3]{-62x^3+6\sqrt{3}\sqrt{35x^6-62e^{2c_1}x^3+27e^{4c_1}}+54e^{2c_1}}} - \frac{x}{3}$$

$$y(x) \to -\frac{1}{12}i\left(\sqrt{3}-i\right)\sqrt[3]{-62x^3+6\sqrt{3}\sqrt{35x^6-62e^{2c_1}x^3+27e^{4c_1}}+54e^{2c_1}} + \frac{i\left(\sqrt{3}+i\right)x^2}{3\sqrt[3]{-62x^3+6\sqrt{3}\sqrt{35x^6-62e^{2c_1}x^3+27e^{4c_1}}+54e^{2c_1}}} - \frac{x}{3}$$

$$y(x) \to \frac{1}{6}\left(\sqrt[3]{6\sqrt{105}\sqrt{x^6}-62x^3} + \frac{2\,2^{2/3}x^2}{\sqrt[3]{3\sqrt{105}\sqrt{x^6}-31x^3}} - 2x\right)$$

$$y(x) \to \frac{1}{12}\left(i\left(\sqrt{3}+i\right)\sqrt[3]{6\sqrt{105}\sqrt{x^6}-62x^3} - \frac{2i2^{2/3}\left(\sqrt{3}-i\right)x^2}{\sqrt[3]{3\sqrt{105}\sqrt{x^6}-31x^3}} - 4x\right)$$

$$y(x) \to \frac{1}{12}\left(\left(-1-i\sqrt{3}\right)\sqrt[3]{6\sqrt{105}\sqrt{x^6}-62x^3} + \frac{2i2^{2/3}\left(\sqrt{3}+i\right)x^2}{\sqrt[3]{3\sqrt{105}\sqrt{x^6}-31x^3}} - 4x\right)$$

5.34 problem 34

Internal problem ID [112]
Internal file name [`OUTPUT/112_Sunday_June_05_2022_01_35_04_AM_92740676/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 34.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{2xy^2 + \left(2x^2y + 4y^3\right) y' = -3x^2}$$

5.34.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(2y\,x^2 + 4y^3\right)\mathrm{d}y &= \left(-2x\,y^2 - 3x^2\right)\mathrm{d}x \\
\left(2x\,y^2 + 3x^2\right)\mathrm{d}x + \left(2y\,x^2 + 4y^3\right)\mathrm{d}y &= 0 \qquad (2A)
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= 2x\,y^2 + 3x^2 \\
N(x, y) &= 2y\,x^2 + 4y^3
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\,y^2 + 3x^2\right) \\
&= 4yx
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(2y\,x^2 + 4y^3\right) \\
&= 4yx
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 2x\, y^2 + 3x^2 \mathrm{d}x$$
$$\phi = x^2\left(y^2 + x\right) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 2y\, x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 2y\, x^2 + 4y^3$. Therefore equation (4) becomes

$$2y\, x^2 + 4y^3 = 2y\, x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 4y^3$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(4y^3\right) \mathrm{d}y$$
$$f(y) = y^4 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x^2\left(y^2 + x\right) + y^4 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^2\left(y^2 + x\right) + y^4$$

Summary

The solution(s) found are the following

$$x^2(y^2+x)+y^4=c_1 \tag{1}$$

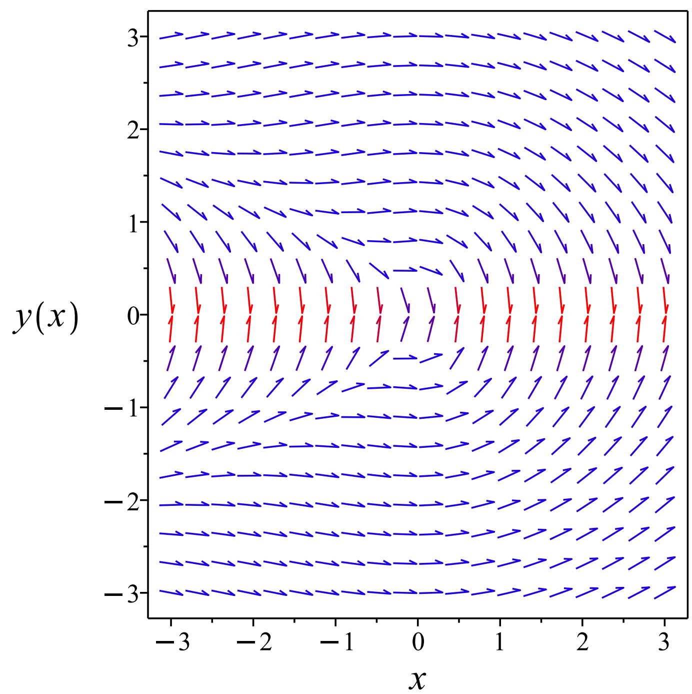


Figure 323: Slope field plot

Verification of solutions

$$x^2(y^2+x)+y^4=c_1$$

Verified OK.

5.34.2 Maple step by step solution

Let's solve

$2xy^2+(2x^2y+4y^3)\,y'=-3x^2$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $4yx = 4yx$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (2x\,y^2 + 3x^2)\,dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x^2y^2 + x^3 + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $2y\,x^2 + 4y^3 = 2y\,x^2 + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = 4y^3$

- Solve for $f_1(y)$

  $f_1(y) = y^4$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x^2y^2 + y^4 + x^3$

- Substitute $F(x, y)$ into the solution of the ODE

  $x^2y^2 + y^4 + x^3 = c_1$

- Solve for y

  $\left\{y = -\frac{\sqrt{-2x^2 - 2\sqrt{x^4 - 4x^3 + 4c_1}}}{2}, y = \frac{\sqrt{-2x^2 - 2\sqrt{x^4 - 4x^3 + 4c_1}}}{2}, y = -\frac{\sqrt{-2x^2 + 2\sqrt{x^4 - 4x^3 + 4c_1}}}{2}, y = \frac{\sqrt{-2x^2 + 2\sqrt{x^4}}}{2}\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 117

```
dsolve(3*x^2+2*x*y(x)^2+(2*x^2*y(x)+4*y(x)^3)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{-2x^2 - 2\sqrt{x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = \frac{\sqrt{-2x^2 - 2\sqrt{x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = -\frac{\sqrt{-2x^2 + 2\sqrt{x^4 - 4x^3 - 4c_1}}}{2}$$
$$y(x) = \frac{\sqrt{-2x^2 + 2\sqrt{x^4 - 4x^3 - 4c_1}}}{2}$$

✓ Solution by Mathematica

Time used: 5.897 (sec). Leaf size: 155

```
DSolve[3*x^2+2*x*y[x]^2+(2*x^2*y[x]+4*y[x]^3)*y'[x]== 0,y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to -\frac{\sqrt{-x^2 - \sqrt{x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-x^2 - \sqrt{x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to -\frac{\sqrt{-x^2 + \sqrt{x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$
$$y(x) \to \frac{\sqrt{-x^2 + \sqrt{x^4 - 4x^3 + 4c_1}}}{\sqrt{2}}$$

5.35 problem 35

Internal problem ID [113]
Internal file name [`OUTPUT/113_Sunday_June_05_2022_01_35_05_AM_36984241/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 35.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\frac{y}{x} + \left(\ln \left(x \right)+y^{2}\right) y^{\prime} = -x^{3}}$$

5.35.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\ln(x) + y^2)\,\mathrm{d}y &= \left(-x^3 - \frac{y}{x}\right)\mathrm{d}x\\ \left(x^3 + \frac{y}{x}\right)\mathrm{d}x + (\ln(x) + y^2)\,\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x^3 + \frac{y}{x}\\ N(x, y) &= \ln(x) + y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^3 + \frac{y}{x}\right)\\ &= \frac{1}{x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\ln(x) + y^2)\\ &= \frac{1}{x}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int x^3+\frac{y}{x}\,\mathrm{d}x$$

$$\phi=\frac{x^4}{4}+\ln(x)\,y+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=\ln(x)+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=\ln(x)+y^2$. Therefore equation (4) becomes

$$\ln(x)+y^2=\ln(x)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=y^2$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int(y^2)\,\mathrm{d}y$$

$$f(y)=\frac{y^3}{3}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^4}{4} + \ln(x)\, y + \frac{y^3}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^4}{4} + \ln(x)\, y + \frac{y^3}{3}$$

Summary

The solution(s) found are the following

$$\frac{x^4}{4} + \ln(x)\, y + \frac{y^3}{3} = c_1 \tag{1}$$

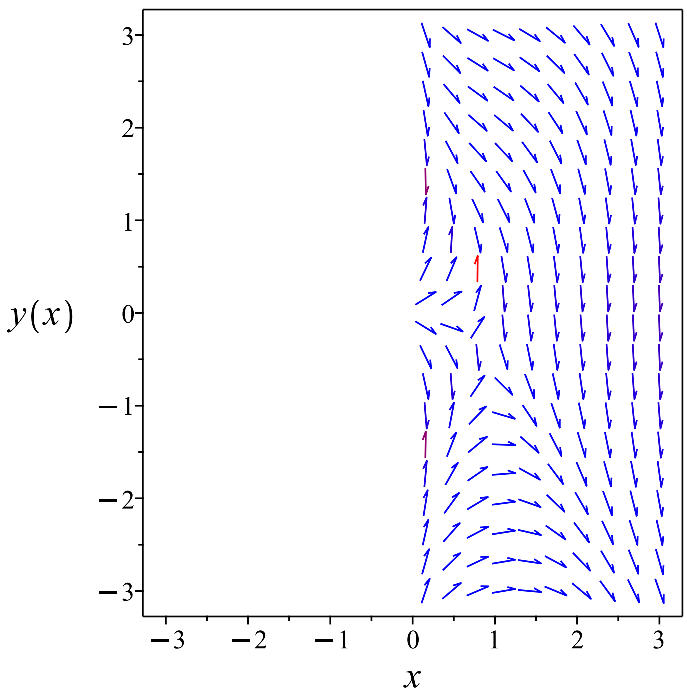


Figure 324: Slope field plot

Verification of solutions

$$\frac{x^4}{4} + \ln(x)\, y + \frac{y^3}{3} = c_1$$

Verified OK.

5.35.2 Maple step by step solution

Let's solve

$\frac{y}{x}+\left(\ln \left(x \right)+y^{2}\right) y^{\prime}=-x^{3}$

- Highest derivative means the order of the ODE is 1

  $y^{\prime}$

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^{2} function

    $F^{\prime}\left(x , y\right)=0$

  ◦ Compute derivative of lhs

    $F^{\prime}\left(x , y\right)+\left(\frac{\partial}{\partial y}F \left(x , y\right)\right) y^{\prime}=0$

  ◦ Evaluate derivatives

    $\frac{1}{x}=\frac{1}{x}$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F \left(x , y\right)=c_{1} , M \left(x , y\right)=F^{\prime}\left(x , y\right) , N \left(x , y\right)=\frac{\partial}{\partial y}F \left(x , y\right)\right]$

- Solve for $F \left(x , y\right)$ by integrating $M \left(x , y\right)$ with respect to x

  $F \left(x , y\right)=\int \left(x^{3}+\frac{y}{x}\right)d x +f_{1}\left(y \right)$

- Evaluate integral

  $F \left(x , y\right)=\frac{x^{4}}{4}+\ln \left(x \right) y +f_{1}\left(y \right)$

- Take derivative of $F \left(x , y\right)$ with respect to y

  $N \left(x , y\right)=\frac{\partial}{\partial y}F \left(x , y\right)$

- Compute derivative

  $\ln \left(x \right)+y^{2}=\ln \left(x \right)+\frac{d}{d y}f_{1}\left(y \right)$

- Isolate for $\frac{d}{d y}f_{1}\left(y \right)$

  $\frac{d}{d y}f_{1}\left(y \right)=y^{2}$

- Solve for $f_{1}\left(y \right)$

  $f_{1}\left(y \right)=\frac{y^{3}}{3}$

- Substitute $f_{1}\left(y \right)$ into equation for $F \left(x , y\right)$

$$F(x,y) = \frac{x^4}{4} + \ln(x)\, y + \frac{y^3}{3}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{x^4}{4} + \ln(x)\, y + \frac{y^3}{3} = c_1$$

- Solve for y

$$\left\{y = \frac{\left(-3x^4+12c_1+\sqrt{64\ln(x)^3+9x^8-72c_1x^4+144c_1^2}\right)^{\frac{1}{3}}}{2} - \frac{2\ln(x)}{\left(-3x^4+12c_1+\sqrt{64\ln(x)^3+9x^8-72c_1x^4+144c_1^2}\right)^{\frac{1}{3}}}, y = -\frac{\left(-3\right.}{}\right.$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 307

```
dsolve(x^3+y(x)/x+(ln(x)+y(x)^2)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{\left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9(x^4 + 4c_1)^2}\right)^{\frac{2}{3}} - 4\ln(x)}{2\left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9(x^4 + 4c_1)^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\left(-\left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9(x^4 + 4c_1)^2}\right)^{\frac{2}{3}} - 4\ln(x)\right)\sqrt{3} - \left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9\right.}{4\left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9(x^4 + 4c_1)^2}\right)^{\frac{1}{3}}}$$

$$y(x) = \frac{i\left(\left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9(x^4 + 4c_1)^2}\right)^{\frac{2}{3}} + 4\ln(x)\right)\sqrt{3} - \left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9(\right.}{4\left(-3x^4 - 12c_1 + \sqrt{64\ln(x)^3 + 9(x^4 + 4c_1)^2}\right)^{\frac{1}{3}}}$$

✓ Solution by Mathematica

Time used: 1.864 (sec). Leaf size: 307

```
DSolve[x^3+y[x]/x+(Log[x]+y[x]^2)*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{-4\log(x) + \left(-3x^4 + \sqrt{64\log^3(x) + 9\left(x^4 - 4c_1\right)^2} + 12c_1\right)^{2/3}}{2\sqrt[3]{-3x^4 + \sqrt{64\log^3(x) + 9\left(x^4 - 4c_1\right)^2} + 12c_1}}$$

$$y(x) \to \frac{i\left(\sqrt{3} + i\right)\left(-3x^4 + \sqrt{64\log^3(x) + 9\left(x^4 - 4c_1\right)^2} + 12c_1\right)^{2/3} + \left(4 + 4i\sqrt{3}\right)\log(x)}{4\sqrt[3]{-3x^4 + \sqrt{64\log^3(x) + 9\left(x^4 - 4c_1\right)^2} + 12c_1}}$$

$$y(x) \to \frac{\left(-1 - i\sqrt{3}\right)\left(-3x^4 + \sqrt{64\log^3(x) + 9\left(x^4 - 4c_1\right)^2} + 12c_1\right)^{2/3} + \left(4 - 4i\sqrt{3}\right)\log(x)}{4\sqrt[3]{-3x^4 + \sqrt{64\log^3(x) + 9\left(x^4 - 4c_1\right)^2} + 12c_1}}$$

5.36 problem 36

Internal problem ID [114]
Internal file name [`OUTPUT/114_Sunday_June_05_2022_01_35_07_AM_5611736/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 36.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{yx}y + (e^{yx}x + 2y)\,y' = -1}$$

5.36.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{yx}x+2y)\,\mathrm{d}y&=(-1-\mathrm{e}^{yx}y)\,\mathrm{d}x\\ (\mathrm{e}^{yx}y+1)\,\mathrm{d}x+(\mathrm{e}^{yx}x+2y)\,\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\mathrm{e}^{yx}y+1\\ N(x,y)&=\mathrm{e}^{yx}x+2y\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(\mathrm{e}^{yx}y+1)\\ &=\mathrm{e}^{yx}(yx+1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(\mathrm{e}^{yx}x+2y)\\ &=\mathrm{e}^{yx}(yx+1)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \mathrm{e}^{yx} y + 1 \,\mathrm{d}x$$

$$\phi = x + \mathrm{e}^{yx} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{yx} x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{yx} x + 2y$. Therefore equation (4) becomes

$$\mathrm{e}^{yx} x + 2y = \mathrm{e}^{yx} x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 2y$$

Integrating the above w.r.t y gives

$$\int f'(y) \,\mathrm{d}y = \int (2y) \,\mathrm{d}y$$

$$f(y) = y^2 + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = x + \mathrm{e}^{yx} + y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x + \mathrm{e}^{yx} + y^2$$

Summary
The solution(s) found are the following

$$x + e^{yx} + y^2 = c_1 \tag{1}$$

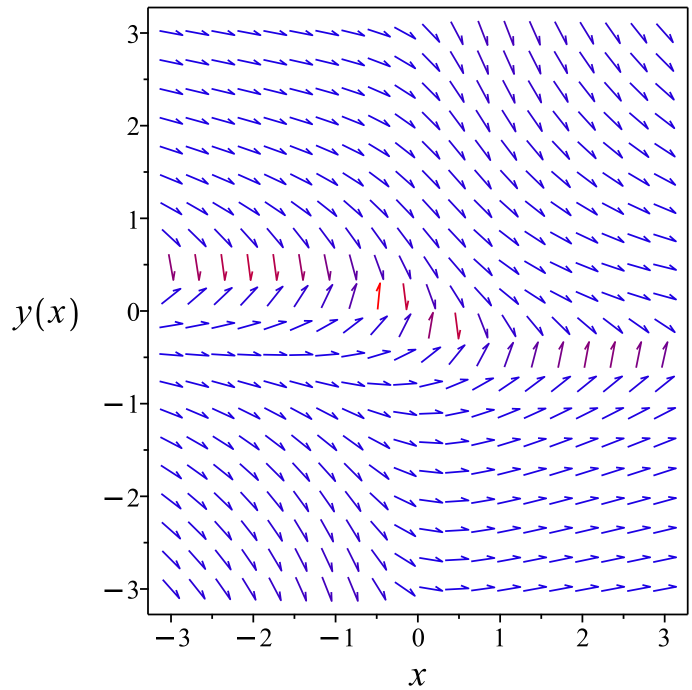


Figure 325: Slope field plot

Verification of solutions

$$x + e^{yx} + y^2 = c_1$$

Verified OK.

5.36.2 Maple step by step solution

Let's solve

$e^{yx} y + (e^{yx} x + 2y) y' = -1$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x, y) = 0$

- Compute derivative of lhs

  $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$

- Evaluate derivatives

  $e^{yx}yx + e^{yx} = e^{yx}yx + e^{yx}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (e^{yx}y + 1)\, dx + f_1(y)$

- Evaluate integral

  $F(x, y) = x + e^{yx} + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$

- Compute derivative

  $e^{yx}x + 2y = e^{yx}x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 2y$

- Solve for $f_1(y)$

  $f_1(y) = y^2$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = x + e^{yx} + y^2$

- Substitute $F(x, y)$ into the solution of the ODE

  $x + e^{yx} + y^2 = c_1$

- Solve for y

  $y = \mathit{RootOf}\left(-x - e^{\_Zx} - \_Z^2 + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(1+exp(x*y(x))*y(x)+(exp(x*y(x))*x+2*y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$e^{xy(x)} + x + y(x)^2 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.216 (sec). Leaf size: 18

```
DSolve[1+Exp[x*y[x]]*y[x]+(Exp[x*y[x]]*x+2*y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[y(x)^2 + e^{xy(x)} + x = c_1, y(x)\right]$$

5.37 problem 37

Internal problem ID [115]
Internal file name [`OUTPUT/115_Sunday_June_05_2022_01_35_08_AM_30353114/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 37.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{\ln(y) + \left(e^y + \frac{x}{y}\right) y' = -\cos(x)}$$

5.37.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\mathrm{e}^y + \frac{x}{y}\right)\mathrm{d}y = (-\ln(y) - \cos(x))\,\mathrm{d}x$$

$$(\cos(x) + \ln(y))\,\mathrm{d}x + \left(\mathrm{e}^y + \frac{x}{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x,y) &= \cos(x) + \ln(y) \\
N(x,y) &= \mathrm{e}^y + \frac{x}{y}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\cos(x) + \ln(y)) \\
&= \frac{1}{y}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^y + \frac{x}{y}\right) \\
&= \frac{1}{y}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \cos(x) + \ln(y) \, \mathrm{d}x$$

$$\phi = \sin(x) + x \ln(y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^y + \frac{x}{y}$. Therefore equation (4) becomes

$$\mathrm{e}^y + \frac{x}{y} = \frac{x}{y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\mathrm{e}^y) \, \mathrm{d}y$$

$$f(y) = \mathrm{e}^y + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x) + x \ln(y) + \mathrm{e}^y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x) + x\ln(y) + e^y$$

Summary

The solution(s) found are the following

$$\sin(x) + x\ln(y) + e^y = c_1 \tag{1}$$

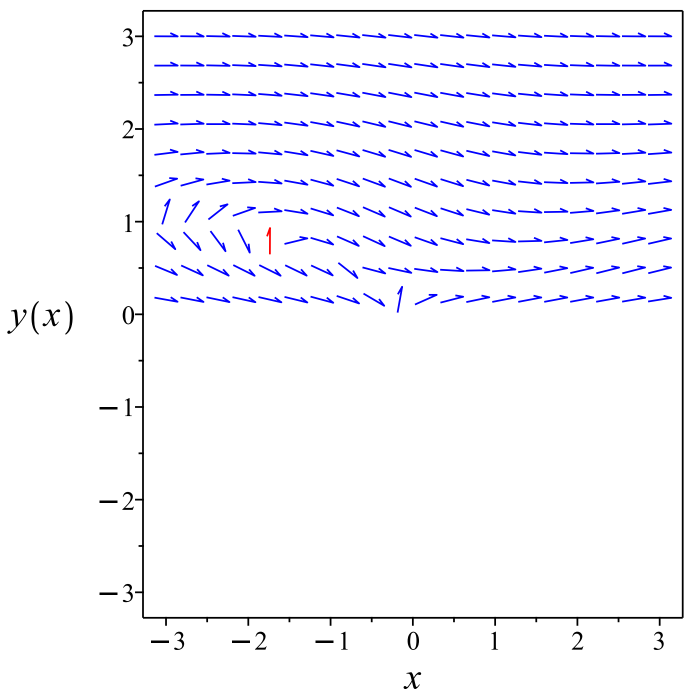


Figure 326: Slope field plot

Verification of solutions

$$\sin(x) + x\ln(y) + e^y = c_1$$

Verified OK.

5.37.2 Maple step by step solution

Let's solve

$\ln(y) + \left(e^{y} + \frac{x}{y}\right) y' = -\cos(x)$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

  - Evaluate derivatives

    $\frac{1}{y} = \frac{1}{y}$

  - Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y} F(x, y)\right]$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (\cos(x) + \ln(y)) dx + f_1(y)$

- Evaluate integral

  $F(x, y) = \sin(x) + x \ln(y) + f_1(y)$

- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y} F(x, y)$

- Compute derivative

  $e^{y} + \frac{x}{y} = \frac{x}{y} + \frac{d}{dy} f_1(y)$

- Isolate for $\frac{d}{dy} f_1(y)$

  $\frac{d}{dy} f_1(y) = e^{y}$

- Solve for $f_1(y)$

  $f_1(y) = e^{y}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x,y) = \sin(x) + x\ln(y) + e^y$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\sin(x) + x\ln(y) + e^y = c_1$$

- Solve for y

$$y = e^{\mathit{RootOf}\left(e^{\_Z} - \ln(-\_Zx + c_1 - \sin(x))\right)}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.078 (sec). Leaf size: 24

```
dsolve(cos(x)+ln(y(x))+(exp(y(x))+x/y(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = e^{\mathrm{RootOf}\left(e^{\_Z} - \ln(-x\_Z - c_1 - \sin(x))\right)}$$

✓ Solution by Mathematica

Time used: 0.36 (sec). Leaf size: 18

```
DSolve[Cos[x]+Log[y[x]]+(Exp[y[x]]+x/y[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions -> Tru
```

$$\mathrm{Solve}\left[e^{y(x)} + x\log(y(x)) + \sin(x) = c_1, y(x)\right]$$

5.38 problem 38

Internal problem ID [116]
Internal file name [`OUTPUT/116_Sunday_June_05_2022_01_35_10_AM_35287607/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 38.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\arctan(y) + \frac{(x+y)\,y'}{1+y^2} = -x$$

5.38.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{x+y}{y^2+1}\right)\mathrm{d}y &= (-x - \arctan(y))\,\mathrm{d}x\\ (x + \arctan(y))\,\mathrm{d}x + \left(\frac{x+y}{y^2+1}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= x + \arctan(y)\\ N(x, y) &= \frac{x+y}{y^2+1}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(x + \arctan(y))\\ &= \frac{1}{y^2+1}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x+y}{y^2+1}\right)\\ &= \frac{1}{y^2+1}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$

$$\int \frac{\partial \phi}{\partial x} \, dx = \int x + \arctan(y) \, dx$$

$$\phi = \frac{x(x + 2\arctan(y))}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2 + 1} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x+y}{y^2+1}$. Therefore equation (4) becomes

$$\frac{x + y}{y^2 + 1} = \frac{x}{y^2 + 1} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{y}{y^2 + 1}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, dy = \int \left(\frac{y}{y^2 + 1}\right) dy$$

$$f(y) = \frac{\ln(y^2 + 1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x(x + 2\arctan(y))}{2} + \frac{\ln(y^2 + 1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x(x + 2\arctan(y))}{2} + \frac{\ln(y^2 + 1)}{2}$$

Summary

The solution(s) found are the following

$$\frac{x(x + 2\arctan(y))}{2} + \frac{\ln(1 + y^2)}{2} = c_1 \tag{1}$$

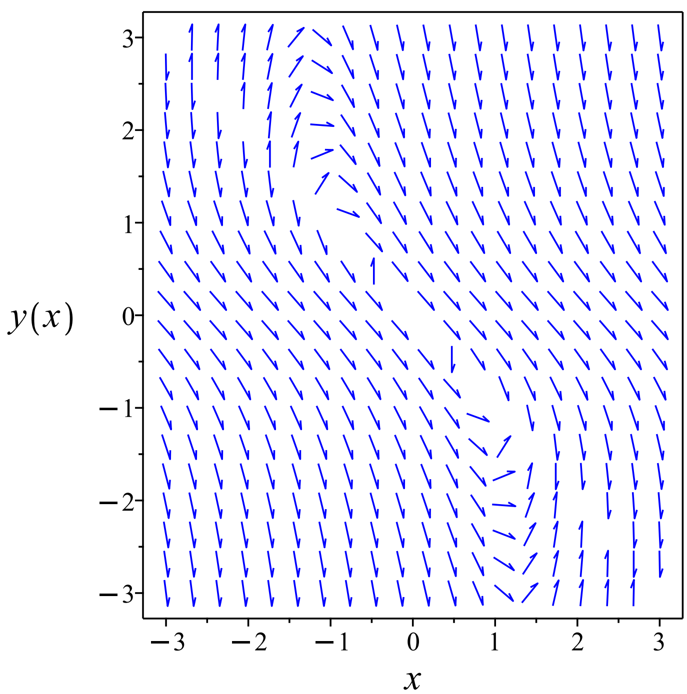


Figure 327: Slope field plot

Verification of solutions

$$\frac{x(x + 2\arctan(y))}{2} + \frac{\ln(1 + y^2)}{2} = c_1$$

Verified OK.

5.38.2 Maple step by step solution

Let's solve

$\arctan(y) + \frac{(x+y)y'}{1+y^2} = -x$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y) = 0$
  - Compute derivative of lhs

    $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right)y' = 0$
  - Evaluate derivatives

    $\frac{1}{y^2+1} = \frac{1}{y^2+1}$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (x + \arctan(y))\, dx + f_1(y)$
- Evaluate integral

  $F(x,y) = \frac{x^2}{2} + \arctan(y)\, x + f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $\frac{x+y}{y^2+1} = \frac{x}{y^2+1} + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = -\frac{x}{y^2+1} + \frac{x+y}{y^2+1}$
- Solve for $f_1(y)$

  $f_1(y) = \frac{\ln(y^2+1)}{2}$
- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x, y) = \frac{x^2}{2} + \arctan(y)\, x + \frac{\ln(y^2+1)}{2}$$

- Substitute $F(x, y)$ into the solution of the ODE

$$\frac{x^2}{2} + \arctan(y)\, x + \frac{\ln(y^2+1)}{2} = c_1$$

- Solve for y

$$y = \tan\left(RootOf(-2\_Zx - x^2 + 2\ln(\cos(\_Z)) + 2c_1)\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve(x+arctan(y(x))+(x+y(x))*diff(y(x),x)/(1+y(x)^2) = 0,y(x), singsol=all)
```

$$y(x) = \tan\left(\mathrm{RootOf}\left(2x\_Z + x^2 - 2\ln(\cos(\_Z)) + 2c_1\right)\right)$$

✓ Solution by Mathematica

Time used: 0.137 (sec). Leaf size: 30

```
DSolve[x+ArcTan[y[x]]+(x+y[x])*y'[x]/(1+y[x]^2) == 0,y[x],x,IncludeSingularSolutions -> True
```

$$\mathrm{Solve}\left[x\arctan(y(x)) + \frac{x^2}{2} + \frac{1}{2}\log\left(y(x)^2+1\right) = c_1, y(x)\right]$$

5.39 problem 39

Internal problem ID [117]
Internal file name [`OUTPUT/117_Sunday_June_05_2022_01_35_11_AM_55785690/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 39.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact, _rational]`

$$\boxed{3y^3x^2 + y^4 + (3x^3y^2 + 4xy^3 + y^4)\,y' = 0}$$

5.39.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x^3y^2 + 4x\,y^3 + y^4)\,\mathrm{d}y &= (-3y^3x^2 - y^4)\,\mathrm{d}x\\ (3y^3x^2 + y^4)\,\mathrm{d}x + (3x^3y^2 + 4x\,y^3 + y^4)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3y^3x^2 + y^4\\ N(x,y) &= 3x^3y^2 + 4x\,y^3 + y^4\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(3y^3x^2 + y^4)\\ &= 9x^2y^2 + 4y^3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x^3y^2 + 4x\,y^3 + y^4)\\ &= 9x^2y^2 + 4y^3\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 3y^3x^2 + y^4 \, \mathrm{d}x$$

$$\phi = y^3 x(x^2 + y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned} \frac{\partial \phi}{\partial y} &= 3y^2 x(x^2 + y) + x y^3 + f'(y) \qquad (4) \\ &= x y^2(3x^2 + 4y) + f'(y) \end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 3x^3y^2 + 4x y^3 + y^4$. Therefore equation (4) becomes

$$3x^3y^2 + 4x y^3 + y^4 = x y^2(3x^2 + 4y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y^4$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y^4) \, \mathrm{d}y$$

$$f(y) = \frac{y^5}{5} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^3 x(x^2 + y) + \frac{y^5}{5} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^3 x(x^2 + y) + \frac{y^5}{5}$$

Summary
The solution(s) found are the following

$$y^3x(x^2+y)+\frac{y^5}{5}=c_1 \tag{1}$$

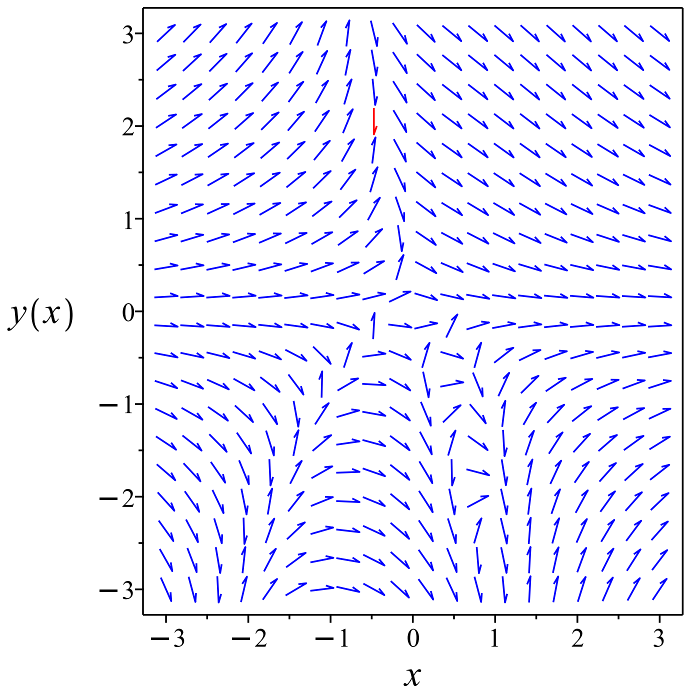


Figure 328: Slope field plot

Verification of solutions

$$y^3x(x^2+y)+\frac{y^5}{5}=c_1$$

Verified OK.

5.39.2 Maple step by step solution

Let's solve

$3y^3x^2+y^4+(3x^3y^2+4xy^3+y^4)\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

- ODE is exact if the lhs is the total derivative of a C^2 function

  $F'(x,y)=0$
- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$
- Evaluate derivatives

  $9x^2y^2+4y^3=9x^2y^2+4y^3$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int (3y^3x^2+y^4)\,dx+f_1(y)$
- Evaluate integral

  $F(x,y)=y^3(x^3+yx)+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative

  $3x^3y^2+4x\,y^3+y^4=3y^2(x^3+yx)+x\,y^3+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=3x^3y^2+3x\,y^3+y^4-3y^2(x^3+yx)$
- Solve for $f_1(y)$

  $f_1(y)=\frac{y^5}{5}$
- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=y^3(x^3+yx)+\frac{y^5}{5}$
- Substitute $F(x,y)$ into the solution of the ODE

  $y^3(x^3+yx)+\frac{y^5}{5}=c_1$
- Solve for y

  $y=\mathit{RootOf}\left(5x^3\_Z^3+\_Z^5+5x\_Z^4-5c_1\right)$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(3*x^2*y(x)^3+y(x)^4+(3*x^3*y(x)^2+4*x*y(x)^3+y(x)^4)*diff(y(x),x) = 0,y(x), singsol=a
```

$$y(x) = 0$$
$$xy(x)^4 + x^3y(x)^3 + \frac{y(x)^5}{5} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 33.636 (sec). Leaf size: 171

```
DSolve[3*x^2*y[x]^3+y[x]^4+(3*x^3*y[x]^2+4*x*y[x]^3+y[x]^4)*y'[x] == 0,y[x],x,IncludeSingula
```

$$y(x) \to 0$$
$$y(x) \to \text{Root}\left[\#1^5 + 5\#1^4x + 5\#1^3x^3 - 5c_1\&, 1\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 5\#1^4x + 5\#1^3x^3 - 5c_1\&, 2\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 5\#1^4x + 5\#1^3x^3 - 5c_1\&, 3\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 5\#1^4x + 5\#1^3x^3 - 5c_1\&, 4\right]$$
$$y(x) \to \text{Root}\left[\#1^5 + 5\#1^4x + 5\#1^3x^3 - 5c_1\&, 5\right]$$
$$y(x) \to 0$$

5.40 problem 40

Internal problem ID [118]
Internal file name [OUTPUT/118_Sunday_June_05_2022_01_35_13_AM_10704552/index.tex]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 40.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{x} \sin(y) + \tan(y) + \left(e^{x} \cos(y) + x \sec(y)^{2}\right) y' = 0}$$

5.40.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\mathrm{e}^x \cos(y) + x \sec(y)^2\right) \mathrm{d}y = \left(-\mathrm{e}^x \sin(y) - \tan(y)\right) \mathrm{d}x$$
$$\left(\mathrm{e}^x \sin(y) + \tan(y)\right) \mathrm{d}x + \left(\mathrm{e}^x \cos(y) + x \sec(y)^2\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = \mathrm{e}^x \sin(y) + \tan(y)$$
$$N(x, y) = \mathrm{e}^x \cos(y) + x \sec(y)^2$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\mathrm{e}^x \sin(y) + \tan(y)\right)\\ &= \mathrm{e}^x \cos(y) + \sec(y)^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^x \cos(y) + x \sec(y)^2\right)\\ &= \mathrm{e}^x \cos(y) + \sec(y)^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$
$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int M \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \mathrm{d}x &= \int \mathrm{e}^{x} \sin(y) + \tan(y) \mathrm{d}x \\
\phi &= \mathrm{e}^{x} \sin(y) + x \tan(y) + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\begin{aligned}
\frac{\partial \phi}{\partial y} &= \mathrm{e}^{x} \cos(y) + x\left(1 + \tan(y)^{2}\right) + f'(y) \qquad (4) \\
&= \mathrm{e}^{x} \cos(y) + x \sec(y)^{2} + f'(y)
\end{aligned}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{x} \cos(y) + x \sec(y)^{2}$. Therefore equation (4) becomes

$$
\mathrm{e}^{x} \cos(y) + x \sec(y)^{2} = \mathrm{e}^{x} \cos(y) + x \sec(y)^{2} + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$
f(y) = c_1
$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$
\phi = \mathrm{e}^{x} \sin(y) + x \tan(y) + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = \mathrm{e}^{x} \sin(y) + x \tan(y)
$$

Summary

The solution(s) found are the following

$$
\mathrm{e}^{x} \sin(y) + x \tan(y) = c_1 \qquad (1)
$$

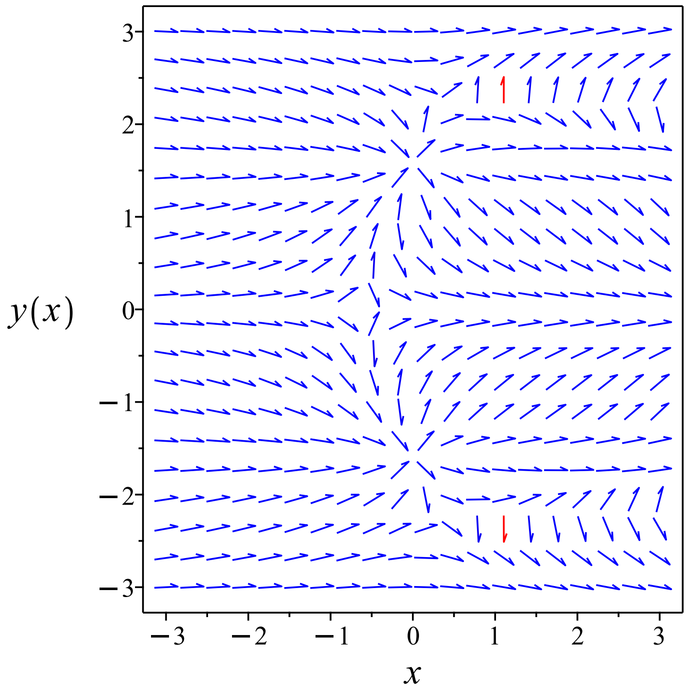


Figure 329: Slope field plot

Verification of solutions

$$e^x \sin(y) + x \tan(y) = c_1$$

Verified OK.

5.40.2 Maple step by step solution

Let's solve

$e^x \sin(y) + \tan(y) + (e^x \cos(y) + x \sec(y)^2) y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ∘ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ∘ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y} F(x, y)\right) y' = 0$

- Evaluate derivatives

  $e^x \cos(y) + 1 + \tan(y)^2 = e^x \cos(y) + \sec(y)^2$

- Simplify

  $e^x \cos(y) + \sec(y)^2 = e^x \cos(y) + \sec(y)^2$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (e^x \sin(y) + \tan(y))\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = e^x \sin(y) + x\tan(y) + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $e^x \cos(y) + x\sec(y)^2 = e^x \cos(y) + x\left(1 + \tan(y)^2\right) + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = x\sec(y)^2 - x\left(1 + \tan(y)^2\right)$

- Solve for $f_1(y)$

  $f_1(y) = 0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = e^x \sin(y) + x\tan(y)$

- Substitute $F(x,y)$ into the solution of the ODE

  $e^x \sin(y) + x\tan(y) = c_1$

- Solve for y

  $y = \arctan\left(\frac{c_1 \mathit{RootOf}\left((e^x)^2 \_Z^4 + 2x e^x \_Z^3 + \left(c_1^2 + x^2 - (e^x)^2\right)\_Z^2 - 2x e^x \_Z - x^2\right)}{\mathit{RootOf}\left((e^x)^2 \_Z^4 + 2x e^x \_Z^3 + \left(c_1^2 + x^2 - (e^x)^2\right)\_Z^2 - 2x e^x \_Z - x^2\right) e^x + x}, \mathit{RootOf}\left((e^x)^2 \_Z^4 + 2x e\right.$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 153

```
dsolve(exp(x)*sin(y(x))+tan(y(x))+(exp(x)*cos(y(x))+x*sec(y(x))^2)*diff(y(x),x) = 0,y(x), si
```

$$y(x) = \arctan\left(-\frac{c_1 \operatorname{RootOf}\left(\_Z^4 e^{2x} + 2x e^x \_Z^3 + \left(c_1^2 + x^2 - e^{2x}\right)\_Z^2 - 2x e^x \_Z - x^2\right)}{\operatorname{RootOf}\left(\_Z^4 e^{2x} + 2x e^x \_Z^3 + \left(c_1^2 + x^2 - e^{2x}\right)\_Z^2 - 2x e^x \_Z - x^2\right) e^x + x}, \operatorname{RootOf}\left(\_Z^4 e^{2x} + 2x e^x \_Z^3 + \left(c_1^2 + x^2 - e^{2x}\right)\_Z^2 - 2x e^x \_Z - x^2\right)\right)$$

✓ Solution by Mathematica

Time used: 60.842 (sec). Leaf size: 5539

```
DSolve[Exp[x]*Sin[y[x]]+Tan[y[x]]+(Exp[x]*Cos[y[x]]+x*Sec[y[x]]^2)*y'[x] == 0,y[x],x,Include
```

Too large to display

5.41 problem 41

Internal problem ID [119]
Internal file name [`OUTPUT/119_Sunday_June_05_2022_01_35_21_AM_68556579/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 41.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

```
[_exact, _rational]
```

$$\boxed{\frac{2x}{y}-\frac{3y^{2}}{x^{4}}+\left(-\frac{x^{2}}{y^{2}}+\frac{1}{\sqrt{y}}+\frac{2y}{x^{3}}\right)y'=0}$$

5.41.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x}=M$$
$$\frac{\partial\phi}{\partial y}=N$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\left(-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}\right)\mathrm{d}y=\left(-\frac{2x}{y}+\frac{3y^2}{x^4}\right)\mathrm{d}x$$

$$\left(\frac{2x}{y}-\frac{3y^2}{x^4}\right)\mathrm{d}x+\left(-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}\right)\mathrm{d}y=0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\frac{2x}{y}-\frac{3y^2}{x^4}\\N(x,y)&=-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(\frac{2x}{y}-\frac{3y^2}{x^4}\right)\\&=-\frac{2x}{y^2}-\frac{6y}{x^4}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}\left(-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}\right)\\&=-\frac{2x}{y^2}-\frac{6y}{x^4}\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int M\,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x=\int \frac{2x}{y}-\frac{3y^2}{x^4}\,\mathrm{d}x$$

$$\phi=\frac{x^5+y^3}{y\,x^3}+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\begin{aligned}\frac{\partial \phi}{\partial y}&=\frac{3y}{x^3}-\frac{x^5+y^3}{y^2x^3}+f'(y) \qquad (4)\\ &=\frac{-x^5+2y^3}{x^3y^2}+f'(y)\end{aligned}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}$. Therefore equation (4) becomes

$$-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}=\frac{-x^5+2y^3}{x^3y^2}+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=\frac{1}{\sqrt{y}}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int\left(\frac{1}{\sqrt{y}}\right)\mathrm{d}y$$

$$f(y)=2\sqrt{y}+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x^5 + y^3}{y\,x^3} + 2\sqrt{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x^5 + y^3}{y\,x^3} + 2\sqrt{y}$$

Summary
The solution(s) found are the following

$$\frac{x^5 + y^3}{y x^3} + 2\sqrt{y} = c_1 \tag{1}$$

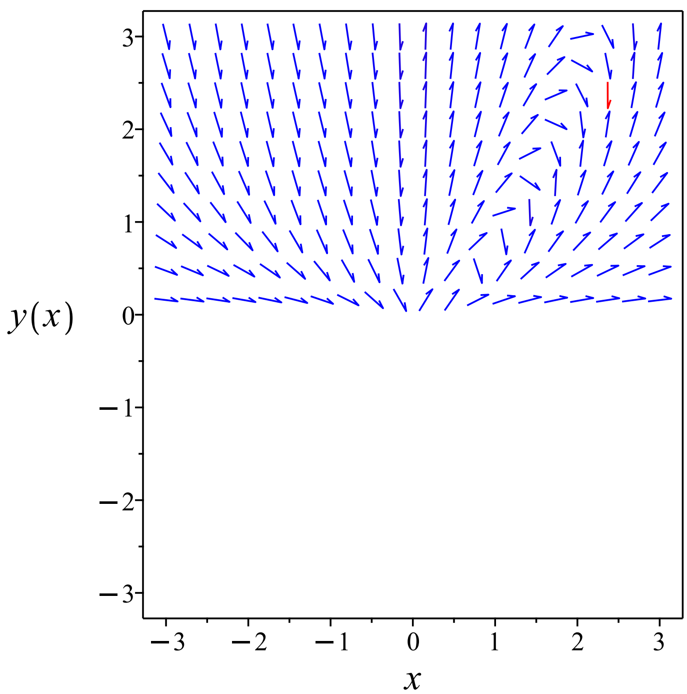


Figure 330: Slope field plot

Verification of solutions

$$\frac{x^5 + y^3}{y x^3} + 2\sqrt{y} = c_1$$

Verified OK.

5.41.2 Maple step by step solution

Let's solve

$\frac{2x}{y}-\frac{3y^2}{x^4}+\left(-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}\right)y'=0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  ◦ Evaluate derivatives

    $-\frac{2x}{y^2}-\frac{6y}{x^4}=-\frac{2x}{y^2}-\frac{6y}{x^4}$

  ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\left(\frac{2x}{y}-\frac{3y^2}{x^4}\right)dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{x^2}{y}+\frac{y^2}{x^3}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $-\frac{x^2}{y^2}+\frac{1}{\sqrt{y}}+\frac{2y}{x^3}=-\frac{x^2}{y^2}+\frac{2y}{x^3}+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=\frac{1}{\sqrt{y}}$

- Solve for $f_1(y)$

  $f_1(y)=2\sqrt{y}$

- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \frac{x^2}{y} + \frac{y^2}{x^3} + 2\sqrt{y}$

- Substitute $F(x, y)$ into the solution of the ODE

  $\frac{x^2}{y} + \frac{y^2}{x^3} + 2\sqrt{y} = c_1$

- Solve for y

  $y = \mathit{RootOf}\left(-c_1x^3\_Z^2 + \_Z^6 + 2x^3\_Z^3 + x^5\right)^2$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 35

```
dsolve(2*x/y(x)-3*y(x)^2/x^4+(-x^2/y(x)^2+1/y(x)^(1/2)+2*y(x)/x^3)*diff(y(x),x) = 0,y(x), si
```

$$\frac{2y(x)^{\frac{3}{2}}x^3 + c_1x^3y(x) + x^5 + y(x)^3}{x^3y(x)} = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[2*x/y[x]-3*y[x]^2/x^4+(-x^2/y[x]^2+1/y[x]^(1/2)+2*y[x]/x^3)*y'[x] == 0,y[x],x,Include
```

Not solved

5.42 problem 42

Internal problem ID [120]
Internal file name [`OUTPUT/120_Sunday_June_05_2022_01_35_35_AM_74389913/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 1.6, Substitution methods and exact equations. Page 74
Problem number: 42.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[[_1st_order, _with_linear_symmetries], _exact, _rational]`

$$\boxed{\frac{2x^{\frac{5}{2}}-3y^{\frac{5}{3}}}{2x^{\frac{5}{2}}y^{\frac{2}{3}}}+\frac{\left(-2x^{\frac{5}{2}}+3y^{\frac{5}{3}}\right)y'}{3x^{\frac{3}{2}}y^{\frac{5}{3}}}=0}$$

5.42.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{3y}{2x} \end{aligned}$$

Where $f(x) = \frac{3}{2x}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{3}{2x}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{3}{2x}\,dx \\
\ln(y) &= \frac{3\ln(x)}{2} + c_1 \\
y &= \mathrm{e}^{\frac{3\ln(x)}{2}+c_1} \\
&= x^{\frac{3}{2}} c_1
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{\frac{3}{2}} c_1 \tag{1}$$

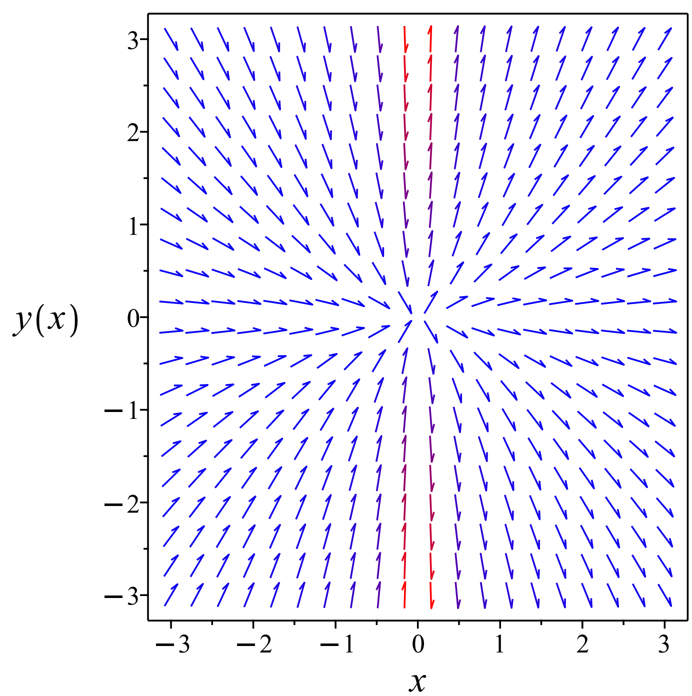


Figure 331: Slope field plot

Verification of solutions

$$y = x^{\frac{3}{2}} c_1$$

Verified OK.

5.42.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{2x}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{3y}{2x} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{2x}dx}$$
$$= \frac{1}{x^{\frac{3}{2}}}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(\frac{y}{x^{\frac{3}{2}}}\right) = 0$$

Integrating gives

$$\frac{y}{x^{\frac{3}{2}}} = c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^{\frac{3}{2}}}$ results in

$$y = x^{\frac{3}{2}}c_1$$

Summary
The solution(s) found are the following

$$y = x^{\frac{3}{2}}c_1 \tag{1}$$

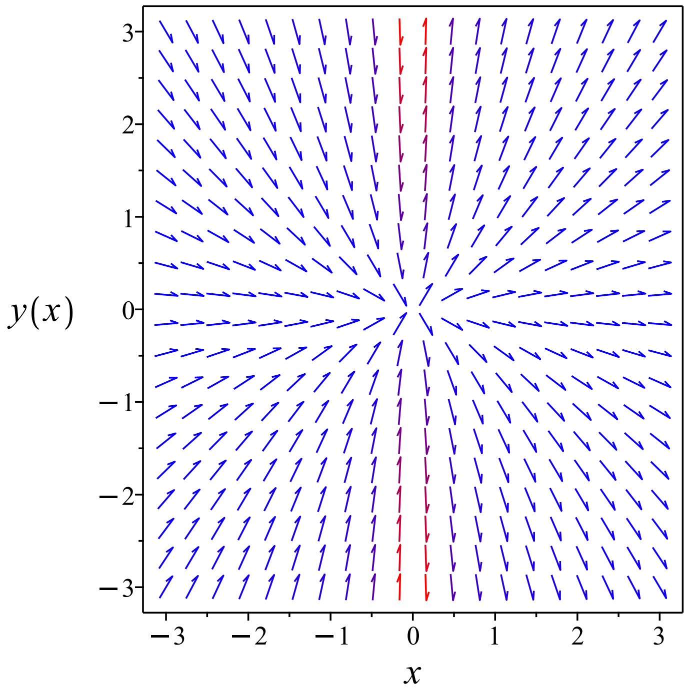


Figure 332: Slope field plot

Verification of solutions

$$y = x^{\frac{3}{2}} c_1$$

Verified OK.

5.42.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$\frac{2x^{\frac{5}{2}} - 3(u(x)\, x)^{\frac{5}{3}}}{2x^{\frac{5}{2}} \left(u\left(x\right) x\right)^{\frac{2}{3}}} + \frac{\left(-2x^{\frac{5}{2}} + 3(u(x)\, x)^{\frac{5}{3}}\right) \left(u'(x)\, x + u(x)\right)}{3x^{\frac{3}{2}} \left(u\left(x\right) x\right)^{\frac{5}{3}}} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x) g(u) \\ &= \frac{u}{2x} \end{aligned}$$

Where $f(x) = \frac{1}{2x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= \frac{1}{2x}\,dx \\
\int \frac{1}{u}\,du &= \int \frac{1}{2x}\,dx \\
\ln(u) &= \frac{\ln(x)}{2} + c_2 \\
u &= \mathrm{e}^{\frac{\ln(x)}{2}+c_2} \\
&= c_2\sqrt{x}
\end{aligned}$$

Therefore the solution y is

$$\begin{aligned}
y &= ux \\
&= x^{\frac{3}{2}}c_2
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{\frac{3}{2}}c_2 \tag{1}$$

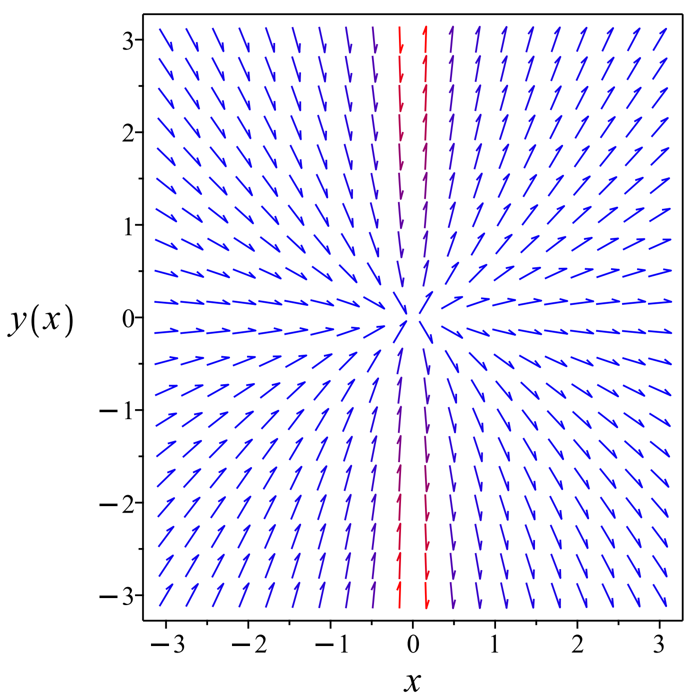


Figure 333: Slope field plot

Verification of solutions

$$y = x^{\frac{3}{2}} c_2$$

Verified OK.

5.42.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{3y}{2x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 240: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^{\frac{3}{2}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^{\frac{3}{2}}} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^{\frac{3}{2}}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{3y}{2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y}{2x^{\frac{5}{2}}} \\ S_y &= \frac{1}{x^{\frac{3}{2}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^{\frac{3}{2}}} = c_1$$

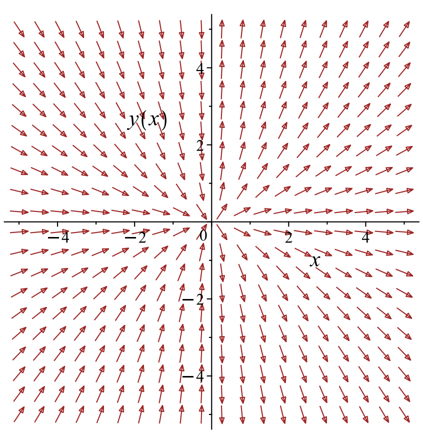
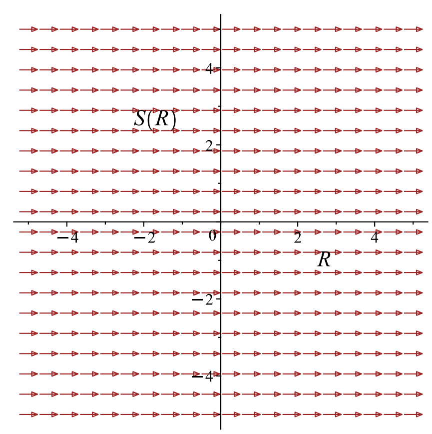
Which simplifies to

$$\frac{y}{x^{\frac{3}{2}}} = c_1$$

Which gives

$$y = x^{\frac{3}{2}} c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{3y}{2x}$ 	$R = x$ $S = \frac{y}{x^{\frac{3}{2}}}$	$\frac{dS}{dR} = 0$ 

Summary

The solution(s) found are the following

$$y = x^{\frac{3}{2}} c_1 \tag{1}$$

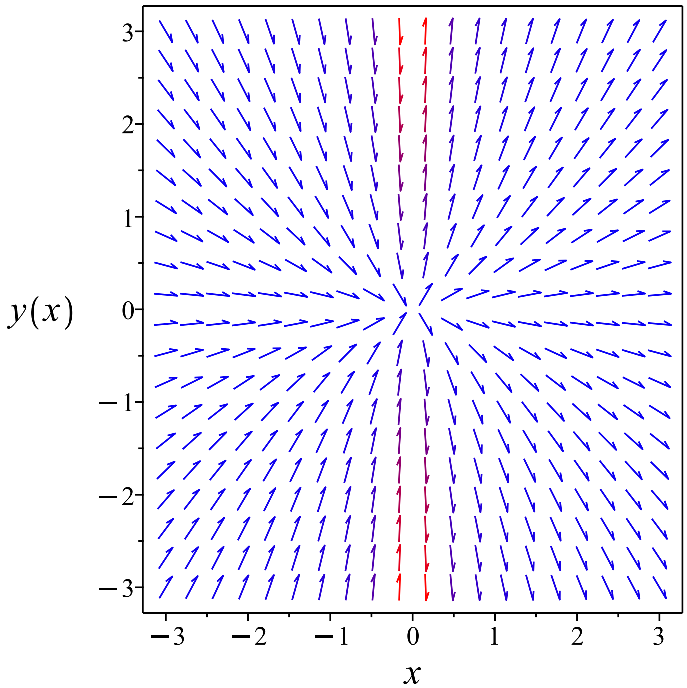


Figure 334: Slope field plot

Verification of solutions

$$y = x^{\frac{3}{2}} c_1$$

Verified OK.

5.42.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{2}{3y}\right)\mathrm{d}y &= \left(\frac{1}{x}\right)\mathrm{d}x\\ \left(-\frac{1}{x}\right)\mathrm{d}x + \left(\frac{2}{3y}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{1}{x}\\ N(x, y) &= \frac{2}{3y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{1}{x}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{2}{3y}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x\\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{1}{x}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{2}{3y}$. Therefore equation (4) becomes

$$\frac{2}{3y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{2}{3y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}\int f'(y)\,\mathrm{d}y &= \int \left(\frac{2}{3y}\right)\mathrm{d}y\\ f(y) &= \frac{2\ln(y)}{3} + c_1\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{2\ln(y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{2\ln(y)}{3}$$

The solution becomes

$$y = e^{\frac{3\ln(x)}{2} + \frac{3c_1}{2}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{3\ln(x)}{2} + \frac{3c_1}{2}} \tag{1}$$

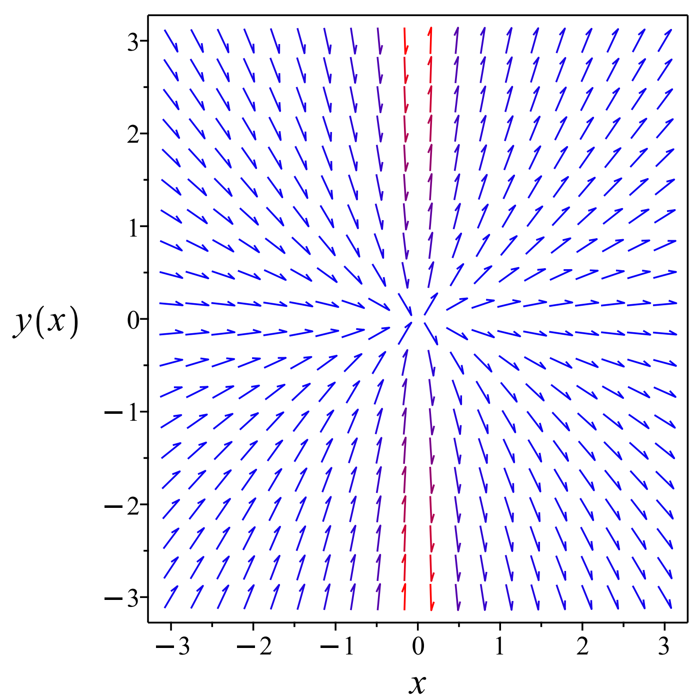


Figure 335: Slope field plot

Verification of solutions

$$y = e^{\frac{3\ln(x)}{2}+\frac{3c_1}{2}}$$

Verified OK.

5.42.6 Maple step by step solution

Let's solve

$$\frac{2x^{\frac{5}{2}}-3y^{\frac{5}{3}}}{2x^{\frac{5}{2}}y^{\frac{2}{3}}}+\frac{\left(-2x^{\frac{5}{2}}+3y^{\frac{5}{3}}\right)y'}{3x^{\frac{3}{2}}y^{\frac{5}{3}}}=0$$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact
  - ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x,y)=0$

  - ◦ Compute derivative of lhs

    $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

  - ◦ Evaluate derivatives

    $-\frac{5}{2x^{\frac{5}{2}}}-\frac{2x^{\frac{5}{2}}-3y^{\frac{5}{3}}}{3x^{\frac{5}{2}}y^{\frac{5}{3}}}=-\frac{5}{3y^{\frac{5}{3}}}-\frac{-2x^{\frac{5}{2}}+3y^{\frac{5}{3}}}{2x^{\frac{5}{2}}y^{\frac{5}{3}}}$

  - ◦ Simplify

    $\frac{-9y^{\frac{5}{3}}-4x^{\frac{5}{2}}}{6x^{\frac{5}{2}}y^{\frac{5}{3}}}=\frac{-9y^{\frac{5}{3}}-4x^{\frac{5}{2}}}{6x^{\frac{5}{2}}y^{\frac{5}{3}}}$

  - ◦ Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int\frac{2x^{\frac{5}{2}}-3y^{\frac{5}{3}}}{2x^{\frac{5}{2}}y^{\frac{2}{3}}}dx+f_1(y)$

- Evaluate integral

  $F(x,y)=\frac{2x+\frac{2y^{\frac{5}{3}}}{x^{\frac{3}{2}}}}{2y^{\frac{2}{3}}}+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

$$N(x,y)=\frac{\partial}{\partial y}F(x,y)$$

- Compute derivative

$$\frac{-2x^{\frac{5}{2}}+3y^{\frac{5}{3}}}{3x^{\frac{3}{2}}y^{\frac{5}{3}}}=-\frac{2x+\frac{2y^{\frac{5}{3}}}{x^{\frac{3}{2}}}}{3y^{\frac{5}{3}}}+\frac{5}{3x^{\frac{3}{2}}}+\frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y)=\frac{-2x^{\frac{5}{2}}+3y^{\frac{5}{3}}}{3x^{\frac{3}{2}}y^{\frac{5}{3}}}+\frac{2x+\frac{2y^{\frac{5}{3}}}{x^{\frac{3}{2}}}}{3y^{\frac{5}{3}}}-\frac{5}{3x^{\frac{3}{2}}}$$

- Solve for $f_1(y)$

$$f_1(y)=0$$

- Substitute $f_1(y)$ into equation for $F(x,y)$

$$F(x,y)=\frac{2x+\frac{2y^{\frac{5}{3}}}{x^{\frac{3}{2}}}}{2y^{\frac{2}{3}}}$$

- Substitute $F(x,y)$ into the solution of the ODE

$$\frac{2x+\frac{2y^{\frac{5}{3}}}{x^{\frac{3}{2}}}}{2y^{\frac{2}{3}}}=c_1$$

- Solve for y

$$y=RootOf\left(-c_1\_Z^2x^{\frac{3}{2}}+\_Z^5+x^{\frac{5}{2}}\right)^3$$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 185

```
dsolve(1/2*(2*x^(5/2)-3*y(x)^(5/3))/x^(5/2)/y(x)^(2/3)+1/3*(-2*x^(5/2)+3*y(x)^(5/3))*diff(y(
```

$$y(x) = \frac{2^{\frac{3}{5}} 3^{\frac{2}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}}}{3}$$

$$y(x) = -\frac{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}+\sqrt{5}+1\right)^{3} 2^{\frac{3}{5}} 3^{\frac{2}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}}}{192}$$

$$y(x) = \frac{\left(i\sqrt{2}\,\sqrt{5-\sqrt{5}}-\sqrt{5}-1\right)^{3} 2^{\frac{3}{5}} 3^{\frac{2}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}}}{192}$$

$$y(x) = -\frac{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}-\sqrt{5}+1\right)^{3} 2^{\frac{3}{5}} 3^{\frac{2}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}}}{192}$$

$$y(x) = \frac{\left(i\sqrt{2}\,\sqrt{5+\sqrt{5}}+\sqrt{5}-1\right)^{3} 2^{\frac{3}{5}} 3^{\frac{2}{5}} \left(x^{\frac{5}{2}}\right)^{\frac{3}{5}}}{192}$$

$$\frac{x}{y\left(x\right)^{\frac{2}{3}}}+\frac{y(x)}{x^{\frac{3}{2}}}+c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 260

```
DSolve[1/2*(2*x^(5/2)-3*y[x]^(5/3))/x^(5/2)/y[x]^(2/3)+1/3*(-2*x^(5/2)+3*y[x]^(5/3))*y'[x]/x
```

$$y(x) \to \left(\frac{2}{3}\right)^{3/5} \left(x^{5/2}\right)^{3/5}$$
$$y(x) \to c_1 x^{3/2}$$
$$y(x) \to -\left(-\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to \left(-\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to -\left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to \left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to -\sqrt[5]{-1}\left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to \sqrt[5]{-1}\left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to -(-1)^{2/5} \left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to (-1)^{2/5} \left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to -(-1)^{4/5} \left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to (-1)^{4/5} \left(\frac{2}{3}\right)^{3/5} x^{3/2}$$
$$y(x) \to \left(\frac{2}{3}\right)^{3/5} \left(x^{5/2}\right)^{3/5}$$

6 Chapter 1 review problems. Page 78

6.1 problem 1

Internal problem ID [121]
Internal file name [`OUTPUT/121_Sunday_June_05_2022_01_35_36_AM_29210918/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 1.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{3y - y'x = -x^3}$$

6.1.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{3}{x}$$
$$q(x) = x^2$$

Hence the ode is

$$y' - \frac{3y}{x} = x^2$$

The integrating factor μ is

$$\mu = e^{\int -\frac{3}{x}dx}$$
$$= \frac{1}{x^3}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(x^2)$$
$$\frac{d}{dx}\left(\frac{y}{x^3}\right) = \left(\frac{1}{x^3}\right)(x^2)$$
$$d\left(\frac{y}{x^3}\right) = \frac{1}{x}\,dx$$

Integrating gives

$$\frac{y}{x^3} = \int \frac{1}{x}\,dx$$
$$\frac{y}{x^3} = \ln(x) + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$y = \ln(x)\,x^3 + c_1 x^3$$

which simplifies to

$$y = x^3(\ln(x) + c_1)$$

Summary

The solution(s) found are the following

$$y = x^3(\ln(x) + c_1) \tag{1}$$

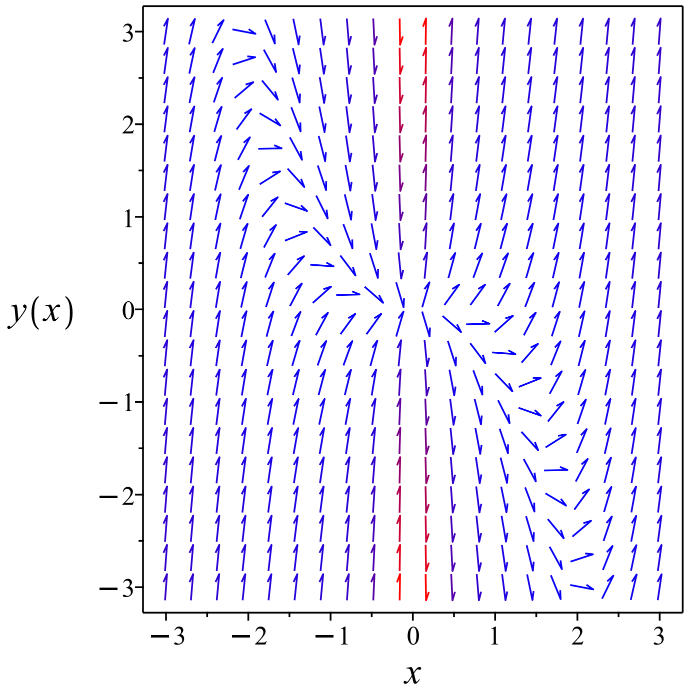


Figure 336: Slope field plot

Verification of solutions

$$y = x^3(\ln(x) + c_1)$$

Verified OK.

6.1.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{x^3 + 3y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 243: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^3
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^3} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{x^3 + 3y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{3y}{x^4} \\ S_y &= \frac{1}{x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^3} = \ln(x) + c_1$$

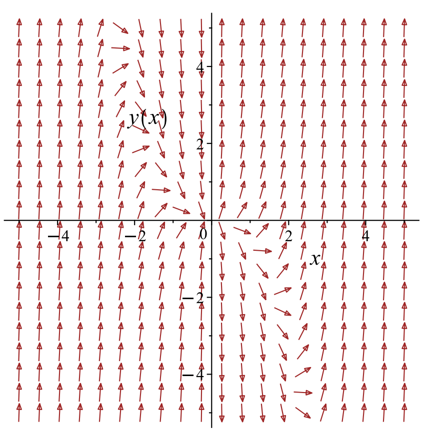
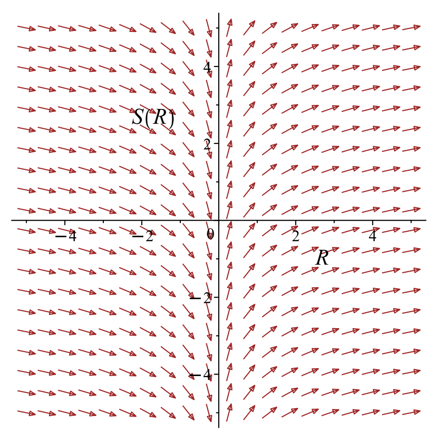
Which simplifies to

$$\frac{y}{x^3} = \ln(x) + c_1$$

Which gives

$$y = x^3(\ln(x) + c_1)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{x^3+3y}{x}$ 	$R = x$ $S = \frac{y}{x^3}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = x^3(\ln(x) + c_1) \tag{1}$$

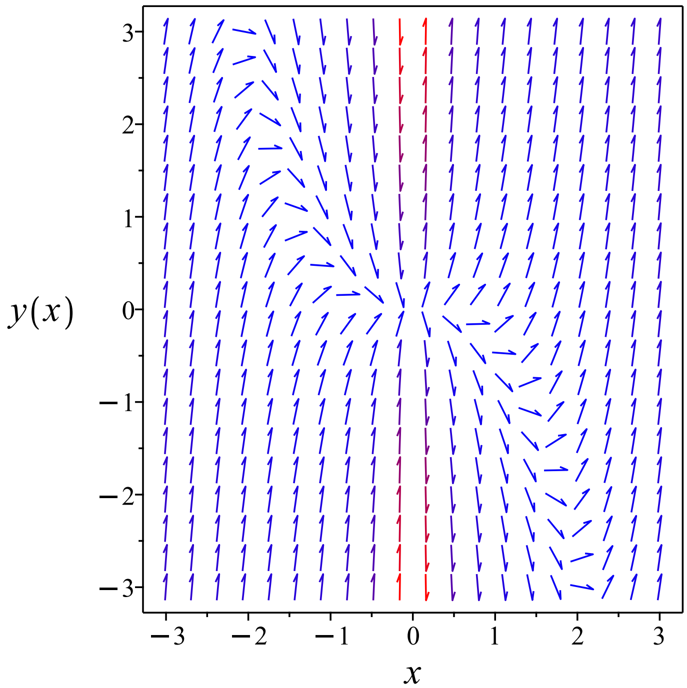


Figure 337: Slope field plot

Verification of solutions

$$y = x^3(\ln(x) + c_1)$$

Verified OK.

6.1.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned} (-x) \, \mathrm{d}y &= \left(-x^3 - 3y\right) \mathrm{d}x \\ \left(x^3 + 3y\right) \mathrm{d}x + (-x) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= x^3 + 3y \\ N(x, y) &= -x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^3 + 3y\right) \\ &= 3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-x) \\ &= -1 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= -\frac{1}{x}((3) - (-1)) \\
&= -\frac{4}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{4}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-4\ln(x)} \\
&= \frac{1}{x^4}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x^4}(x^3 + 3y) \\
&= \frac{x^3 + 3y}{x^4}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x^4}(-x) \\
&= -\frac{1}{x^3}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{x^3 + 3y}{x^4}\right) + \left(-\frac{1}{x^3}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \overline{M} \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{x^3 + 3y}{x^4} \, \mathrm{d}x$$

$$\phi = -\frac{y}{x^3} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{1}{x^3} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{x^3}$. Therefore equation (4) becomes

$$-\frac{1}{x^3} = -\frac{1}{x^3} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{y}{x^3} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{y}{x^3} + \ln(x)$$

The solution becomes

$$y = (\ln(x) - c_1)\, x^3$$

Summary

The solution(s) found are the following

$$y = (\ln(x) - c_1)\, x^3 \tag{1}$$

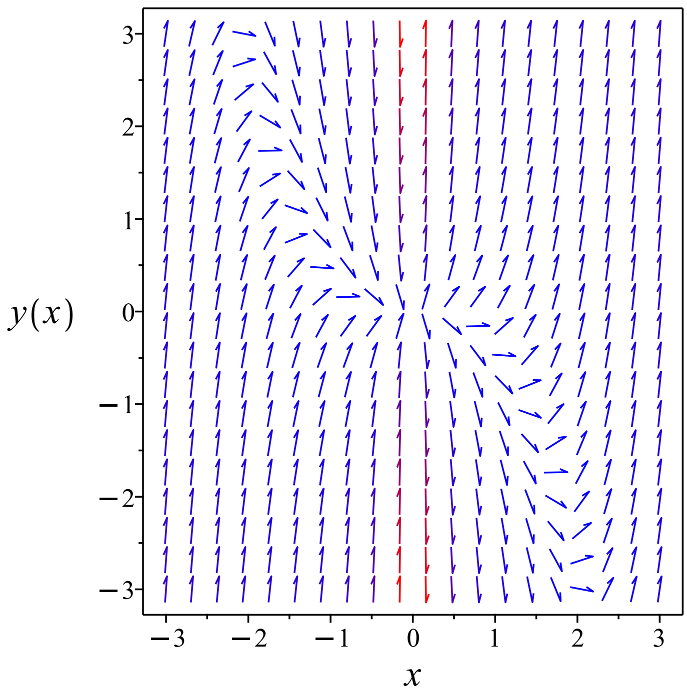


Figure 338: Slope field plot

Verification of solutions

$$y = (\ln(x) - c_1)\, x^3$$

Verified OK.

6.1.4 Maple step by step solution

Let's solve

$3y - y'x = -x^3$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = \frac{3y}{x} + x^2$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' - \frac{3y}{x} = x^2$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' - \frac{3y}{x}\right) = \mu(x)\, x^2$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' - \frac{3y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = -\frac{3\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = \frac{1}{x^3}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x)\, x^2 dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x)\, x^2 dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)x^2 dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \frac{1}{x^3}$

  $y = x^3\left(\int \frac{1}{x} dx + c_1\right)$

- Evaluate the integrals on the rhs

  $y = x^3(\ln(x) + c_1)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(x^3+3*y(x)-x*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = (\ln(x) + c_1)\, x^3$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 14

```
DSolve[x^3+3*y[x]-x*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^3(\log(x) + c_1)$$

6.2 problem 2

Internal problem ID [122]
Internal file name [`OUTPUT/122_Sunday_June_05_2022_01_35_36_AM_37178771/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 2.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3y^2 + xy^2 - y'x^2 = 0}$$

6.2.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y^2(x+3)}{x^2} \end{aligned}$$

Where $f(x) = \frac{x+3}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = \frac{x+3}{x^2}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int \frac{x+3}{x^2}\,dx$$

$$-\frac{1}{y} = \ln(x) - \frac{3}{x} + c_1$$

Which results in

$$y = -\frac{x}{-3 + x\ln(x) + c_1 x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{-3 + x\ln(x) + c_1 x} \tag{1}$$

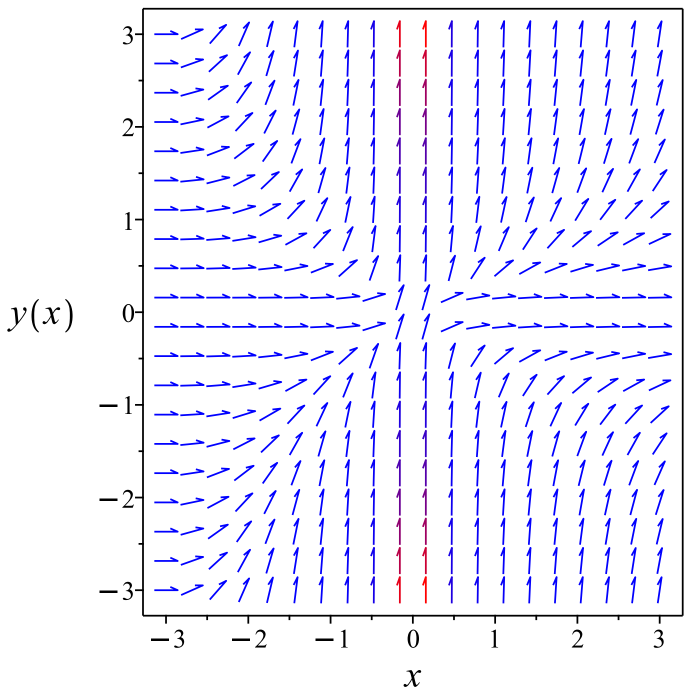


Figure 339: Slope field plot

Verification of solutions

$$y = -\frac{x}{-3 + x\ln(x) + c_1 x}$$

Verified OK.

6.2.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y^2(x+3)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 246: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{x^2}{x+3} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{x^2}{x+3}} dx
\end{aligned}
$$

Which results in

$$
S = \ln(x) - \frac{3}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = \frac{y^2(x+3)}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{x+3}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x \ln(x) - 3}{x} = -\frac{1}{y} + c_1
$$

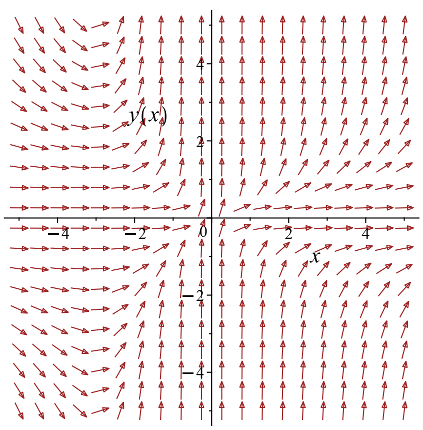
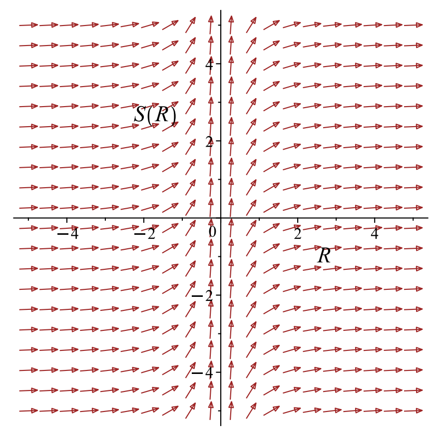
Which simplifies to

$$
\frac{x \ln(x) - 3}{x} = -\frac{1}{y} + c_1
$$

Which gives

$$
y = -\frac{x}{x \ln(x) - c_1 x - 3}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y^2(x+3)}{x^2}$ 	$R = y$ $S = \frac{x \ln(x) - 3}{x}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = -\frac{x}{x \ln(x) - c_1 x - 3} \tag{1}$$

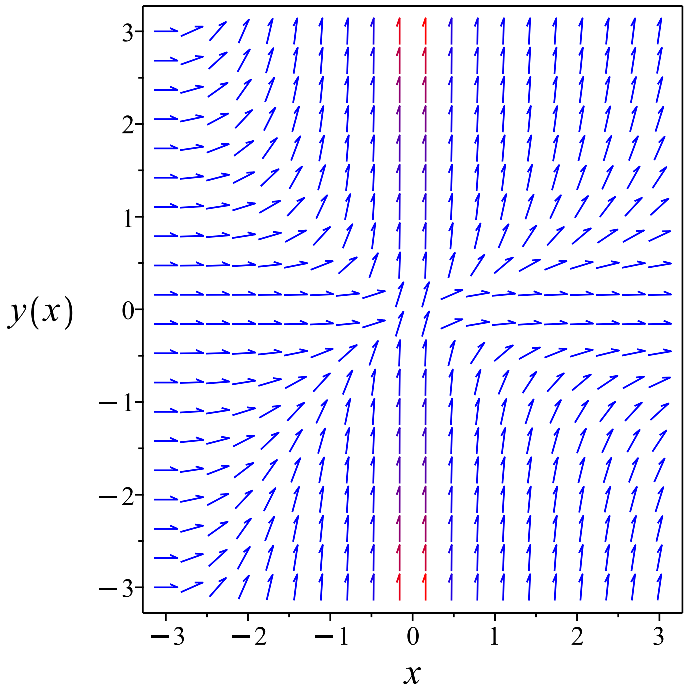


Figure 340: Slope field plot

Verification of solutions

$$y = -\frac{x}{x \ln(x) - c_1 x - 3}$$

Verified OK.

6.2.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2}\right) \mathrm{d}y = \left(\frac{x+3}{x^2}\right) \mathrm{d}x$$
$$\left(-\frac{x+3}{x^2}\right) \mathrm{d}x + \left(\frac{1}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x+3}{x^2}$$
$$N(x, y) = \frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{x+3}{x^2}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x+3}{x^2} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\ln(x) + \frac{3}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2}$. Therefore equation (4) becomes

$$
\frac{1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{y^2}\right)\mathrm{d}y$$

$$f(y) = -\frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(x) + \frac{3}{x} - \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(x) + \frac{3}{x} - \frac{1}{y}$$

The solution becomes

$$y = -\frac{x}{-3 + x\ln(x) + c_1 x}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{-3 + x\ln(x) + c_1 x} \tag{1}$$

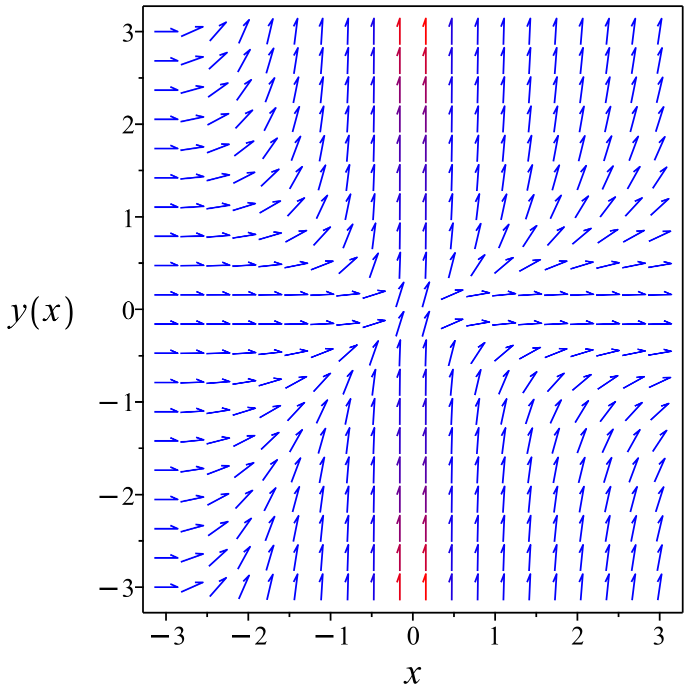


Figure 341: Slope field plot

Verification of solutions

$$y = -\frac{x}{-3 + x\ln(x) + c_1 x}$$

Verified OK.

6.2.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y^2(x+3)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y^2}{x} + \frac{3y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = \frac{x+3}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{\frac{(x+3)u}{x^2}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - \left(f_2' + f_1 f_2\right) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{2(x+3)}{x^3} + \frac{1}{x^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
\frac{(x+3)\, u''(x)}{x^2} - \left(-\frac{2(x+3)}{x^3} + \frac{1}{x^2}\right) u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \left(\ln(x) - \frac{3}{x}\right) c_2
$$

The above shows that

$$
u'(x) = \frac{(x+3)\, c_2}{x^2}
$$

Using the above in (1) gives the solution

$$
y = -\frac{c_2}{c_1 + \left(\ln(x) - \frac{3}{x}\right) c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = -\frac{x}{x \ln(x) + c_3 x - 3}
$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{x \ln(x) + c_3 x - 3} \tag{1}$$

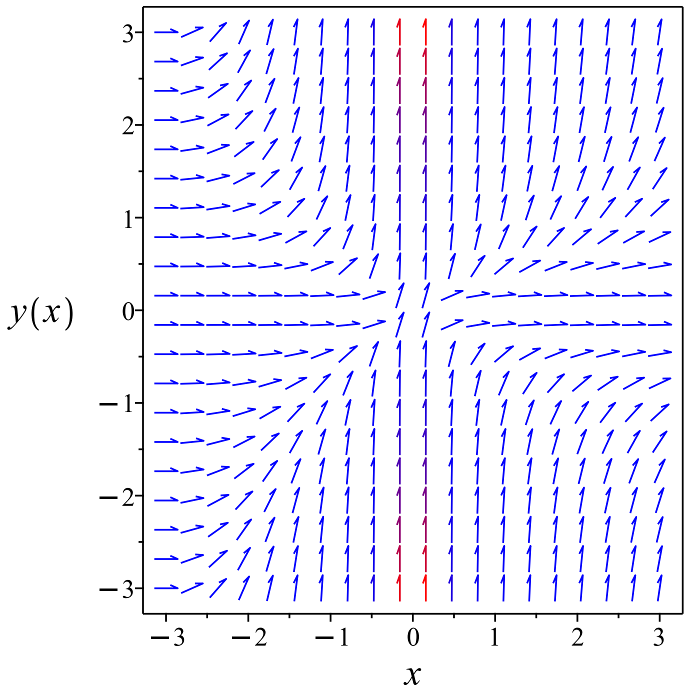


Figure 342: Slope field plot

Verification of solutions

$$y = -\frac{x}{x \ln(x) + c_3 x - 3}$$

Verified OK.

6.2.5 Maple step by step solution

Let's solve

$3y^2 + xy^2 - y'x^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = \frac{x+3}{x^2}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int \frac{x+3}{x^2} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = \ln(x) - \frac{3}{x} + c_1$

- Solve for y

  $y = -\frac{x}{-3 + x\ln(x) + c_1 x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(3*y(x)^2+x*y(x)^2-x^2*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{x}{3 - x\ln(x) + c_1 x}$$

✓ Solution by Mathematica

Time used: 0.13 (sec). Leaf size: 25

```
DSolve[3*y[x]^2+x*y[x]^2-x^2*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{x\log(x) + c_1 x - 3}$$
$$y(x) \to 0$$

6.3 problem 3

Internal problem ID [123]
Internal file name [`OUTPUT/123_Sunday_June_05_2022_01_35_37_AM_12516135/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{yx + y^2 - y'x^2 = 0}$$

6.3.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u(x)\,x^2 + u(x)^2\,x^2 - (u'(x)\,x + u(x))\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u^2}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\,du = \frac{1}{x}\,dx$$
$$\int \frac{1}{u^2}\,du = \int \frac{1}{x}\,dx$$
$$-\frac{1}{u} = \ln(x) + c_2$$

The solution is

$$-\frac{1}{u(x)} - \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} - \ln(x) - c_2 = 0$$
$$-\frac{x}{y} - \ln(x) - c_2 = 0$$

Summary

The solution(s) found are the following

$$-\frac{x}{y} - \ln(x) - c_2 = 0 \tag{1}$$

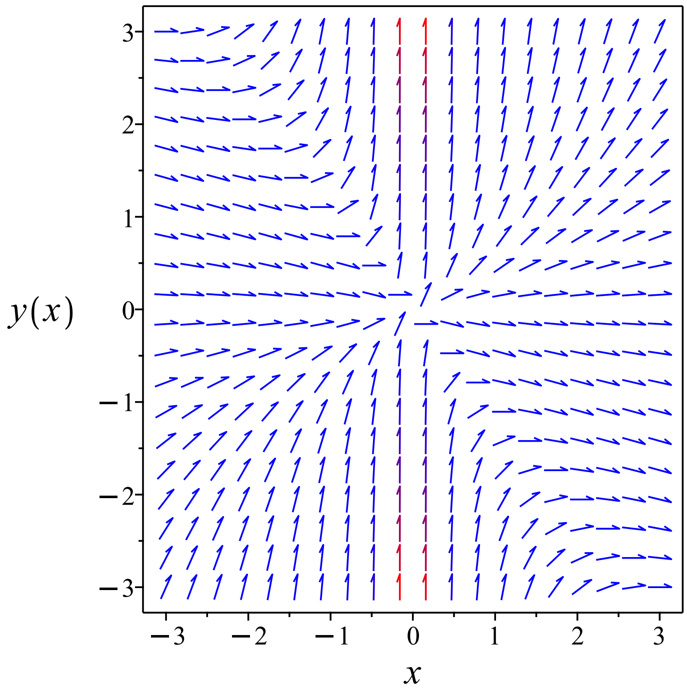


Figure 343: Slope field plot

Verification of solutions

$$-\frac{x}{y} - \ln(x) - c_2 = 0$$

Verified OK.

6.3.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x+y)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 249: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int b f(x) dx - h(x)}}{g(x)}$	$\frac{f(x) e^{-\int b f(x) dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^2}{x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^2}{x}} dy \end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(x + y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{1}{y} \\ S_y &= \frac{x}{y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = \ln(x) + c_1$$

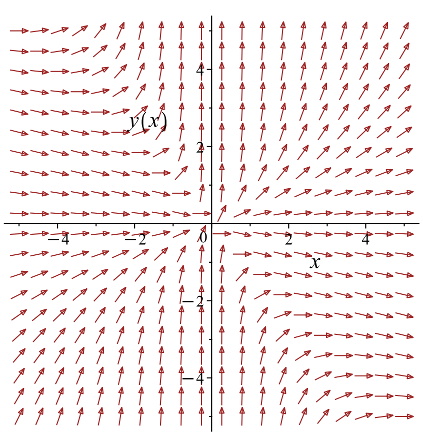
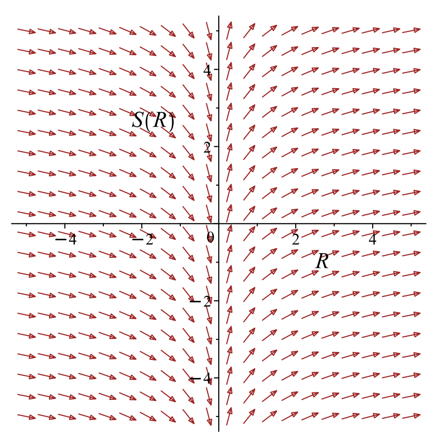
Which simplifies to

$$-\frac{x}{y} = \ln(x) + c_1$$

Which gives

$$y = -\frac{x}{\ln(x) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(x+y)}{x^2}$ 	$R = x$ $S = -\frac{x}{y}$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary
The solution(s) found are the following

$$y = -\frac{x}{\ln(x) + c_1} \tag{1}$$

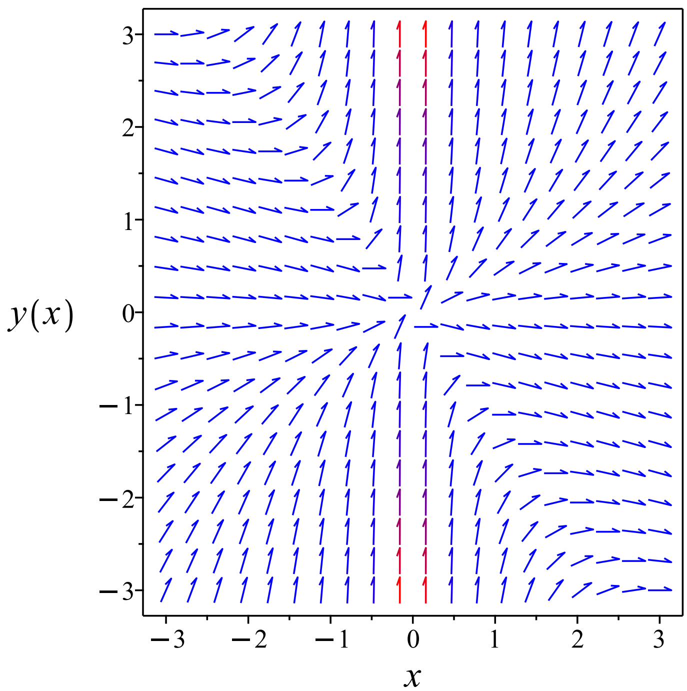


Figure 344: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) + c_1}$$

Verified OK.

6.3.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= \frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} + \frac{1}{x^2} \\ w' &= -\frac{w}{x} - \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -\frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int \frac{1}{x} dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu w) &= (\mu)\left(-\frac{1}{x^2}\right) \\ \frac{d}{dx}(wx) &= (x)\left(-\frac{1}{x^2}\right) \\ d(wx) &= \left(-\frac{1}{x}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} wx &= \int -\frac{1}{x} dx \\ wx &= -\ln(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{-\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{-\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{-\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{-\ln(x) + c_1} \tag{1}$$

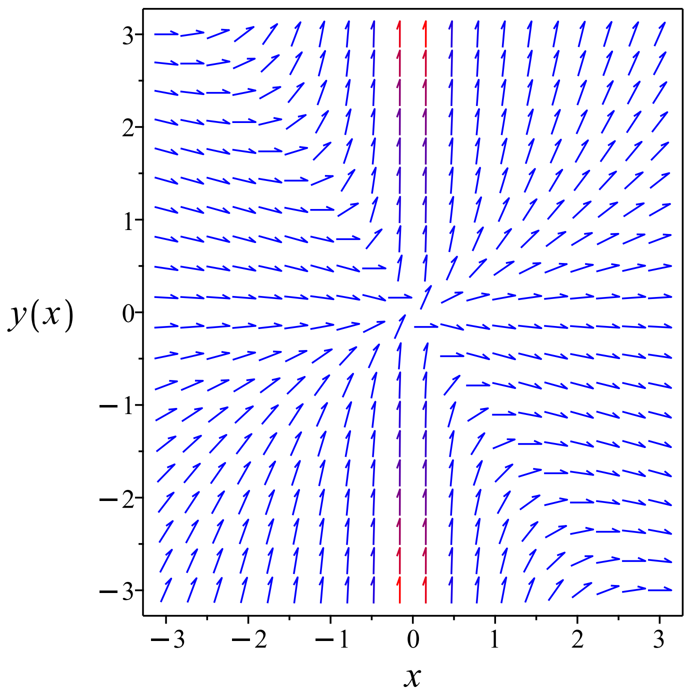


Figure 345: Slope field plot

Verification of solutions

$$y = \frac{x}{-\ln(x) + c_1}$$

Verified OK.

6.3.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-x^2)\,\mathrm{d}y &= (-yx - y^2)\,\mathrm{d}x\\ (yx + y^2)\,\mathrm{d}x + (-x^2)\,\mathrm{d}y &= 0 \qquad \text{(2A)}\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= yx + y^2\\ N(x,y) &= -x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(yx + y^2\right)\\ &= x + 2y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-x^2\right)\\ &= -2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{x y^2}$ is an integrating factor. Therefore by multiplying $M = yx + y^2$ and $N = -x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{yx + y^2}{x y^2}\\ N &= -\frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{x}{y^2}\right)\mathrm{d}y &= \left(-\frac{yx + y^2}{x\,y^2}\right)\mathrm{d}x\\ \left(\frac{yx + y^2}{x\,y^2}\right)\mathrm{d}x + \left(-\frac{x}{y^2}\right)\mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \frac{yx + y^2}{x\,y^2}\\ N(x, y) &= -\frac{x}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{yx + y^2}{x\,y^2}\right)\\ &= -\frac{1}{y^2}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{x}{y^2}\right)\\ &= -\frac{1}{y^2}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int \frac{yx + y^2}{x\, y^2} \mathrm{d}x$$
$$\phi = \ln (x) + \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -\frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{x}{y^2}$. Therefore equation (4) becomes

$$-\frac{x}{y^2} = -\frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \ln (x) + \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \ln(x) + \frac{x}{y}$$

The solution becomes

$$y = -\frac{x}{\ln(x) - c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\ln(x) - c_1} \tag{1}$$

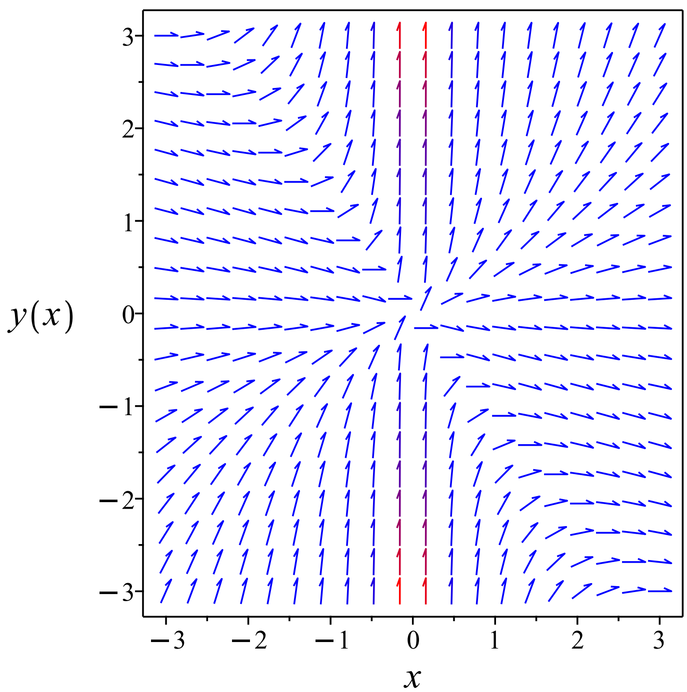


Figure 346: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) - c_1}$$

Verified OK.

6.3.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(x+y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= \frac{1}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x) + c_1$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 x}{c_2 \ln(x) + c_1}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x}{\ln(x) + c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{\ln(x) + c_3} \tag{1}$$

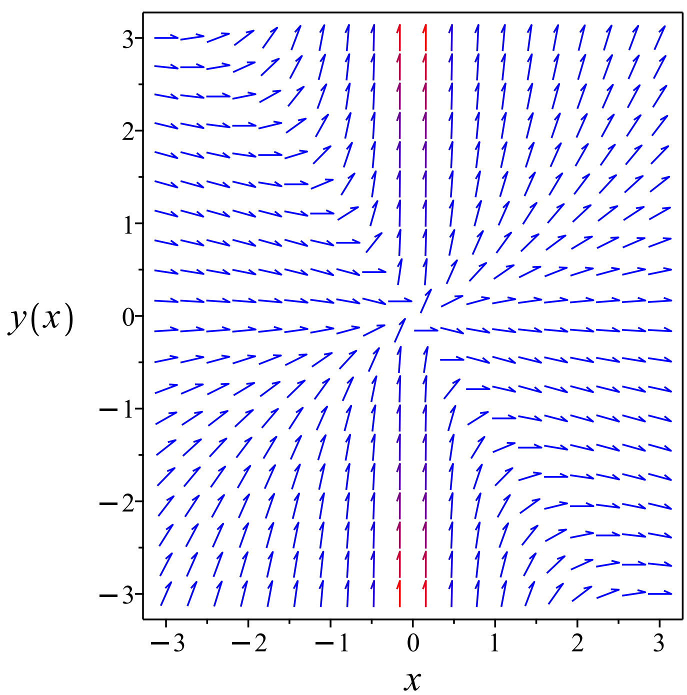


Figure 347: Slope field plot

Verification of solutions

$$y = -\frac{x}{\ln(x) + c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x*y(x)+y(x)^2-x^2*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{x}{c_1 - \ln(x)}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 21

```
DSolve[x*y[x]+y[x]^2-x^2*y'[x]== 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{-\log(x) + c_1}$$
$$y(x) \to 0$$

6.4 problem 4

Internal problem ID [124]
Internal file name [OUTPUT/124_Sunday_June_05_2022_01_35_38_AM_97682275/index.tex]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{2xy^3 + \left(\sin\left(y\right) + 3x^2y^2\right)y' = -\mathrm{e}^x}$$

6.4.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\sin(y) + 3x^2y^2\right)\mathrm{d}y &= \left(-\mathrm{e}^x - 2x\,y^3\right)\mathrm{d}x\\ \left(2x\,y^3 + \mathrm{e}^x\right)\mathrm{d}x + \left(\sin(y) + 3x^2y^2\right)\mathrm{d}y &= 0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 2x\,y^3 + \mathrm{e}^x\\ N(x,y) &= \sin(y) + 3x^2y^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(2x\,y^3 + \mathrm{e}^x\right)\\ &= 6x\,y^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\sin(y) + 3x^2y^2\right)\\ &= 6x\,y^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int 2x \, y^3 + \mathrm{e}^x \, \mathrm{d}x$$

$$\phi = y^3 x^2 + \mathrm{e}^x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3x^2 y^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \sin(y) + 3x^2 y^2$. Therefore equation (4) becomes

$$\sin(y) + 3x^2 y^2 = 3x^2 y^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \sin(y)$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\sin(y)) \, \mathrm{d}y$$

$$f(y) = -\cos(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = y^3 x^2 + \mathrm{e}^x - \cos(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^3 x^2 + \mathrm{e}^x - \cos(y)$$

Summary

The solution(s) found are the following

$$y^3x^2 + e^x - \cos(y) = c_1 \tag{1}$$

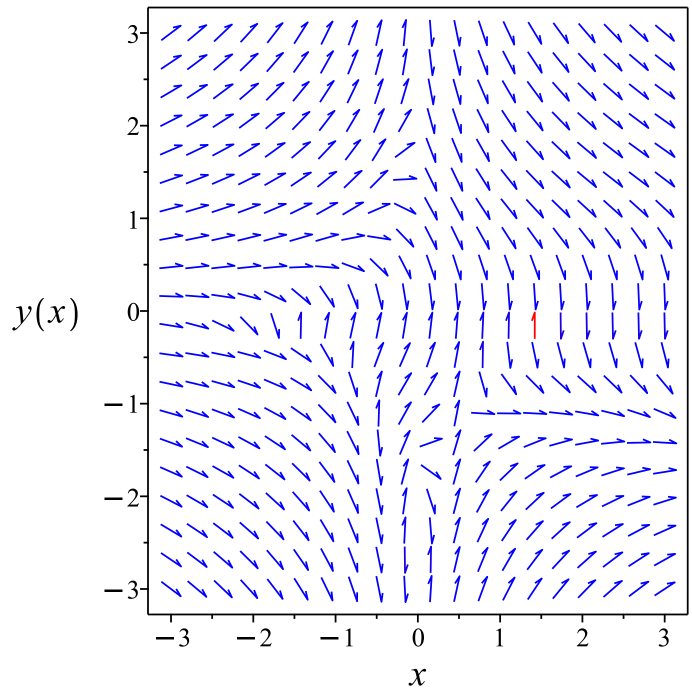


Figure 348: Slope field plot

Verification of solutions

$$y^3x^2 + e^x - \cos(y) = c_1$$

Verified OK.

6.4.2 Maple step by step solution

Let's solve

$2xy^3 + (\sin(y) + 3x^2y^2)\, y' = -e^x$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  - ODE is exact if the lhs is the total derivative of a C^2 function

$$F'(x, y) = 0$$

- Compute derivative of lhs

$$F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$$

- Evaluate derivatives

$$6x\,y^2 = 6x\,y^2$$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

$$\left[F(x, y) = c_1, M(x, y) = F'(x, y)\,, N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$$

- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

$$F(x, y) = \int (2x\,y^3 + \mathrm{e}^x)\,dx + f_1(y)$$

- Evaluate integral

$$F(x, y) = y^3x^2 + \mathrm{e}^x + f_1(y)$$

- Take derivative of $F(x, y)$ with respect to y

$$N(x, y) = \frac{\partial}{\partial y}F(x, y)$$

- Compute derivative

$$\sin(y) + 3x^2y^2 = 3x^2y^2 + \frac{d}{dy}f_1(y)$$

- Isolate for $\frac{d}{dy}f_1(y)$

$$\frac{d}{dy}f_1(y) = \sin(y)$$

- Solve for $f_1(y)$

$$f_1(y) = -\cos(y)$$

- Substitute $f_1(y)$ into equation for $F(x, y)$

$$F(x, y) = y^3x^2 + \mathrm{e}^x - \cos(y)$$

- Substitute $F(x, y)$ into the solution of the ODE

$$y^3x^2 + \mathrm{e}^x - \cos(y) = c_1$$

- Solve for y

$$y = \mathit{RootOf}\left(-x^2\_Z^3 + c_1 + \cos(\_Z) - \mathrm{e}^x\right)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 20

```
dsolve(exp(x)+2*x*y(x)^3+(sin(y(x))+3*x^2*y(x)^2)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$x^2y(x)^3 + e^x - \cos(y(x)) + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.255 (sec). Leaf size: 23

```
DSolve[Exp[x]+2*x*y[x]^3+(Sin[y[x]]+3*x^2*y[x]^2)*y'[x]== 0,y[x],x,IncludeSingularSolutions
```

$$\text{Solve}\left[x^2y(x)^3 - \cos(y(x)) + e^x = c_1, y(x)\right]$$

6.5 problem 5

Internal problem ID [125]
Internal file name [`OUTPUT/125_Sunday_June_05_2022_01_35_40_AM_3110566/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3y + x^4 y' - 2yx = 0}$$

6.5.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{y(-3+2x)}{x^4} \end{aligned}$$

Where $f(x) = \frac{-3+2x}{x^4}$ and $g(y) = y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y}\,dy &= \frac{-3+2x}{x^4}\,dx \\
\int \frac{1}{y}\,dy &= \int \frac{-3+2x}{x^4}\,dx \\
\ln(y) &= \frac{1}{x^3} - \frac{1}{x^2} + c_1 \\
y &= \mathrm{e}^{\frac{1}{x^3}-\frac{1}{x^2}+c_1} \\
&= c_1 \mathrm{e}^{\frac{1}{x^3}-\frac{1}{x^2}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{1}{x^3}-\frac{1}{x^2}} \tag{1}$$

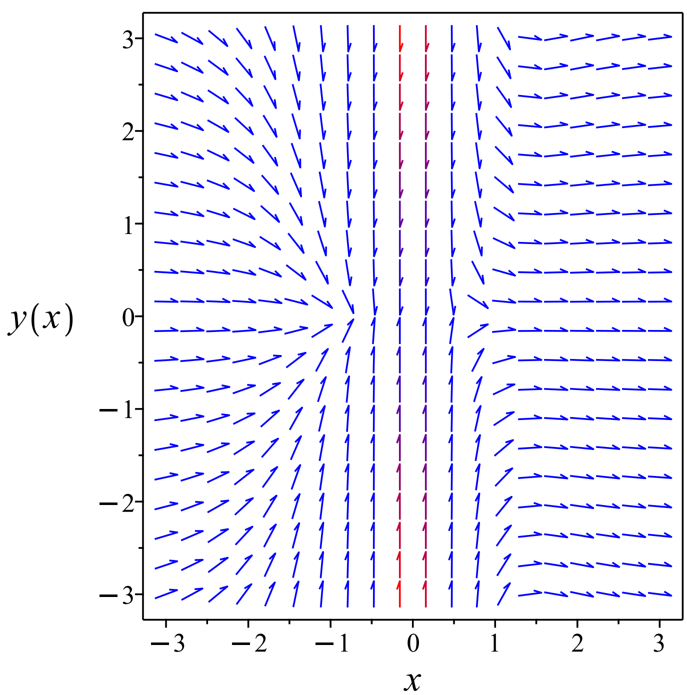


Figure 349: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{\frac{1}{x^3}-\frac{1}{x^2}}$$

Verified OK.

6.5.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{-3+2x}{x^4}$$
$$q(x) = 0$$

Hence the ode is

$$y' - \frac{y(-3+2x)}{x^4} = 0$$

The integrating factor μ is

$$\mu = e^{\int -\frac{-3+2x}{x^4}dx}$$
$$= e^{\frac{1}{x^2}-\frac{1}{x^3}}$$

The ode becomes

$$\frac{d}{dx}\mu y = 0$$
$$\frac{d}{dx}\left(e^{\frac{1}{x^2}-\frac{1}{x^3}}y\right) = 0$$

Integrating gives

$$e^{\frac{1}{x^2}-\frac{1}{x^3}}y = c_1$$

Dividing both sides by the integrating factor $\mu = e^{\frac{1}{x^2}-\frac{1}{x^3}}$ results in

$$y = c_1 e^{\frac{1-x}{x^3}}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{\frac{1-x}{x^3}} \tag{1}$$

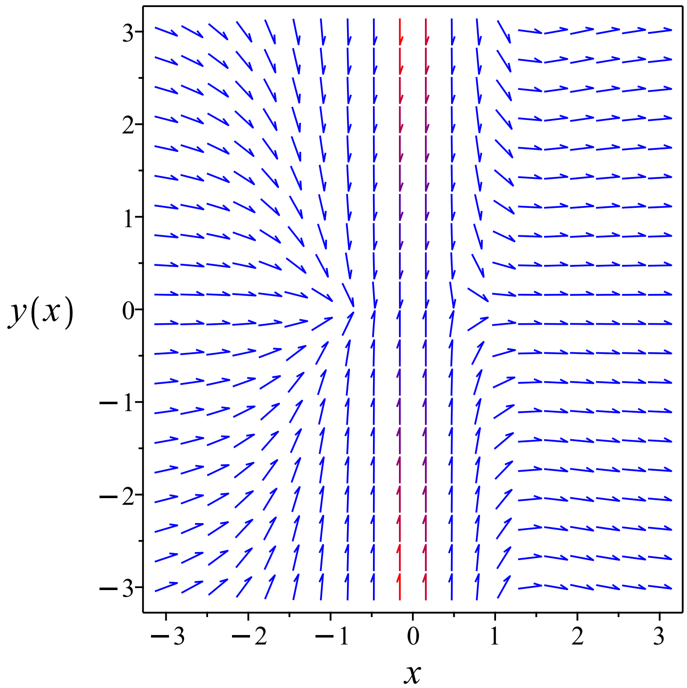


Figure 350: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{1-x}{x^3}}$$

Verified OK.

6.5.3 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$3u(x)\,x + x^4(u'(x)\,x + u(x)) - 2u(x)\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u(x^3 - 2x + 3)}{x^4} \end{aligned}$$

Where $f(x) = -\frac{x^3-2x+3}{x^4}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\,du &= -\frac{x^3-2x+3}{x^4}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{x^3-2x+3}{x^4}\,dx \\
\ln(u) &= \frac{1}{x^3} - \ln(x) - \frac{1}{x^2} + c_2 \\
u &= \mathrm{e}^{\frac{1}{x^3}-\ln(x)-\frac{1}{x^2}+c_2} \\
&= c_2 \mathrm{e}^{\frac{1}{x^3}-\ln(x)-\frac{1}{x^2}}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_2 \mathrm{e}^{\frac{1}{x^3}} \mathrm{e}^{-\frac{1}{x^2}}}{x}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= ux \\
&= c_2 \mathrm{e}^{\frac{1}{x^3}} \mathrm{e}^{-\frac{1}{x^2}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \mathrm{e}^{\frac{1}{x^3}} \mathrm{e}^{-\frac{1}{x^2}} \tag{1}
$$

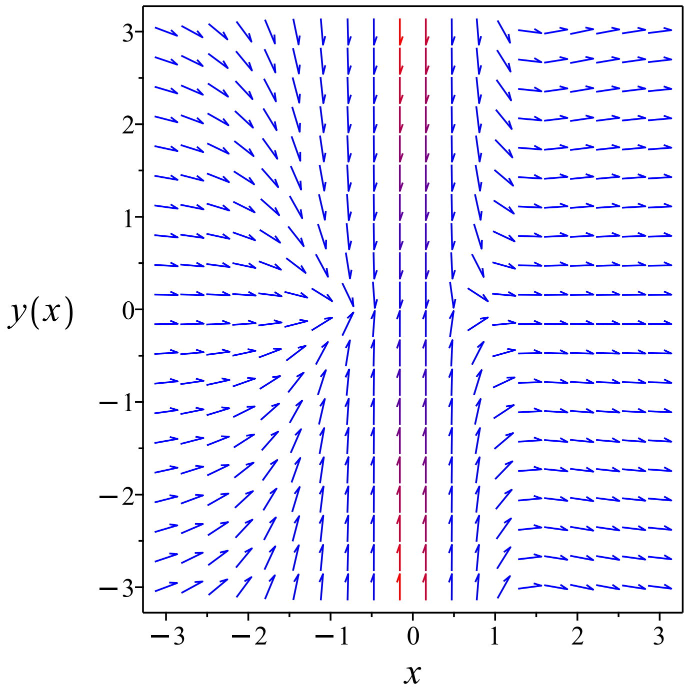


Figure 351: Slope field plot

Verification of solutions

$$y = c_2 e^{\frac{1}{x^3}} e^{-\frac{1}{x^2}}$$

Verified OK.

6.5.4 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(-3 + 2x)}{x^4}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 252: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= e^{\frac{1}{x^3}-\frac{1}{x^2}}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{\frac{1}{x^3} - \frac{1}{x^2}}} dy \end{aligned}$$

Which results in

$$S = \mathrm{e}^{\frac{1}{x^2} - \frac{1}{x^3}} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(-3 + 2x)}{x^4}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y(-2x + 3)\,\mathrm{e}^{\frac{x-1}{x^3}}}{x^4} \\ S_y &= \mathrm{e}^{\frac{x-1}{x^3}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{\frac{x-1}{x^3}} = c_1$$

Which simplifies to

$$y\,\mathrm{e}^{\frac{x-1}{x^3}} = c_1$$

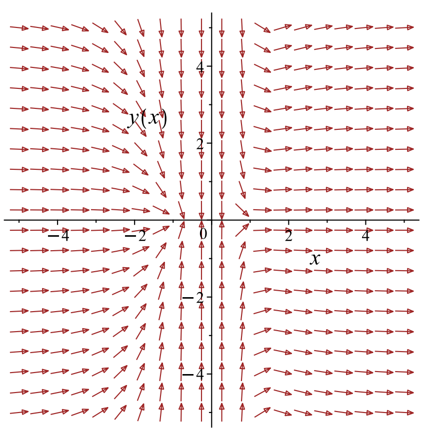
Which gives

$$y = c_1 \mathrm{e}^{-\frac{x-1}{x^3}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>

$\frac{dy}{dx} = \frac{y(-3+2x)}{x^4}$



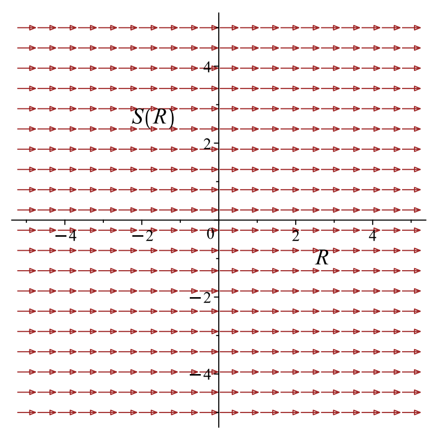
</td><td>

$R = x$

$S = y\,\mathrm{e}^{\frac{x-1}{x^3}}$

</td><td>

$\frac{dS}{dR} = 0$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x-1}{x^3}} \tag{1}$$

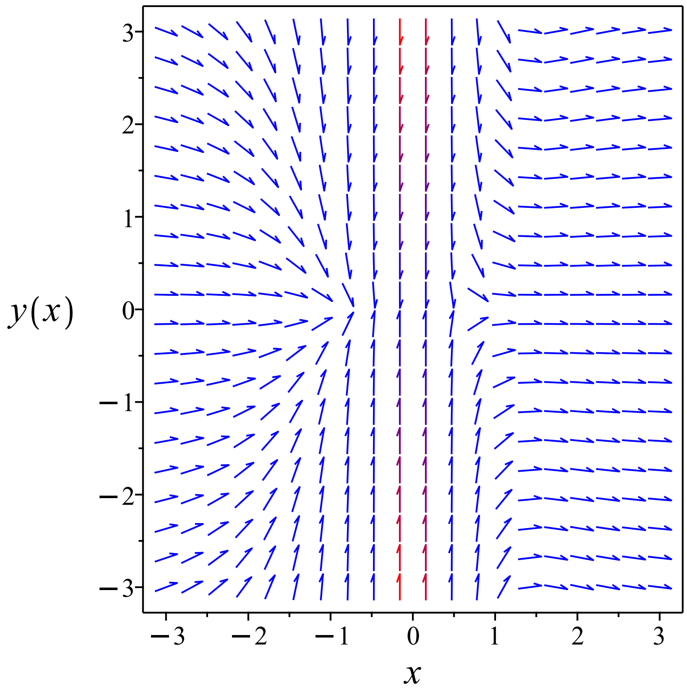


Figure 352: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x-1}{x^3}}$$

Verified OK.

6.5.5 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx} \phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y}\right) \mathrm{d}y = \left(\frac{-3 + 2x}{x^4}\right) \mathrm{d}x$$
$$\left(-\frac{-3 + 2x}{x^4}\right) \mathrm{d}x + \left(\frac{1}{y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{-3 + 2x}{x^4}$$
$$N(x, y) = \frac{1}{y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{-3 + 2x}{x^4}\right) \\ &= 0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int -\frac{-3 + 2x}{x^4} \mathrm{d}x$$
$$\phi = \frac{x - 1}{x^3} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y}$. Therefore equation (4) becomes

$$\frac{1}{y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \mathrm{d}y = \int \left(\frac{1}{y}\right) \mathrm{d}y$$
$$f(y) = \ln(y) + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{x-1}{x^3} + \ln(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{x-1}{x^3} + \ln(y)$$

The solution becomes

$$y = e^{\frac{c_1 x^3 - x + 1}{x^3}}$$

Summary

The solution(s) found are the following

$$y = e^{\frac{c_1 x^3 - x + 1}{x^3}} \tag{1}$$

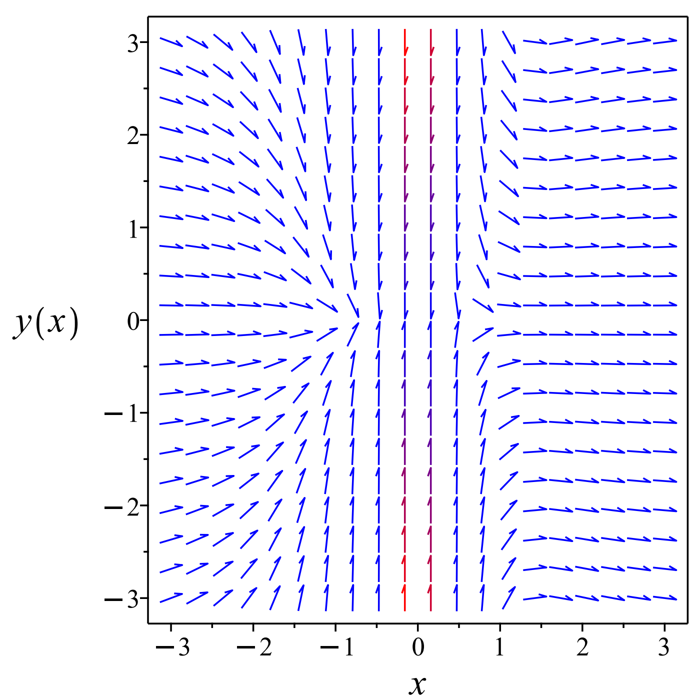


Figure 353: Slope field plot

Verification of solutions

$$y = e^{\frac{c_1 x^3 - x + 1}{x^3}}$$

Verified OK.

6.5.6 Maple step by step solution

Let's solve

$3y + x^4 y' - 2yx = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y} = \frac{-3+2x}{x^4}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y} dx = \int \frac{-3+2x}{x^4} dx + c_1$

- Evaluate integral

  $\ln(y) = \frac{1}{x^3} - \frac{1}{x^2} + c_1$

- Solve for y

  $y = e^{\frac{c_1 x^3 - x + 1}{x^3}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 16

```
dsolve(3*y(x)+x^4*diff(y(x),x) = 2*x*y(x),y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{1-x}{x^3}}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 24

```
DSolve[3*y[x]+x^4*y'[x] == 2*x*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{\frac{1-x}{x^3}}$$
$$y(x) \to 0$$

6.6 problem 6

Internal problem ID [126]
Internal file name [`OUTPUT/126_Sunday_June_05_2022_01_35_41_AM_56216560/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 6.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{2xy^2 + y'x^2 - y^2 = 0}$$

6.6.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2(2x-1)}{x^2} \end{aligned}$$

Where $f(x) = -\frac{2x-1}{x^2}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = -\frac{2x-1}{x^2}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -\frac{2x-1}{x^2}\,dx$$

$$-\frac{1}{y} = -2\ln(x) - \frac{1}{x} + c_1$$

Which results in

$$y = \frac{x}{2x\ln(x) - c_1 x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{2x\ln(x) - c_1 x + 1} \tag{1}$$

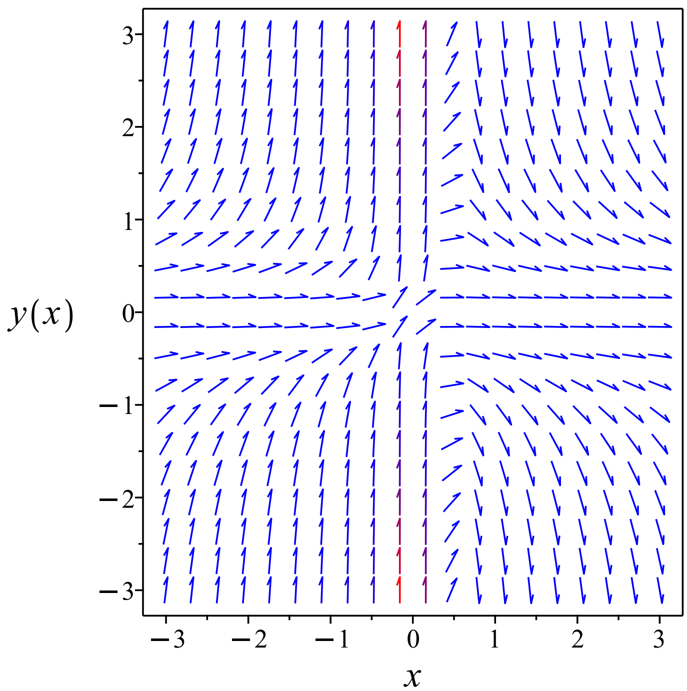


Figure 354: Slope field plot

Verification of solutions

$$y = \frac{x}{2x\ln(x) - c_1 x + 1}$$

Verified OK.

6.6.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\frac{y^2(2x-1)}{x^2} \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 255: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= -\frac{x^2}{2x-1} \\
\eta(x,y) &= 0
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{x^2}{2x-1}} dx
\end{aligned}
$$

Which results in

$$
S = -2\ln(x) - \frac{1}{x}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{y^2(2x-1)}{x^2}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-2x+1}{x^2} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-2x\ln(x) - 1}{x} = -\frac{1}{y} + c_1$$

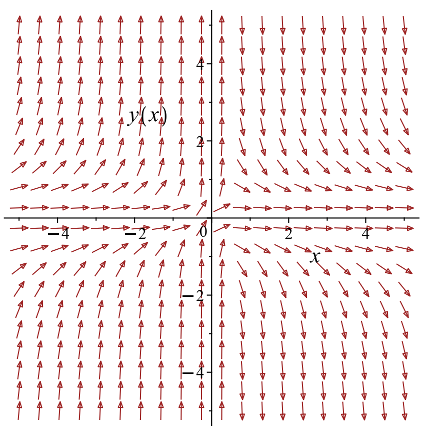
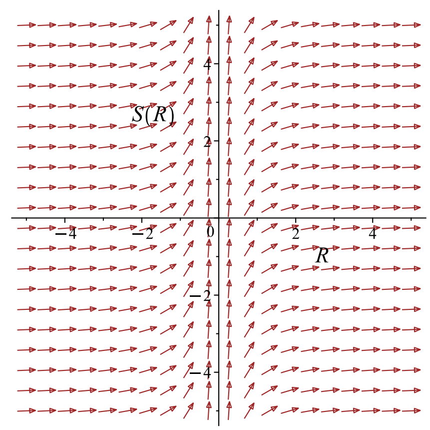
Which simplifies to

$$\frac{-2x\ln(x) - 1}{x} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{x}{1 + 2x\ln(x) + c_1 x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2(2x-1)}{x^2}$ 	$R = y$ $S = \frac{-2x\ln(x)-1}{x}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{x}{1 + 2x\ln(x) + c_1 x} \tag{1}$$

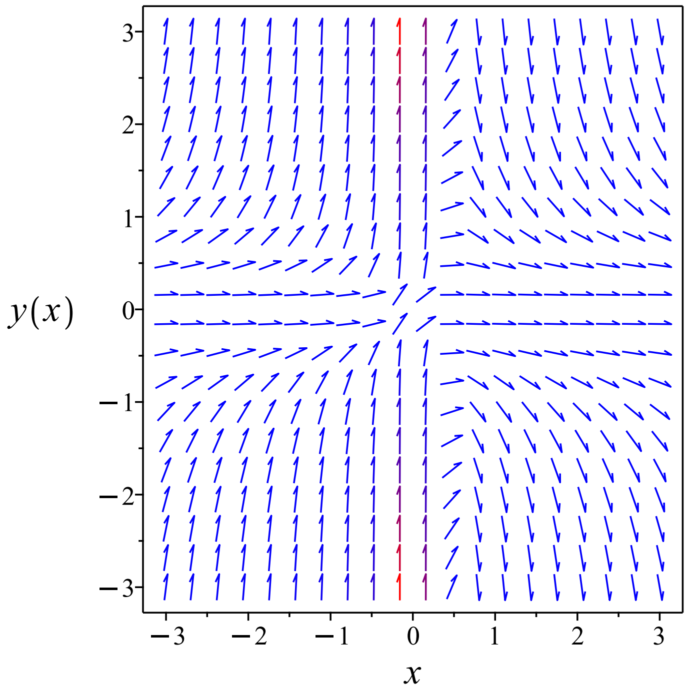


Figure 355: Slope field plot

Verification of solutions

$$y = \frac{x}{1 + 2x \ln(x) + c_1 x}$$

Verified OK.

6.6.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y^2}\right) \mathrm{d}y = \left(\frac{2x - 1}{x^2}\right) \mathrm{d}x$$
$$\left(-\frac{2x - 1}{x^2}\right) \mathrm{d}x + \left(-\frac{1}{y^2}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{2x - 1}{x^2}$$
$$N(x, y) = -\frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{2x - 1}{x^2}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int M\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int -\frac{2x-1}{x^2}\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -2\ln(x) - \frac{1}{x} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$
-\frac{1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^2}\right) \mathrm{d}y$$
$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -2\ln(x) - \frac{1}{x} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2\ln(x) - \frac{1}{x} + \frac{1}{y}$$

The solution becomes

$$y = \frac{x}{1 + 2x\ln(x) + c_1 x}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{1 + 2x\ln(x) + c_1 x} \tag{1}$$

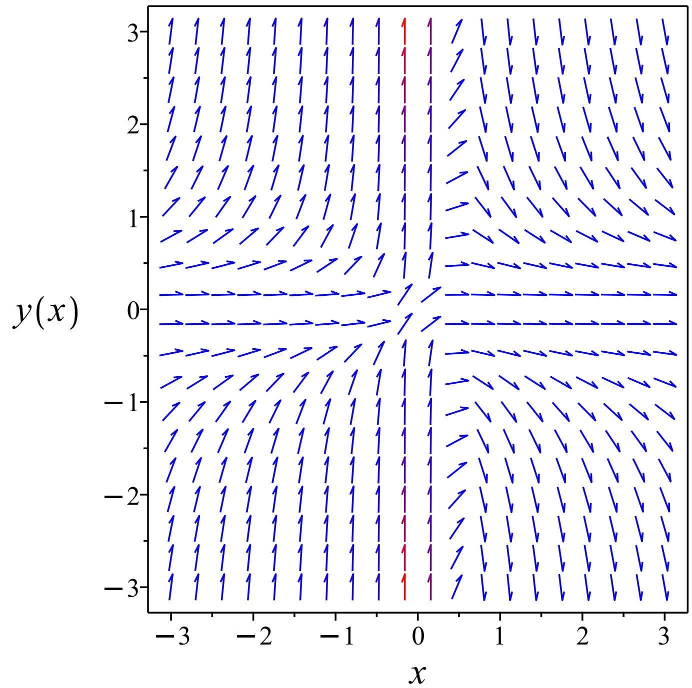


Figure 356: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x}{1 + 2x\ln(x) + c_1 x}$$

Verified OK.

6.6.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2(2x - 1)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{2y^2}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\frac{2x-1}{x^2}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{(2x-1)u}{x^2}}
\end{aligned}
\tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= \frac{4x-2}{x^3} - \frac{2}{x^2} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{(2x-1)\, u''(x)}{x^2} - \left(\frac{4x-2}{x^3} - \frac{2}{x^2}\right) u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \left(2\ln(x) + \frac{1}{x}\right) c_2
$$

The above shows that

$$
u'(x) = \frac{(2x-1)\, c_2}{x^2}
$$

Using the above in (1) gives the solution

$$
y = \frac{c_2}{c_1 + \left(2\ln(x) + \frac{1}{x}\right) c_2}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$
y = \frac{x}{2x\ln(x) + c_3 x + 1}
$$

Summary
The solution(s) found are the following

$$y = \frac{x}{2x \ln(x) + c_3 x + 1} \tag{1}$$

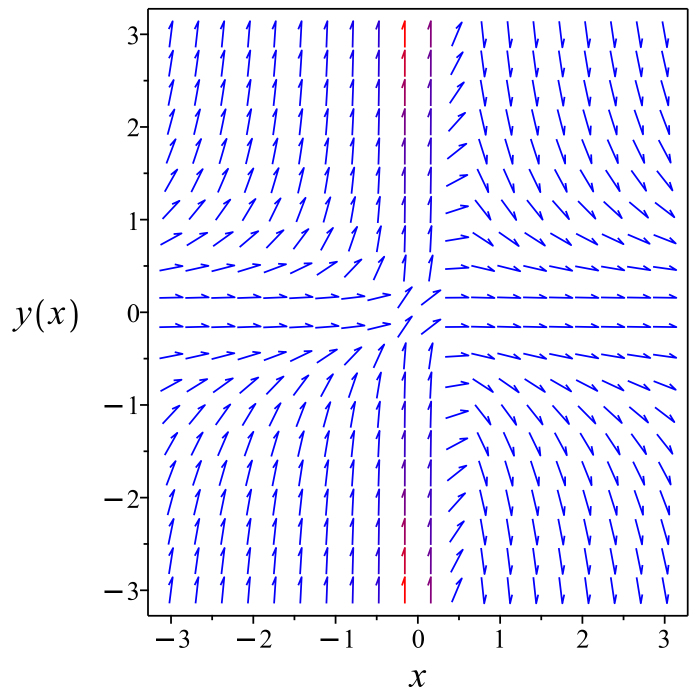


Figure 357: Slope field plot

Verification of solutions

$$y = \frac{x}{2x \ln(x) + c_3 x + 1}$$

Verified OK.

6.6.5 Maple step by step solution

Let's solve

$2xy^2 + y'x^2 - y^2 = 0$

- Highest derivative means the order of the ODE is 1

y'

- Separate variables

$\frac{y'}{y^2} = -\frac{2x-1}{x^2}$

- Integrate both sides with respect to x

$\int \frac{y'}{y^2}dx = \int -\frac{2x-1}{x^2}dx + c_1$

- Evaluate integral

$-\frac{1}{y} = -2\ln(x) - \frac{1}{x} + c_1$

- Solve for y

$y = \frac{x}{2x\ln(x)-c_1x+1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(2*x*y(x)^2+x^2*diff(y(x),x) = y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x}{1 + 2x\ln(x) + c_1x}$$

✓ Solution by Mathematica

Time used: 0.131 (sec). Leaf size: 26

```
DSolve[2*x*y[x]^2+x^2*y'[x] == y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{2x\log(x) + c_1(-x) + 1}$$
$$y(x) \to 0$$

6.7 problem 7

Internal problem ID [127]
Internal file name [`OUTPUT/127_Sunday_June_05_2022_01_35_41_AM_15266296/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2x^2y + x^3y' = 1}$$

6.7.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x}$$
$$q(x) = \frac{1}{x^3}$$

Hence the ode is

$$y' + \frac{2y}{x} = \frac{1}{x^3}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{2}{x}dx} \\ &= x^2\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{x^3}\right) \\ \frac{d}{dx}(y\,x^2) &= (x^2)\left(\frac{1}{x^3}\right) \\ d(y\,x^2) &= \frac{1}{x}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}y\,x^2 &= \int \frac{1}{x}\,dx \\ y\,x^2 &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$y = \frac{\ln(x)}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$y = \frac{\ln(x) + c_1}{x^2}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{\ln(x) + c_1}{x^2} \tag{1}$$

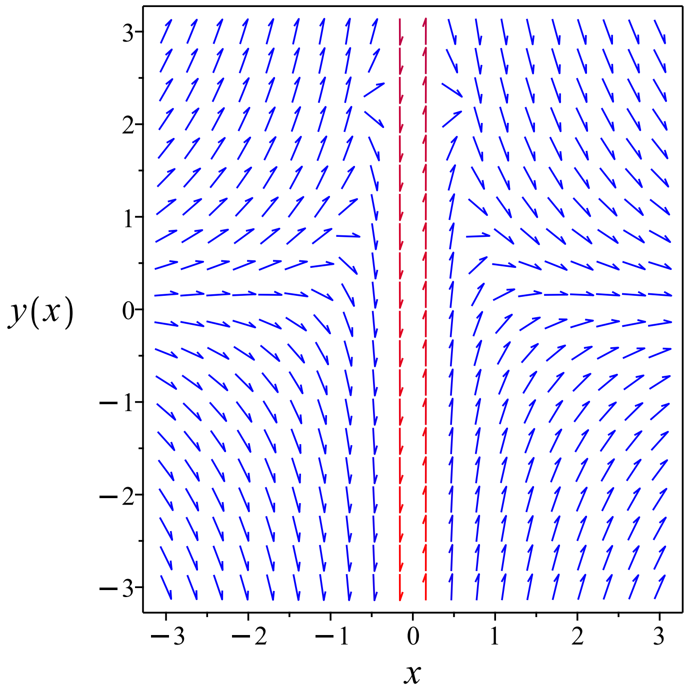


Figure 358: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) + c_1}{x^2}$$

Verified OK.

6.7.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y\,x^2 - 1}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 258: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^2}} dy \end{aligned}$$

Which results in

$$S = y\, x^2$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y\, x^2 - 1}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2yx \\ S_y &= x^2 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$x^2 y = \ln(x) + c_1$$

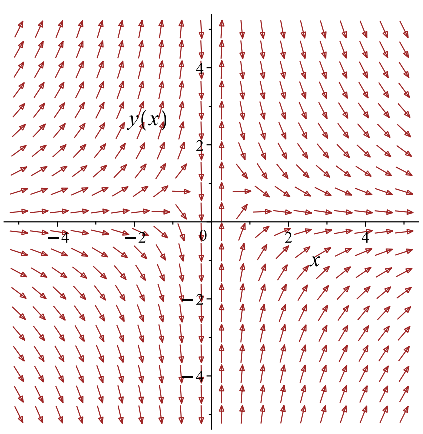
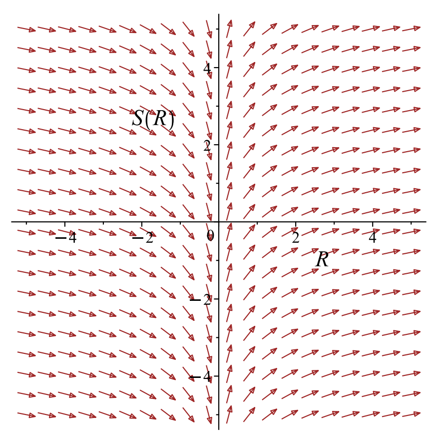
Which simplifies to

$$x^2 y = \ln(x) + c_1$$

Which gives

$$y = \frac{\ln(x) + c_1}{x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y\,x^2-1}{x^3}$ 	$R = x$ $S = y\,x^2$	$\frac{dS}{dR} = \frac{1}{R}$ 

Summary

The solution(s) found are the following

$$y = \frac{\ln(x) + c_1}{x^2} \tag{1}$$

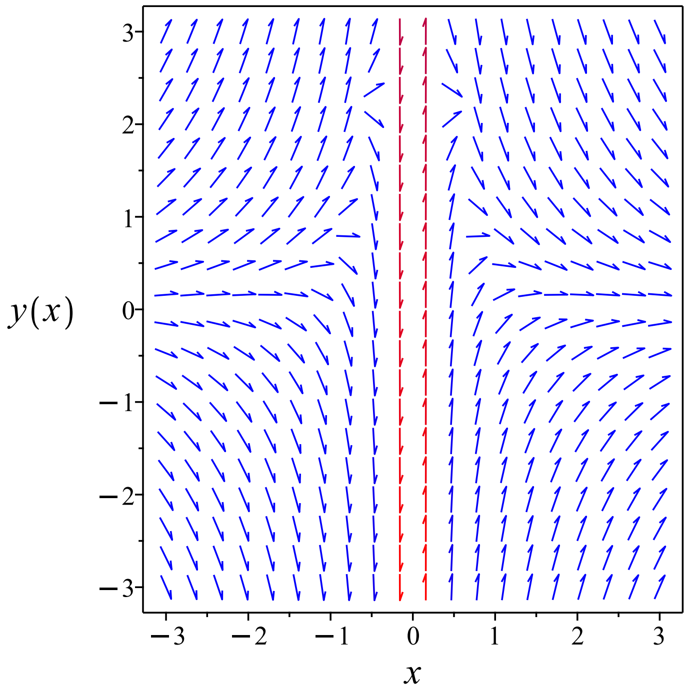


Figure 359: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) + c_1}{x^2}$$

Verified OK.

6.7.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^3)\, \mathrm{d}y &= (-2y\, x^2 + 1)\, \mathrm{d}x\\ (2y\, x^2 - 1)\, \mathrm{d}x + (x^3)\, \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= 2y\, x^2 - 1\\ N(x, y) &= x^3\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y\, x^2 - 1)\\ &= 2x^2\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^3)\\ &= 3x^2\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^3}\left(\left(2x^2\right) - \left(3x^2\right)\right) \\
&= -\frac{1}{x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\ln(x)} \\
&= \frac{1}{x}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x}\left(2y\,x^2 - 1\right) \\
&= \frac{2y\,x^2 - 1}{x}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x}\left(x^3\right) \\
&= x^2
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{2y\,x^2 - 1}{x}\right) + \left(x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \overline{M} \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int \frac{2y\,x^2 - 1}{x} \,\mathrm{d}x$$
$$\phi = y\,x^2 - \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x^2 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^2$. Therefore equation (4) becomes

$$x^2 = x^2 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,x^2 - \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,x^2 - \ln(x)$$

The solution becomes

$$y = \frac{\ln(x) + c_1}{x^2}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(x) + c_1}{x^2} \tag{1}$$

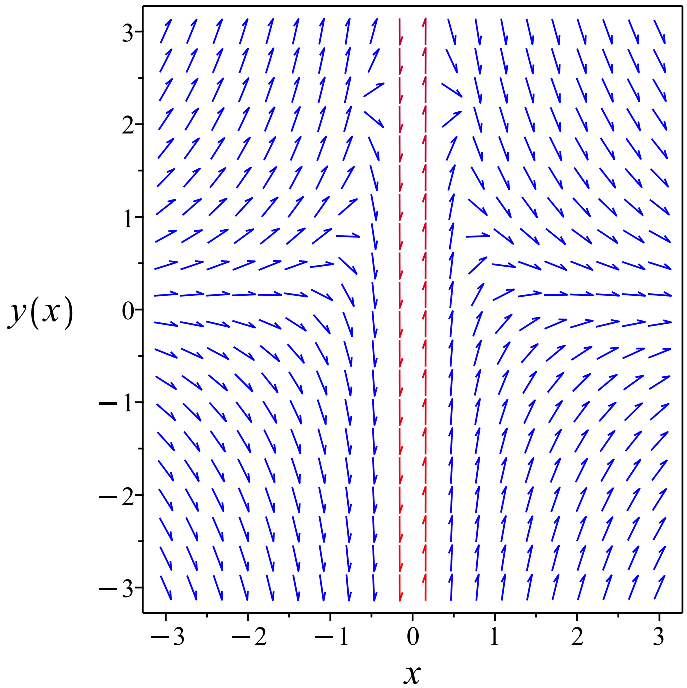


Figure 360: Slope field plot

Verification of solutions

$$y = \frac{\ln(x) + c_1}{x^2}$$

Verified OK.

6.7.4 Maple step by step solution

Let's solve

$2x^2y + x^3y' = 1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{2y}{x} + \frac{1}{x^3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x} = \frac{1}{x^3}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \frac{\mu(x)}{x^3}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x)\left(y' + \frac{2y}{x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \frac{\mu(x)}{x^3} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \frac{\mu(x)}{x^3} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{\mu(x)}{x^3} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^2$

  $y = \frac{\int \frac{1}{x} dx + c_1}{x^2}$

- Evaluate the integrals on the rhs

  $y = \frac{\ln(x) + c_1}{x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(2*x^2*y(x)+x^3*diff(y(x),x) = 1,y(x), singsol=all)
```

$$y(x) = \frac{\ln (x) + c_1}{x^2}$$

✓ Solution by Mathematica

Time used: 0.026 (sec). Leaf size: 14

```
DSolve[2*x^2*y[x]+x^3*y'[x] == 1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log(x) + c_1}{x^2}$$

6.8 problem 8

Internal problem ID [128]
Internal file name [`OUTPUT/128_Sunday_June_05_2022_01_35_42_AM_40316007/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 8.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{2yx + y'x^2 - y^2 = 0}$$

6.8.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2u(x)\,x^2 + (u'(x)\,x + u(x))\,x^2 - u(x)^2\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{u(u-3)}{x} \end{aligned}$$

Where $f(x) = \frac{1}{x}$ and $g(u) = u(u-3)$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u(u-3)}\,du &= \frac{1}{x}\,dx \\
\int \frac{1}{u(u-3)}\,du &= \int \frac{1}{x}\,dx \\
\frac{\ln(u-3)}{3} - \frac{\ln(u)}{3} &= \ln(x) + c_2
\end{aligned}
$$

The above can be written as

$$
\begin{aligned}
\left(\frac{1}{3}\right)(\ln(u-3) - \ln(u)) &= \ln(x) + 2c_2 \\
\ln(u-3) - \ln(u) &= (3)(\ln(x) + 2c_2) \\
&= 3\ln(x) + 6c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
e^{\ln(u-3)-\ln(u)} = e^{3\ln(x)+3c_2}
$$

Which simplifies to

$$
\begin{aligned}
\frac{u-3}{u} &= 3c_2 x^3 \\
&= c_3 x^3
\end{aligned}
$$

Therefore the solution y is

$$
\begin{aligned}
y &= xu \\
&= -\frac{3x}{c_3 x^3 - 1}
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = -\frac{3x}{c_3 x^3 - 1} \tag{1}
$$

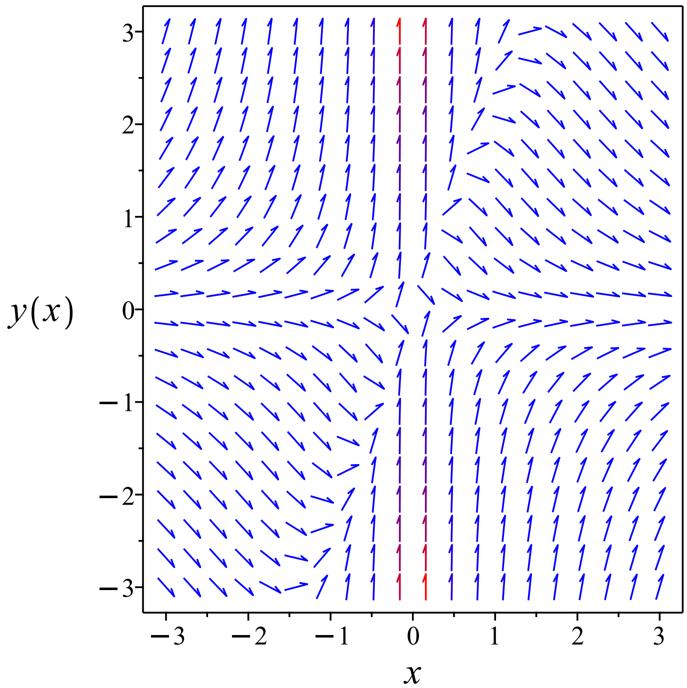


Figure 361: Slope field plot

Verification of solutions

$$y = -\frac{3x}{c_3 x^3 - 1}$$

Verified OK.

6.8.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(-2x + y)}{x^2}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 261: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2y^2
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 y^2} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{x^2 y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{y(-2x + y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{2}{x^3 y} \\ S_y &= \frac{1}{x^2 y^2} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x^4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{3R^3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{x^2 y} = -\frac{1}{3x^3} + c_1$$

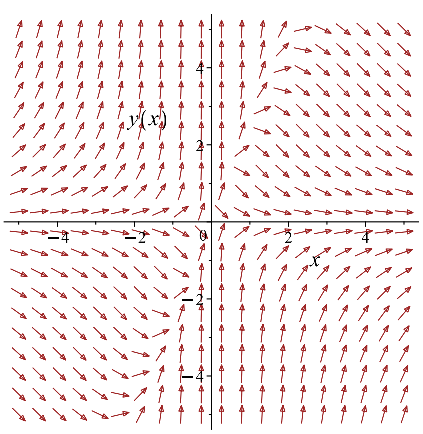
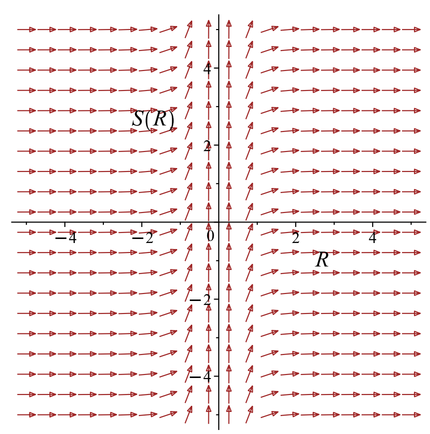
Which simplifies to

$$-\frac{1}{x^2 y} = -\frac{1}{3x^3} + c_1$$

Which gives

$$y = -\frac{3x}{3c_1 x^3 - 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{y(-2x+y)}{x^2}$ 	$R = x$ $S = -\frac{1}{x^2 y}$	$\frac{dS}{dR} = \frac{1}{R^4}$ 

Summary
The solution(s) found are the following

$$y = -\frac{3x}{3c_1x^3 - 1} \tag{1}$$

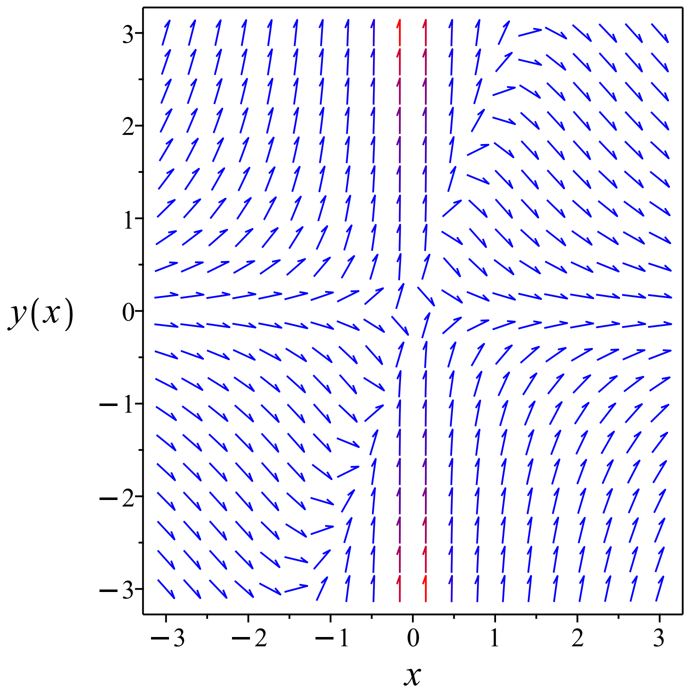


Figure 362: Slope field plot

Verification of solutions

$$y = -\frac{3x}{3c_1x^3 - 1}$$

Verified OK.

6.8.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(-2x + y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x}y + \frac{1}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{2}{x} \\ f_1(x) &= \frac{1}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = -\frac{2}{yx} + \frac{1}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= -\frac{2w(x)}{x} + \frac{1}{x^2} \\ w' &= \frac{2w}{x} - \frac{1}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = -\frac{1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -\frac{2}{x}dx} \\ &= \frac{1}{x^2} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{w}{x^2}\right) &= \left(\frac{1}{x^2}\right)\left(-\frac{1}{x^2}\right) \\ \mathrm{d}\left(\frac{w}{x^2}\right) &= \left(-\frac{1}{x^4}\right) \mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{w}{x^2} &= \int -\frac{1}{x^4} \,\mathrm{d}x \\ \frac{w}{x^2} &= \frac{1}{3x^3} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = \frac{1}{3x} + c_1 x^2$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{1}{3x} + c_1 x^2$$

Or

$$y = \frac{1}{\frac{1}{3x} + c_1 x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\frac{1}{3x} + c_1 x^2} \tag{1}$$

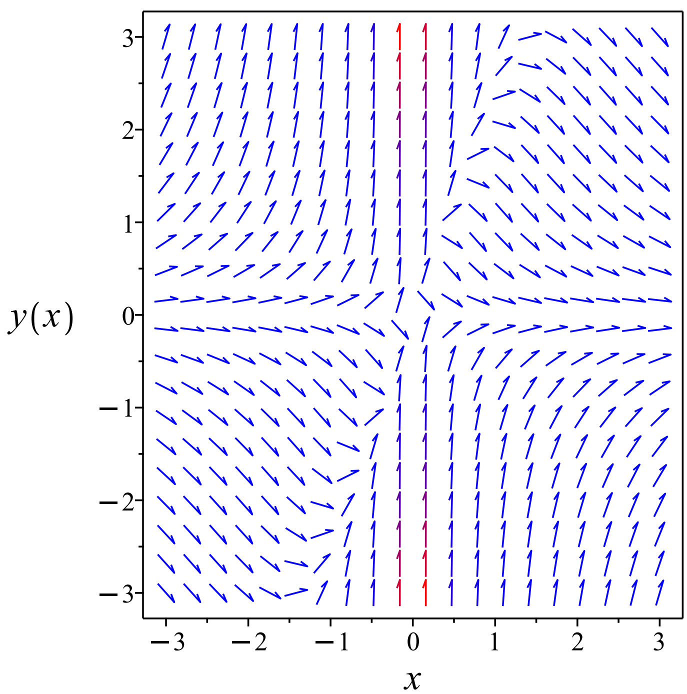


Figure 363: Slope field plot

Verification of solutions

$$y = \frac{1}{\frac{1}{3x} + c_1 x^2}$$

Verified OK.

6.8.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(-2x+y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -\frac{2y}{x} + \frac{y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = -\frac{2}{x}$ and $f_2(x) = \frac{1}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{2}{x^3} \\ f_1 f_2 &= -\frac{2}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{u''(x)}{x^2} + \frac{4u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{c_2}{x^3}$$

The above shows that

$$u'(x) = -\frac{3c_2}{x^4}$$

Using the above in (1) gives the solution

$$y = \frac{3c_2}{x^2 \left(c_1 + \frac{c_2}{x^3}\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{3x}{c_3 x^3 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{3x}{c_3 x^3 + 1} \tag{1}$$

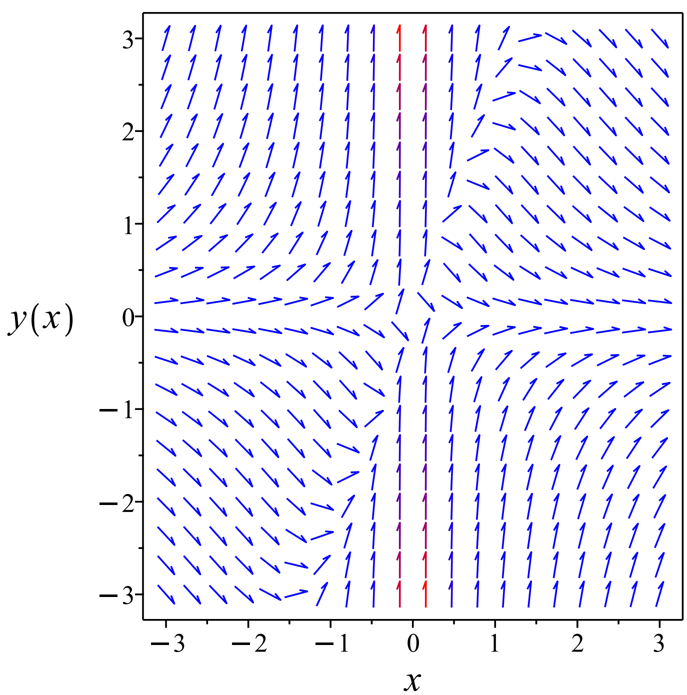


Figure 364: Slope field plot

Verification of solutions

$$y = \frac{3x}{c_3 x^3 + 1}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(2*x*y(x)+x^2*diff(y(x),x) = y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{3x}{3c_1 x^3 + 1}$$

✓ Solution by Mathematica

Time used: 0.122 (sec). Leaf size: 24

```
DSolve[2*x*y[x]+x^2*y'[x] == y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{3x}{1 + 3c_1 x^3}$$
$$y(x) \to 0$$

6.9 problem 9

Internal problem ID [129]
Internal file name [`OUTPUT/129_Sunday_June_05_2022_01_35_43_AM_63988582/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 9.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{2y + y'x - 6x^2\sqrt{y} = 0}$$

6.9.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{2(-3x^2\sqrt{y} + y)}{x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 263: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{\sqrt{y}}{x} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{\sqrt{y}}{x}} dy \end{aligned}$$

Which results in

$$S = 2x\sqrt{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{2(-3x^2\sqrt{y} + y)}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 2\sqrt{y} \\ S_y &= \frac{x}{\sqrt{y}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 6x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 6R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$2x\sqrt{y} = 2x^3 + c_1$$

Which simplifies to

$$2x\sqrt{y} = 2x^3 + c_1$$

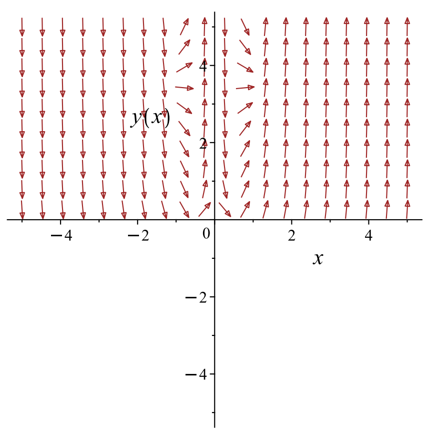
Which gives

$$y = \frac{(2x^3 + c_1)^2}{4x^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>

$\frac{dy}{dx} = -\frac{2(-3x^2\sqrt{y}+y)}{x}$



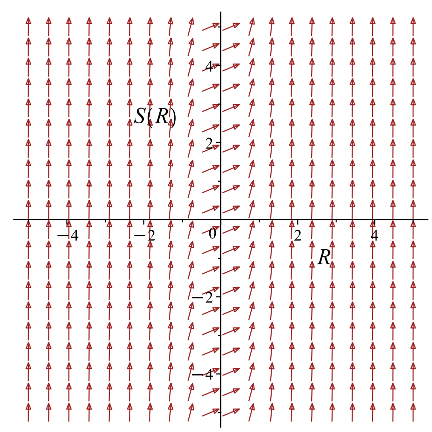
</td><td>

$R = x$

$S = 2x\sqrt{y}$

</td><td>

$\frac{dS}{dR} = 6R^2$



</td></tr>
</table>

Summary

The solution(s) found are the following

$$y = \frac{(2x^3 + c_1)^2}{4x^2} \tag{1}$$

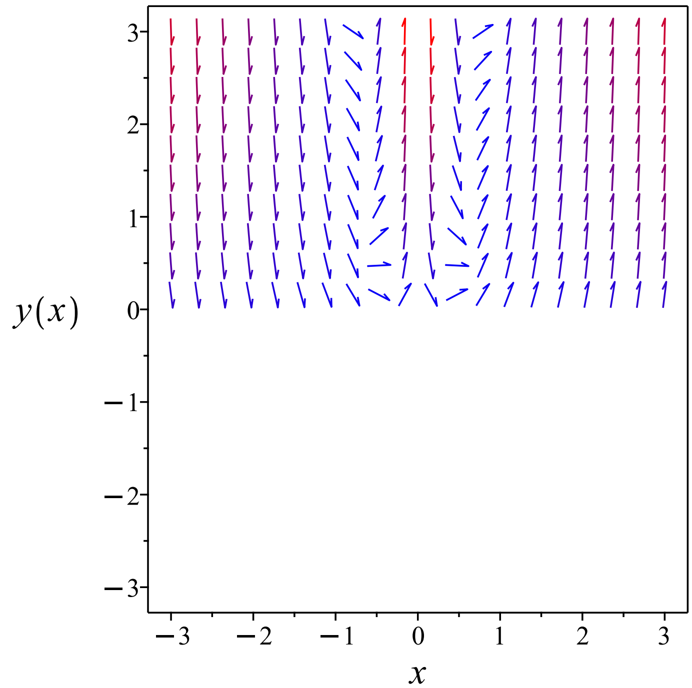


Figure 365: Slope field plot

Verification of solutions

$$y = \frac{(2x^3 + c_1)^2}{4x^2}$$

Verified OK.

6.9.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{2(-3x^2\sqrt{y} + y)}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{2}{x}y + 6x\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$
\begin{aligned}
f_0(x) &= -\frac{2}{x} \\
f_1(x) &= 6x \\
n &= \frac{1}{2}
\end{aligned}
$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$
y' \frac{1}{\sqrt{y}} = -\frac{2\sqrt{y}}{x} + 6x \tag{4}
$$

Let

$$
\begin{aligned}
w &= y^{1-n} \\
&= \sqrt{y}
\end{aligned} \tag{5}
$$

Taking derivative of equation (5) w.r.t x gives

$$
w' = \frac{1}{2\sqrt{y}} y' \tag{6}
$$

Substituting equations (5) and (6) into equation (4) gives

$$
\begin{aligned}
2w'(x) &= -\frac{2w(x)}{x} + 6x \\
w' &= -\frac{w}{x} + 3x
\end{aligned} \tag{7}
$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
w'(x) + p(x)w(x) = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= 3x
\end{aligned}
$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = 3x$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)(3x) \\ \frac{d}{dx}(wx) &= (x)(3x) \\ d(wx) &= \left(3x^2\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}wx &= \int 3x^2 \, dx \\ wx &= x^3 + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = x^2 + \frac{c_1}{x}$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = x^2 + \frac{c_1}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{y} = x^2 + \frac{c_1}{x} \tag{1}$$

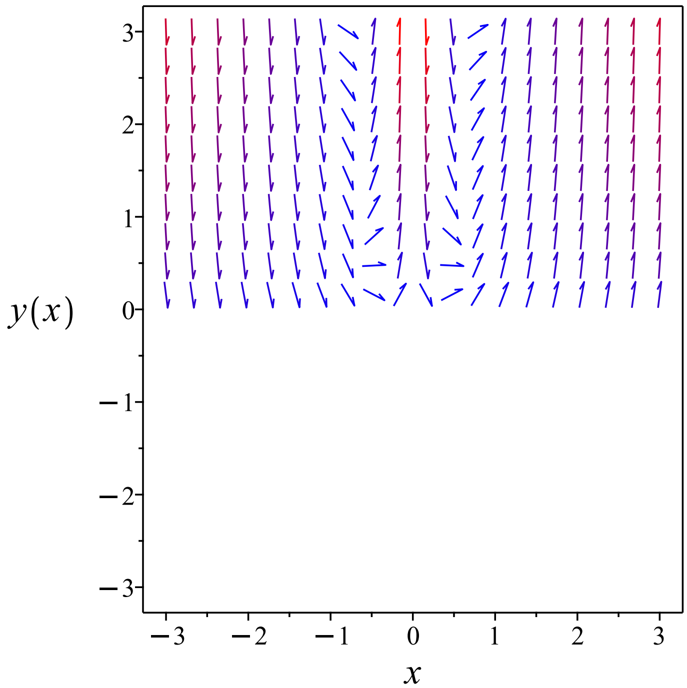


Figure 366: Slope field plot

Verification of solutions

$$\sqrt{y} = x^2 + \frac{c_1}{x}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(2*y(x)+x*diff(y(x),x) = 6*x^2*y(x)^(1/2),y(x), singsol=all)
```

$$\frac{-x^{3}+\sqrt{y\left(x\right)}\, x-c_{1}}{x}=0$$

✓ Solution by Mathematica

Time used: 0.15 (sec). Leaf size: 17

```
DSolve[2*y[x]+x*y'[x] == 6*x^2*y[x]^(1/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to \frac{\left(x^{3}+c_{1}\right)^{2}}{x^{2}}$$

6.10 problem 10

Internal problem ID [130]
Internal file name [`OUTPUT/130_Sunday_June_05_2022_01_35_44_AM_52766153/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 10.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - y^2 - x^2y^2 = x^2 + 1}$$

6.10.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= (y^2+1)(x^2+1) \end{aligned}$$

Where $f(x) = x^2 + 1$ and $g(y) = y^2 + 1$. Integrating both sides gives

$$\frac{1}{y^2+1}\,dy = x^2 + 1\,dx$$

$$\int \frac{1}{y^2+1}\,dy = \int x^2 + 1\,dx$$

$$\arctan(y) = \frac{1}{3}x^3 + x + c_1$$

Which results in

$$y = \tan\left(\frac{1}{3}x^3 + x + c_1\right)$$

Summary
The solution(s) found are the following

$$y = \tan\left(\frac{1}{3}x^3 + x + c_1\right) \tag{1}$$

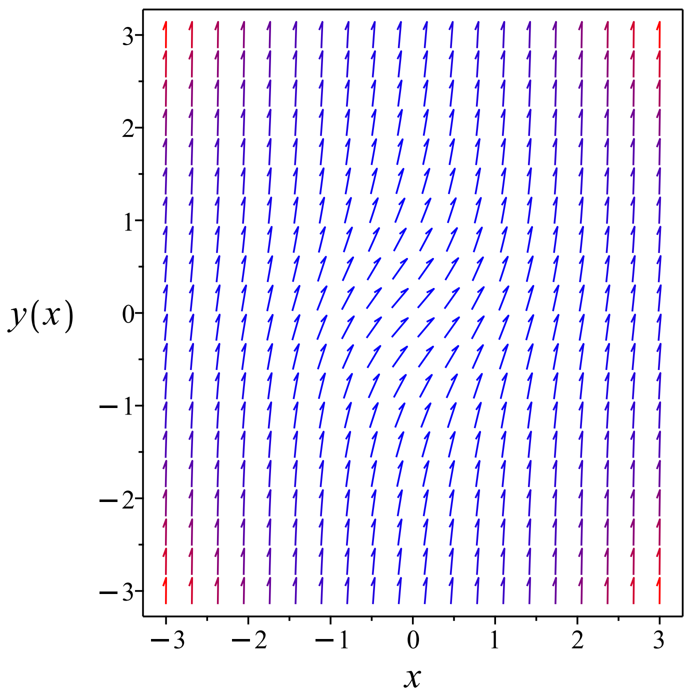


Figure 367: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{1}{3}x^3 + x + c_1\right)$$

Verified OK.

6.10.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = x^2y^2 + x^2 + y^2 + 1$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 265: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= \frac{1}{x^2 + 1} \\
\eta(x, y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x^2+1}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{1}{3} x^3 + x
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$
\omega(x, y) = x^2 y^2 + x^2 + y^2 + 1
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x^2 + 1 \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2+1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2+1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \arctan(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{1}{3}x^3 + x = \arctan(y) + c_1$$

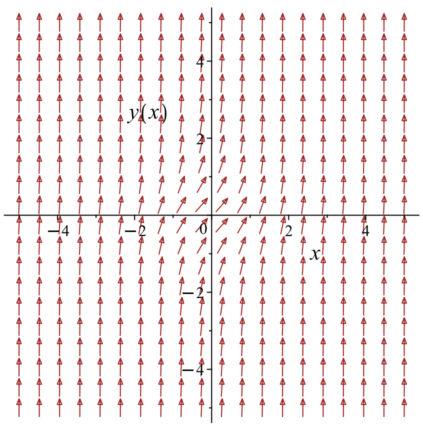
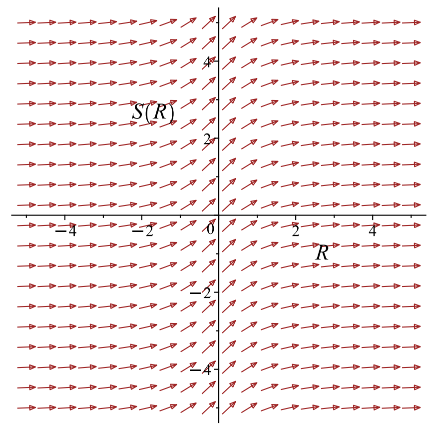
Which simplifies to

$$\frac{1}{3}x^3 + x = \arctan(y) + c_1$$

Which gives

$$y = -\tan\left(-\frac{1}{3}x^3 + c_1 - x\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2y^2 + x^2 + y^2 + 1$ 	$R = y$ $S = \frac{1}{3}x^3 + x$	$\frac{dS}{dR} = \frac{1}{R^2+1}$ 

Summary

The solution(s) found are the following

$$y = -\tan\left(-\frac{1}{3}x^3 + c_1 - x\right) \tag{1}$$

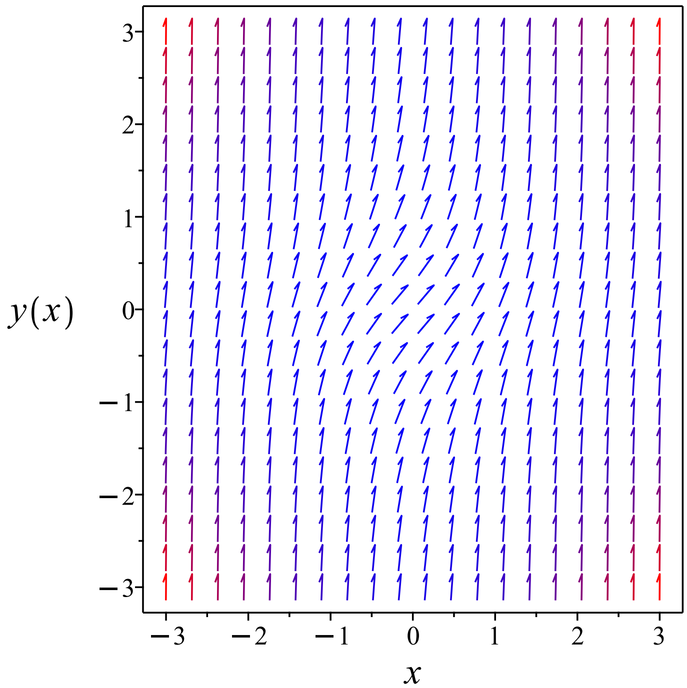


Figure 368: Slope field plot

Verification of solutions

$$y = -\tan\left(-\frac{1}{3}x^3 + c_1 - x\right)$$

Verified OK.

6.10.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{y^2 + 1}\right) \mathrm{d}y = \left(x^2 + 1\right) \mathrm{d}x$$
$$\left(-x^2 - 1\right) \mathrm{d}x + \left(\frac{1}{y^2 + 1}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2 - 1$$
$$N(x, y) = \frac{1}{y^2 + 1}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2 - 1\right)\\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{y^2+1}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 - 1 \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{1}{3}x^3 - x + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^2+1}$. Therefore equation (4) becomes

$$
\frac{1}{y^2+1} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{y^2+1}
$$

Integrating the above w.r.t y gives

$$
\begin{aligned}
\int f'(y) \, \mathrm{d}y &= \int \left(\frac{1}{y^2+1}\right) \mathrm{d}y \\
f(y) &= \arctan(y) + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} - x + \arctan(y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} - x + \arctan(y)$$

The solution becomes

$$y = \tan\left(\frac{1}{3}x^3 + x + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \tan\left(\frac{1}{3}x^3 + x + c_1\right) \tag{1}$$

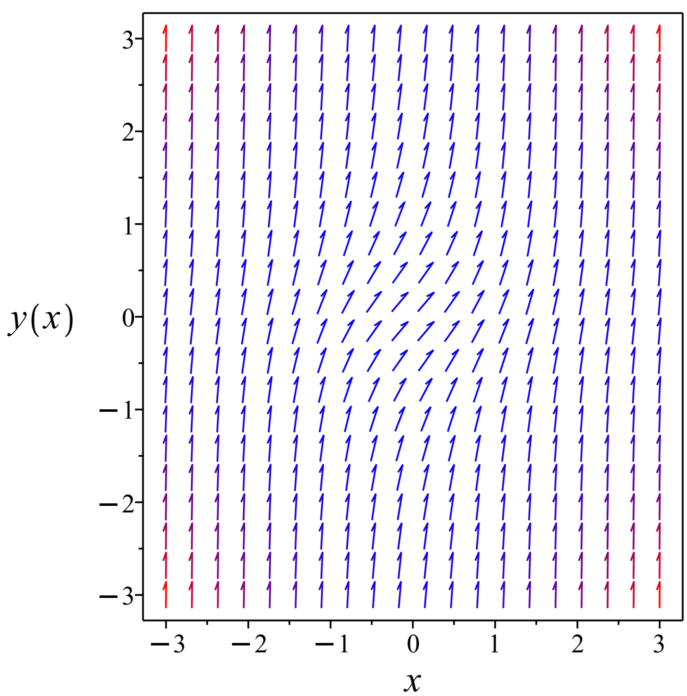


Figure 369: Slope field plot

Verification of solutions

$$y = \tan\left(\frac{1}{3}x^3 + x + c_1\right)$$

Verified OK.

6.10.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2y^2 + x^2 + y^2 + 1 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2y^2 + x^2 + y^2 + 1$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2 + 1$, $f_1(x) = 0$ and $f_2(x) = x^2 + 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{(x^2 + 1)\, u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2)\, u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 2x \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= (x^2 + 1)^3 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$(x^2 + 1)\, u''(x) - 2xu'(x) + (x^2 + 1)^3\, u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \sin\left(\frac{1}{3}x^3 + x\right) + c_2 \cos\left(\frac{1}{3}x^3 + x\right)$$

The above shows that

$$u'(x) = (x^2 + 1)\left(c_1 \cos\left(\frac{1}{3}x^3 + x\right) - c_2 \sin\left(\frac{1}{3}x^3 + x\right)\right)$$

Using the above in (1) gives the solution

$$y = -\frac{c_1 \cos\left(\frac{1}{3}x^3 + x\right) - c_2 \sin\left(\frac{1}{3}x^3 + x\right)}{c_1 \sin\left(\frac{1}{3}x^3 + x\right) + c_2 \cos\left(\frac{1}{3}x^3 + x\right)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{-c_3 \cos\left(\frac{1}{3}x^3 + x\right) + \sin\left(\frac{1}{3}x^3 + x\right)}{c_3 \sin\left(\frac{1}{3}x^3 + x\right) + \cos\left(\frac{1}{3}x^3 + x\right)}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_3 \cos\left(\frac{1}{3}x^3 + x\right) + \sin\left(\frac{1}{3}x^3 + x\right)}{c_3 \sin\left(\frac{1}{3}x^3 + x\right) + \cos\left(\frac{1}{3}x^3 + x\right)} \tag{1}$$

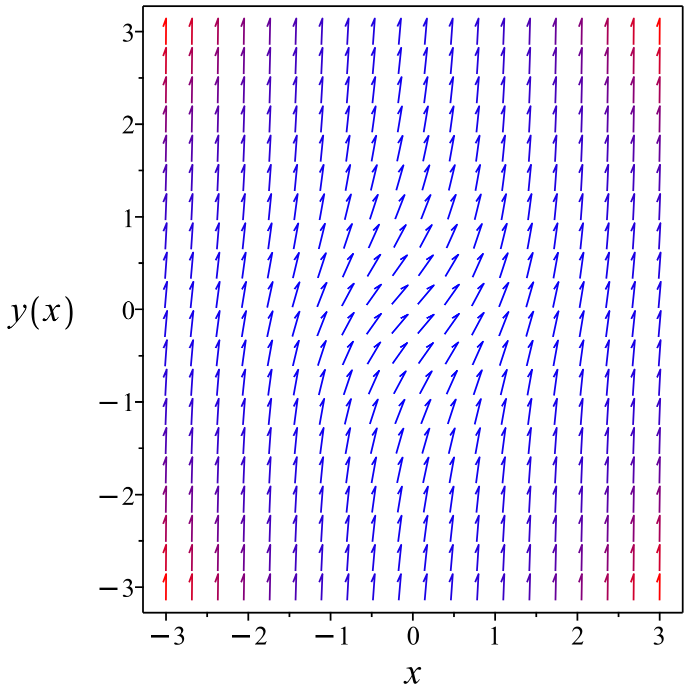


Figure 370: Slope field plot

Verification of solutions

$$y = \frac{-c_3 \cos\left(\frac{1}{3}x^3 + x\right) + \sin\left(\frac{1}{3}x^3 + x\right)}{c_3 \sin\left(\frac{1}{3}x^3 + x\right) + \cos\left(\frac{1}{3}x^3 + x\right)}$$

Verified OK.

6.10.5 Maple step by step solution

Let's solve

$y' - y^2 - x^2y^2 = x^2 + 1$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y^2} = x^2 + 1$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y^2} dx = \int (x^2 + 1) dx + c_1$

- Evaluate integral

$\arctan(y) = \frac{1}{3}x^3 + x + c_1$

- Solve for y

  $y = \tan\left(\frac{1}{3}x^3 + x + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
<- separable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve(diff(y(x),x) = 1+x^2+y(x)^2+x^2*y(x)^2,y(x), singsol=all)
```

$$y(x) = \tan\left(\frac{1}{3}x^3 + c_1 + x\right)$$

✓ Solution by Mathematica

Time used: 0.186 (sec). Leaf size: 17

```
DSolve[y'[x] == 1+x^2+y[x]^2+x^2*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \tan\left(\frac{x^3}{3} + x + c_1\right)$$

6.11 problem 11

Internal problem ID [131]
Internal file name [`OUTPUT/131_Sunday_June_05_2022_01_35_44_AM_50146025/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "bernoulli", "exactByInspection", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y'x^2 - yx - 3y^2 = 0}$$

6.11.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$(u'(x)\,x + u(x))\,x^2 - u(x)\,x^2 - 3u(x)^2\,x^2 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{3u^2}{x} \end{aligned}$$

Where $f(x) = \frac{3}{x}$ and $g(u) = u^2$. Integrating both sides gives

$$\frac{1}{u^2}\, du = \frac{3}{x}\, dx$$
$$\int \frac{1}{u^2}\, du = \int \frac{3}{x}\, dx$$
$$-\frac{1}{u} = 3\ln(x) + c_2$$

The solution is

$$-\frac{1}{u(x)} - 3\ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x}{y} - 3\ln(x) - c_2 = 0$$
$$-\frac{x}{y} - 3\ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$-\frac{x}{y} - 3\ln(x) - c_2 = 0 \tag{1}$$

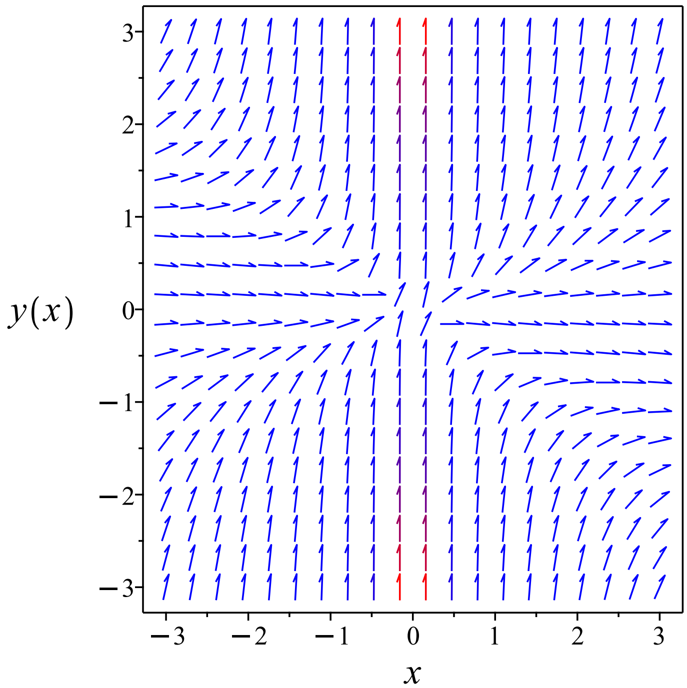


Figure 371: Slope field plot

Verification of solutions

$$-\frac{x}{y} - 3\ln(x) - c_2 = 0$$

Verified OK.

6.11.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{y(x+3y)}{x^2}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 268: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^2}{x} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{y^2}{x}} dy
\end{aligned}$$

Which results in

$$S = -\frac{x}{y}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{y(x + 3y)}{x^2}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= -\frac{1}{y} \\
S_y &= \frac{x}{y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{3}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{3}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 3\ln(R) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x}{y} = 3\ln(x) + c_1$$

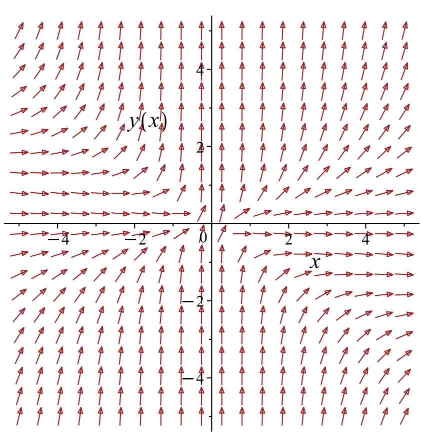
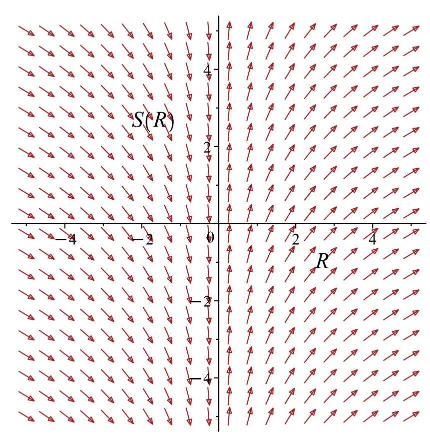
Which simplifies to

$$-\frac{x}{y} = 3\ln(x) + c_1$$

Which gives

$$y = -\frac{x}{3\ln(x) + c_1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = \frac{y(x+3y)}{x^2}$
</td><td>$R = x$
$S = -\frac{x}{y}$</td><td>$\frac{dS}{dR} = \frac{3}{R}$
</td></tr>
</table>

Summary
The solution(s) found are the following

$$y = -\frac{x}{3\ln(x) + c_1} \tag{1}$$

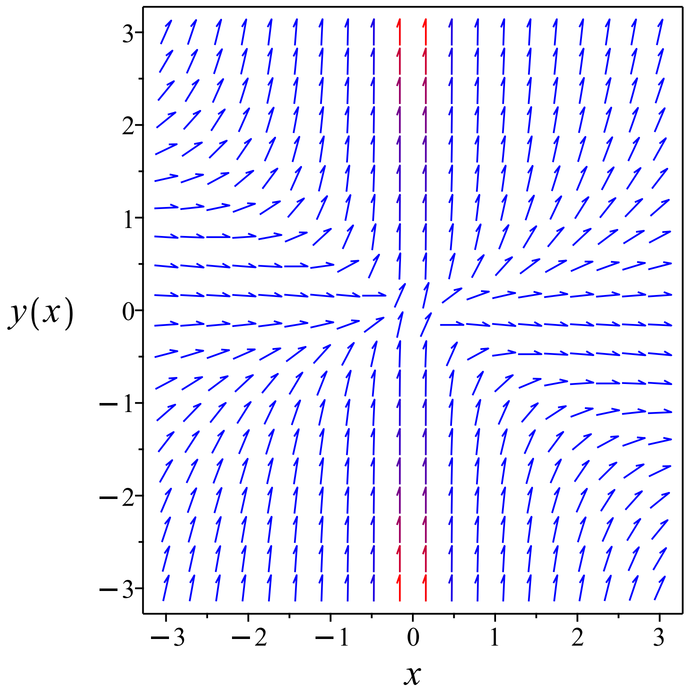


Figure 372: Slope field plot

Verification of solutions

$$y = -\frac{x}{3\ln(x) + c_1}$$

Verified OK.

6.11.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{y(x+3y)}{x^2} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y + \frac{3}{x^2}y^2 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= \frac{3}{x^2} \\ n &= 2 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^2$ gives

$$y'\frac{1}{y^2} = \frac{1}{yx} + \frac{3}{x^2} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{1}{y^2}y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -w'(x) &= \frac{w(x)}{x} + \frac{3}{x^2} \\ w' &= -\frac{w}{x} - \frac{3}{x^2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{3}{x^2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -\frac{3}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{3}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(wx) &= (x)\left(-\frac{3}{x^2}\right) \\ \mathrm{d}(wx) &= \left(-\frac{3}{x}\right)\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} wx &= \int -\frac{3}{x}\,\mathrm{d}x \\ wx &= -3\ln(x) + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{3\ln(x)}{x} + \frac{c_1}{x}$$

which simplifies to

$$w(x) = \frac{-3\ln(x) + c_1}{x}$$

Replacing w in the above by $\frac{1}{y}$ using equation (5) gives the final solution.

$$\frac{1}{y} = \frac{-3\ln(x) + c_1}{x}$$

Or

$$y = \frac{x}{-3\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{-3\ln(x) + c_1} \tag{1}$$

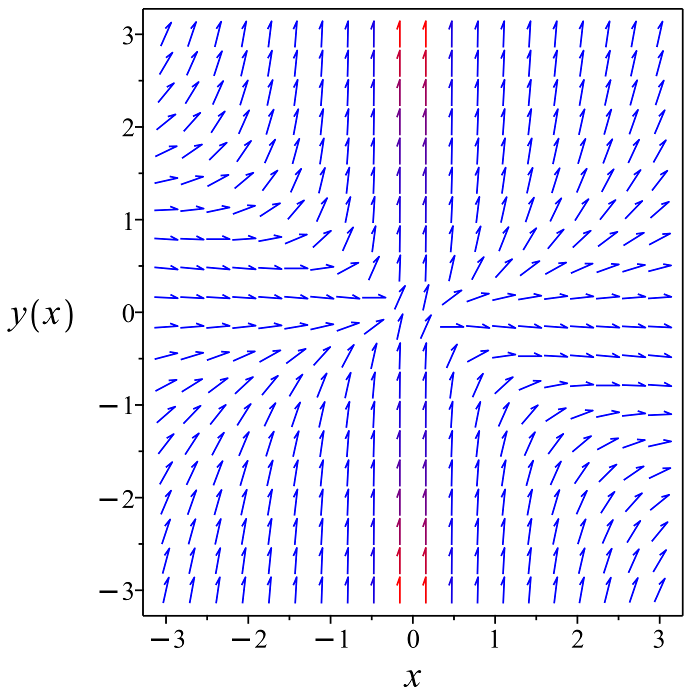


Figure 373: Slope field plot

Verification of solutions

$$y = \frac{x}{-3\ln(x) + c_1}$$

Verified OK.

6.11.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2)\,\mathrm{d}y&=(yx+3y^2)\,\mathrm{d}x\\ (-yx-3y^2)\,\mathrm{d}x+(x^2)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-yx-3y^2\\ N(x,y)&=x^2\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-yx - 3y^2) \\ &= -x - 6y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. By inspection $\frac{1}{xy^2}$ is an integrating factor. Therefore by multiplying $M = -3y^2 - yx$ and $N = x^2$ by this integrating factor the ode becomes exact. The new M, N are

$$\begin{aligned}M &= \frac{-3y^2 - yx}{xy^2} \\ N &= \frac{x}{y^2}\end{aligned}$$

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}
\left(\frac{x}{y^2}\right)\mathrm{d}y &= \left(-\frac{-yx - 3y^2}{x\,y^2}\right)\mathrm{d}x \\
\left(\frac{-yx - 3y^2}{x\,y^2}\right)\mathrm{d}x + \left(\frac{x}{y^2}\right)\mathrm{d}y &= 0 \qquad \text{(2A)}
\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}
M(x, y) &= \frac{-yx - 3y^2}{x\,y^2} \\
N(x, y) &= \frac{x}{y^2}
\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{-yx - 3y^2}{x\,y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x}{y^2}\right) \\
&= \frac{1}{y^2}
\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \frac{-yx - 3y^2}{x\, y^2} \, \mathrm{d}x$$

$$\phi = -3 \ln(x) - \frac{x}{y} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{x}{y^2} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{x}{y^2}$. Therefore equation (4) becomes

$$\frac{x}{y^2} = \frac{x}{y^2} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -3 \ln(x) - \frac{x}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -3\ln(x) - \frac{x}{y}$$

The solution becomes

$$y = -\frac{x}{3\ln(x) + c_1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{3\ln(x) + c_1} \tag{1}$$

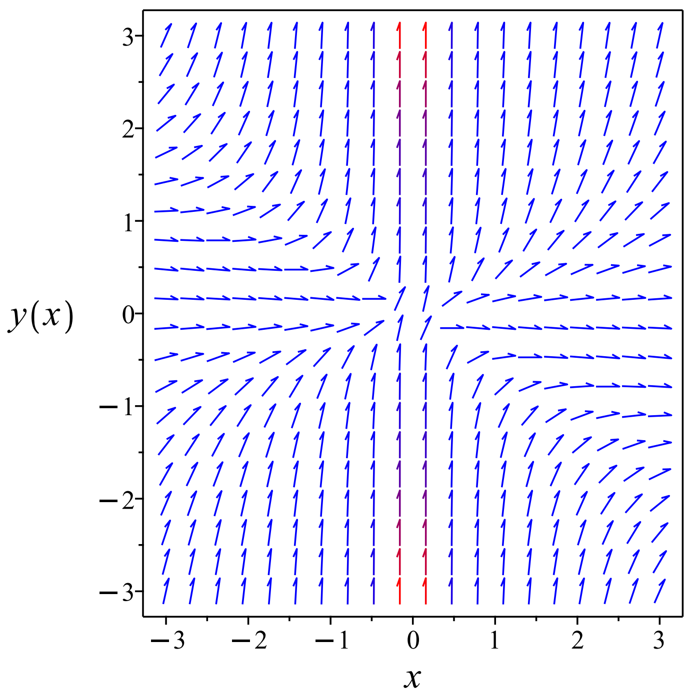


Figure 374: Slope field plot

Verification of solutions

$$y = -\frac{x}{3\ln(x) + c_1}$$

Verified OK.

6.11.5 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= \frac{y(x+3y)}{x^2} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = \frac{y}{x} + \frac{3y^2}{x^2}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = \frac{1}{x}$ and $f_2(x) = \frac{3}{x^2}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{\frac{3u}{x^2}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -\frac{6}{x^3} \\ f_1 f_2 &= \frac{3}{x^3} \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$\frac{3u''(x)}{x^2} + \frac{3u'(x)}{x^3} = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_2 \ln(x) + c_1$$

The above shows that

$$u'(x) = \frac{c_2}{x}$$

Using the above in (1) gives the solution

$$y = -\frac{c_2 x}{3 (c_2 \ln (x) + c_1)}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = -\frac{x}{3 \ln (x) + 3c_3}$$

Summary

The solution(s) found are the following

$$y = -\frac{x}{3 \ln (x) + 3c_3} \tag{1}$$

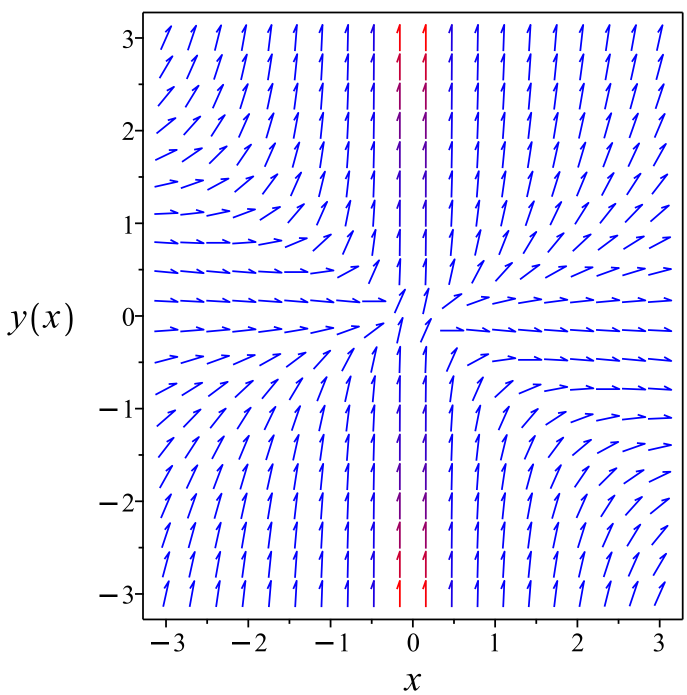


Figure 375: Slope field plot

Verification of solutions

$$y = -\frac{x}{3 \ln (x) + 3c_3}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x) = x*y(x)+3*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x}{-3 \ln (x) + c_1}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 21

```
DSolve[x^2*y'[x] == x*y[x]+3*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x}{-3 \log(x) + c_1}$$
$$y(x) \to 0$$

6.12 problem 12

Internal problem ID [132]
Internal file name [`OUTPUT/132_Sunday_June_05_2022_01_35_45_AM_67072539/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _exact, _rational, [_Abel, `2nd
   type`, `class B`]]
```

$$\boxed{6xy^3 + 2y^4 + (9x^2y^2 + 8xy^3)\, y' = 0}$$

6.12.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$6x^4u(x)^3 + 2u(x)^4\, x^4 + (9x^4u(x)^2 + 8x^4u(x)^3)\,(u'(x)\, x + u(x)) = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{5u(2u+3)}{x\,(8u+9)} \end{aligned}$$

Where $f(x) = -\frac{5}{x}$ and $g(u) = \frac{u(2u+3)}{8u+9}$. Integrating both sides gives

$$\frac{1}{\frac{u(2u+3)}{8u+9}}\,du = -\frac{5}{x}\,dx$$

$$\int \frac{1}{\frac{u(2u+3)}{8u+9}}\,du = \int -\frac{5}{x}\,dx$$

$$\ln(2u+3) + 3\ln(u) = -5\ln(x) + c_2$$

Raising both side to exponential gives

$$e^{\ln(2u+3)+3\ln(u)} = e^{-5\ln(x)+c_2}$$

Which simplifies to

$$2u^4 + 3u^3 = \frac{c_3}{x^5}$$

Therefore the solution y is

$$\begin{aligned} y &= xu \\ &= \operatorname{RootOf}\left(2x\_Z^4 + 3x^2\_Z^3 - c_3\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \operatorname{RootOf}\left(2x\_Z^4 + 3x^2\_Z^3 - c_3\right) \tag{1}$$

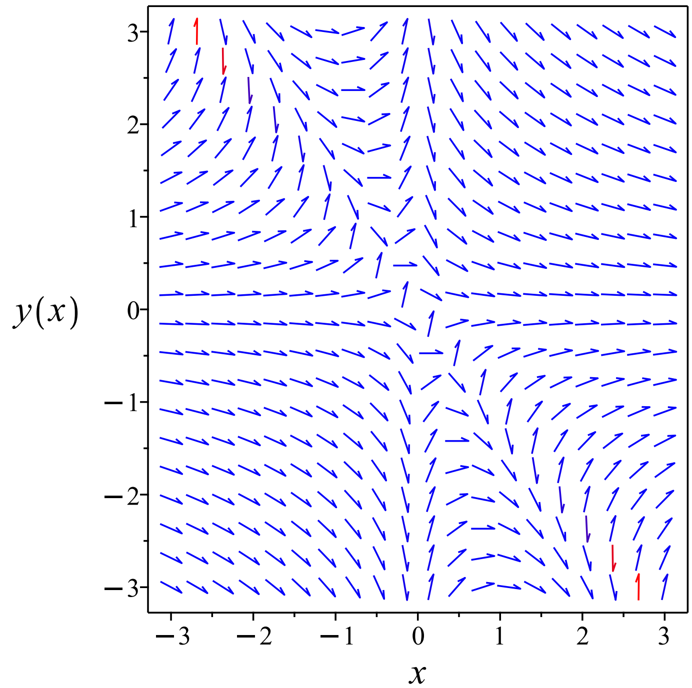


Figure 376: Slope field plot

Verification of solutions

$$y = \mathrm{RootOf}\left(2x\_Z^4 + 3x^2\_Z^3 - c_3\right)$$

Verified OK.

6.12.2 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$y' = -\frac{2y(3x+y)}{x\,(9x+8y)}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$
$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned}&b_2 - \frac{2y(3x+y)(b_3-a_2)}{x(9x+8y)} - \frac{4y^2(3x+y)^2 a_3}{x^2(9x+8y)^2}\\ &- \left(-\frac{6y}{x(9x+8y)} + \frac{2y(3x+y)}{x^2(9x+8y)} + \frac{18y(3x+y)}{x(9x+8y)^2}\right)(xa_2 + ya_3 + a_1)\\ &- \left(-\frac{2(3x+y)}{x(9x+8y)} - \frac{2y}{x(9x+8y)} + \frac{16y(3x+y)}{x(9x+8y)^2}\right)(xb_2 + yb_3 + b_1) = 0\end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\frac{135x^4b_2 + 180x^3yb_2 + 30x^2y^2a_2 - 90x^2y^2a_3 + 80x^2y^2b_2 - 30x^2y^2b_3 - 60x\,y^3a_3 - 20y^4a_3 + 54x^3b_1 - 54x^2}{x^2(9x+8y)^2} = 0$$

Setting the numerator to zero gives

$$\begin{aligned}&135x^4b_2 + 180x^3yb_2 + 30x^2y^2a_2 - 90x^2y^2a_3 + 80x^2y^2b_2 - 30x^2y^2b_3 - 60x\,y^3a_3\\ &- 20y^4a_3 + 54x^3b_1 - 54x^2ya_1 + 36x^2yb_1 - 36x\,y^2a_1 + 16x\,y^2b_1 - 16y^3a_1 = 0\end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned}&30a_2v_1^2v_2^2 - 90a_3v_1^2v_2^2 - 60a_3v_1v_2^3 - 20a_3v_2^4 + 135b_2v_1^4 + 180b_2v_1^3v_2 + 80b_2v_1^2v_2^2\\ &- 30b_3v_1^2v_2^2 - 54a_1v_1^2v_2 - 36a_1v_1v_2^2 - 16a_1v_2^3 + 54b_1v_1^3 + 36b_1v_1^2v_2 + 16b_1v_1v_2^2 = 0\end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$\begin{aligned}
&135b_2v_1^4 + 180b_2v_1^3v_2 + 54b_1v_1^3 + (30a_2 - 90a_3 + 80b_2 - 30b_3)\, v_1^2v_2^2 \\
&+ (-54a_1 + 36b_1)\, v_1^2v_2 - 60a_3v_1v_2^3 + (-36a_1 + 16b_1)\, v_1v_2^2 - 20a_3v_2^4 - 16a_1v_2^3 = 0
\end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-16a_1 &= 0 \\
-60a_3 &= 0 \\
-20a_3 &= 0 \\
54b_1 &= 0 \\
135b_2 &= 0 \\
180b_2 &= 0 \\
-54a_1 + 36b_1 &= 0 \\
-36a_1 + 16b_1 &= 0 \\
30a_2 - 90a_3 + 80b_2 - 30b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= b_3 \\
a_3 &= 0 \\
b_1 &= 0 \\
b_2 &= 0 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\begin{aligned}
\xi &= x \\
\eta &= y
\end{aligned}$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x,y)\,\xi \\
&= y - \left(-\frac{2y(3x+y)}{x\,(9x+8y)}\right)(x) \\
&= \frac{15yx + 10y^2}{9x+8y} \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$
R = x
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\eta} dy \\
&= \int \frac{1}{\frac{15yx+10y^2}{9x+8y}} dy
\end{aligned}
$$

Which results in

$$
S = \frac{3\ln(y)}{5} + \frac{\ln(3x+2y)}{5}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\frac{2y(3x+y)}{x\,(9x+8y)}
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{3}{15x + 10y} \\
S_y &= \frac{9x + 8y}{15yx + 10y^2}
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = -\frac{1}{5x} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = -\frac{1}{5R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{\ln(R)}{5} + c_1 \tag{4}
$$

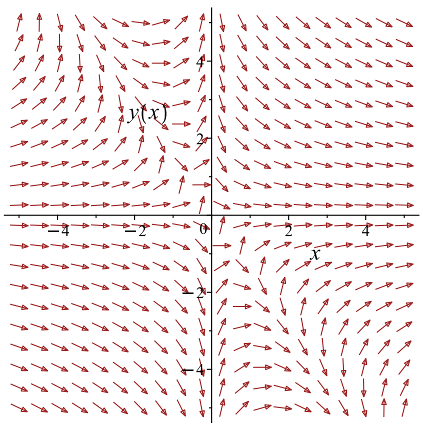
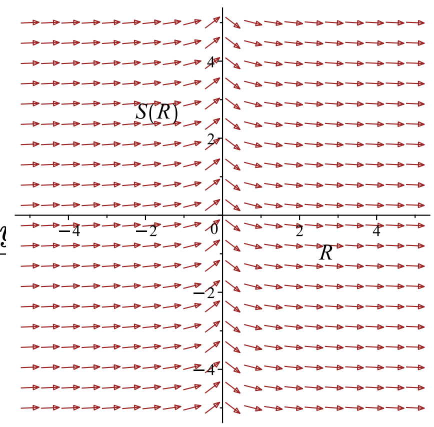
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{3\ln(y)}{5} + \frac{\ln(3x + 2y)}{5} = -\frac{\ln(x)}{5} + c_1
$$

Which simplifies to

$$
\frac{3\ln(y)}{5} + \frac{\ln(3x + 2y)}{5} = -\frac{\ln(x)}{5} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y(3x+y)}{x(9x+8y)}$ 	$R = x$ $S = \frac{3\ln(y)}{5} + \frac{\ln(3x+2y)}{5}$	$\frac{dS}{dR} = -\frac{1}{5R}$ 

Summary

The solution(s) found are the following

$$\frac{3\ln(y)}{5} + \frac{\ln(3x+2y)}{5} = -\frac{\ln(x)}{5} + c_1 \tag{1}$$

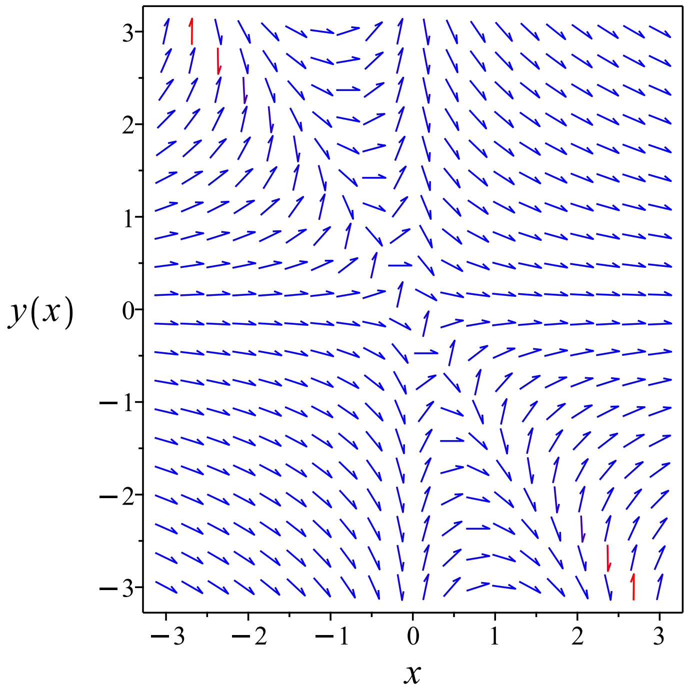


Figure 377: Slope field plot

Verification of solutions

$$\frac{3 \ln(y)}{5} + \frac{\ln(3x + 2y)}{5} = -\frac{\ln(x)}{5} + c_1$$

Verified OK.

6.12.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(9x^2y^2 + 8x\,y^3\right) \mathrm{d}y = \left(-6x\,y^3 - 2y^4\right) \mathrm{d}x$$
$$\left(6x\,y^3 + 2y^4\right) \mathrm{d}x + \left(9x^2y^2 + 8x\,y^3\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 6x\,y^3 + 2y^4$$
$$N(x, y) = 9x^2y^2 + 8x\,y^3$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(6x\,y^3 + 2y^4\right) \\ &= 18x\,y^2 + 8y^3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(9x^2y^2 + 8x\,y^3\right) \\ &= 18x\,y^2 + 8y^3 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 6x\, y^3 + 2y^4 \,\mathrm{d}x$$
$$\phi = y^3 x(3x + 2y) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 3y^2 x(3x + 2y) + 2x\, y^3 + f'(y) \tag{4}$$
$$= x\, y^2(9x + 8y) + f'(y)$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 9x^2 y^2 + 8x\, y^3$. Therefore equation (4) becomes

$$9x^2 y^2 + 8x\, y^3 = x\, y^2(9x + 8y) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y^3 x(3x + 2y) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y^3 x(3x + 2y)$$

Summary

The solution(s) found are the following

$$y^3 x(3x + 2y) = c_1 \tag{1}$$

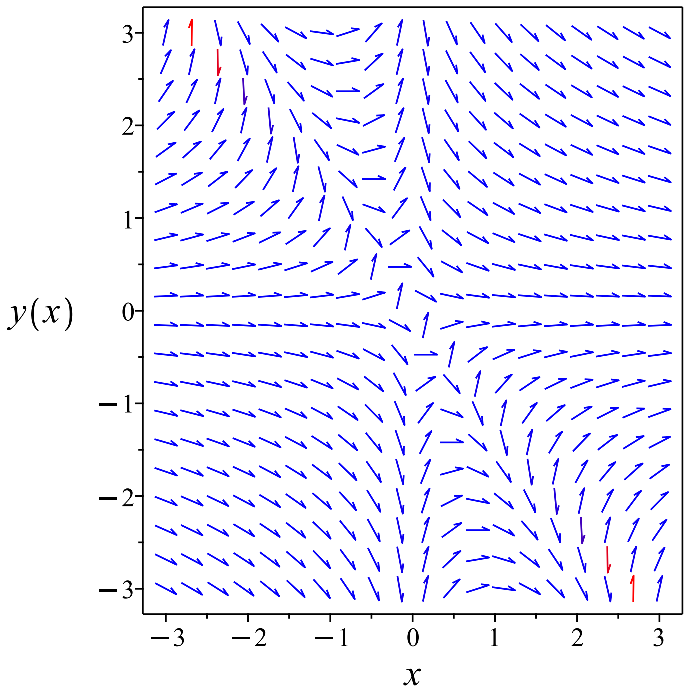


Figure 378: Slope field plot

Verification of solutions

$$y^3 x(3x + 2y) = c_1$$

Verified OK.

6.12.4 Maple step by step solution

Let's solve

$6xy^3 + 2y^4 + (9x^2y^2 + 8xy^3)\, y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ∘ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y)=0$

- Compute derivative of lhs

  $F'(x,y)+\left(\frac{\partial}{\partial y}F(x,y)\right)y'=0$

- Evaluate derivatives

  $18x\,y^2+8y^3=18x\,y^2+8y^3$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y)=\int(6x\,y^3+2y^4)\,dx+f_1(y)$

- Evaluate integral

  $F(x,y)=2y^3\left(\frac{3}{2}x^2+yx\right)+f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $9x^2y^2+8x\,y^3=6y^2\left(\frac{3}{2}x^2+yx\right)+2x\,y^3+\frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y)=9x^2y^2+6x\,y^3-6y^2\left(\frac{3}{2}x^2+yx\right)$

- Solve for $f_1(y)$

  $f_1(y)=0$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y)=2y^3\left(\frac{3}{2}x^2+yx\right)$

- Substitute $F(x,y)$ into the solution of the ODE

  $2y^3\left(\frac{3}{2}x^2+yx\right)=c_1$

- Solve for y

  $y=RootOf\left(2x\_Z^4+3x^2\_Z^3-c_1\right)$

Maple trace

```
`Classification methods on request
Methods to be used are: [exact]
----------------------------
* Tackling ODE using method: exact
--- Trying classification methods ---
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 25

```
dsolve(6*x*y(x)^3+2*y(x)^4+(9*x^2*y(x)^2+8*x*y(x)^3)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = 0$$
$$3x^2 y(x)^3 + 2xy(x)^4 + c_1 = 0$$

✓ Solution by Mathematica

Time used: 60.142 (sec). Leaf size: 1714

```
DSolve[6*x*y[x]^3+2*y[x]^4+(9*x^2*y[x]^2+8*x*y[x]^3)*y'[x] == 0,y[x],x,IncludeSingularSoluti
```

$$y(x) \to 0$$

$$y(x) \to \frac{1}{2}\sqrt{\frac{9x^2}{16} - \frac{4\sqrt[3]{\frac{2}{3}}e^{c_1}}{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}} + \frac{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}}{2\sqrt[3]{2}3^{2/3}x}} - \frac{1}{2}\sqrt{\frac{9x^2}{8} + \frac{4\sqrt[3]{\frac{2}{3}}e^{c_1}}{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}} - \frac{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x}}{2\sqrt[3]{2}3^{2/3}x}} - \frac{3x}{8}$$

$$y(x) \to \frac{1}{2}\sqrt{\frac{9x^2}{16} - \frac{4\sqrt[3]{\frac{2}{3}}e^{c_1}}{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}} + \frac{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}}{2\sqrt[3]{2}3^{2/3}x}} + \frac{1}{2}\sqrt{\frac{9x^2}{8} + \frac{4\sqrt[3]{\frac{2}{3}}e^{c_1}}{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}} - \frac{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x}}{2\sqrt[3]{2}3^{2/3}x}} - \frac{3x}{8}$$

$$y(x) \to -\frac{1}{2}\sqrt{\frac{9x^2}{16} - \frac{4\sqrt[3]{\frac{2}{3}}e^{c_1}}{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}} + \frac{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x}}{2\sqrt[3]{2}3^{2/3}x}} - \frac{1}{2}\sqrt{\frac{9x^2}{8} + \frac{4\sqrt[3]{\frac{2}{3}}e^{c_1}}{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}x^4}} - \frac{\sqrt[3]{\sqrt{3}\sqrt{e^{2c_1}x^3(2187x^5+2048e^{c_1})} - 81e^{c_1}}}{2\sqrt[3]{2}3^{2/3}x}} - \frac{3x}{8}$$

6.13 problem 13

Internal problem ID [133]
Internal file name [`OUTPUT/133_Sunday_June_05_2022_01_35_47_AM_87580713/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 13.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"unknown"**

Maple gives the following as the ode type

```
[`y=_G(x,y')`]
```

Unable to solve or complete the solution.

$$\boxed{y' - y^2 - x^2y^4 = x^2 + 1}$$

Unable to determine ODE type.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
Looking for potential symmetries
trying inverse_Riccati
trying an equivalence to an Abel ODE
differential order: 1; trying a linearization to 2nd order
--- trying a change of variables {x -> y(x), y(x) -> x}
differential order: 1; trying a linearization to 2nd order
trying 1st order ODE linearizable_by_differentiation
--- Trying Lie symmetry methods, 1st order ---
`, `-> Computing symmetries using: way = 3
`, `-> Computing symmetries using: way = 4
`, `-> Computing symmetries using: way = 5
trying symmetry patterns for 1st order ODEs
-> trying a symmetry pattern of the form [F(x)*G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)*G(y)]
-> trying symmetry patterns of the forms [F(x),G(y)] and [G(y),F(x)]
-> trying a symmetry pattern of the form [F(x),G(x)]
-> trying a symmetry pattern of the form [F(y),G(y)]
-> trying a symmetry pattern of the form [F(x)+G(y), 0]
-> trying a symmetry pattern of the form [0, F(x)+G(y)]
-> trying a symmetry pattern of the form [F(x),G(x)*y+H(x)]
-> trying a symmetry pattern of conformal type`
```

✗ Solution by Maple

```
dsolve(diff(y(x),x) = 1+x^2+y(x)^2+x^2*y(x)^4,y(x), singsol=all)
```

No solution found

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[y'[x] == 1+x^2+y[x]^2+x^2*y[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

Not solved

6.14 problem 14

Internal problem ID [134]
Internal file name [`OUTPUT/134_Sunday_June_05_2022_01_35_47_AM_15054587/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 14.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{x^3y' - x^2y + y^3 = 0}$$

6.14.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\, x$ on the above ode results in new ode in $u(x)$

$$x^3(u'(x)\, x + u(x)) - x^3u(x) + u(x)^3\, x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^3}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^3$. Integrating both sides gives

$$\frac{1}{u^3}\, du = -\frac{1}{x}\, dx$$

$$\int \frac{1}{u^3} du = \int -\frac{1}{x} dx$$
$$-\frac{1}{2u^2} = -\ln(x) + c_2$$

The solution is

$$-\frac{1}{2u(x)^2} + \ln(x) - c_2 = 0$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$-\frac{x^2}{2y^2} + \ln(x) - c_2 = 0$$
$$-\frac{x^2}{2y^2} + \ln(x) - c_2 = 0$$

Summary
The solution(s) found are the following

$$-\frac{x^2}{2y^2} + \ln(x) - c_2 = 0 \tag{1}$$

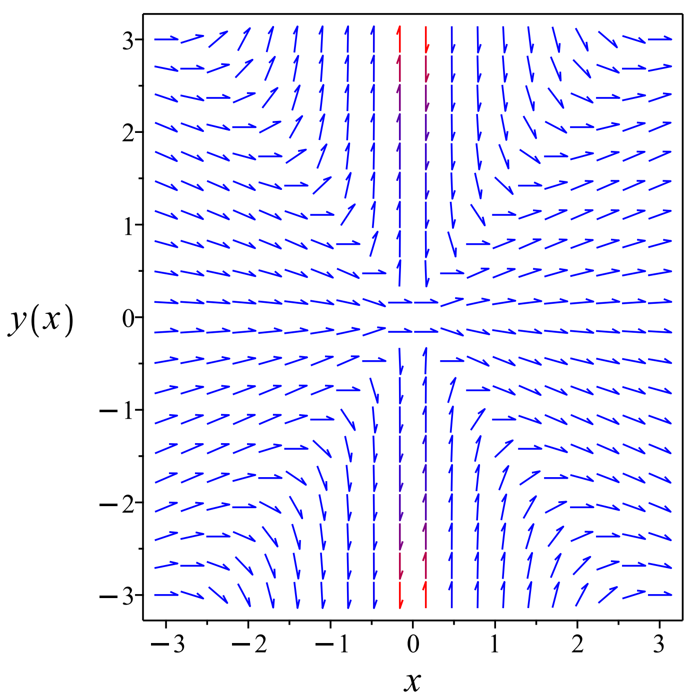


Figure 379: Slope field plot

Verification of solutions

$$-\frac{x^2}{2y^2} + \ln(x) - c_2 = 0$$

Verified OK.

6.14.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y(-x^2 + y^2)}{x^3}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 271: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{y^3}{x^2}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^2}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^2}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-x^2 + y^2)}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{x}{y^2} \\ S_y &= \frac{x^2}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\ln(R) + c_1 \tag{4}$$

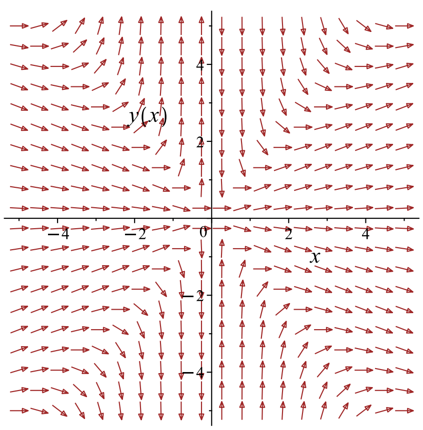
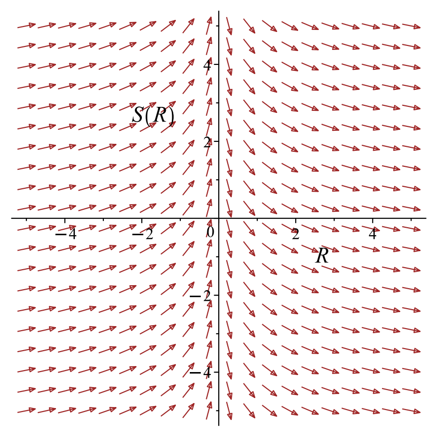
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^2}{2y^2} = -\ln(x) + c_1$$

Which simplifies to

$$-\frac{x^2}{2y^2} = -\ln(x) + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-x^2+y^2)}{x^3}$ 	$R = x$ $S = -\frac{x^2}{2y^2}$	$\frac{dS}{dR} = -\frac{1}{R}$ 

Summary
The solution(s) found are the following

$$-\frac{x^2}{2y^2} = -\ln(x) + c_1 \tag{1}$$

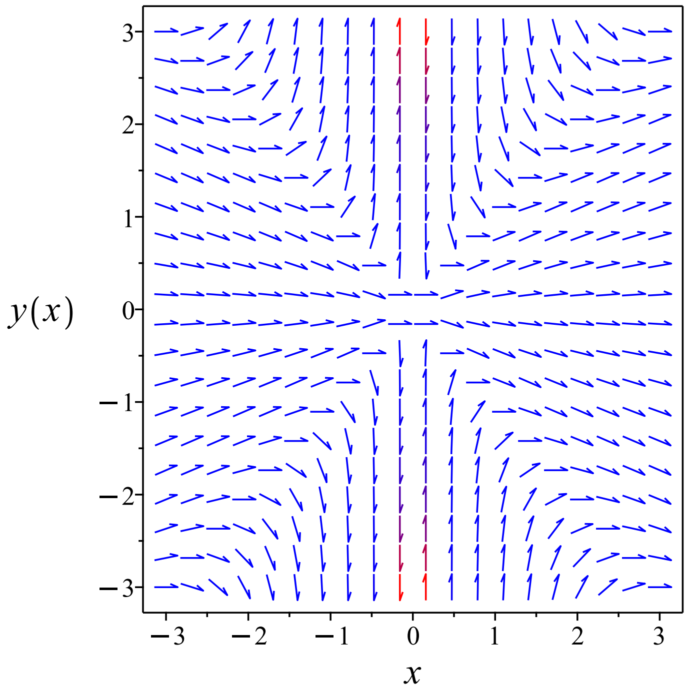


Figure 380: Slope field plot

Verification of solutions

$$-\frac{x^2}{2y^2} = -\ln(x) + c_1$$

Verified OK.

6.14.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-x^2+y^2)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{1}{x}y - \frac{1}{x^3}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{1}{x} \\ f_1(x) &= -\frac{1}{x^3} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{1}{x\, y^2} - \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= \frac{w(x)}{x} - \frac{1}{x^3} \\ w' &= -\frac{2w}{x} + \frac{2}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= \frac{2}{x^3} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{2w(x)}{x} = \frac{2}{x^3}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{2}{x}dx} \\
&= x^2
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(\frac{2}{x^3}\right) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left(w\,x^2\right) &= \left(x^2\right)\left(\frac{2}{x^3}\right) \\
\mathrm{d}\left(w\,x^2\right) &= \left(\frac{2}{x}\right)\,\mathrm{d}x
\end{aligned}$$

Integrating gives

$$\begin{aligned}
w\,x^2 &= \int \frac{2}{x}\,\mathrm{d}x \\
w\,x^2 &= 2\ln(x) + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x^2$ results in

$$w(x) = \frac{2\ln(x)}{x^2} + \frac{c_1}{x^2}$$

which simplifies to

$$w(x) = \frac{2\ln(x) + c_1}{x^2}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{2\ln(x) + c_1}{x^2}$$

Solving for y gives

$$\begin{aligned}
y(x) &= \frac{x}{\sqrt{2\ln(x) + c_1}} \\
y(x) &= -\frac{x}{\sqrt{2\ln(x) + c_1}}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x}{\sqrt{2\ln(x) + c_1}} \tag{1}$$

$$y = -\frac{x}{\sqrt{2\ln(x) + c_1}} \tag{2}$$

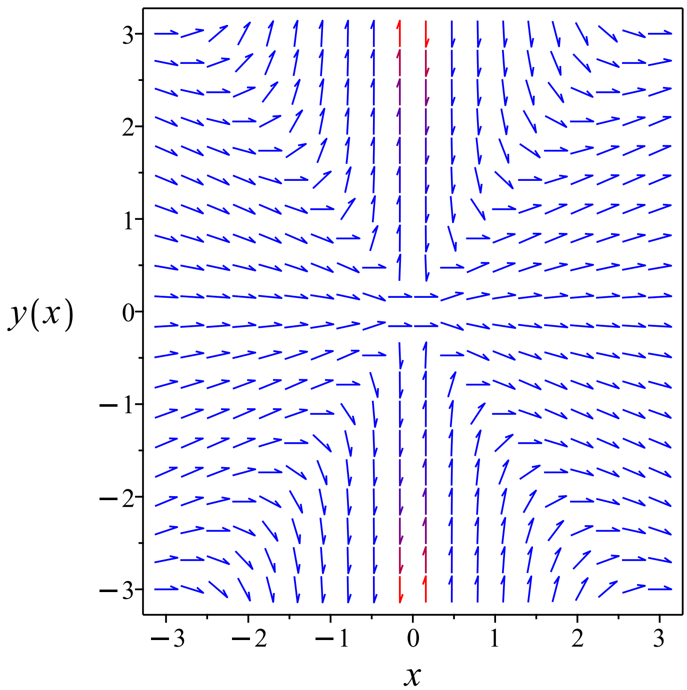


Figure 381: Slope field plot

Verification of solutions

$$y = \frac{x}{\sqrt{2\ln(x) + c_1}}$$

Verified OK.

$$y = -\frac{x}{\sqrt{2\ln(x) + c_1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 28

```
dsolve(x^3*diff(y(x),x) = x^2*y(x)-y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{x}{\sqrt{2\ln(x) + c_1}}$$
$$y(x) = -\frac{x}{\sqrt{2\ln(x) + c_1}}$$

✓ Solution by Mathematica

Time used: 0.166 (sec). Leaf size: 41

```
DSolve[x^3*y'[x] == x^2*y[x]-y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x}{\sqrt{2\log(x) + c_1}}$$
$$y(x) \to \frac{x}{\sqrt{2\log(x) + c_1}}$$
$$y(x) \to 0$$

6.15 problem 15

Internal problem ID [135]
Internal file name [`OUTPUT/135_Sunday_June_05_2022_01_35_49_AM_25014521/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 15.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_linear, `class A`]]
```

$$\boxed{3y + y' = 3x^2 e^{-3x}}$$

6.15.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 3x^2 e^{-3x} \end{aligned}$$

Hence the ode is

$$3y + y' = 3x^2 e^{-3x}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int 3dx} \\ &= e^{3x} \end{aligned}$$

The ode becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) = (\mu)\left(3x^2 \mathrm{e}^{-3x}\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y\,\mathrm{e}^{3x}\right) = \left(\mathrm{e}^{3x}\right)\left(3x^2 \mathrm{e}^{-3x}\right)$$
$$\mathrm{d}\left(y\,\mathrm{e}^{3x}\right) = \left(3x^2\right)\,\mathrm{d}x$$

Integrating gives

$$y\,\mathrm{e}^{3x} = \int 3x^2\,\mathrm{d}x$$
$$y\,\mathrm{e}^{3x} = x^3 + c_1$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{3x}$ results in

$$y = \mathrm{e}^{-3x}x^3 + c_1 \mathrm{e}^{-3x}$$

which simplifies to

$$y = \mathrm{e}^{-3x}\left(x^3 + c_1\right)$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}\left(x^3 + c_1\right) \tag{1}$$

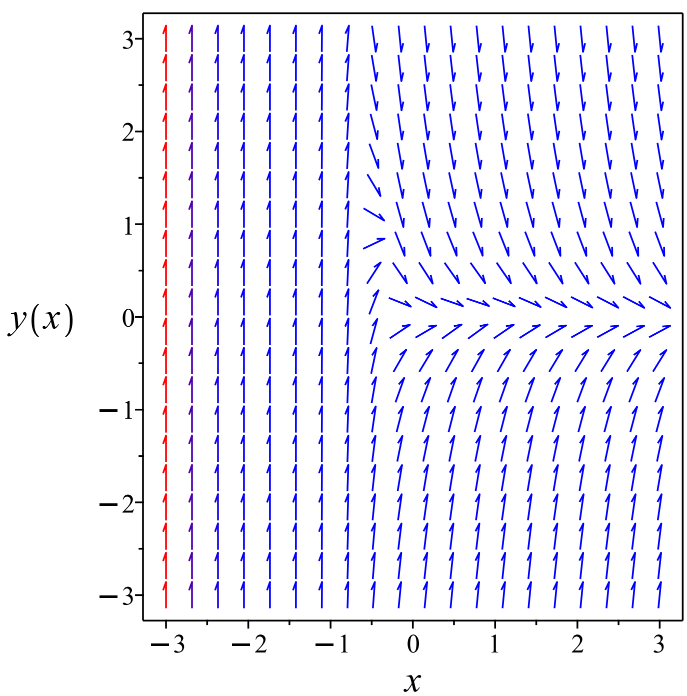


Figure 382: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^3 + c_1)$$

Verified OK.

6.15.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -3(y\,e^{3x} - x^2)\,e^{-3x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 273: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= e^{-3x}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\mathrm{e}^{-3x}} dy \end{aligned}$$

Which results in

$$S = y\,\mathrm{e}^{3x}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -3\left(y\,\mathrm{e}^{3x} - x^2\right)\mathrm{e}^{-3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\,\mathrm{e}^{3x} \\ S_y &= \mathrm{e}^{3x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3x^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3R^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = R^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$y\,\mathrm{e}^{3x} = x^3 + c_1$$

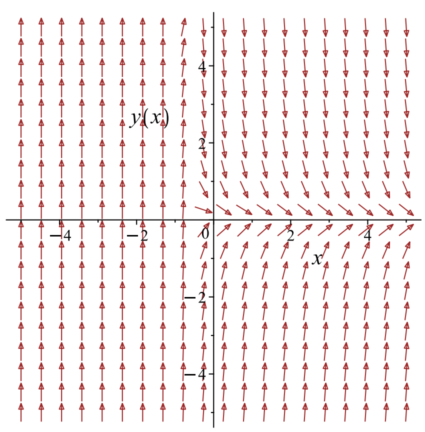
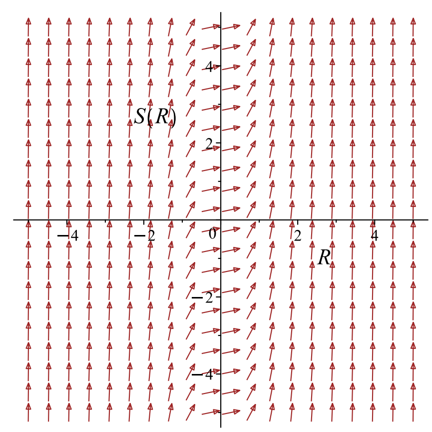
Which simplifies to

$$y\,\mathrm{e}^{3x} = x^3 + c_1$$

Which gives

$$y = \mathrm{e}^{-3x}\left(x^3 + c_1\right)$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -3(y\,\mathrm{e}^{3x} - x^2)\,\mathrm{e}^{-3x}$ 	$R = x$ $S = y\,\mathrm{e}^{3x}$	$\frac{dS}{dR} = 3R^2$ 

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-3x}\left(x^3 + c_1\right) \tag{1}$$

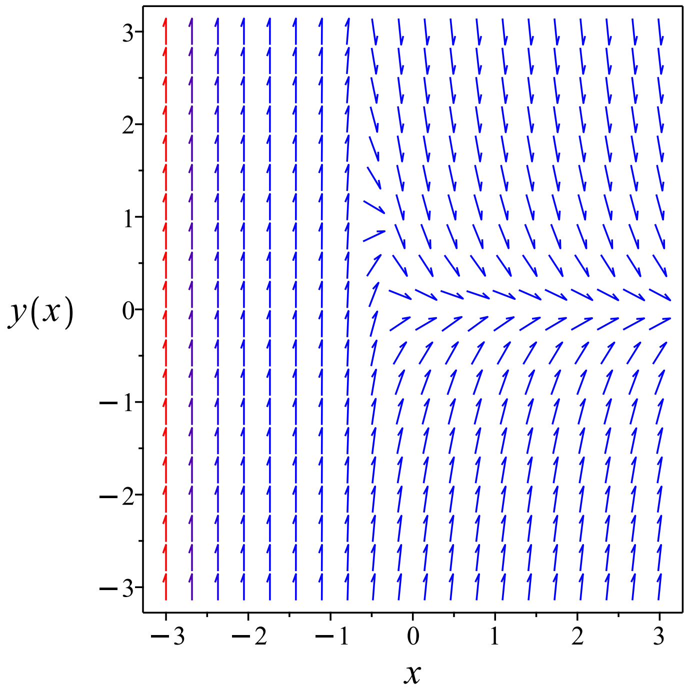


Figure 383: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^3 + c_1)$$

Verified OK.

6.15.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\mathrm{e}^{3x}\right)\mathrm{d}y = \left(-3y\,\mathrm{e}^{3x} + 3x^2\right)\mathrm{d}x$$
$$\left(3y\,\mathrm{e}^{3x} - 3x^2\right)\mathrm{d}x + \left(\mathrm{e}^{3x}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = 3y\,\mathrm{e}^{3x} - 3x^2$$
$$N(x, y) = \mathrm{e}^{3x}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y\,\mathrm{e}^{3x} - 3x^2\right) \\ &= 3\,\mathrm{e}^{3x}\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\mathrm{e}^{3x}\right) \\ &= 3\,\mathrm{e}^{3x}\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int M \,\mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x = \int 3y\,\mathrm{e}^{3x} - 3x^2 \,\mathrm{d}x$$

$$\phi = y\,\mathrm{e}^{3x} - x^3 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{3x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{3x}$. Therefore equation (4) becomes

$$\mathrm{e}^{3x} = \mathrm{e}^{3x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = y\,\mathrm{e}^{3x} - x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = y\,\mathrm{e}^{3x} - x^3$$

The solution becomes

$$y = e^{-3x}(x^3 + c_1)$$

Summary
The solution(s) found are the following

$$y = e^{-3x}(x^3 + c_1) \tag{1}$$

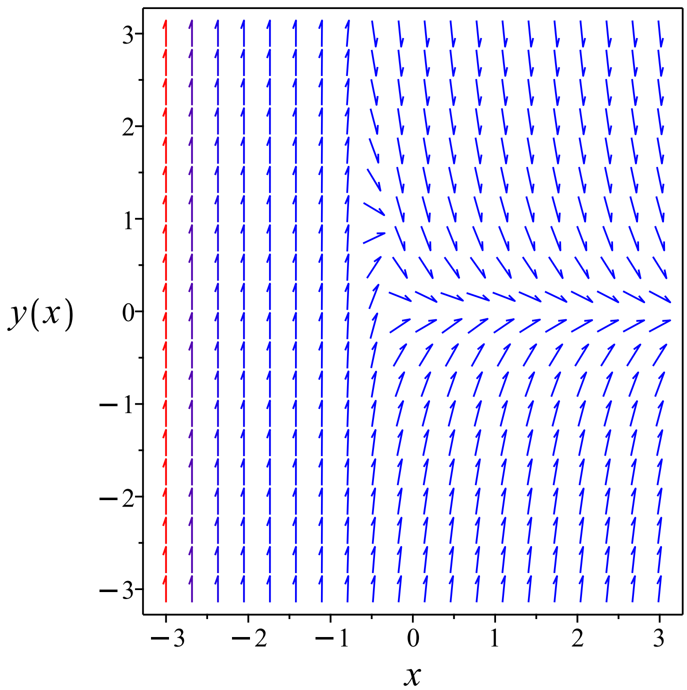


Figure 384: Slope field plot

Verification of solutions

$$y = e^{-3x}(x^3 + c_1)$$

Verified OK.

6.15.4 Maple step by step solution

Let's solve

$3y + y' = \frac{3x^2}{e^{3x}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -3y + \frac{3x^2}{e^{3x}}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $3y + y' = \frac{3x^2}{e^{3x}}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)(3y + y') = \frac{3\mu(x)x^2}{e^{3x}}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x) y)$

  $\mu(x)(3y + y') = \mu'(x) y + \mu(x) y'$

- Isolate $\mu'(x)$

  $\mu'(x) = 3\mu(x)$

- Solve to find the integrating factor

  $\mu(x) = (e^{3x})^2 e^{-3x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x) y)\right) dx = \int \frac{3\mu(x)x^2}{e^{3x}} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x) y = \int \frac{3\mu(x)x^2}{e^{3x}} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{3\mu(x)x^2}{e^{3x}} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (e^{3x})^2 e^{-3x}$

  $y = \frac{\int 3x^2 e^{-3x} e^{3x} dx + c_1}{(e^{3x})^2 e^{-3x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^3 + c_1}{(e^{3x})^2 e^{-3x}}$

- Simplify

$$y = e^{-3x}(x^3 + c_1)$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(3*y(x)+diff(y(x),x) = 3*x^2/exp(3*x),y(x), singsol=all)
```

$$y(x) = (x^3 + c_1)\, e^{-3x}$$

✓ Solution by Mathematica

Time used: 0.057 (sec). Leaf size: 17

```
DSolve[3*y[x]+y'[x] == 3*x^2/Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(x^3 + c_1)$$

6.16 problem 16

Internal problem ID [136]
Internal file name [`OUTPUT/136_Sunday_June_05_2022_01_35_49_AM_86607590/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 16.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"riccati", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class C`], _Riccati]
```

$$\boxed{2yx + y' - y^2 = x^2}$$

6.16.1 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$\begin{aligned} y' &= x^2 - 2yx + y^2 \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$\xi = xa_2 + ya_3 + a_1 \tag{1E}$$

$$\eta = xb_2 + yb_3 + b_1 \tag{2E}$$

Where the unknown coefficients are

$$\{a_1, a_2, a_3, b_1, b_2, b_3\}$$

Substituting equations (1E,2E) and ω into (A) gives

$$\begin{aligned} & b_2+\left(x^2-2yx+y^2\right)(b_3-a_2)-\left(x^2-2yx+y^2\right)^2 a_3 \\ & -(2x-2y)(xa_2+ya_3+a_1)-(-2x+2y)(xb_2+yb_3+b_1)=0 \end{aligned} \tag{5E}$$

Putting the above in normal form gives

$$\begin{aligned} & -x^4a_3+4x^3ya_3-6x^2y^2a_3+4x\,y^3a_3-y^4a_3-3x^2a_2+2x^2b_2+x^2b_3+4xya_2 \\ & -2xya_3-2xyb_2-y^2a_2+2y^2a_3-y^2b_3-2xa_1+2xb_1+2ya_1-2yb_1+b_2=0 \end{aligned}$$

Setting the numerator to zero gives

$$\begin{aligned} & -x^4a_3+4x^3ya_3-6x^2y^2a_3+4x\,y^3a_3-y^4a_3-3x^2a_2+2x^2b_2+x^2b_3+4xya_2 \\ & -2xya_3-2xyb_2-y^2a_2+2y^2a_3-y^2b_3-2xa_1+2xb_1+2ya_1-2yb_1+b_2=0 \end{aligned} \tag{6E}$$

Looking at the above PDE shows the following are all the terms with $\{x,y\}$ in them.

$$\{x,y\}$$

The following substitution is now made to be able to collect on all terms with $\{x,y\}$ in them

$$\{x=v_1,y=v_2\}$$

The above PDE (6E) now becomes

$$\begin{aligned} & -a_3v_1^4+4a_3v_1^3v_2-6a_3v_1^2v_2^2+4a_3v_1v_2^3-a_3v_2^4-3a_2v_1^2+4a_2v_1v_2-a_2v_2^2-2a_3v_1v_2 \\ & +2a_3v_2^2+2b_2v_1^2-2b_2v_1v_2+b_3v_1^2-b_3v_2^2-2a_1v_1+2a_1v_2+2b_1v_1-2b_1v_2+b_2=0 \end{aligned} \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1,v_2\}$$

Equation (7E) now becomes

$$\begin{aligned} & -a_3v_1^4+4a_3v_1^3v_2-6a_3v_1^2v_2^2+(-3a_2+2b_2+b_3)\,v_1^2 \\ & +4a_3v_1v_2^3+(4a_2-2a_3-2b_2)\,v_1v_2+(-2a_1+2b_1)\,v_1 \\ & -a_3v_2^4+(-a_2+2a_3-b_3)\,v_2^2+(2a_1-2b_1)\,v_2+b_2=0 \end{aligned} \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$
\begin{aligned}
b_2 &= 0 \\
-6a_3 &= 0 \\
-a_3 &= 0 \\
4a_3 &= 0 \\
-2a_1 + 2b_1 &= 0 \\
2a_1 - 2b_1 &= 0 \\
-3a_2 + 2b_2 + b_3 &= 0 \\
-a_2 + 2a_3 - b_3 &= 0 \\
4a_2 - 2a_3 - 2b_2 &= 0
\end{aligned}
$$

Solving the above equations for the unknowns gives

$$
\begin{aligned}
a_1 &= b_1 \\
a_2 &= 0 \\
a_3 &= 0 \\
b_1 &= b_1 \\
b_2 &= 0 \\
b_3 &= 0
\end{aligned}
$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$
\begin{aligned}
\xi &= 1 \\
\eta &= 1
\end{aligned}
$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$
\begin{aligned}
\eta &= \eta - \omega(x, y)\,\xi \\
&= 1 - \left(x^2 - 2yx + y^2\right)(1) \\
&= -x^2 + 2yx - y^2 + 1 \\
\xi &= 0
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{-x^2 + 2yx - y^2 + 1} dy \end{aligned}$$

Which results in

$$S = -\frac{\ln(-1 - x + y)}{2} + \frac{\ln(-x + y + 1)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = x^2 - 2yx + y^2$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{(x - y + 1)(x - y - 1)} \\ S_y &= -\frac{1}{(x - y + 1)(x - y - 1)} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -1 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -1$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{\ln(-1-x+y)}{2} + \frac{\ln(1-x+y)}{2} = -x + c_1$$

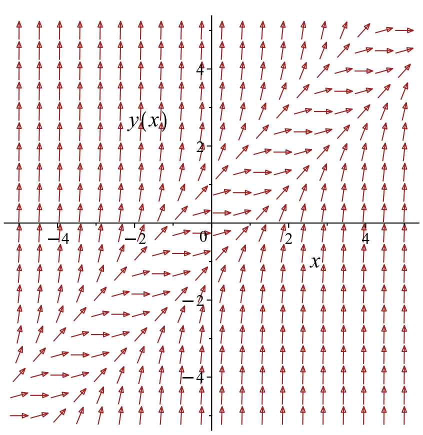
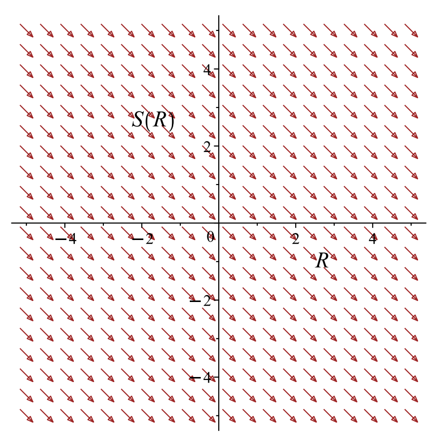
Which simplifies to

$$-\frac{\ln(-1-x+y)}{2} + \frac{\ln(1-x+y)}{2} = -x + c_1$$

Which gives

$$y = \frac{x\,\mathrm{e}^{-2x+2c_1} + \mathrm{e}^{-2x+2c_1} - x + 1}{-1 + \mathrm{e}^{-2x+2c_1}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = x^2 - 2yx + y^2$ 	$R = x$ $S = -\frac{\ln(-1-x+y)}{2} +$	$\frac{dS}{dR} = -1$ 

Summary

The solution(s) found are the following

$$y = \frac{x\,e^{-2x+2c_1} + e^{-2x+2c_1} - x + 1}{-1 + e^{-2x+2c_1}} \tag{1}$$

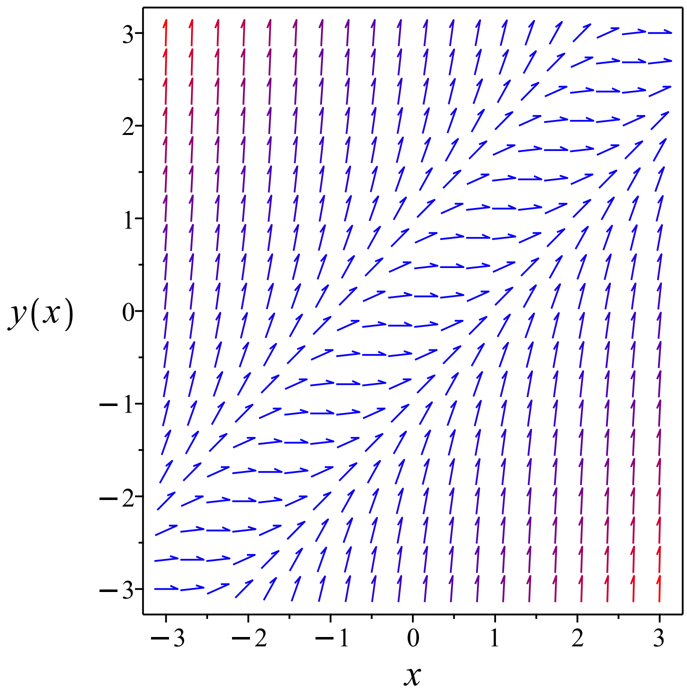


Figure 385: Slope field plot

Verification of solutions

$$y = \frac{x\,e^{-2x+2c_1} + e^{-2x+2c_1} - x + 1}{-1 + e^{-2x+2c_1}}$$

Verified OK.

6.16.2 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= x^2 - 2yx + y^2 \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = x^2 - 2yx + y^2$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = x^2$, $f_1(x) = -2x$ and $f_2(x) = 1$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{u} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= 0 \\ f_1 f_2 &= -2x \\ f_2^2 f_0 &= x^2 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$u''(x) + 2xu'(x) + x^2 u(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 \mathrm{e}^{-\frac{x(-2+x)}{2}} + c_2 \mathrm{e}^{-\frac{x(2+x)}{2}}$$

The above shows that

$$u'(x) = -c_1(x-1)\,\mathrm{e}^{-\frac{x(-2+x)}{2}} - c_2 \mathrm{e}^{-\frac{x(2+x)}{2}} (x+1)$$

Using the above in (1) gives the solution

$$y = -\frac{-c_1(x-1)\,\mathrm{e}^{-\frac{x(-2+x)}{2}} - c_2 \mathrm{e}^{-\frac{x(2+x)}{2}} (x+1)}{c_1 \mathrm{e}^{-\frac{x(-2+x)}{2}} + c_2 \mathrm{e}^{-\frac{x(2+x)}{2}}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{c_3(x-1)\,\mathrm{e}^{-\frac{x(-2+x)}{2}} + (x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{c_3\mathrm{e}^{-\frac{x(-2+x)}{2}} + \mathrm{e}^{-\frac{x(2+x)}{2}}}$$

Summary
The solution(s) found are the following

$$y = \frac{c_3(x-1)\,\mathrm{e}^{-\frac{x(-2+x)}{2}} + (x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{c_3\mathrm{e}^{-\frac{x(-2+x)}{2}} + \mathrm{e}^{-\frac{x(2+x)}{2}}} \tag{1}$$

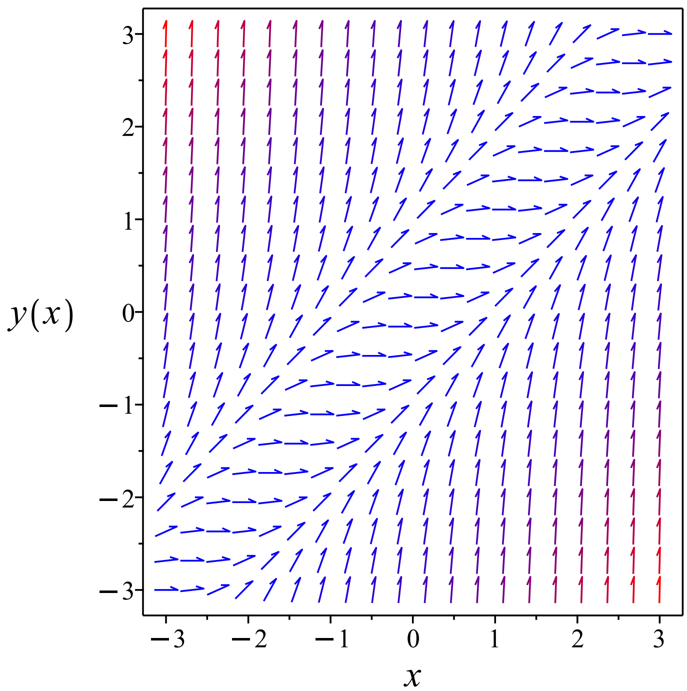


Figure 386: Slope field plot

Verification of solutions

$$y = \frac{c_3(x-1)\,\mathrm{e}^{-\frac{x(-2+x)}{2}} + (x+1)\,\mathrm{e}^{-\frac{x(2+x)}{2}}}{c_3\mathrm{e}^{-\frac{x(-2+x)}{2}} + \mathrm{e}^{-\frac{x(2+x)}{2}}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous C
1st order, trying the canonical coordinates of the invariance group
   -> Calling odsolve with the ODE`, diff(y(x), x) = 1, y(x)`      *** Sublevel 2 ***
      Methods for first order ODEs:
      --- Trying classification methods ---
      trying a quadrature
      trying 1st order linear
      <- 1st order linear successful
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve(diff(y(x),x) = x^2-2*x*y(x)+y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{c_1(x-1)\,e^{2x} - x - 1}{-1 + e^{2x}c_1}$$

✓ Solution by Mathematica

Time used: 0.127 (sec). Leaf size: 29

```
DSolve[y'[x] == x^2-2*x*y[x]+y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x + \frac{1}{\frac{1}{2} + c_1 e^{2x}} - 1$$
$$y(x) \to x - 1$$

6.17 problem 17

Internal problem ID [137]
Internal file name [`OUTPUT/137_Sunday_June_05_2022_01_35_50_AM_8857866/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 17.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{yx}y + (e^{y} + e^{yx}x)\, y' = -e^{x}}$$

6.17.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x} &= M\\ \frac{\partial\phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^{y}+\mathrm{e}^{yx}x)\,\mathrm{d}y&=(-\mathrm{e}^{x}-\mathrm{e}^{yx}y)\,\mathrm{d}x\\(\mathrm{e}^{x}+\mathrm{e}^{yx}y)\,\mathrm{d}x+(\mathrm{e}^{y}+\mathrm{e}^{yx}x)\,\mathrm{d}y&=0\end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=\mathrm{e}^{x}+\mathrm{e}^{yx}y\\N(x,y)&=\mathrm{e}^{y}+\mathrm{e}^{yx}x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(\mathrm{e}^{x}+\mathrm{e}^{yx}y)\\&=\mathrm{e}^{yx}(yx+1)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(\mathrm{e}^{y}+\mathrm{e}^{yx}x)\\&=\mathrm{e}^{yx}(yx+1)\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=M \tag{1}$$

$$\frac{\partial\phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int \mathrm{e}^{x} + \mathrm{e}^{yx} y \, \mathrm{d}x$$

$$\phi = \mathrm{e}^{yx} + \mathrm{e}^{x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \mathrm{e}^{yx} x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \mathrm{e}^{y} + \mathrm{e}^{yx} x$. Therefore equation (4) becomes

$$\mathrm{e}^{y} + \mathrm{e}^{yx} x = \mathrm{e}^{yx} x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \mathrm{e}^{y}$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (\mathrm{e}^{y}) \, \mathrm{d}y$$

$$f(y) = \mathrm{e}^{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \mathrm{e}^{yx} + \mathrm{e}^{x} + \mathrm{e}^{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \mathrm{e}^{yx} + \mathrm{e}^{x} + \mathrm{e}^{y}$$

Summary

The solution(s) found are the following

$$e^{yx} + e^{x} + e^{y} = c_1 \tag{1}$$

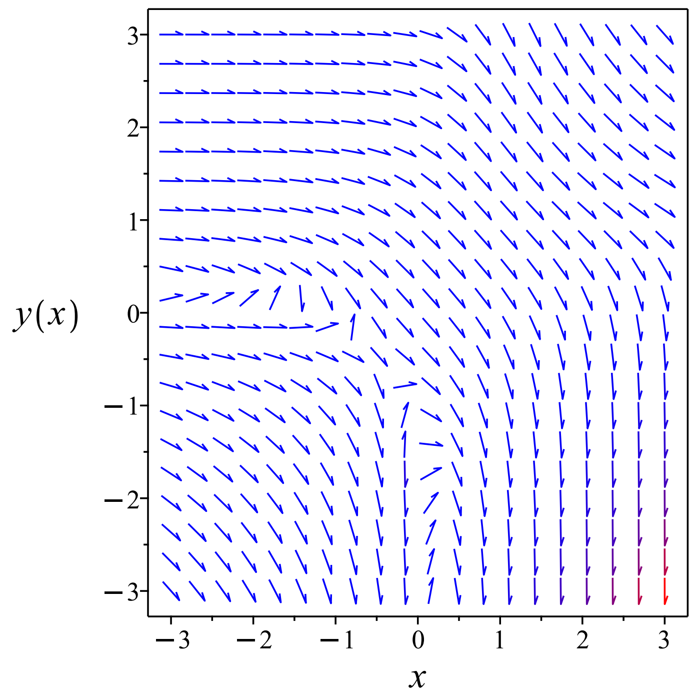


Figure 387: Slope field plot

Verification of solutions

$$e^{yx} + e^{x} + e^{y} = c_1$$

Verified OK.

6.17.2 Maple step by step solution

Let's solve

$e^{yx}y + (e^{y} + e^{yx}x)\, y' = -e^{x}$

- Highest derivative means the order of the ODE is 1

  y'

- □ Check if ODE is exact

  - ∘ ODE is exact if the lhs is the total derivative of a C^2 function

$F'(x,y) = 0$

- Compute derivative of lhs

  $F'(x,y) + \left(\frac{\partial}{\partial y}F(x,y)\right) y' = 0$

- Evaluate derivatives

  $e^{yx}yx + e^{yx} = e^{yx}yx + e^{yx}$

- Condition met, ODE is exact

- Exact ODE implies solution will be of this form

  $\left[F(x,y) = c_1, M(x,y) = F'(x,y), N(x,y) = \frac{\partial}{\partial y}F(x,y)\right]$

- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x

  $F(x,y) = \int (e^x + e^{yx}y)\, dx + f_1(y)$

- Evaluate integral

  $F(x,y) = e^{yx} + e^x + f_1(y)$

- Take derivative of $F(x,y)$ with respect to y

  $N(x,y) = \frac{\partial}{\partial y}F(x,y)$

- Compute derivative

  $e^y + e^{yx}x = e^{yx}x + \frac{d}{dy}f_1(y)$

- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = e^y$

- Solve for $f_1(y)$

  $f_1(y) = e^y$

- Substitute $f_1(y)$ into equation for $F(x,y)$

  $F(x,y) = e^{yx} + e^x + e^y$

- Substitute $F(x,y)$ into the solution of the ODE

  $e^{yx} + e^x + e^y = c_1$

- Solve for y

  $y = \mathit{RootOf}\left(-e^{\_Zx} - e^x - e^{\_Z} + c_1\right)$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(exp(x)+exp(x*y(x))*y(x)+(exp(y(x))+exp(x*y(x))*x)*diff(y(x),x) = 0,y(x), singsol=all)
```

$$e^{xy(x)} + e^{x} + e^{y(x)} + c_1 = 0$$

✓ Solution by Mathematica

Time used: 0.214 (sec). Leaf size: 20

```
DSolve[Exp[x]+Exp[x*y[x]]*y[x]+(Exp[y[x]]+Exp[x*y[x]]*x)*y'[x] == 0,y[x],x,IncludeSingularSo
```

$$\text{Solve}\left[e^{y(x)} + e^{xy(x)} + e^{x} = c_1, y(x)\right]$$

6.18 problem 18

Internal problem ID [138]
Internal file name [`OUTPUT/138_Sunday_June_05_2022_01_35_52_AM_85319751/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 18.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{2x^2y - x^3y' - y^3 = 0}$$

6.18.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$2x^3u(x) - x^3(u'(x)\,x + u(x)) - u(x)^3\,x^3 = 0$$

In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= -\frac{u^3-u}{x}\end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u^3 - u$. Integrating both sides gives

$$\frac{1}{u^3-u}\,du = -\frac{1}{x}\,dx$$

$$\int \frac{1}{u^3-u}\,du = \int -\frac{1}{x}\,dx$$

$$-\ln(u)+\frac{\ln(u+1)}{2}+\frac{\ln(u-1)}{2} = -\ln(x)+c_2$$

Raising both side to exponential gives

$$e^{-\ln(u)+\frac{\ln(u+1)}{2}+\frac{\ln(u-1)}{2}} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\frac{\sqrt{u+1}\,\sqrt{u-1}}{u} = \frac{c_3}{x}$$

The solution is

$$\frac{\sqrt{u(x)+1}\,\sqrt{u(x)-1}}{u(x)} = \frac{c_3}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\frac{x\sqrt{1+\frac{y}{x}}\,\sqrt{\frac{y}{x}-1}}{y} = \frac{c_3}{x}$$

$$\frac{x\sqrt{\frac{x+y}{x}}\,\sqrt{\frac{-x+y}{x}}}{y} = \frac{c_3}{x}$$

Summary

The solution(s) found are the following

$$\frac{x\sqrt{\frac{x+y}{x}}\,\sqrt{\frac{-x+y}{x}}}{y} = \frac{c_3}{x} \tag{1}$$

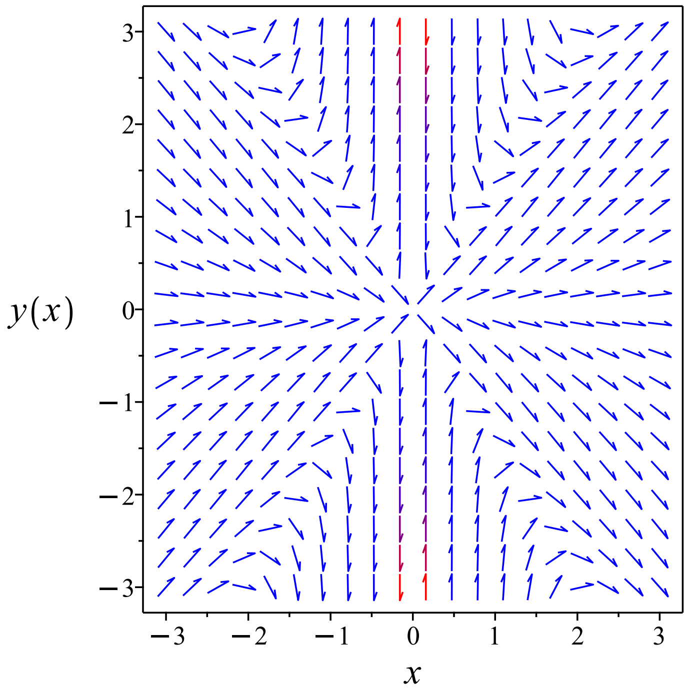


Figure 388: Slope field plot

Verification of solutions

$$\frac{x\sqrt{\frac{x+y}{x}}\,\sqrt{\frac{-x+y}{x}}}{y}=\frac{c_3}{x}$$

Verified OK.

6.18.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y'=-\frac{y(-2x^2+y^2)}{x^3}$$
$$y'=\omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x+\omega(\eta_y-\xi_x)-\omega^2\xi_y-\omega_x\xi-\omega_y\eta=0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ,η

Table 277: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \frac{y^3}{x^4}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{y^3}{x^4}} dy \end{aligned}$$

Which results in

$$S = -\frac{x^4}{2y^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{y(-2x^2 + y^2)}{x^3}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2x^3}{y^2} \\ S_y &= \frac{x^4}{y^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{R^2}{2} + c_1 \tag{4}$$

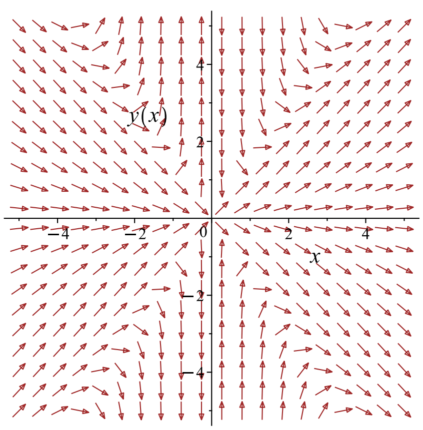
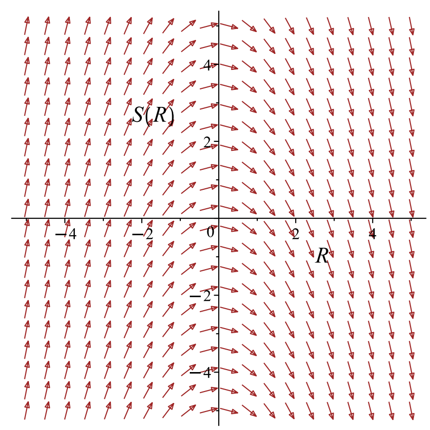
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{x^4}{2y^2} = -\frac{x^2}{2} + c_1$$

Which simplifies to

$$-\frac{x^4}{2y^2} = -\frac{x^2}{2} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(-2x^2+y^2)}{x^3}$ 	$R = x$ $S = -\frac{x^4}{2y^2}$	$\frac{dS}{dR} = -R$ 

Summary
The solution(s) found are the following

$$-\frac{x^4}{2y^2} = -\frac{x^2}{2} + c_1 \tag{1}$$

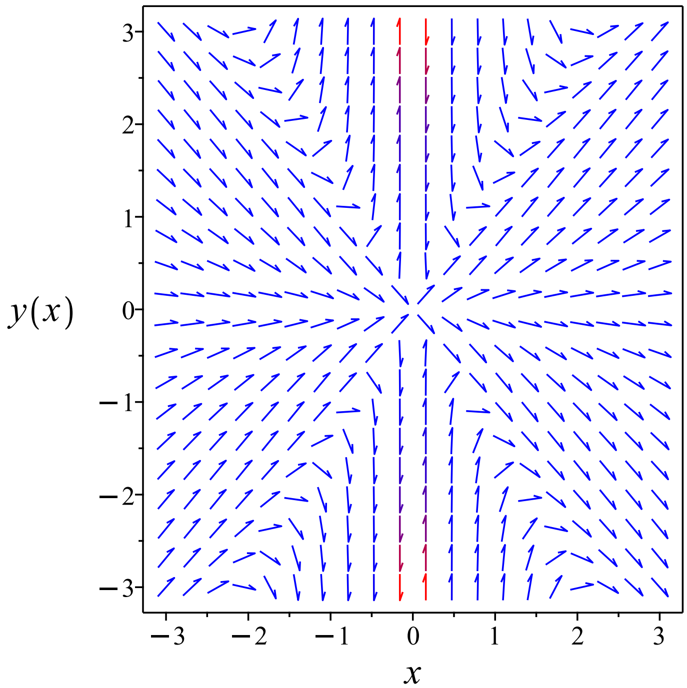


Figure 389: Slope field plot

Verification of solutions

$$-\frac{x^4}{2y^2} = -\frac{x^2}{2} + c_1$$

Verified OK.

6.18.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(-2x^2+y^2)}{x^3} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{2}{x}y - \frac{1}{x^3}y^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{2}{x} \\ f_1(x) &= -\frac{1}{x^3} \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = \frac{2}{x\, y^2} - \frac{1}{x^3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= \frac{2w(x)}{x} - \frac{1}{x^3} \\ w' &= -\frac{4w}{x} + \frac{2}{x^3} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{4}{x} \\ q(x) &= \frac{2}{x^3} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{4w(x)}{x} = \frac{2}{x^3}$$

The integrating factor μ is

$$\mu = e^{\int \frac{4}{x}dx}$$
$$= x^4$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(\frac{2}{x^3}\right)$$
$$\frac{d}{dx}(x^4 w) = (x^4)\left(\frac{2}{x^3}\right)$$
$$d(x^4 w) = (2x)\, dx$$

Integrating gives

$$x^4 w = \int 2x\, dx$$
$$x^4 w = x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = x^4$ results in

$$w(x) = \frac{1}{x^2} + \frac{c_1}{x^4}$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$\frac{1}{y^2} = \frac{1}{x^2} + \frac{c_1}{x^4}$$

Solving for y gives

$$y(x) = \frac{x^2}{\sqrt{x^2 + c_1}}$$
$$y(x) = -\frac{x^2}{\sqrt{x^2 + c_1}}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{\sqrt{x^2 + c_1}} \tag{1}$$
$$y = -\frac{x^2}{\sqrt{x^2 + c_1}} \tag{2}$$

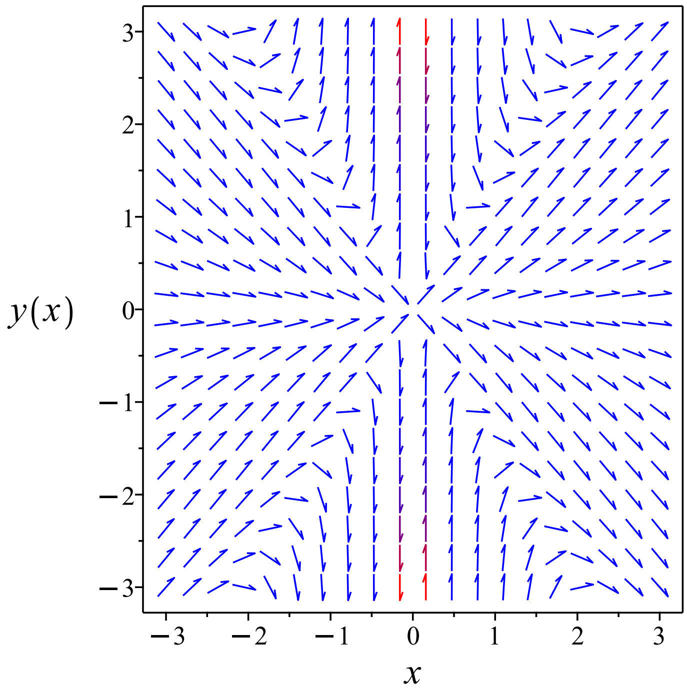


Figure 390: Slope field plot

Verification of solutions

$$y = \frac{x^2}{\sqrt{x^2 + c_1}}$$

Verified OK.

$$y = -\frac{x^2}{\sqrt{x^2 + c_1}}$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(2*x^2*y(x)-x^3*diff(y(x),x) = y(x)^3,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{\sqrt{x^2 + c_1}}$$
$$y(x) = -\frac{x^2}{\sqrt{x^2 + c_1}}$$

✓ Solution by Mathematica

Time used: 0.194 (sec). Leaf size: 43

```
DSolve[2*x^2*y[x]-x^3*y'[x] == y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{x^2}{\sqrt{x^2 + c_1}}$$
$$y(x) \to \frac{x^2}{\sqrt{x^2 + c_1}}$$
$$y(x) \to 0$$

6.19 problem 19

Internal problem ID [139]
Internal file name [`OUTPUT/139_Sunday_June_05_2022_01_35_53_AM_56391137/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 19.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{3y^2x^5 + x^3y' - 2y^2 = 0}$$

6.19.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2(3x^5-2)}{x^3} \end{aligned}$$

Where $f(x) = -\frac{3x^5-2}{x^3}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = -\frac{3x^5-2}{x^3}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -\frac{3x^5-2}{x^3}\,dx$$

$$-\frac{1}{y} = -x^3 - \frac{1}{x^2} + c_1$$

Which results in

$$y = -\frac{x^2}{-x^5 + c_1 x^2 - 1}$$

Summary

The solution(s) found are the following

$$y = -\frac{x^2}{-x^5 + c_1 x^2 - 1} \tag{1}$$

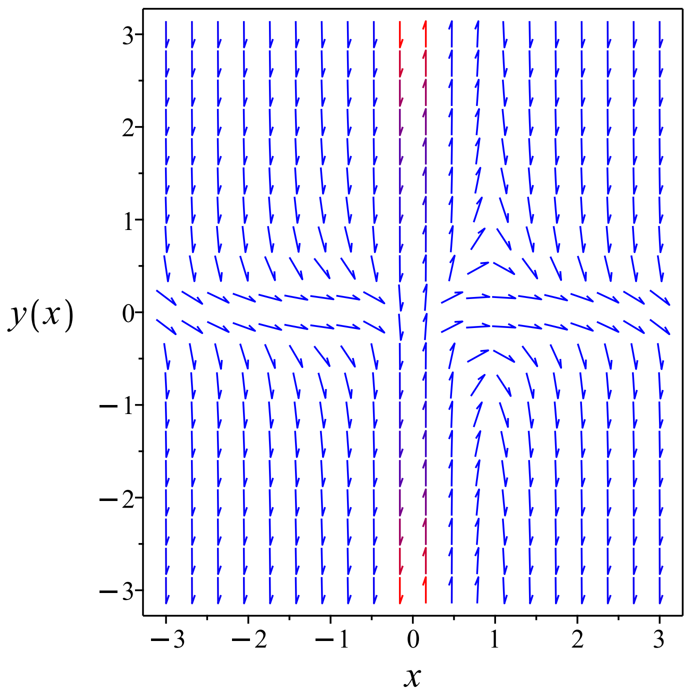


Figure 391: Slope field plot

Verification of solutions

$$y = -\frac{x^2}{-x^5 + c_1 x^2 - 1}$$

Verified OK.

6.19.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2(3x^5-2)}{x^3}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 279: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}\xi(x,y) &= -\frac{x^3}{3x^5-2} \\ \eta(x,y) &= 0 \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\xi} dx \\ &= \int \frac{1}{-\frac{x^3}{3x^5-2}} dx \end{aligned}$$

Which results in

$$S = -x^3 - \frac{1}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2(3x^5-2)}{x^3}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-3x^5 + 2}{x^3} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{-x^5 - 1}{x^2} = -\frac{1}{y} + c_1$$

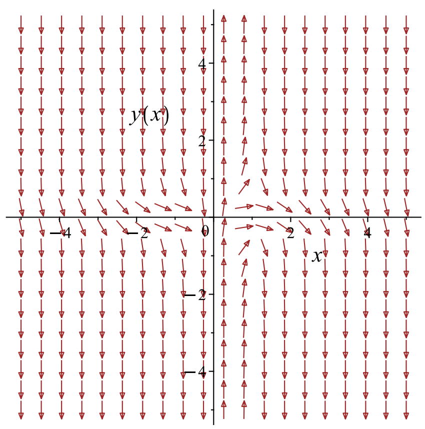
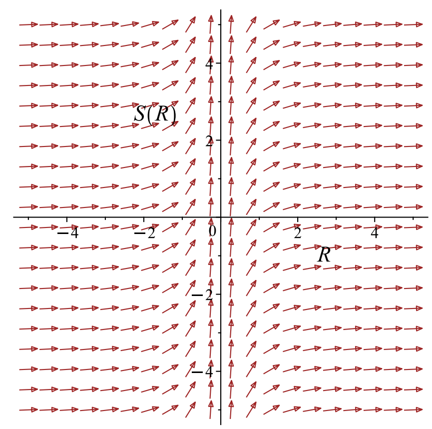
Which simplifies to

$$\frac{-x^5 - 1}{x^2} = -\frac{1}{y} + c_1$$

Which gives

$$y = \frac{x^2}{x^5 + c_1 x^2 + 1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2(3x^5-2)}{x^3}$ 	$R = y$ $S = \frac{-x^5-1}{x^2}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{x^2}{x^5 + c_1 x^2 + 1} \tag{1}$$

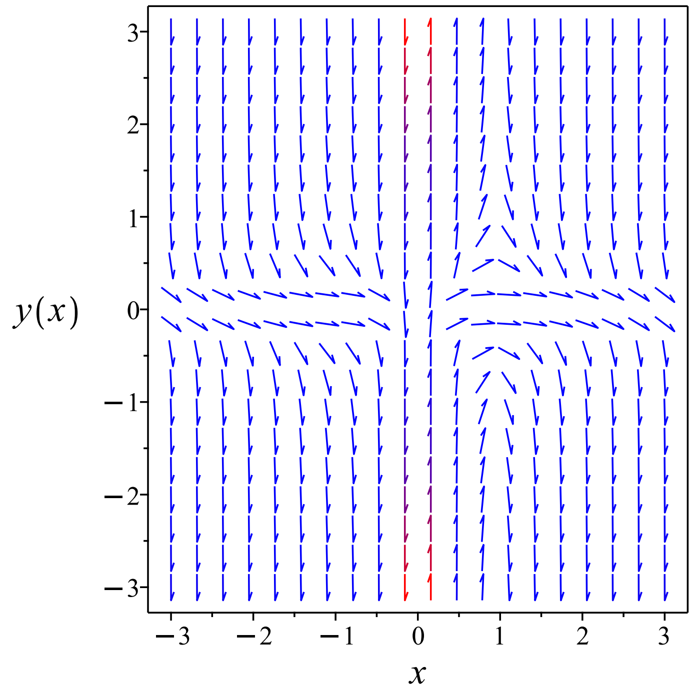


Figure 392: Slope field plot

Verification of solutions

$$y = \frac{x^2}{x^5 + c_1 x^2 + 1}$$

Verified OK.

6.19.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(-\frac{1}{y^2}\right) \mathrm{d}y &= \left(\frac{3x^5 - 2}{x^3}\right) \mathrm{d}x \\ \left(-\frac{3x^5 - 2}{x^3}\right) \mathrm{d}x + \left(-\frac{1}{y^2}\right) \mathrm{d}y &= 0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= -\frac{3x^5 - 2}{x^3} \\ N(x, y) &= -\frac{1}{y^2}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{3x^5 - 2}{x^3}\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right)\\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \qquad (1)\\
\frac{\partial \phi}{\partial y} &= N \qquad (2)
\end{aligned}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int M \,\mathrm{d}x\\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int -\frac{3x^5 - 2}{x^3} \,\mathrm{d}x\\
\phi &= \frac{-x^5 - 1}{x^2} + f(y) \qquad (3)
\end{aligned}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \qquad (4)
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$
-\frac{1}{y^2} = 0 + f'(y) \qquad (5)
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^2}\right)\mathrm{d}y$$
$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-x^5 - 1}{x^2} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-x^5 - 1}{x^2} + \frac{1}{y}$$

The solution becomes

$$y = \frac{x^2}{x^5 + c_1 x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{x^5 + c_1 x^2 + 1} \tag{1}$$

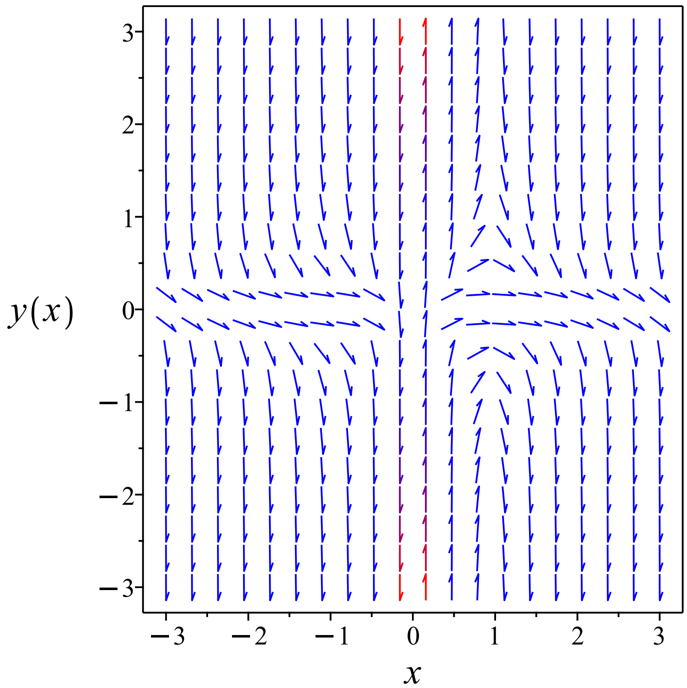


Figure 393: Slope field plot

Verification of solutions

$$y = \frac{x^2}{x^5 + c_1 x^2 + 1}$$

Verified OK.

6.19.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2(3x^5 - 2)}{x^3} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -3x^2y^2 + \frac{2y^2}{x^3}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\frac{3x^5-2}{x^3}$. Let

$$\begin{aligned} y &= \frac{-u'}{f_2 u} \\ &= \frac{-u'}{-\frac{(3x^5-2)u}{x^3}} \end{aligned} \tag{1}$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}$$

But

$$\begin{aligned} f_2' &= -15x + \frac{9x^5-6}{x^4} \\ f_1 f_2 &= 0 \\ f_2^2 f_0 &= 0 \end{aligned}$$

Substituting the above terms back in equation (2) gives

$$-\frac{(3x^5-2)\, u''(x)}{x^3} - \left(-15x + \frac{9x^5-6}{x^4}\right) u'(x) = 0$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$u(x) = c_1 + \frac{(x^5+1)\, c_2}{x^2}$$

The above shows that

$$u'(x) = \frac{c_2(3x^5-2)}{x^3}$$

Using the above in (1) gives the solution

$$y = \frac{c_2}{c_1 + \frac{(x^5+1)c_2}{x^2}}$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{x^2}{x^5 + c_3 x^2 + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2}{x^5 + c_3 x^2 + 1} \tag{1}$$

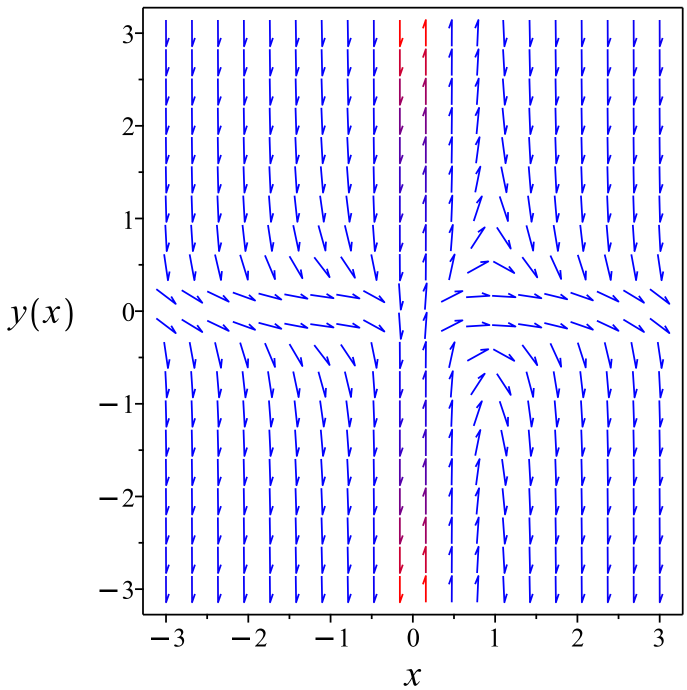


Figure 394: Slope field plot

Verification of solutions

$$y = \frac{x^2}{x^5 + c_3 x^2 + 1}$$

Verified OK.

6.19.5 Maple step by step solution

Let's solve

$3y^2x^5 + x^3y' - 2y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

$\frac{y'}{y^2} = -\frac{3x^5-2}{x^3}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2} dx = \int -\frac{3x^5-2}{x^3} dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -x^3 - \frac{1}{x^2} + c_1$

- Solve for y

  $y = -\frac{x^2}{-x^5+c_1x^2-1}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(3*x^5*y(x)^2+x^3*diff(y(x),x) = 2*y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{x^2}{x^5 + c_1x^2 + 1}$$

✓ Solution by Mathematica

Time used: 0.143 (sec). Leaf size: 28

```
DSolve[3*x^5*y[x]^2+x^3*y'[x] == 2*y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^2}{x^5 - c_1x^2 + 1}$$
$$y(x) \to 0$$

6.20 problem 20

Internal problem ID [140]
Internal file name [`OUTPUT/140_Sunday_June_05_2022_01_35_53_AM_10000095/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 20.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{3y + y'x = \frac{3}{x^{\frac{3}{2}}}}$$

6.20.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{3}{x}$$
$$q(x) = \frac{3}{x^{\frac{5}{2}}}$$

Hence the ode is

$$y' + \frac{3y}{x} = \frac{3}{x^{\frac{5}{2}}}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{3}{x}dx} \\
&= x^3
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{3}{x^{\frac{5}{2}}}\right) \\
\frac{d}{dx}\left(y\,x^3\right) &= \left(x^3\right)\left(\frac{3}{x^{\frac{5}{2}}}\right) \\
d\left(y\,x^3\right) &= \left(3\sqrt{x}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
y\,x^3 &= \int 3\sqrt{x}\,dx \\
y\,x^3 &= 2x^{\frac{3}{2}} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x^3$ results in

$$
y = \frac{2}{x^{\frac{3}{2}}} + \frac{c_1}{x^3}
$$

Summary

The solution(s) found are the following

$$
y = \frac{2}{x^{\frac{3}{2}}} + \frac{c_1}{x^3} \tag{1}
$$

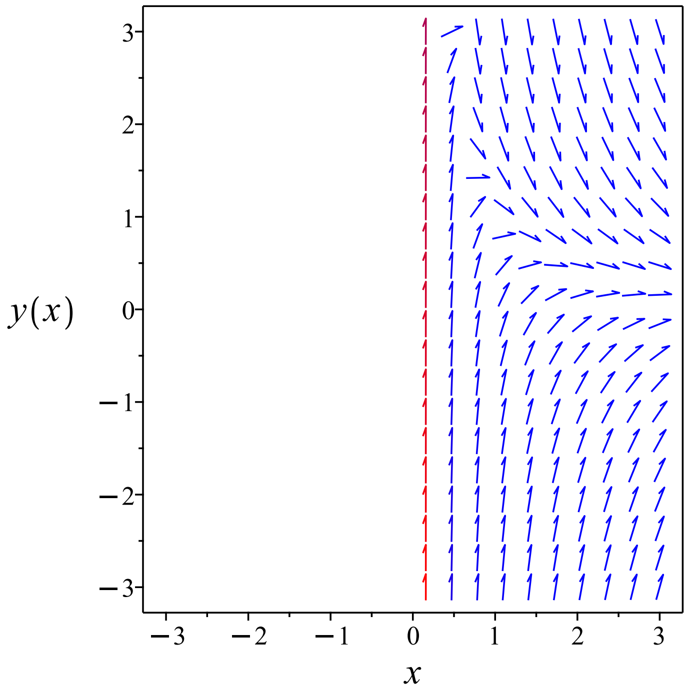


Figure 395: Slope field plot

Verification of solutions

$$y = \frac{2}{x^{\frac{3}{2}}} + \frac{c_1}{x^3}$$

Verified OK.

6.20.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3\left(y\,x^{\frac{3}{2}} - 1\right)}{x^{\frac{5}{2}}}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 282: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$	$\frac{a_1 b_2 x - a_2 b_1 x - b_1 c_2 + b_2 c_1}{a_1 b_2 - a_2 b_1}$	$\frac{a_1 b_2 y - a_2 b_1 y - a_1 c_2 - a_2 c_1}{a_1 b_2 - a_2 b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1) f(x) dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x^3}} dy \end{aligned}$$

Which results in

$$S = y\, x^3$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3\left(y\, x^{\frac{3}{2}} - 1\right)}{x^{\frac{5}{2}}}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= 3y\, x^2 \\ S_y &= x^3 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3\sqrt{x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3\sqrt{R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 2R^{\frac{3}{2}} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx^3 = c_1 + 2x^{\frac{3}{2}}$$

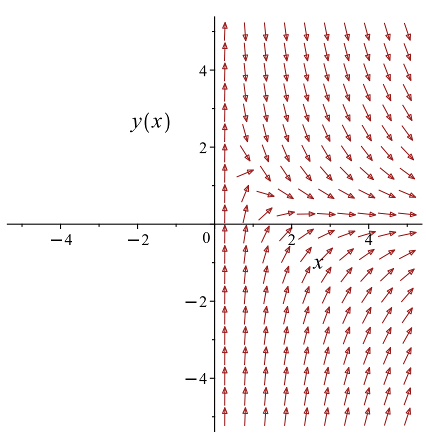
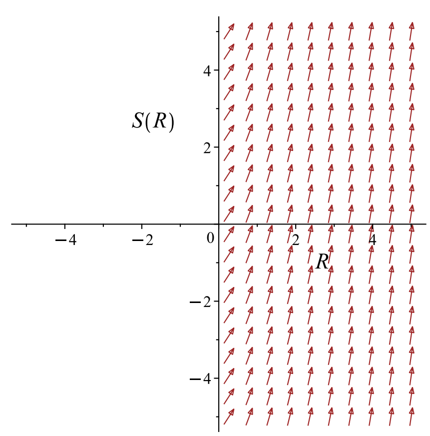
Which simplifies to

$$yx^3 = c_1 + 2x^{\frac{3}{2}}$$

Which gives

$$y = \frac{c_1 + 2x^{\frac{3}{2}}}{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3\left(y\,x^{\frac{3}{2}}-1\right)}{x^{\frac{5}{2}}}$ 	$R = x$ $S = y\,x^3$	$\frac{dS}{dR} = 3\sqrt{R}$ 

Summary
The solution(s) found are the following

$$y = \frac{c_1 + 2x^{\frac{3}{2}}}{x^3} \tag{1}$$

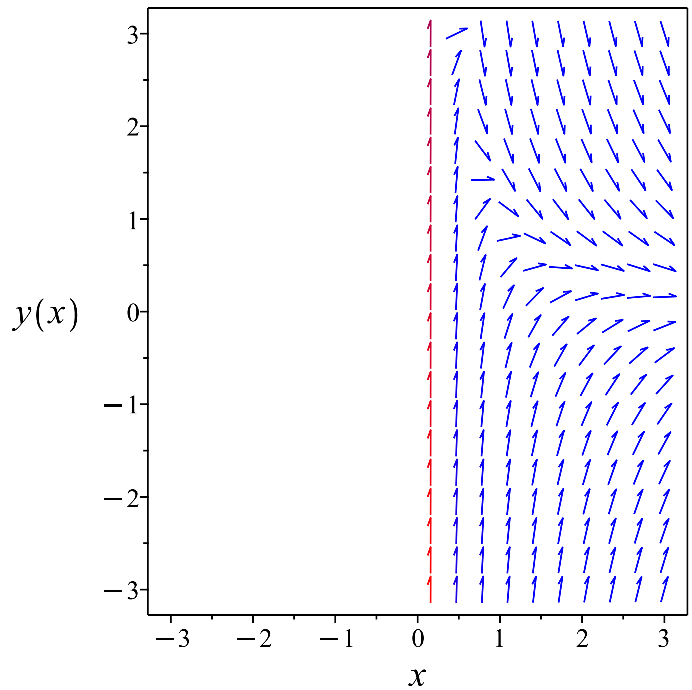


Figure 396: Slope field plot

Verification of solutions

$$y = \frac{c_1 + 2x^{\frac{3}{2}}}{x^3}$$

Verified OK.

6.20.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the

ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x)\,\mathrm{d}y = \left(-3y + \frac{3}{x^{\frac{3}{2}}}\right)\mathrm{d}x$$

$$\left(3y - \frac{3}{x^{\frac{3}{2}}}\right)\mathrm{d}x + (x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y) &= 3y - \frac{3}{x^{\frac{3}{2}}} \\ N(x,y) &= x\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(3y - \frac{3}{x^{\frac{3}{2}}}\right)\\ &= 3\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x)\\ &= 1\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)\\ &= \frac{1}{x}((3) - (1))\\ &= \frac{2}{x}\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int A\,\mathrm{d}x}\\ &= e^{\int \frac{2}{x}\,\mathrm{d}x}\end{aligned}$$

The result of integrating gives

$$\begin{aligned}\mu &= e^{2\ln(x)}\\ &= x^2\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}\overline{M} &= \mu M\\ &= x^2\left(3y - \frac{3}{x^{\frac{3}{2}}}\right)\\ &= 3\left(y - \frac{1}{x^{\frac{3}{2}}}\right)x^2\end{aligned}$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= x^2(x) \\
&= x^3
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(3\left(y - \frac{1}{x^{\frac{3}{2}}}\right)x^2\right) + \left(x^3\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int 3\left(y - \frac{1}{x^{\frac{3}{2}}}\right)x^2\,\mathrm{d}x
\end{aligned}
$$

$$
\phi = -2x^{\frac{3}{2}} + y\,x^3 + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = x^3 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x^3$. Therefore equation (4) becomes

$$
x^3 = x^3 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = 0
$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -2x^{\frac{3}{2}} + y\,x^3 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -2x^{\frac{3}{2}} + y\,x^3$$

The solution becomes

$$y = \frac{c_1 + 2x^{\frac{3}{2}}}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 + 2x^{\frac{3}{2}}}{x^3} \tag{1}$$

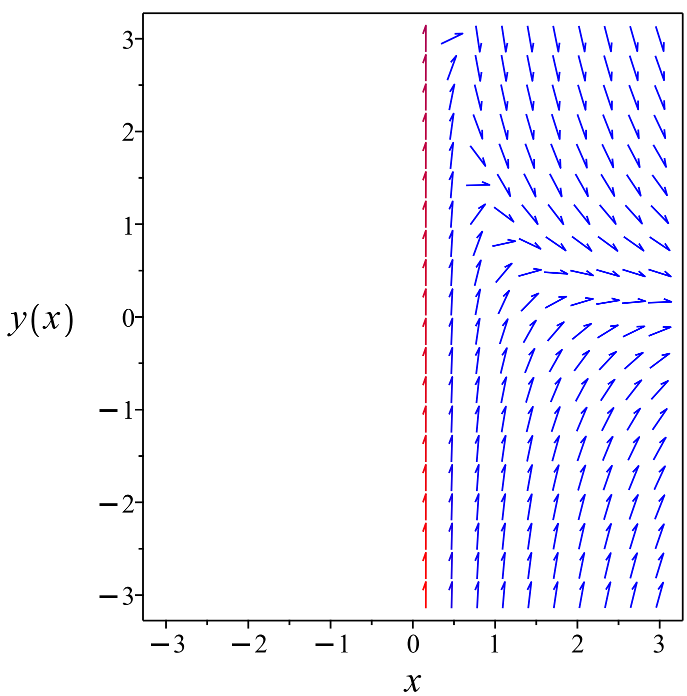


Figure 397: Slope field plot

Verification of solutions

$$y = \frac{c_1 + 2x^{\frac{3}{2}}}{x^3}$$

Verified OK.

6.20.4 Maple step by step solution

Let's solve

$3y + y'x = \frac{3}{x^{\frac{3}{2}}}$

- Highest derivative means the order of the ODE is 1

  y'
- Isolate the derivative

  $y' = -\frac{3y}{x} + \frac{3}{x^{\frac{5}{2}}}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{3y}{x} = \frac{3}{x^{\frac{5}{2}}}$
- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \frac{3\mu(x)}{x^{\frac{5}{2}}}$
- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{3y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$
- Isolate $\mu'(x)$

  $\mu'(x) = \frac{3\mu(x)}{x}$
- Solve to find the integrating factor

  $\mu(x) = x^3$
- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{3\mu(x)}{x^{\frac{5}{2}}} dx + c_1$
- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{3\mu(x)}{x^{\frac{5}{2}}} dx + c_1$
- Solve for y

  $y = \frac{\int \frac{3\mu(x)}{x^{\frac{5}{2}}} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x^3$

  $y = \frac{\int 3\sqrt{x}dx + c_1}{x^3}$
- Evaluate the integrals on the rhs

  $y = \frac{c_1 + 2x^{\frac{3}{2}}}{x^3}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(3*y(x)+x*diff(y(x),x) = 3/x^(3/2),y(x), singsol=all)
```

$$y(x) = \frac{2x^{\frac{3}{2}} + c_1}{x^3}$$

✓ Solution by Mathematica

Time used: 0.033 (sec). Leaf size: 19

```
DSolve[3*y[x]+x*y'[x]== 3/x^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2x^{3/2} + c_1}{x^3}$$

6.21 problem 21

Internal problem ID [141]
Internal file name [`OUTPUT/141_Sunday_June_05_2022_01_35_54_AM_47359439/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 21.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y(x-1)+\left(x^2-1\right)y'=1}$$

6.21.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$p(x)=\frac{1}{x+1}$$
$$q(x)=\frac{1}{x^2-1}$$

Hence the ode is

$$y'+\frac{y}{x+1}=\frac{1}{x^2-1}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x+1}dx} \\
&= x+1
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{1}{x^2-1}\right) \\
\frac{d}{dx}((x+1)\, y) &= (x+1)\left(\frac{1}{x^2-1}\right) \\
d((x+1)\, y) &= \frac{1}{x-1}\, dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
(x+1)\, y &= \int \frac{1}{x-1}\, dx \\
(x+1)\, y &= \ln(x-1) + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x+1$ results in

$$
y = \frac{\ln(x-1)}{x+1} + \frac{c_1}{x+1}
$$

which simplifies to

$$
y = \frac{\ln(x-1) + c_1}{x+1}
$$

Summary

The solution(s) found are the following

$$
y = \frac{\ln(x-1) + c_1}{x+1} \tag{1}
$$

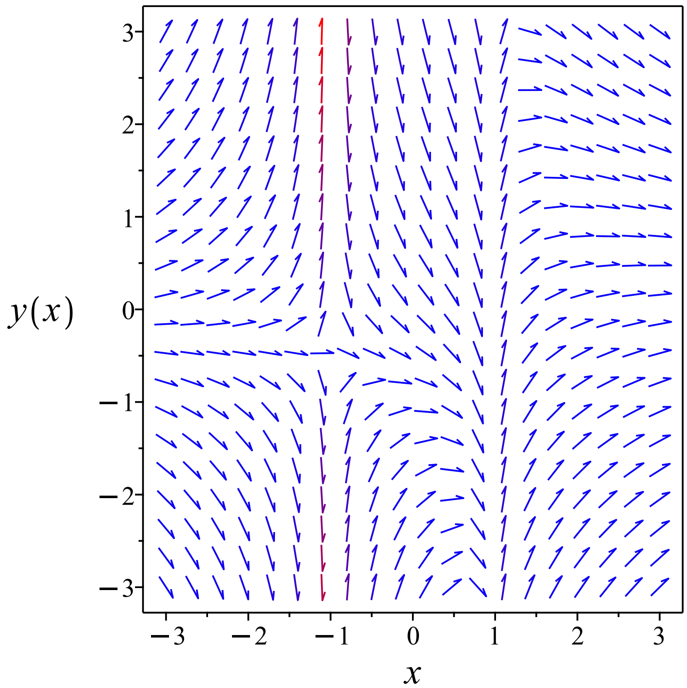


Figure 398: Slope field plot

Verification of solutions

$$y = \frac{\ln(x-1)+c_1}{x+1}$$

Verified OK.

6.21.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{yx-y-1}{x^2-1}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 285: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\xi(x, y) = 0$$
$$\eta(x, y) = \frac{1}{x+1} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x+1}} dy \end{aligned}$$

Which results in

$$S = (x + 1)\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = -\frac{yx - y - 1}{x^2 - 1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x + 1 \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{1}{x - 1} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{1}{R - 1}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = \ln(R-1) + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x+1)\,y = \ln(x-1) + c_1$$

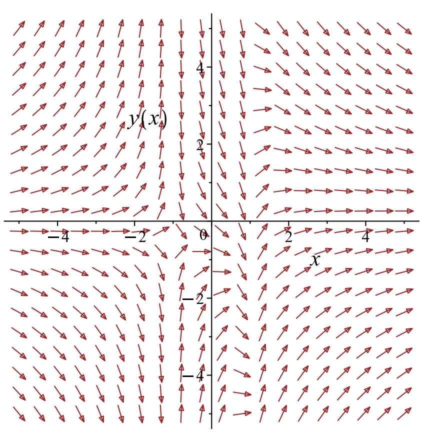
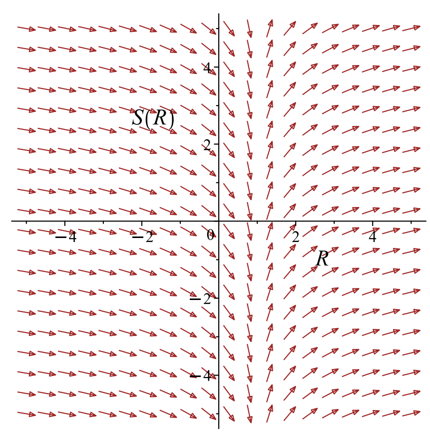
Which simplifies to

$$(x+1)\,y = \ln(x-1) + c_1$$

Which gives

$$y = \frac{\ln(x-1) + c_1}{x+1}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{yx-y-1}{x^2-1}$ 	$R = x$ $S = (x+1)\,y$	$\frac{dS}{dR} = \frac{1}{R-1}$ 

Summary
The solution(s) found are the following

$$y = \frac{\ln(x-1) + c_1}{x+1} \tag{1}$$

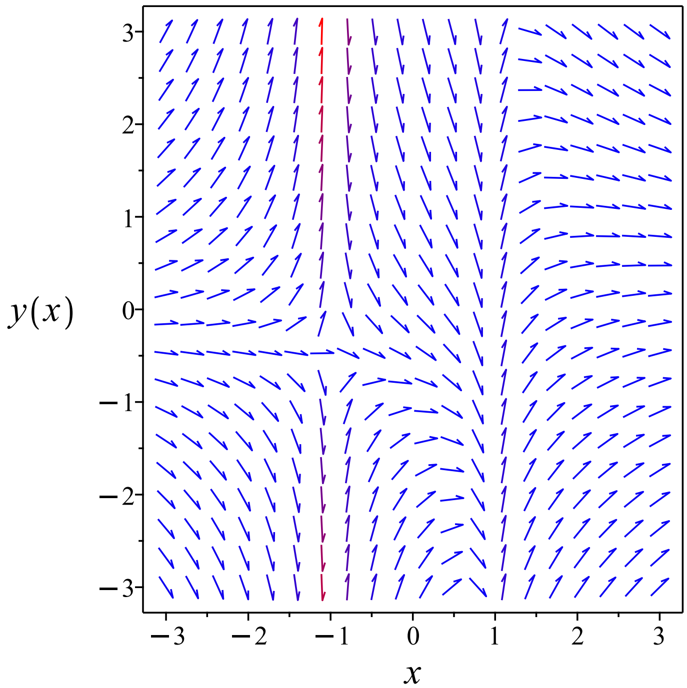


Figure 399: Slope field plot

Verification of solutions

$$y = \frac{\ln(x-1) + c_1}{x+1}$$

Verified OK.

6.21.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(x^2 - 1)\,\mathrm{d}y &= (-y(x - 1) + 1)\,\mathrm{d}x \\ (-1 + y(x - 1))\,\mathrm{d}x + (x^2 - 1)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -1 + y(x - 1)$$
$$N(x, y) = x^2 - 1$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-1 + y(x - 1)) \\ &= x - 1\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x^2 - 1) \\ &= 2x\end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$
\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{x^2 - 1}((x - 1) - (2x)) \\
&= -\frac{1}{x - 1}
\end{aligned}
$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{x-1} \, \mathrm{d}x}
\end{aligned}
$$

The result of integrating gives

$$
\begin{aligned}
\mu &= e^{-\ln(x-1)} \\
&= \frac{1}{x - 1}
\end{aligned}
$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$
\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{x - 1}(-1 + y(x - 1)) \\
&= \frac{-1 + y(x - 1)}{x - 1}
\end{aligned}
$$

And

$$
\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{x - 1}(x^2 - 1) \\
&= x + 1
\end{aligned}
$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$
\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{-1 + y(x - 1)}{x - 1}\right) + (x + 1)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}
$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \overline{M} \,\mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \,\mathrm{d}x &= \int \frac{-1 + y(x-1)}{x-1} \,\mathrm{d}x \\
\phi &= yx - \ln(x-1) + f(y) \qquad (3)
\end{aligned}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x + 1$. Therefore equation (4) becomes

$$x + 1 = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 1$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y) \,\mathrm{d}y &= \int (1) \,\mathrm{d}y \\
f(y) &= y + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = yx - \ln(x-1) + y + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = yx - \ln(x-1) + y$$

The solution becomes

$$y = \frac{\ln(x-1) + c_1}{x+1}$$

Summary
The solution(s) found are the following

$$y = \frac{\ln(x-1) + c_1}{x+1} \tag{1}$$

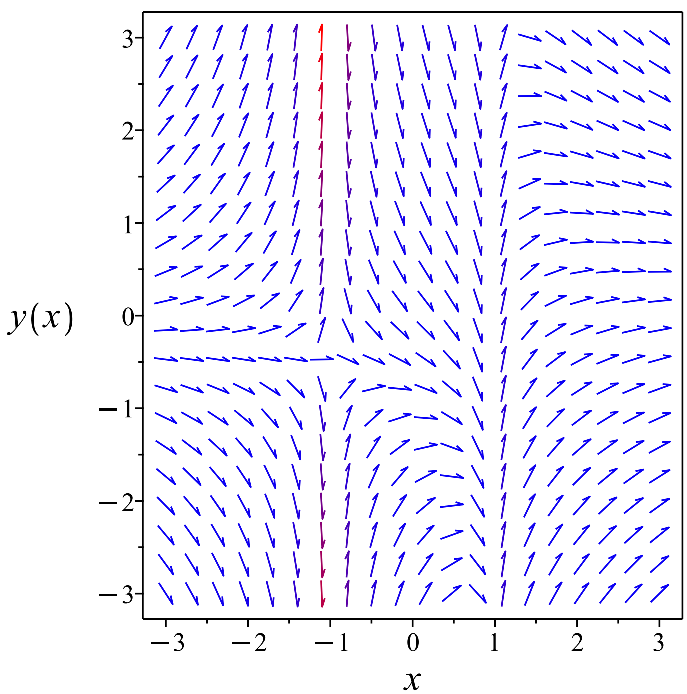


Figure 400: Slope field plot

Verification of solutions

$$y = \frac{\ln(x-1) + c_1}{x+1}$$

Verified OK.

6.21.4 Maple step by step solution

Let's solve

$y(x-1)+\left(x^{2}-1\right)y'=1$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y'=-\frac{y}{x+1}+\frac{1}{x^{2}-1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y'+\frac{y}{x+1}=\frac{1}{x^{2}-1}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y'+\frac{y}{x+1}\right)=\frac{\mu(x)}{x^{2}-1}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y'+\frac{y}{x+1}\right)=\mu'(x)\,y+\mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x)=\frac{\mu(x)}{x+1}$

- Solve to find the integrating factor

  $\mu(x)=\frac{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}{\sqrt{-(x-1)(x+1)}}$

- Integrate both sides with respect to x

  $\int\left(\frac{d}{dx}(\mu(x)\,y)\right)dx=\int\frac{\mu(x)}{x^{2}-1}dx+c_{1}$

- Evaluate the integral on the lhs

  $\mu(x)\,y=\int\frac{\mu(x)}{x^{2}-1}dx+c_{1}$

- Solve for y

  $y=\frac{\int\frac{\mu(x)}{x^{2}-1}dx+c_{1}}{\mu(x)}$

- Substitute $\mu(x)=\frac{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}{\sqrt{-(x-1)(x+1)}}$

  $y=\frac{\sqrt{-(x-1)(x+1)}\left(\int\frac{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}{\sqrt{-(x-1)(x+1)}\,\left(x^{2}-1\right)}dx+c_{1}\right)}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}$

- Evaluate the integrals on the rhs

$$y = \frac{\sqrt{-(x-1)(x+1)}\left(\frac{\ln(x-1)\sqrt{x+1}\,\sqrt{x-1}}{\sqrt{-(x-1)(x+1)}}+c_1\right)}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}$$

- Simplify

$$y = \frac{\ln(x-1)\sqrt{x+1}\,\sqrt{x-1}+c_1\sqrt{-x^2+1}}{\sqrt{x-1}\,(x+1)^{\frac{3}{2}}}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve((-1+x)*y(x)+(x^2-1)*diff(y(x),x) = 1,y(x), singsol=all)
```

$$y(x) = \frac{\ln(x-1)+c_1}{x+1}$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 18

```
DSolve[(-1+x)*y[x]+(x^2-1)*y'[x] == 1,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\log(x-1)+c_1}{x+1}$$

6.22 problem 22

Internal problem ID [142]
Internal file name [`OUTPUT/142_Sunday_June_05_2022_01_35_55_AM_28775600/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 22.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{y'x - 12x^4y^{\frac{2}{3}} - 6y = 0}$$

6.22.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{12x^4y^{\frac{2}{3}} + 6y}{x}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 288: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned} \xi(x,y) &= 0 \\ \eta(x,y) &= x^2 y^{\frac{2}{3}} \end{aligned} \tag{A1}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2 y^{\frac{2}{3}}} dy \end{aligned}$$

Which results in

$$S = \frac{3y^{\frac{1}{3}}}{x^2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{12x^4 y^{\frac{2}{3}} + 6y}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{6y^{\frac{1}{3}}}{x^3} \\ S_y &= \frac{1}{x^2 y^{\frac{2}{3}}} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 12x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 12R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = 6R^2 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{3y^{\frac{1}{3}}}{x^2} = 6x^2 + c_1$$

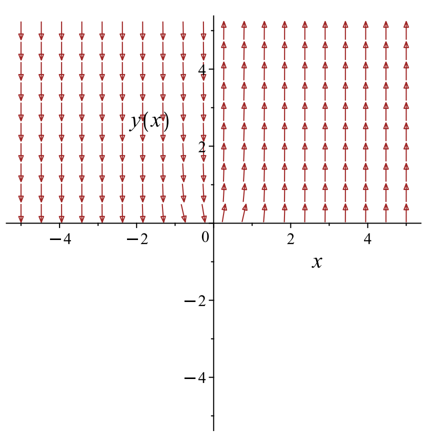
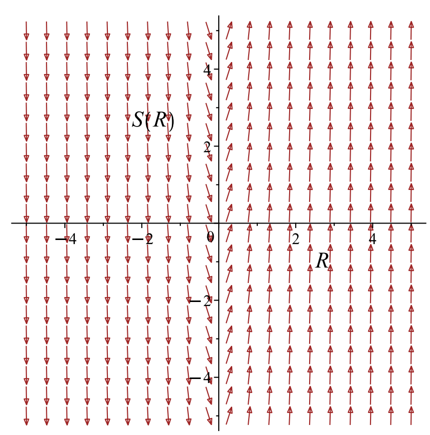
Which simplifies to

$$\frac{3y^{\frac{1}{3}}}{x^2} = 6x^2 + c_1$$

Which gives

$$y = 8x^{12} + 4x^{10}c_1 + \frac{2}{3}x^8c_1^2 + \frac{1}{27}c_1^3x^6$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{12x^4y^{\frac{2}{3}}+6y}{x}$ 	$R = x$ $S = \frac{3y^{\frac{1}{3}}}{x^2}$	$\frac{dS}{dR} = 12R$ 

Summary

The solution(s) found are the following

$$y = 8x^{12} + 4x^{10}c_1 + \frac{2}{3}x^8c_1^2 + \frac{1}{27}c_1^3x^6 \tag{1}$$

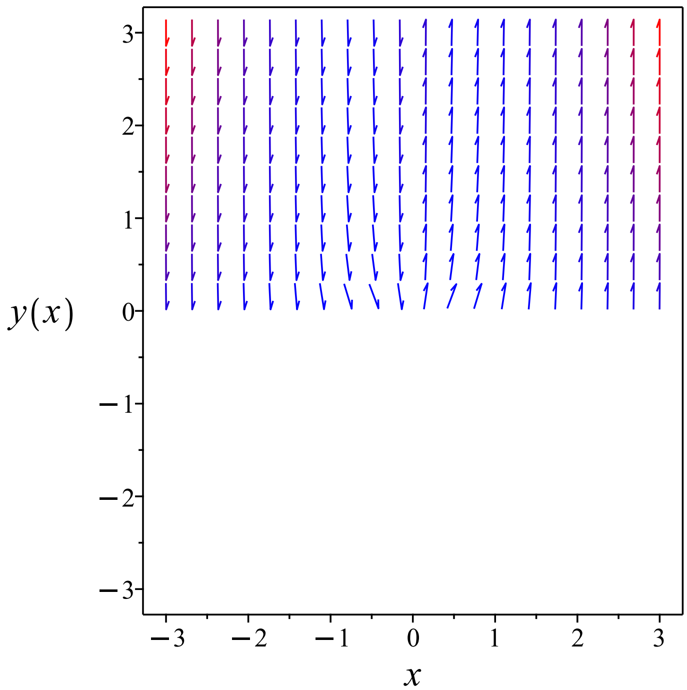


Figure 401: Slope field plot

Verification of solutions

$$y = 8x^{12} + 4x^{10}c_1 + \frac{2}{3}x^8c_1^2 + \frac{1}{27}c_1^3x^6$$

Verified OK.

6.22.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= \frac{12x^4y^{\frac{2}{3}} + 6y}{x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = \frac{6}{x}y + 12x^3y^{\frac{2}{3}} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= \frac{6}{x} \\ f_1(x) &= 12x^3 \\ n &= \frac{2}{3} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^{\frac{2}{3}}$ gives

$$y' \frac{1}{y^{\frac{2}{3}}} = \frac{6y^{\frac{1}{3}}}{x} + 12x^3 \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^{\frac{1}{3}} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{3y^{\frac{2}{3}}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 3w'(x) &= \frac{6w(x)}{x} + 12x^3 \\ w' &= \frac{2w}{x} + 4x^3 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$p(x) = -\frac{2}{x}$$
$$q(x) = 4x^3$$

Hence the ode is

$$w'(x) - \frac{2w(x)}{x} = 4x^3$$

The integrating factor μ is

$$\mu = e^{\int -\frac{2}{x}dx}$$
$$= \frac{1}{x^2}$$

The ode becomes

$$\frac{d}{dx}(\mu w) = (\mu)\left(4x^3\right)$$
$$\frac{d}{dx}\left(\frac{w}{x^2}\right) = \left(\frac{1}{x^2}\right)\left(4x^3\right)$$
$$d\left(\frac{w}{x^2}\right) = (4x)\,dx$$

Integrating gives

$$\frac{w}{x^2} = \int 4x\,dx$$
$$\frac{w}{x^2} = 2x^2 + c_1$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2}$ results in

$$w(x) = 2x^4 + c_1x^2$$

Replacing w in the above by $y^{\frac{1}{3}}$ using equation (5) gives the final solution.

$$y^{\frac{1}{3}} = 2x^4 + c_1x^2$$

Summary

The solution(s) found are the following

$$y^{\frac{1}{3}} = 2x^4 + c_1x^2 \tag{1}$$

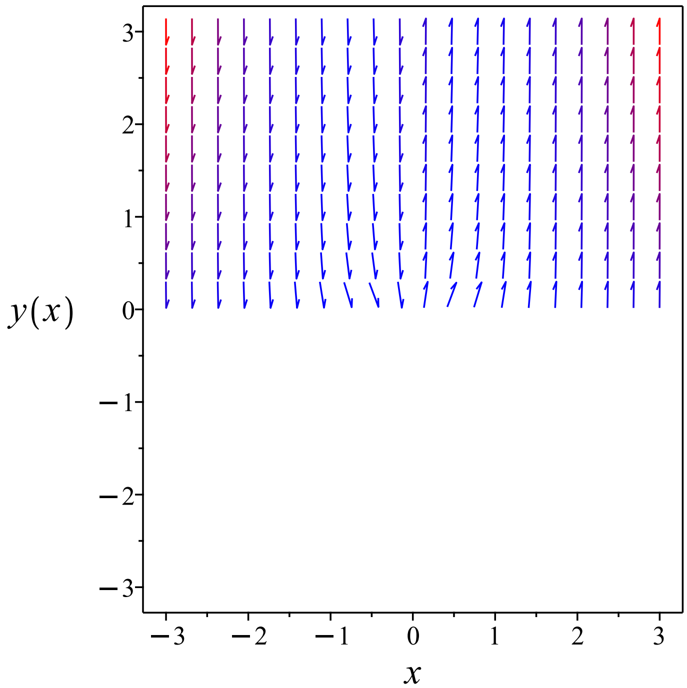


Figure 402: Slope field plot

Verification of solutions

$$y^{\frac{1}{3}} = 2x^4 + c_1 x^2$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x*diff(y(x),x) = 12*x^4*y(x)^(2/3)+6*y(x),y(x), singsol=all)
```

$$-2x^4 - c_1 x^2 + y(x)^{\frac{1}{3}} = 0$$

✓ Solution by Mathematica

Time used: 0.161 (sec). Leaf size: 19

```
DSolve[x*y'[x]== 12*x^4*y[x]^(2/3)+6*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^6 (2x^2 + c_1)^3$$

6.23 problem 23

Internal problem ID [143]
Internal file name [OUTPUT/143_Sunday_June_05_2022_01_35_56_AM_10395588/index.tex]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 23.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[_exact]`

$$\boxed{e^{y} + \cos(x)\, y + (e^{y} x + \sin(x))\, y' = 0}$$

6.23.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(\mathrm{e}^y x + \sin(x))\,\mathrm{d}y &= (-\cos(x)\,y - \mathrm{e}^y)\,\mathrm{d}x \\ (\mathrm{e}^y + \cos(x)\,y)\,\mathrm{d}x + (\mathrm{e}^y x + \sin(x))\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x, y) &= \mathrm{e}^y + \cos(x)\,y \\ N(x, y) &= \mathrm{e}^y x + \sin(x)\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(\mathrm{e}^y + \cos(x)\,y) \\ &= \mathrm{e}^y + \cos(x)\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(\mathrm{e}^y x + \sin(x)) \\ &= \mathrm{e}^y + \cos(x)\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, dx = \int M \, dx$$
$$\int \frac{\partial \phi}{\partial x} \, dx = \int e^{y} + \cos(x) \, y \, dx$$
$$\phi = \sin(x) \, y + e^{y} x + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = e^{y} x + \sin(x) + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = e^{y} x + \sin(x)$. Therefore equation (4) becomes

$$e^{y} x + \sin(x) = e^{y} x + \sin(x) + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sin(x) \, y + e^{y} x + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sin(x) \, y + e^{y} x$$

The solution becomes

$$y = -\operatorname{LambertW}\left(\frac{x \, e^{\frac{c_1}{\sin(x)}}}{\sin(x)}\right) + \frac{c_1}{\sin(x)}$$

Summary

The solution(s) found are the following

$$y = -\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{\frac{c_1}{\sin(x)}}}{\sin(x)}\right)+\frac{c_1}{\sin(x)} \tag{1}$$

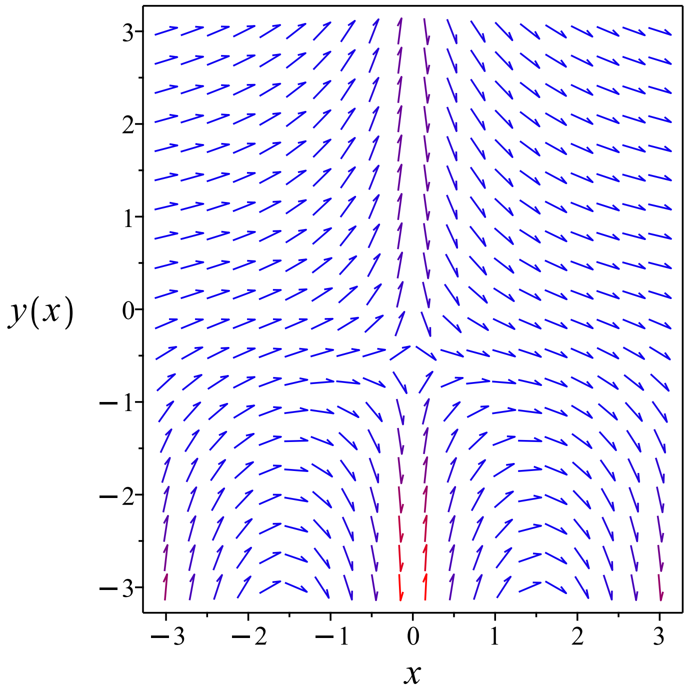


Figure 403: Slope field plot

Verification of solutions

$$y = -\mathrm{LambertW}\left(\frac{x\,\mathrm{e}^{\frac{c_1}{\sin(x)}}}{\sin(x)}\right)+\frac{c_1}{\sin(x)}$$

Verified OK.

6.23.2 Maple step by step solution

Let's solve

$\mathrm{e}^{y}+\cos(x)\,y+(\mathrm{e}^{y}x+\sin(x))\,y'=0$

- Highest derivative means the order of the ODE is 1

  y'

- ☐ Check if ODE is exact
  - ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$
  - Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right) y' = 0$
  - Evaluate derivatives

    $e^y + \cos(x) = e^y + \cos(x)$
  - Condition met, ODE is exact
- Exact ODE implies solution will be of this form

  $\left[F(x, y) = c_1, M(x, y) = F'(x, y), N(x, y) = \frac{\partial}{\partial y}F(x, y)\right]$
- Solve for $F(x, y)$ by integrating $M(x, y)$ with respect to x

  $F(x, y) = \int (e^y + \cos(x)\, y)\, dx + f_1(y)$
- Evaluate integral

  $F(x, y) = \sin(x)\, y + e^y x + f_1(y)$
- Take derivative of $F(x, y)$ with respect to y

  $N(x, y) = \frac{\partial}{\partial y}F(x, y)$
- Compute derivative

  $e^y x + \sin(x) = \sin(x) + e^y x + \frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$

  $\frac{d}{dy}f_1(y) = 0$
- Solve for $f_1(y)$

  $f_1(y) = 0$
- Substitute $f_1(y)$ into equation for $F(x, y)$

  $F(x, y) = \sin(x)\, y + e^y x$
- Substitute $F(x, y)$ into the solution of the ODE

  $\sin(x)\, y + e^y x = c_1$
- Solve for y

  $y = -\mathit{LambertW}\left(\frac{x\, e^{\frac{c_1}{\sin(x)}}}{\sin(x)}\right) + \frac{c_1}{\sin(x)}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying Chini
differential order: 1; looking for linear symmetries
trying exact
<- exact successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve(exp(y(x))+cos(x)*y(x)+(exp(y(x))*x+sin(x))*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = -\operatorname{LambertW}\left(\csc\left(x\right) \mathrm{e}^{-\csc(x) c_1} x\right) - \csc\left(x\right) c_1$$

✓ Solution by Mathematica

Time used: 4.553 (sec). Leaf size: 25

```
DSolve[Exp[y[x]]+Cos[x]*y[x]+(Exp[y[x]]*x+Sin[x])*y'[x] == 0,y[x],x,IncludeSingularSolutions
```

$$y(x) \to c_1 \csc(x) - W\left(x \csc(x) e^{c_1 \csc(x)}\right)$$

6.24 problem 24

Internal problem ID [144]
Internal file name [`OUTPUT/144_Sunday_June_05_2022_01_35_59_AM_24068941/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 24.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "riccati", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{9x^2y^2 + x^{\frac{3}{2}}y' - y^2 = 0}$$

6.24.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= -\frac{y^2(9x^2-1)}{x^{\frac{3}{2}}} \end{aligned}$$

Where $f(x) = -\frac{9x^2-1}{x^{\frac{3}{2}}}$ and $g(y) = y^2$. Integrating both sides gives

$$\frac{1}{y^2}\,dy = -\frac{9x^2-1}{x^{\frac{3}{2}}}\,dx$$

$$\int \frac{1}{y^2}\,dy = \int -\frac{9x^2-1}{x^{\frac{3}{2}}}\,dx$$

$$
-\frac{1}{y} = -\frac{2(3x^2+1)}{\sqrt{x}} + c_1
$$

Which results in

$$
y = -\frac{\sqrt{x}}{-2+\sqrt{x}\, c_1 - 6x^2}
$$

Summary

The solution(s) found are the following

$$
y = -\frac{\sqrt{x}}{-2+\sqrt{x}\, c_1 - 6x^2} \tag{1}
$$

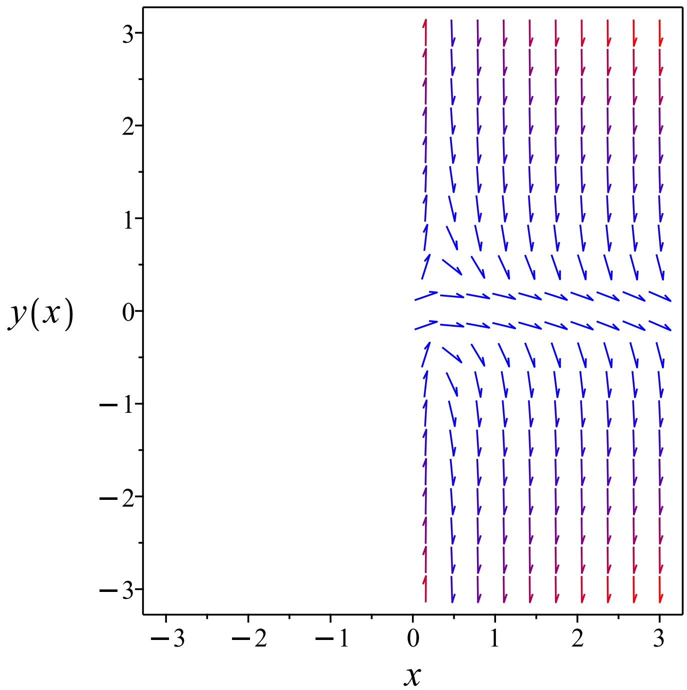


Figure 404: Slope field plot

Verification of solutions

$$
y = -\frac{\sqrt{x}}{-2+\sqrt{x}\, c_1 - 6x^2}
$$

Verified OK.

6.24.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{y^2(9x^2-1)}{x^{\frac{3}{2}}}$$
$$y' = \omega(x,y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \qquad \text{(A)}$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 291: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int(n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$\begin{aligned}
\xi(x,y) &= -\frac{x^{\frac{3}{2}}}{9x^2-1} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$R = y$$

S is found from

$$\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{-\frac{x^{\frac{3}{2}}}{9x^2-1}} dx
\end{aligned}$$

Which results in

$$S = -\frac{2(3x^2+1)}{\sqrt{x}}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \qquad (2)$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y^2(9x^2-1)}{x^{\frac{3}{2}}}$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \frac{-9x^2+1}{x^{\frac{3}{2}}} \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^2} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^2}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\frac{1}{R} + c_1 \tag{4}
$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{-6x^2-2}{\sqrt{x}} = -\frac{1}{y} + c_1
$$

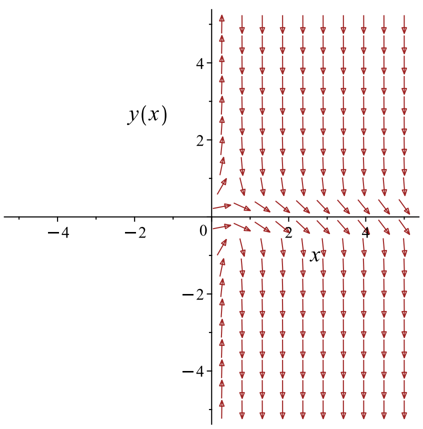
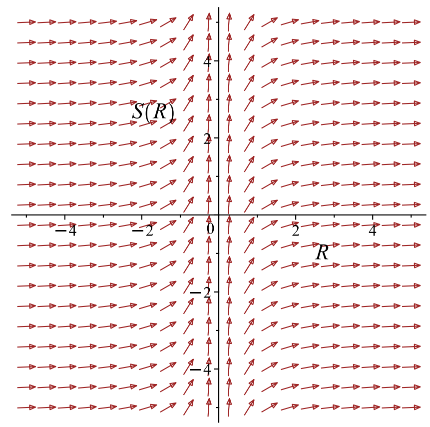
Which simplifies to

$$
\frac{-6x^2-2}{\sqrt{x}} = -\frac{1}{y} + c_1
$$

Which gives

$$
y = \frac{\sqrt{x}}{2+\sqrt{x}\, c_1 + 6x^2}
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y^2(9x^2-1)}{x^{\frac{3}{2}}}$ 	$R = y$ $S = \frac{-6x^2 - 2}{\sqrt{x}}$	$\frac{dS}{dR} = \frac{1}{R^2}$ 

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x}}{2 + \sqrt{x}\, c_1 + 6x^2} \tag{1}$$

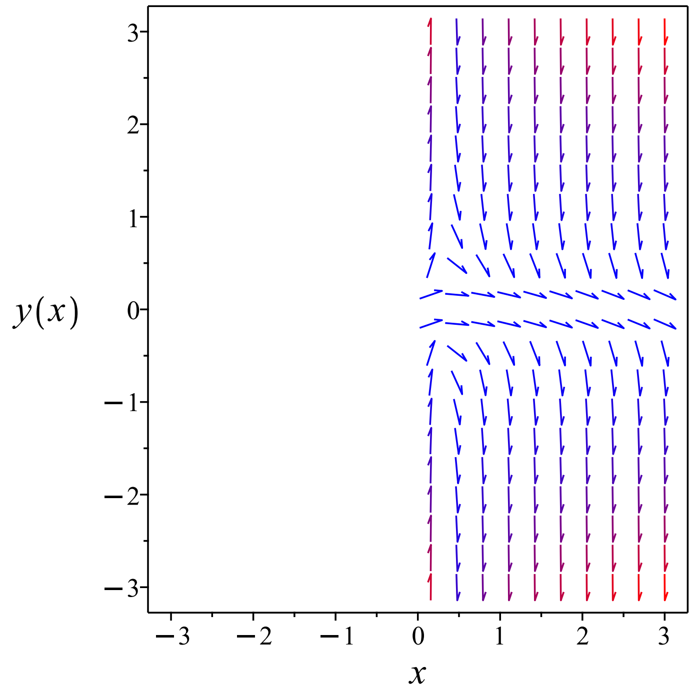


Figure 405: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x}}{2 + \sqrt{x}\, c_1 + 6x^2}$$

Verified OK.

6.24.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(-\frac{1}{y^2}\right)\mathrm{d}y = \left(\frac{9x^2-1}{x^{\frac{3}{2}}}\right)\mathrm{d}x$$
$$\left(-\frac{9x^2-1}{x^{\frac{3}{2}}}\right)\mathrm{d}x + \left(-\frac{1}{y^2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -\frac{9x^2-1}{x^{\frac{3}{2}}}$$
$$N(x,y) = -\frac{1}{y^2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{9x^2-1}{x^{\frac{3}{2}}}\right)$$
$$= 0$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(-\frac{1}{y^2}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{9x^2 - 1}{x^{\frac{3}{2}}} \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = \frac{-6x^2 - 2}{\sqrt{x}} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -\frac{1}{y^2}$. Therefore equation (4) becomes

$$
-\frac{1}{y^2} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = -\frac{1}{y^2}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(-\frac{1}{y^2}\right)\mathrm{d}y$$

$$f(y) = \frac{1}{y} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = \frac{-6x^2 - 2}{\sqrt{x}} + \frac{1}{y} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \frac{-6x^2 - 2}{\sqrt{x}} + \frac{1}{y}$$

The solution becomes

$$y = \frac{\sqrt{x}}{2 + \sqrt{x}\,c_1 + 6x^2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x}}{2 + \sqrt{x}\,c_1 + 6x^2} \tag{1}$$

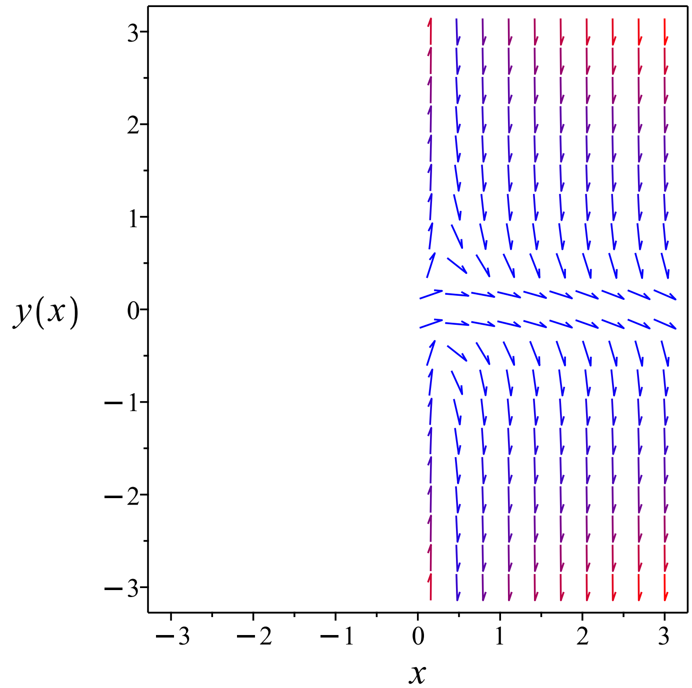


Figure 406: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x}}{2 + \sqrt{x}\, c_1 + 6x^2}$$

Verified OK.

6.24.4 Solving as riccati ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{y^2(9x^2 - 1)}{x^{\frac{3}{2}}} \end{aligned}$$

This is a Riccati ODE. Comparing the ODE to solve

$$y' = -9\sqrt{x}\, y^2 + \frac{y^2}{x^{\frac{3}{2}}}$$

With Riccati ODE standard form

$$y' = f_0(x) + f_1(x)y + f_2(x)y^2$$

Shows that $f_0(x) = 0$, $f_1(x) = 0$ and $f_2(x) = -\frac{9x^2-1}{x^{\frac{3}{2}}}$. Let

$$
\begin{aligned}
y &= \frac{-u'}{f_2 u} \\
&= \frac{-u'}{-\frac{(9x^2-1)u}{x^{\frac{3}{2}}}}
\end{aligned} \tag{1}
$$

Using the above substitution in the given ODE results (after some simplification)in a second order ODE to solve for $u(x)$ which is

$$
f_2 u''(x) - (f_2' + f_1 f_2) u'(x) + f_2^2 f_0 u(x) = 0 \tag{2}
$$

But

$$
\begin{aligned}
f_2' &= -\frac{18}{\sqrt{x}} + \frac{\frac{27x^2}{2} - \frac{3}{2}}{x^{\frac{5}{2}}} \\
f_1 f_2 &= 0 \\
f_2^2 f_0 &= 0
\end{aligned}
$$

Substituting the above terms back in equation (2) gives

$$
-\frac{(9x^2-1) u''(x)}{x^{\frac{3}{2}}} - \left(-\frac{18}{\sqrt{x}} + \frac{\frac{27x^2}{2} - \frac{3}{2}}{x^{\frac{5}{2}}}\right) u'(x) = 0
$$

Solving the above ODE (this ode solved using Maple, not this program), gives

$$
u(x) = c_1 + \frac{(3x^2+1) c_2}{\sqrt{x}}
$$

The above shows that

$$
u'(x) = \frac{c_2(9x^2-1)}{2x^{\frac{3}{2}}}
$$

Using the above in (1) gives the solution

$$
y = \frac{c_2}{2c_1 + \frac{2(3x^2+1)c_2}{\sqrt{x}}}
$$

Dividing both numerator and denominator by c_1 gives, after renaming the constant $\frac{c_2}{c_1} = c_3$ the following solution

$$y = \frac{\sqrt{x}}{2c_3\sqrt{x} + 6x^2 + 2}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{x}}{2c_3\sqrt{x} + 6x^2 + 2} \tag{1}$$

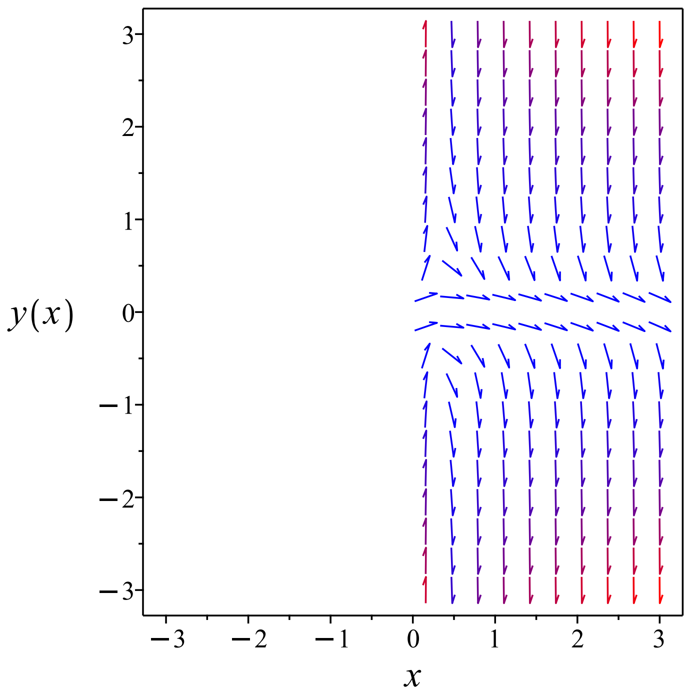


Figure 407: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{x}}{2c_3\sqrt{x} + 6x^2 + 2}$$

Verified OK.

6.24.5 Maple step by step solution

Let's solve

$9x^2y^2 + x^{\frac{3}{2}}y' - y^2 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y^2} = -\frac{(3x+1)(3x-1)}{x^{\frac{3}{2}}}$

- Integrate both sides with respect to x

  $\int \frac{y'}{y^2}dx = \int -\frac{(3x+1)(3x-1)}{x^{\frac{3}{2}}}dx + c_1$

- Evaluate integral

  $-\frac{1}{y} = -\frac{2(3x^2+1)}{\sqrt{x}} + c_1$

- Solve for y

  $y = -\frac{\sqrt{x}}{-2+\sqrt{x}\,c_1-6x^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve(9*x^2*y(x)^2+x^(3/2)*diff(y(x),x) = y(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{\sqrt{x}}{2 + 6x^2 + c_1\sqrt{x}}$$

✓ Solution by Mathematica

Time used: 0.168 (sec). Leaf size: 34

```
DSolve[9*x^2*y[x]^2+x^(3/2)*y'[x] == y[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\sqrt{x}}{6x^2 - c_1\sqrt{x} + 2}$$
$$y(x) \to 0$$

6.25 problem 25

Internal problem ID [145]
Internal file name [`OUTPUT/145_Sunday_June_05_2022_01_36_00_AM_10031191/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 25.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "homogeneousTypeMapleC", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{2y + (x+1)\, y' = 3 + 3x}$$

6.25.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{2}{x+1}$$
$$q(x) = 3$$

Hence the ode is

$$y' + \frac{2y}{x+1} = 3$$

The integrating factor μ is

$$\mu = e^{\int \frac{2}{x+1}dx}$$
$$= (x+1)^2$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)(3)$$
$$\frac{d}{dx}\left((x+1)^2 y\right) = \left((x+1)^2\right)(3)$$
$$d\left((x+1)^2 y\right) = \left(3(x+1)^2\right) dx$$

Integrating gives

$$(x+1)^2 y = \int 3(x+1)^2 \, dx$$
$$(x+1)^2 y = (x+1)^3 + c_1$$

Dividing both sides by the integrating factor $\mu = (x+1)^2$ results in

$$y = x + 1 + \frac{c_1}{(x+1)^2}$$

Summary

The solution(s) found are the following

$$y = x + 1 + \frac{c_1}{(x+1)^2} \tag{1}$$

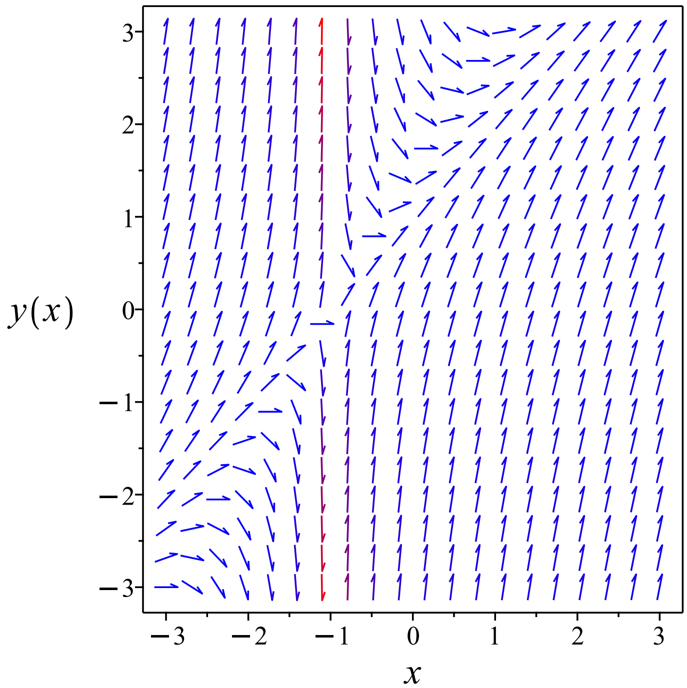


Figure 408: Slope field plot

Verification of solutions

$$y = x + 1 + \frac{c_1}{(x+1)^2}$$

Verified OK.

6.25.2 Solving as homogeneousTypeMapleC ode

Let $Y = y + y_0$ and $X = x + x_0$ then the above is transformed to new ode in $Y(X)$

$$\frac{d}{dX}Y(X) = -\frac{-3 - 3X - 3x_0 + 2Y(X) + 2y_0}{X + x_0 + 1}$$

Solving for possible values of x_0 and y_0 which makes the above ode a homogeneous ode results in

$$x_0 = -1$$
$$y_0 = 0$$

Using these values now it is possible to easily solve for $Y(X)$. The above ode now becomes

$$\frac{d}{dX}Y(X) = -\frac{-3X + 2Y(X)}{X}$$

In canonical form, the ODE is

$$\begin{aligned} Y' &= F(X,Y) \\ &= -\frac{-3X+2Y}{X} \end{aligned} \tag{1}$$

An ode of the form $Y' = \frac{M(X,Y)}{N(X,Y)}$ is called homogeneous if the functions $M(X,Y)$ and $N(X,Y)$ are both homogeneous functions and of the same order. Recall that a function $f(X,Y)$ is homogeneous of order n if

$$f(t^n X, t^n Y) = t^n f(X,Y)$$

In this case, it can be seen that both $M = 3X - 2Y$ and $N = X$ are both homogeneous and of the same order $n = 1$. Therefore this is a homogeneous ode. Since this ode is homogeneous, it is converted to separable ODE using the substitution $u = \frac{Y}{X}$, or $Y = uX$. Hence

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{\mathrm{d}u}{\mathrm{d}X}X + u$$

Applying the transformation $Y = uX$ to the above ODE in (1) gives

$$\begin{aligned} \frac{\mathrm{d}u}{\mathrm{d}X}X + u &= 3 - 2u \\ \frac{\mathrm{d}u}{\mathrm{d}X} &= \frac{3 - 3u(X)}{X} \end{aligned}$$

Or

$$\frac{d}{dX}u(X) - \frac{3 - 3u(X)}{X} = 0$$

Or

$$\left(\frac{d}{dX}u(X)\right)X + 3u(X) - 3 = 0$$

Which is now solved as separable in $u(X)$. Which is now solved in $u(X)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(X,u) \\ &= f(X)g(u) \\ &= \frac{3 - 3u}{X} \end{aligned}$$

Where $f(X) = \frac{1}{X}$ and $g(u) = 3 - 3u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{3 - 3u}\,du &= \frac{1}{X}\,dX \\ \int \frac{1}{3 - 3u}\,du &= \int \frac{1}{X}\,dX \\ -\frac{\ln(-1+u)}{3} &= \ln(X) + c_2 \end{aligned}$$

Raising both side to exponential gives

$$\frac{1}{(-1+u)^{\frac{1}{3}}} = e^{\ln(X)+c_2}$$

Which simplifies to

$$\frac{1}{(-1+u)^{\frac{1}{3}}} = c_3 X$$

Now u in the above solution is replaced back by Y using $u = \frac{Y}{X}$ which results in the solution

$$Y(X) = \frac{(c_3^3 e^{3c_2} X^3 + 1) e^{-3c_2}}{X^2 c_3^3}$$

Using the solution for $Y(X)$

$$Y(X) = \frac{(c_3^3 e^{3c_2} X^3 + 1) e^{-3c_2}}{X^2 c_3^3}$$

And replacing back terms in the above solution using

$$\begin{aligned} Y &= y + y_0 \\ X &= x + x_0 \end{aligned}$$

Or

$$\begin{aligned} Y &= y \\ X &= x - 1 \end{aligned}$$

Then the solution in y becomes

$$y = \frac{\left(c_3^3 e^{3c_2} (x+1)^3 + 1\right) e^{-3c_2}}{(x+1)^2 c_3^3}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_3^3 e^{3c_2} (x+1)^3 + 1\right) e^{-3c_2}}{(x+1)^2 c_3^3} \tag{1}$$

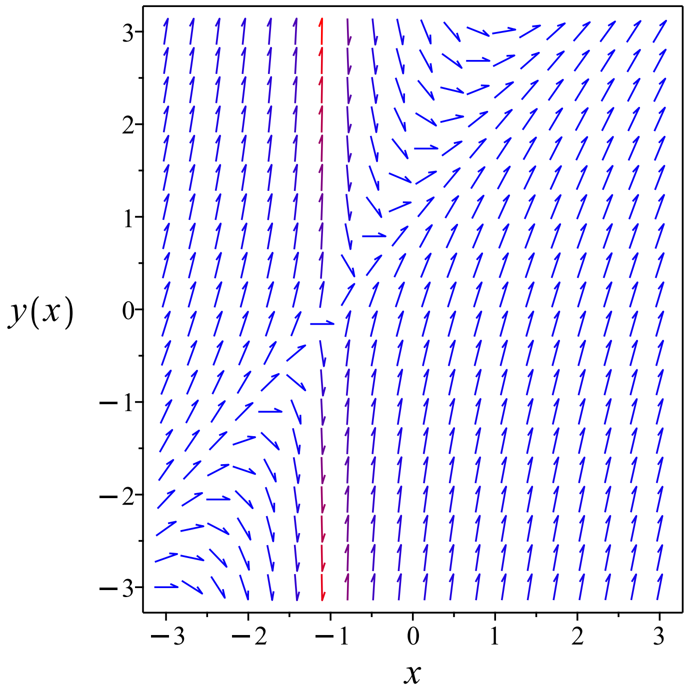


Figure 409: Slope field plot

Verification of solutions

$$y = \frac{\left(c_3^3 \mathrm{e}^{3c_2} (x+1)^3 + 1\right) \mathrm{e}^{-3c_2}}{(x+1)^2 c_3^3}$$

Verified OK.

6.25.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{2y - 3 - 3x}{x+1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 294: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{(x+1)^2}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{(x+1)^2}} dy\end{aligned}$$

Which results in

$$S = (x+1)^2 y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{2y - 3 - 3x}{x+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}R_x &= 1 \\ R_y &= 0 \\ S_x &= 2(x+1)\, y \\ S_y &= (x+1)^2\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 3(x+1)^2 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 3(R+1)^2$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = (R+1)^3 + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$(x+1)^2 y = (x+1)^3 + c_1$$

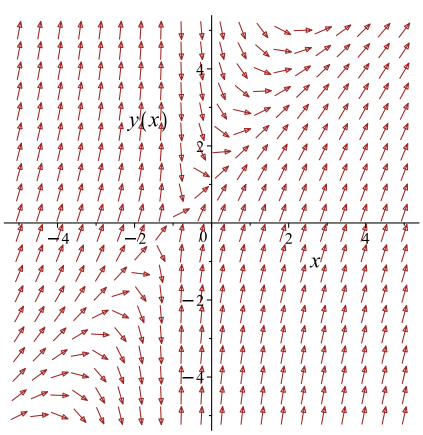
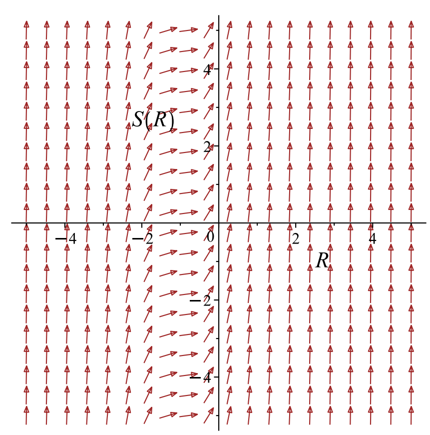
Which simplifies to

$$(x+1)^2 y = (x+1)^3 + c_1$$

Which gives

$$y = \frac{x^3 + 3x^2 + c_1 + 3x + 1}{(x+1)^2}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{2y-3-3x}{x+1}$ 	$R = x$ $S = (x+1)^2 y$	$\frac{dS}{dR} = 3(R+1)^2$ 

Summary
The solution(s) found are the following

$$y = \frac{x^3 + 3x^2 + c_1 + 3x + 1}{(x+1)^2} \tag{1}$$

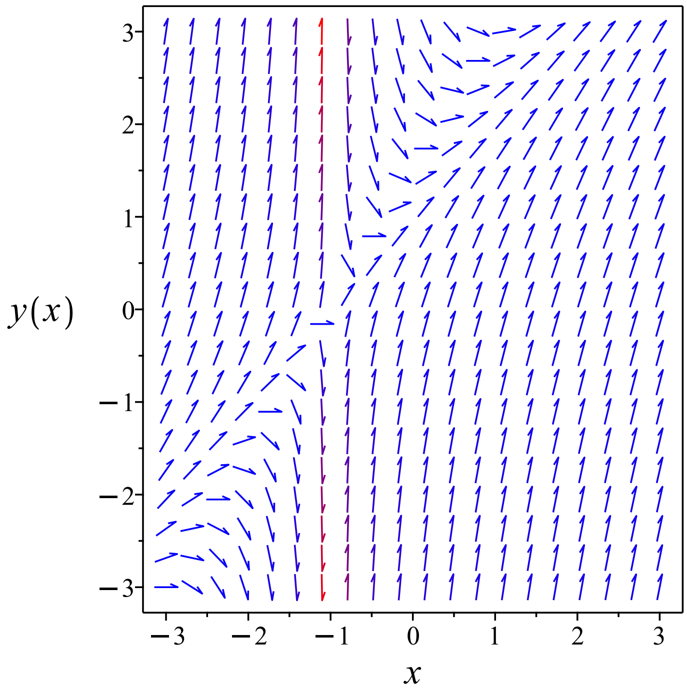


Figure 410: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3x^2 + c_1 + 3x + 1}{(x + 1)^2}$$

Verified OK.

6.25.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
(x + 1) \, \mathrm{d}y &= (-2y + 3 + 3x) \, \mathrm{d}x \\
(2y - 3 - 3x) \, \mathrm{d}x + (x + 1) \, \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= 2y - 3 - 3x \\
N(x, y) &= x + 1
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(2y - 3 - 3x) \\
&= 2
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(x + 1) \\
&= 1
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is not exact. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{x+1}((2) - (1)) \\ &= \frac{1}{x+1} \end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int A \, \mathrm{d}x} \\ &= e^{\int \frac{1}{x+1} \, \mathrm{d}x} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{\ln(x+1)} \\ &= x+1 \end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned} \overline{M} &= \mu M \\ &= x+1(2y-3-3x) \\ &= (x+1)(2y-3-3x) \end{aligned}$$

And

$$\begin{aligned} \overline{N} &= \mu N \\ &= x+1(x+1) \\ &= (x+1)^2 \end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned} \overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ ((x+1)(2y-3-3x)) + \left((x+1)^2\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x}=\overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y}=\overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int\overline{M}\,\mathrm{d}x$$
$$\int\frac{\partial\phi}{\partial x}\,\mathrm{d}x=\int(x+1)\,(2y-3-3x)\,\mathrm{d}x$$
$$\phi=-\left(x^2+(-y+3)\,x-2y+3\right)x+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y}=-(-x-2)\,x+f'(y) \tag{4}$$
$$=x(2+x)+f'(y)$$

But equation (2) says that $\frac{\partial\phi}{\partial y}=(x+1)^2$. Therefore equation (4) becomes

$$(x+1)^2=x(2+x)+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y)=1$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y=\int(1)\,\mathrm{d}y$$
$$f(y)=y+c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi=-\left(x^2+(-y+3)\,x-2y+3\right)x+y+c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\left(x^2 + (-y + 3)\, x - 2y + 3\right) x + y$$

The solution becomes

$$y = \frac{x^3 + 3x^2 + c_1 + 3x}{x^2 + 2x + 1}$$

Summary

The solution(s) found are the following

$$y = \frac{x^3 + 3x^2 + c_1 + 3x}{x^2 + 2x + 1} \tag{1}$$

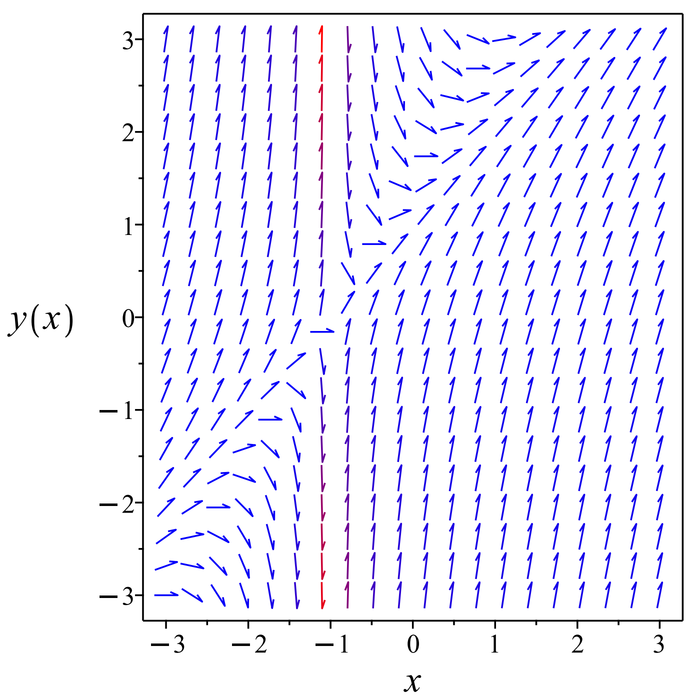


Figure 411: Slope field plot

Verification of solutions

$$y = \frac{x^3 + 3x^2 + c_1 + 3x}{x^2 + 2x + 1}$$

Verified OK.

6.25.5 Maple step by step solution

Let's solve

$2y + (x+1)\, y' = 3 + 3x$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = 3 - \frac{2y}{x+1}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{2y}{x+1} = 3$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{2y}{x+1}\right) = 3\mu(x)$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{2y}{x+1}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{2\mu(x)}{x+1}$

- Solve to find the integrating factor

  $\mu(x) = (x+1)^2$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int 3\mu(x)\, dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int 3\mu(x)\, dx + c_1$

- Solve for y

  $y = \frac{\int 3\mu(x)dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = (x+1)^2$

  $y = \frac{\int 3(x+1)^2 dx + c_1}{(x+1)^2}$

- Evaluate the integrals on the rhs

  $y = \frac{(x+1)^3 + c_1}{(x+1)^2}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(2*y(x)+(1+x)*diff(y(x),x) = 3+3*x,y(x), singsol=all)
```

$$y(x) = x + 1 + \frac{c_1}{(x+1)^2}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 25

```
DSolve[2*y[x]+(1+x)*y'[x] == 3+3*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{x^3 + 3x^2 + 3x + c_1}{(x+1)^2}$$

6.26 problem 26

Internal problem ID [146]
Internal file name [`OUTPUT/146_Sunday_June_05_2022_01_36_01_AM_19870957/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 26.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact"**

Maple gives the following as the ode type

`[[_homogeneous, `class G`], _exact, _rational]`

$$\boxed{9\sqrt{x}\, y^{\frac{4}{3}} - 12x^{\frac{1}{5}} y^{\frac{3}{2}} + \left(8x^{\frac{3}{2}} y^{\frac{1}{3}} - 15x^{\frac{6}{5}}\sqrt{y}\right) y' = 0}$$

6.26.1 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M \\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(8x^{\frac{3}{2}}y^{\frac{1}{3}} - 15x^{\frac{6}{5}}\sqrt{y}\right)\mathrm{d}y = \left(-9\sqrt{x}\,y^{\frac{4}{3}} + 12x^{\frac{1}{5}}y^{\frac{3}{2}}\right)\mathrm{d}x$$

$$\left(9\sqrt{x}\,y^{\frac{4}{3}} - 12x^{\frac{1}{5}}y^{\frac{3}{2}}\right)\mathrm{d}x + \left(8x^{\frac{3}{2}}y^{\frac{1}{3}} - 15x^{\frac{6}{5}}\sqrt{y}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= 9\sqrt{x}\,y^{\frac{4}{3}} - 12x^{\frac{1}{5}}y^{\frac{3}{2}} \\ N(x, y) &= 8x^{\frac{3}{2}}y^{\frac{1}{3}} - 15x^{\frac{6}{5}}\sqrt{y} \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(9\sqrt{x}\,y^{\frac{4}{3}} - 12x^{\frac{1}{5}}y^{\frac{3}{2}}\right) \\ &= 12\sqrt{x}\,y^{\frac{1}{3}} - 18x^{\frac{1}{5}}\sqrt{y} \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(8x^{\frac{3}{2}}y^{\frac{1}{3}} - 15x^{\frac{6}{5}}\sqrt{y}\right) \\ &= 12\sqrt{x}\,y^{\frac{1}{3}} - 18x^{\frac{1}{5}}\sqrt{y} \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial\phi}{\partial x} = M \tag{1}$$

$$\frac{\partial\phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int M \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \mathrm{d}x = \int 9\sqrt{x}\, y^{\frac{4}{3}} - 12x^{\frac{1}{5}} y^{\frac{3}{2}} \mathrm{d}x$$

$$\phi = 6y^{\frac{4}{3}} x^{\frac{3}{2}} - 10y^{\frac{3}{2}} x^{\frac{6}{5}} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 8x^{\frac{3}{2}} y^{\frac{1}{3}} - 15x^{\frac{6}{5}} \sqrt{y} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = 8x^{\frac{3}{2}} y^{\frac{1}{3}} - 15x^{\frac{6}{5}} \sqrt{y}$. Therefore equation (4) becomes

$$8x^{\frac{3}{2}} y^{\frac{1}{3}} - 15x^{\frac{6}{5}} \sqrt{y} = 8x^{\frac{3}{2}} y^{\frac{1}{3}} - 15x^{\frac{6}{5}} \sqrt{y} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = 6y^{\frac{4}{3}} x^{\frac{3}{2}} - 10y^{\frac{3}{2}} x^{\frac{6}{5}} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = 6y^{\frac{4}{3}} x^{\frac{3}{2}} - 10y^{\frac{3}{2}} x^{\frac{6}{5}}$$

Summary

The solution(s) found are the following

$$6y^{\frac{4}{3}} x^{\frac{3}{2}} - 10y^{\frac{3}{2}} x^{\frac{6}{5}} = c_1 \tag{1}$$

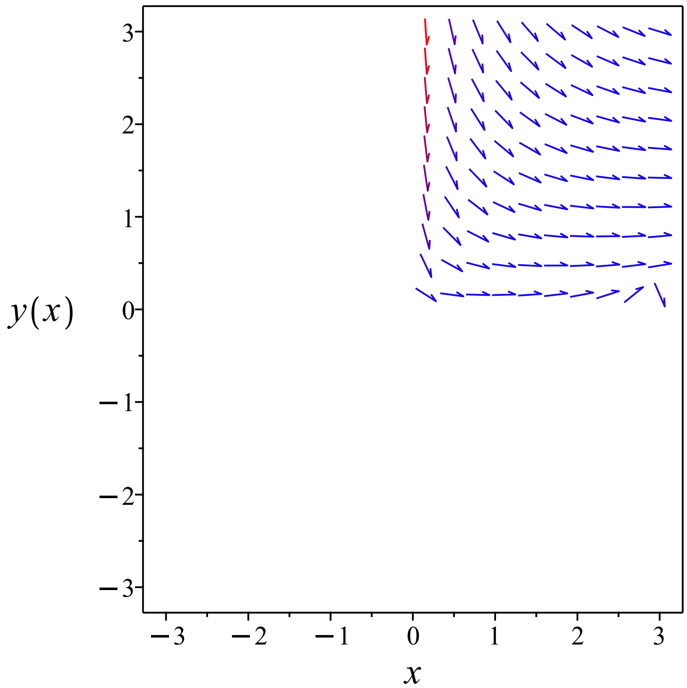


Figure 412: Slope field plot

Verification of solutions

$$6y^{\frac{4}{3}}x^{\frac{3}{2}} - 10y^{\frac{3}{2}}x^{\frac{6}{5}} = c_1$$

Verified OK.

6.26.2 Maple step by step solution

Let's solve

$9\sqrt{x}\, y^{\frac{4}{3}} - 12x^{\frac{1}{5}}y^{\frac{3}{2}} + \left(8x^{\frac{3}{2}}y^{\frac{1}{3}} - 15x^{\frac{6}{5}}\sqrt{y}\right)y' = 0$

- Highest derivative means the order of the ODE is 1

  y'

□ Check if ODE is exact

  ◦ ODE is exact if the lhs is the total derivative of a C^2 function

    $F'(x, y) = 0$

  ◦ Compute derivative of lhs

    $F'(x, y) + \left(\frac{\partial}{\partial y}F(x, y)\right)y' = 0$

- Evaluate derivatives
  $12\sqrt{x}\, y^{\frac{1}{3}}-18x^{\frac{1}{5}}\sqrt{y}=12\sqrt{x}\, y^{\frac{1}{3}}-18x^{\frac{1}{5}}\sqrt{y}$
- Condition met, ODE is exact

- Exact ODE implies solution will be of this form
  $\left[F(x,y)=c_1, M(x,y)=F'(x,y), N(x,y)=\frac{\partial}{\partial y}F(x,y)\right]$
- Solve for $F(x,y)$ by integrating $M(x,y)$ with respect to x
  $F(x,y)=\int \left(9\sqrt{x}\, y^{\frac{4}{3}}-12x^{\frac{1}{5}}y^{\frac{3}{2}}\right)dx+f_1(y)$
- Evaluate integral
  $F(x,y)=6y^{\frac{4}{3}}x^{\frac{3}{2}}-10y^{\frac{3}{2}}x^{\frac{6}{5}}+f_1(y)$
- Take derivative of $F(x,y)$ with respect to y
  $N(x,y)=\frac{\partial}{\partial y}F(x,y)$
- Compute derivative
  $8x^{\frac{3}{2}}y^{\frac{1}{3}}-15x^{\frac{6}{5}}\sqrt{y}=8x^{\frac{3}{2}}y^{\frac{1}{3}}-15x^{\frac{6}{5}}\sqrt{y}+\frac{d}{dy}f_1(y)$
- Isolate for $\frac{d}{dy}f_1(y)$
  $\frac{d}{dy}f_1(y)=0$
- Solve for $f_1(y)$
  $f_1(y)=0$
- Substitute $f_1(y)$ into equation for $F(x,y)$
  $F(x,y)=6y^{\frac{4}{3}}x^{\frac{3}{2}}-10y^{\frac{3}{2}}x^{\frac{6}{5}}$
- Substitute $F(x,y)$ into the solution of the ODE
  $6y^{\frac{4}{3}}x^{\frac{3}{2}}-10y^{\frac{3}{2}}x^{\frac{6}{5}}=c_1$
- Solve for y
  $y=\mathit{RootOf}\left(10\_Z^9x^{\frac{6}{5}}-6\_Z^8x^{\frac{3}{2}}+c_1\right)^6$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous G
1st order, trying the canonical coordinates of the invariance group
<- 1st order, canonical coordinates successful
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 43

```
dsolve(9*x^(1/2)*y(x)^(4/3)-12*x^(1/5)*y(x)^(3/2)+(8*x^(3/2)*y(x)^(1/3)-15*x^(6/5)*y(x)^(1/2
```

$$125y(x)^{\frac{9}{2}} x^{\frac{18}{5}} - 225y(x)^{\frac{13}{3}} x^{\frac{39}{10}} + 135y(x)^{\frac{25}{6}} x^{\frac{21}{5}} - 27y(x)^4 x^{\frac{9}{2}} - c_1 = 0$$

✗ Solution by Mathematica

Time used: 0.0 (sec). Leaf size: 0

```
DSolve[9*x^(1/2)*y[x]^(4/3)-12*x^(1/5)*y[x]^(3/2)+(8*x^(3/2)*y[x]^(1/3)-15*x^(6/5)*y[x]^(1/2
```

Timed out

6.27 problem 27

Internal problem ID [147]
Internal file name [`OUTPUT/147_Sunday_June_05_2022_01_36_02_AM_15229146/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 27.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"bernoulli", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class G`], _rational, _Bernoulli]
```

$$\boxed{3y + x^3y^4 + 3y'x = 0}$$

6.27.1 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$\begin{aligned} y' &= -\frac{y(y^3x^3 + 3)}{3x} \\ y' &= \omega(x, y) \end{aligned}$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2\xi_y - \omega_x\xi - \omega_y\eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 298: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^3 y^4
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^3 y^4} dy \end{aligned}$$

Which results in

$$S = -\frac{1}{3x^3 y^3}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{y(y^3 x^3 + 3)}{3x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{1}{y^3 x^4} \\ S_y &= \frac{1}{x^3 y^4} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{1}{3x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{1}{3R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{\ln(R)}{3} + c_1 \tag{4}$$

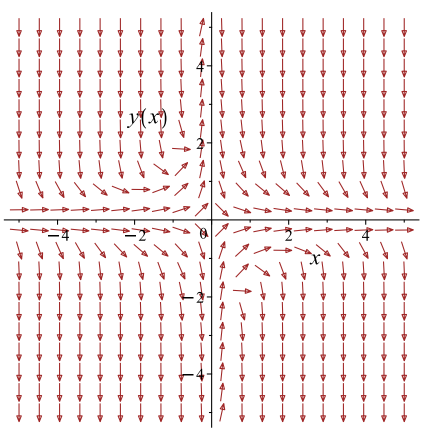
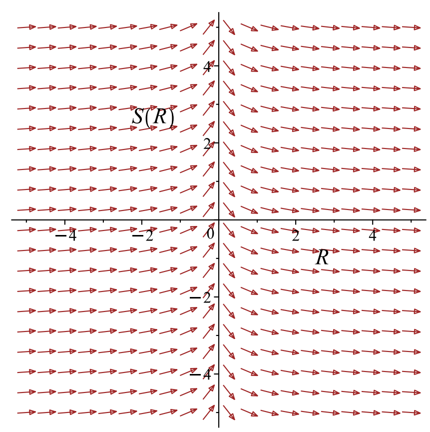
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$-\frac{1}{3x^3y^3} = -\frac{\ln(x)}{3} + c_1$$

Which simplifies to

$$-\frac{1}{3x^3y^3} = -\frac{\ln(x)}{3} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{y(y^3x^3+3)}{3x}$ 	$R = x$ $S = -\frac{1}{3x^3y^3}$	$\frac{dS}{dR} = -\frac{1}{3R}$ 

Summary

The solution(s) found are the following

$$-\frac{1}{3x^3y^3} = -\frac{\ln(x)}{3} + c_1 \tag{1}$$

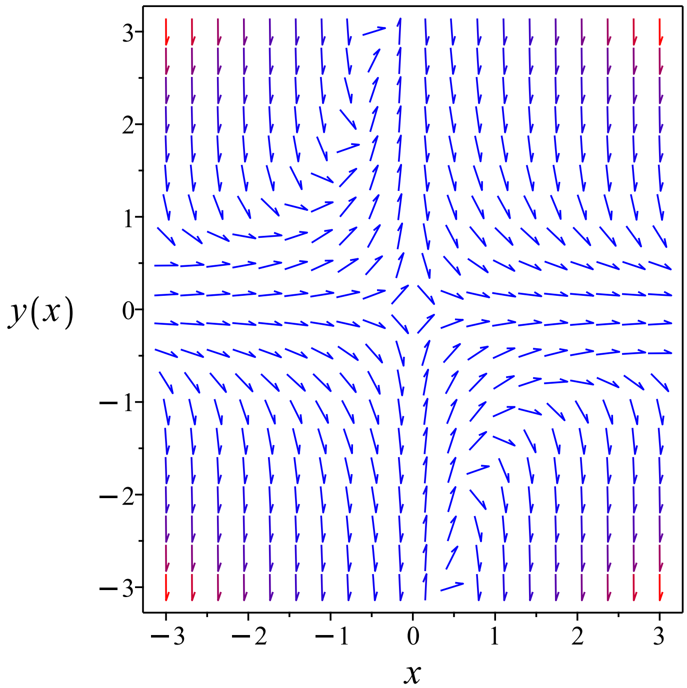


Figure 413: Slope field plot

Verification of solutions

$$-\frac{1}{3x^3y^3} = -\frac{\ln(x)}{3} + c_1$$

Verified OK.

6.27.2 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\frac{y(y^3x^3+3)}{3x} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{x}y - \frac{x^2}{3}y^4 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{x} \\ f_1(x) &= -\frac{x^2}{3} \\ n &= 4 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^4$ gives

$$y' \frac{1}{y^4} = -\frac{1}{x\, y^3} - \frac{x^2}{3} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^3} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{3}{y^4} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{3} &= -\frac{w(x)}{x} - \frac{x^2}{3} \\ w' &= \frac{3w}{x} + x^2 \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x) w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= x^2 \end{aligned}$$

Hence the ode is

$$w'(x) - \frac{3w(x)}{x} = x^2$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{3}{x}dx} \\ &= \frac{1}{x^3}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu w) &= (\mu)\left(x^2\right) \\ \frac{d}{dx}\left(\frac{w}{x^3}\right) &= \left(\frac{1}{x^3}\right)\left(x^2\right) \\ d\left(\frac{w}{x^3}\right) &= \frac{1}{x}\,dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{w}{x^3} &= \int \frac{1}{x}\,dx \\ \frac{w}{x^3} &= \ln(x) + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^3}$ results in

$$w(x) = \ln(x)\,x^3 + c_1x^3$$

which simplifies to

$$w(x) = x^3(\ln(x) + c_1)$$

Replacing w in the above by $\frac{1}{y^3}$ using equation (5) gives the final solution.

$$\frac{1}{y^3} = x^3(\ln(x) + c_1)$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{1}{(\ln(x) + c_1)^{\frac{1}{3}}\,x} \\ y(x) &= \frac{i\sqrt{3} - 1}{2(\ln(x) + c_1)^{\frac{1}{3}}\,x} \\ y(x) &= -\frac{1 + i\sqrt{3}}{2(\ln(x) + c_1)^{\frac{1}{3}}\,x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{1}{\left(\ln\left(x\right)+c_1\right)^{\frac{1}{3}} x} \tag{1}$$

$$y = \frac{i\sqrt{3}-1}{2\left(\ln\left(x\right)+c_1\right)^{\frac{1}{3}} x} \tag{2}$$

$$y = -\frac{1+i\sqrt{3}}{2\left(\ln\left(x\right)+c_1\right)^{\frac{1}{3}} x} \tag{3}$$

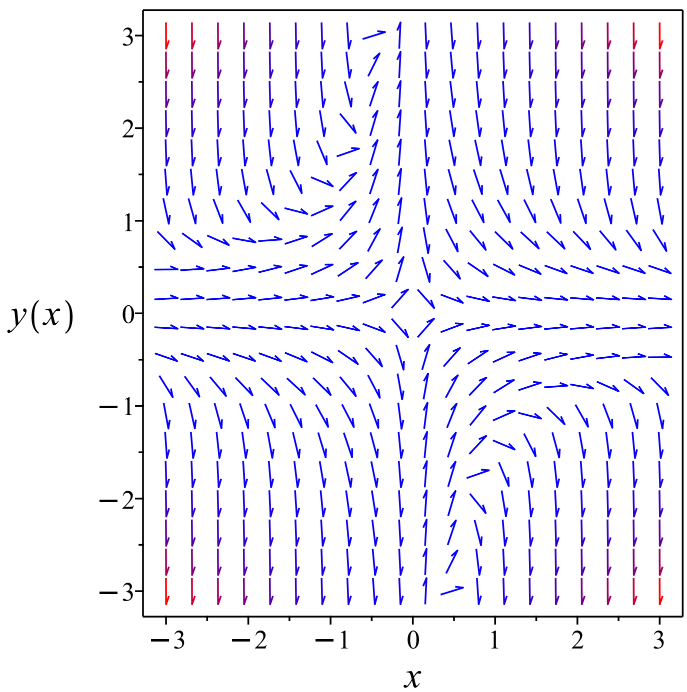


Figure 414: Slope field plot

Verification of solutions

$$y = \frac{1}{\left(\ln \left(x\right)+c_{1}\right)^{\frac{1}{3}} x}$$

Verified OK.

$$y = \frac{i \sqrt{3}-1}{2 \left(\ln \left(x\right)+c_{1}\right)^{\frac{1}{3}} x}$$

Verified OK.

$$y = -\frac{1+i \sqrt{3}}{2 \left(\ln \left(x\right)+c_{1}\right)^{\frac{1}{3}} x}$$

Verified OK.

6.27.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= M\\ \frac{\partial \phi}{\partial y} &= N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition

$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(3x)\,\mathrm{d}y &= \left(-x^3y^4 - 3y\right)\mathrm{d}x \\ \left(x^3y^4 + 3y\right)\mathrm{d}x + (3x)\,\mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x,y) &= x^3y^4 + 3y \\ N(x,y) &= 3x \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(x^3y^4 + 3y\right) \\ &= 4y^3x^3 + 3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(3x) \\ &= 3 \end{aligned}$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned} A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\ &= \frac{1}{3x}\left(\left(4y^3x^3 + 3\right) - (3)\right) \\ &= \frac{4y^3x^2}{3} \end{aligned}$$

Since A depends on y, it can not be used to obtain an integrating factor. We will now try a second method to find an integrating factor. Let

$$\begin{aligned} B &= \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \\ &= \frac{1}{x^3y^4 + 3y}\left((3) - \left(4y^3x^3 + 3\right)\right) \\ &= -\frac{4y^2x^3}{y^3x^3 + 3} \end{aligned}$$

Since B depends on x, it can not be used to obtain an integrating factor.We will now try a third method to find an integrating factor. Let

$$R = \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN}$$

R is now checked to see if it is a function of only $t = xy$. Therefore

$$\begin{aligned} R &= \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{xM - yN} \\ &= \frac{(3) - (4y^3x^3 + 3)}{x\left(x^3y^4 + 3y\right) - y\left(3x\right)} \\ &= -\frac{4}{yx} \end{aligned}$$

Replacing all powers of terms xy by t gives

$$R = -\frac{4}{t}$$

Since R depends on t only, then it can be used to find an integrating factor. Let the integrating factor be μ then

$$\begin{aligned} \mu &= e^{\int R \,\mathrm{d}t} \\ &= e^{\int \left(-\frac{4}{t}\right) \mathrm{d}t} \end{aligned}$$

The result of integrating gives

$$\begin{aligned} \mu &= e^{-4\ln(t)} \\ &= \frac{1}{t^4} \end{aligned}$$

Now t is replaced back with xy giving

$$\mu = \frac{1}{y^4x^4}$$

Multiplying M and N by this integrating factor gives new M and new N which are called $\overline{M}$ and $\overline{N}$ so not to confuse them with the original M and N

$$\begin{aligned}\overline{M} &= \mu M \\ &= \frac{1}{y^4x^4}\left(x^3y^4 + 3y\right) \\ &= \frac{y^3x^3 + 3}{x^4y^3}\end{aligned}$$

And

$$\begin{aligned}\overline{N} &= \mu N \\ &= \frac{1}{y^4x^4}(3x) \\ &= \frac{3}{x^3y^4}\end{aligned}$$

A modified ODE is now obtained from the original ODE, which is exact and can solved. The modified ODE is

$$\begin{aligned}\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \left(\frac{y^3x^3 + 3}{x^4y^3}\right) + \left(\frac{3}{x^3y^4}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = \overline{M} \tag{1}$$

$$\frac{\partial \phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \overline{M}\,\mathrm{d}x \\ \int \frac{\partial \phi}{\partial x}\,\mathrm{d}x &= \int \frac{y^3x^3 + 3}{x^4y^3}\,\mathrm{d}x\end{aligned}$$

$$\phi = -\frac{1}{x^3y^3} + \ln(x) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = \frac{3}{x^3y^4} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{3}{x^3 y^4}$. Therefore equation (4) becomes

$$\frac{3}{x^3 y^4} = \frac{3}{x^3 y^4} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{1}{x^3 y^3} + \ln(x) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{1}{x^3 y^3} + \ln(x)$$

Summary

The solution(s) found are the following

$$-\frac{1}{x^3 y^3} + \ln(x) = c_1 \tag{1}$$

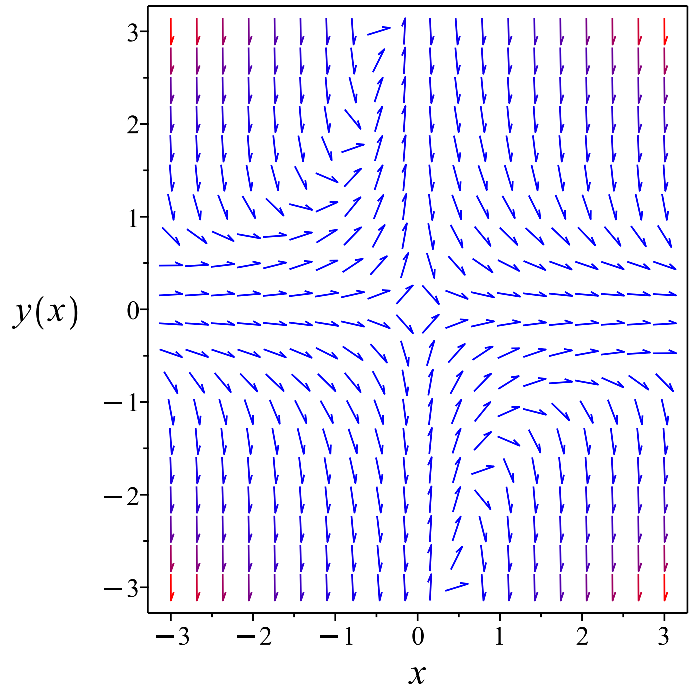


Figure 415: Slope field plot

Verification of solutions

$$-\frac{1}{x^3y^3}+\ln(x)=c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 58

```
dsolve(3*y(x)+x^3*y(x)^4+3*x*diff(y(x),x) = 0,y(x), singsol=all)
```

$$y(x) = \frac{1}{\left(\ln \left(x \right)+c_{1}\right)^{\frac{1}{3}} x}$$
$$y(x) = -\frac{1+i\sqrt{3}}{2 \left(\ln \left(x \right)+c_{1}\right)^{\frac{1}{3}} x}$$
$$y(x) = \frac{i\sqrt{3}-1}{2 \left(\ln \left(x \right)+c_{1}\right)^{\frac{1}{3}} x}$$

✓ Solution by Mathematica

Time used: 0.437 (sec). Leaf size: 70

```
DSolve[3*y[x]+x^3*y[x]^4+3*x*x*y'[x] == 0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{\sqrt[3]{x^3(\log(x)+c_1)}}$$
$$y(x) \to -\frac{\sqrt[3]{-1}}{\sqrt[3]{x^3(\log(x)+c_1)}}$$
$$y(x) \to \frac{(-1)^{2/3}}{\sqrt[3]{x^3(\log(x)+c_1)}}$$
$$y(x) \to 0$$

6.28 problem 28

Internal problem ID [148]
Internal file name [`OUTPUT/148_Sunday_June_05_2022_01_36_03_AM_61601935/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 28.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y + y'x = 2\,\mathrm{e}^{2x}}$$

6.28.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{x}$$
$$q(x) = \frac{2\,\mathrm{e}^{2x}}{x}$$

Hence the ode is

$$y' + \frac{y}{x} = \frac{2\,\mathrm{e}^{2x}}{x}$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int \frac{1}{x}dx} \\
&= x
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2\,e^{2x}}{x}\right) \\
\frac{d}{dx}(yx) &= (x)\left(\frac{2\,e^{2x}}{x}\right) \\
d(yx) &= \left(2\,e^{2x}\right)\,dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
yx &= \int 2\,e^{2x}\,dx \\
yx &= e^{2x} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$
y = \frac{e^{2x}}{x} + \frac{c_1}{x}
$$

which simplifies to

$$
y = \frac{e^{2x} + c_1}{x}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = \frac{e^{2x} + c_1}{x} \tag{1}
$$

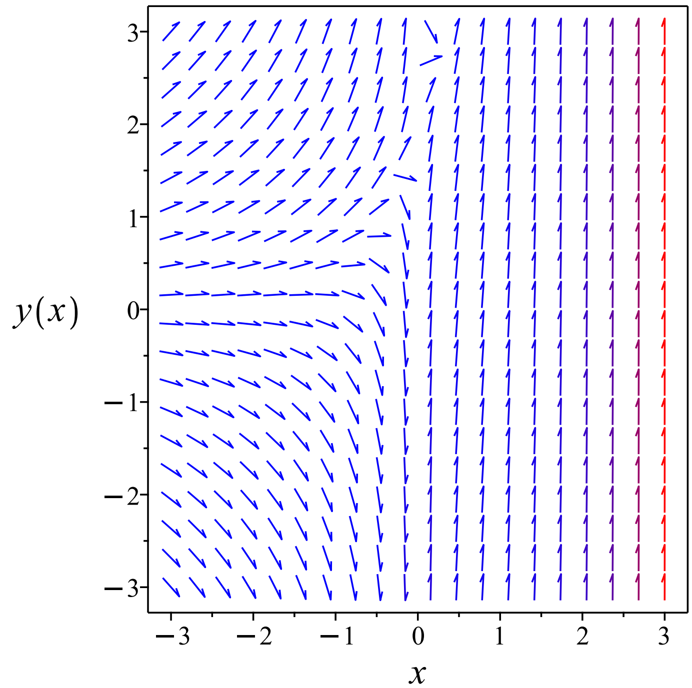


Figure 416: Slope field plot

<u>Verification of solutions</u>

$$
y = \frac{e^{2x} + c_1}{x}
$$

Verified OK.

6.28.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-y + 2\,e^{2x}}{x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 300: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{x}
\end{aligned}
\tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{x}} dy \end{aligned}$$

Which results in

$$S = yx$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y + 2\,\mathrm{e}^{2x}}{x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= y \\ S_y &= x \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 2\,\mathrm{e}^{2x} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 2\,\mathrm{e}^{2R}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = e^{2R} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$yx = e^{2x} + c_1$$

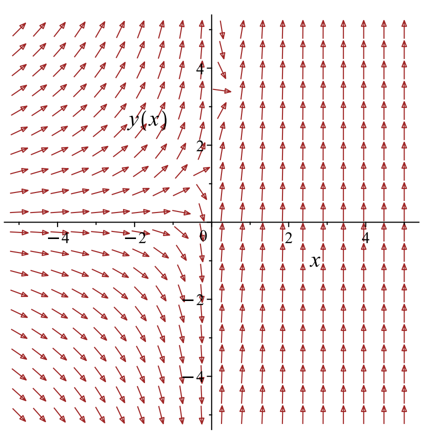
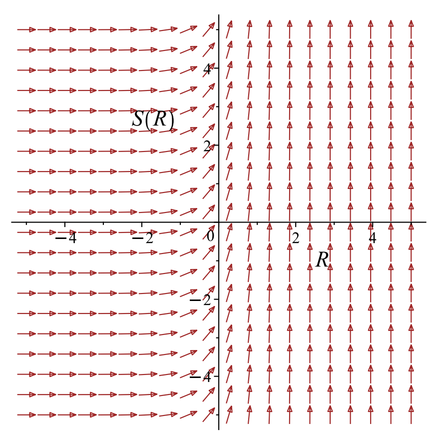
Which simplifies to

$$yx = e^{2x} + c_1$$

Which gives

$$y = \frac{e^{2x} + c_1}{x}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+2e^{2x}}{x}$ 	$R = x$ $S = yx$	$\frac{dS}{dR} = 2e^{2R}$ 

Summary

The solution(s) found are the following

$$y = \frac{e^{2x} + c_1}{x} \tag{1}$$

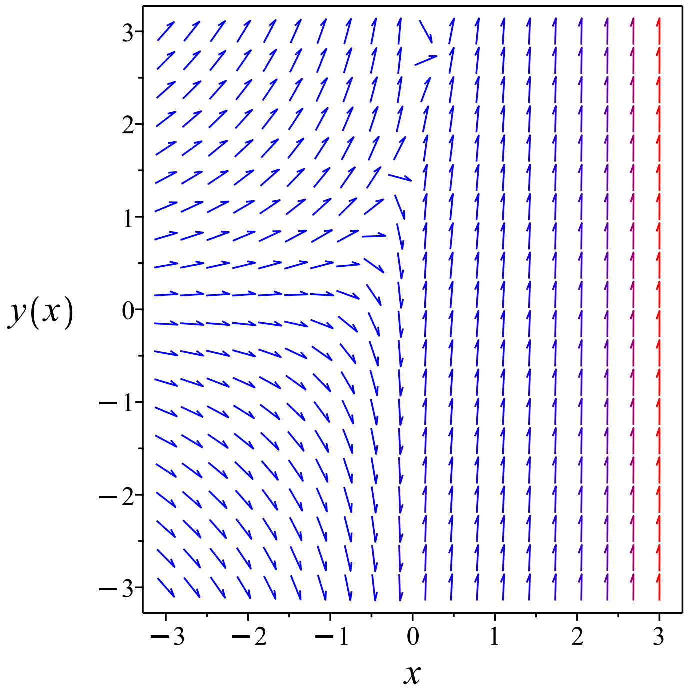


Figure 417: Slope field plot

Verification of solutions

$$y = \frac{e^{2x} + c_1}{x}$$

Verified OK.

6.28.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(x) \,\mathrm{d}y = \left(-y + 2\,\mathrm{e}^{2x}\right) \mathrm{d}x$$
$$\left(y - 2\,\mathrm{e}^{2x}\right) \mathrm{d}x + (x) \,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = y - 2\,\mathrm{e}^{2x}$$
$$N(x, y) = x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y - 2\,\mathrm{e}^{2x}\right)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(x)$$
$$= 1$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int y - 2\,\mathrm{e}^{2x} \, \mathrm{d}x$$

$$\phi = yx - \mathrm{e}^{2x} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = x$. Therefore equation (4) becomes

$$x = x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = yx - \mathrm{e}^{2x} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = yx - \mathrm{e}^{2x}$$

The solution becomes

$$y = \frac{e^{2x} + c_1}{x}$$

Summary

The solution(s) found are the following

$$y = \frac{e^{2x} + c_1}{x} \tag{1}$$

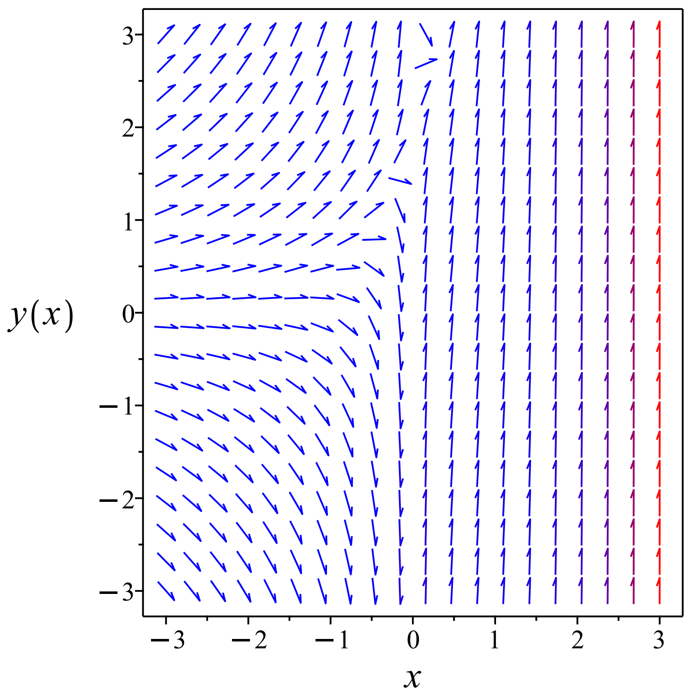


Figure 418: Slope field plot

Verification of solutions

$$y = \frac{e^{2x} + c_1}{x}$$

Verified OK.

6.28.4 Maple step by step solution

Let's solve

$y + y'x = 2\,\mathrm{e}^{2x}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{x} + \frac{2\,\mathrm{e}^{2x}}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{x} = \frac{2\,\mathrm{e}^{2x}}{x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \frac{2\mu(x)\mathrm{e}^{2x}}{x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\,y)$

  $\mu(x)\left(y' + \frac{y}{x}\right) = \mu'(x)\,y + \mu(x)\,y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{x}$

- Solve to find the integrating factor

  $\mu(x) = x$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\,y)\right) dx = \int \frac{2\mu(x)\mathrm{e}^{2x}}{x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\,y = \int \frac{2\mu(x)\mathrm{e}^{2x}}{x} dx + c_1$

- Solve for y

  $y = \frac{\int \frac{2\mu(x)\mathrm{e}^{2x}}{x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = x$

  $y = \frac{\int 2\,\mathrm{e}^{2x} dx + c_1}{x}$

- Evaluate the integrals on the rhs

  $y = \frac{\mathrm{e}^{2x} + c_1}{x}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(y(x)+x*diff(y(x),x) = 2*exp(2*x),y(x), singsol=all)
```

$$y(x) = \frac{e^{2x} + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.043 (sec). Leaf size: 17

```
DSolve[y[x]+x*y'[x] == 2*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{2x} + c_1}{x}$$

6.29 problem 29

Internal problem ID [149]
Internal file name [`OUTPUT/149_Sunday_June_05_2022_01_36_04_AM_15895275/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 29.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"linear", "exactWithIntegrationFactor", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_linear]`

$$\boxed{y + (1+2x)\, y' = (1+2x)^{\frac{3}{2}}}$$

6.29.1 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = \frac{1}{1+2x}$$
$$q(x) = \sqrt{1+2x}$$

Hence the ode is

$$y' + \frac{y}{1+2x} = \sqrt{1+2x}$$

The integrating factor μ is

$$\mu = e^{\int \frac{1}{1+2x}dx}$$
$$= \sqrt{1+2x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\sqrt{1+2x}\right)$$
$$\frac{d}{dx}\left(\sqrt{1+2x}\,y\right) = \left(\sqrt{1+2x}\right)\left(\sqrt{1+2x}\right)$$
$$d\left(\sqrt{1+2x}\,y\right) = (1+2x)\;dx$$

Integrating gives

$$\sqrt{1+2x}\,y = \int 1+2x\,dx$$
$$\sqrt{1+2x}\,y = x^2+x+c_1$$

Dividing both sides by the integrating factor $\mu = \sqrt{1+2x}$ results in

$$y = \frac{x^2+x}{\sqrt{1+2x}} + \frac{c_1}{\sqrt{1+2x}}$$

which simplifies to

$$y = \frac{x^2+c_1+x}{\sqrt{1+2x}}$$

Summary
The solution(s) found are the following

$$y = \frac{x^2+c_1+x}{\sqrt{1+2x}} \tag{1}$$

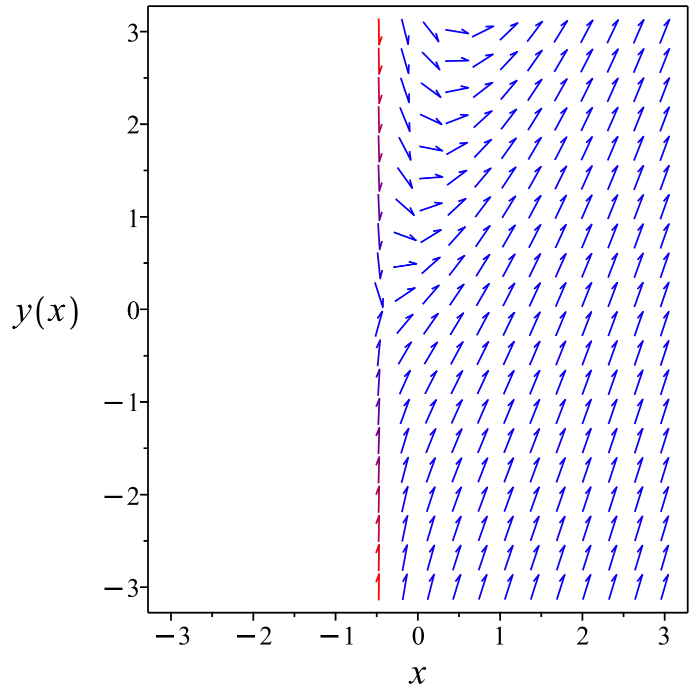


Figure 419: Slope field plot

Verification of solutions

$$
y = \frac{x^2 + c_1 + x}{\sqrt{1 + 2x}}
$$

Verified OK.

6.29.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{-y + (1 + 2x)^{\frac{3}{2}}}{1 + 2x} \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 303: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{\sqrt{1 + 2x}}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{\sqrt{1+2x}}} dy \end{aligned}$$

Which results in

$$S = \sqrt{1+2x}\, y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = \frac{-y + (1+2x)^{\frac{3}{2}}}{1+2x}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y}{\sqrt{1+2x}} \\ S_y &= \sqrt{1+2x} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 1 + 2x \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 1 + 2R$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = R^2 + R + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\sqrt{1+2x}\, y = x^2 + c_1 + x$$

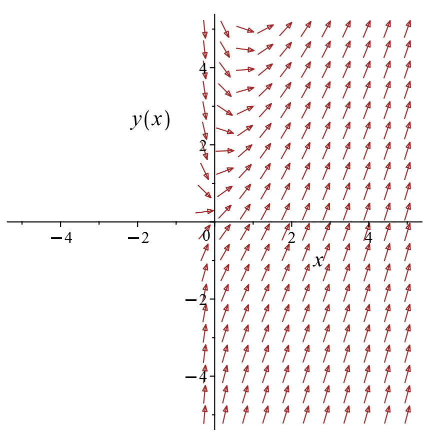
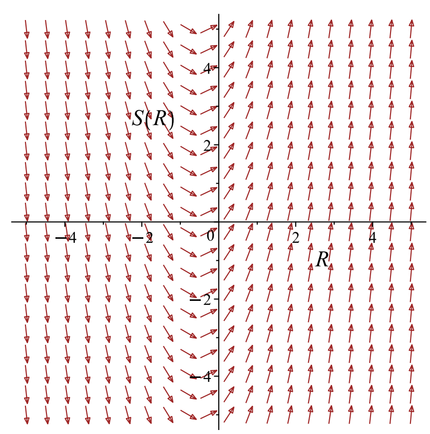
Which simplifies to

$$\sqrt{1+2x}\, y = x^2 + c_1 + x$$

Which gives

$$y = \frac{x^2 + c_1 + x}{\sqrt{1+2x}}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{-y+(1+2x)^{\frac{3}{2}}}{1+2x}$ 	$R = x$ $S = \sqrt{1+2x}\, y$	$\frac{dS}{dR} = 1 + 2R$ 

Summary

The solution(s) found are the following

$$y = \frac{x^2 + c_1 + x}{\sqrt{1+2x}} \tag{1}$$

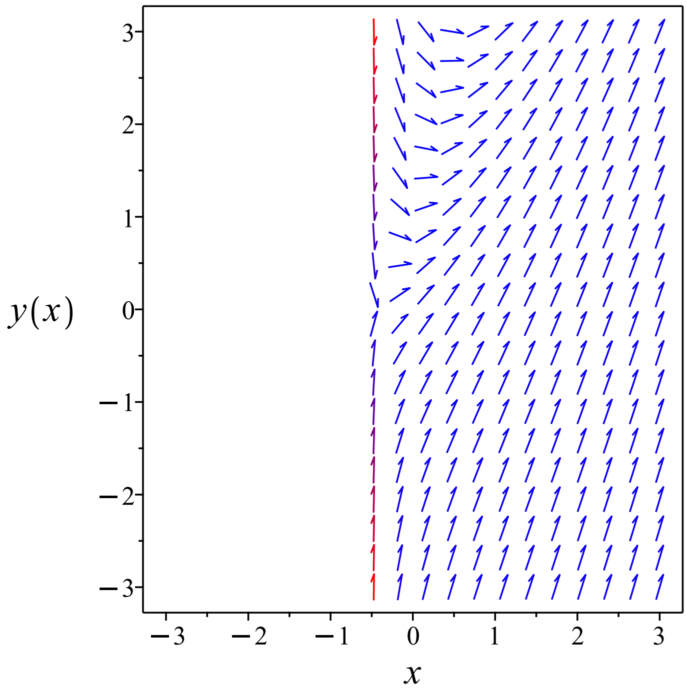


Figure 420: Slope field plot

<u>Verification of solutions</u>

$$y = \frac{x^2 + c_1 + x}{\sqrt{1 + 2x}}$$

Verified OK.

6.29.3 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x + N(x,y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$(1+2x)\,\mathrm{d}y = \left(-y + (1+2x)^{\frac{3}{2}}\right)\mathrm{d}x$$
$$\left(-(1+2x)^{\frac{3}{2}} + y\right)\mathrm{d}x + (1+2x)\,\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x,y) = -(1+2x)^{\frac{3}{2}} + y$$
$$N(x,y) = 1+2x$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(-(1+2x)^{\frac{3}{2}} + y\right)$$
$$= 1$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(1+2x)$$
$$= 2$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, then the ODE is <u>not exact</u>. Since the ODE is not exact, we will try to find an integrating factor to make it exact. Let

$$\begin{aligned}
A &= \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \\
&= \frac{1}{1+2x}((1) - (2)) \\
&= -\frac{1}{1+2x}
\end{aligned}$$

Since A does not depend on y, then it can be used to find an integrating factor. The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int A \, \mathrm{d}x} \\
&= e^{\int -\frac{1}{1+2x} \, \mathrm{d}x}
\end{aligned}$$

The result of integrating gives

$$\begin{aligned}
\mu &= e^{-\frac{\ln(1+2x)}{2}} \\
&= \frac{1}{\sqrt{1+2x}}
\end{aligned}$$

M and N are multiplied by this integrating factor, giving new M and new N which are called $\overline{M}$ and $\overline{N}$ for now so not to confuse them with the original M and N.

$$\begin{aligned}
\overline{M} &= \mu M \\
&= \frac{1}{\sqrt{1+2x}}\left(-(1+2x)^{\frac{3}{2}} + y\right) \\
&= \frac{(-1-2x)\sqrt{1+2x} + y}{\sqrt{1+2x}}
\end{aligned}$$

And

$$\begin{aligned}
\overline{N} &= \mu N \\
&= \frac{1}{\sqrt{1+2x}}(1+2x) \\
&= \sqrt{1+2x}
\end{aligned}$$

Now a modified ODE is ontained from the original ODE, which is exact and can be solved. The modified ODE is

$$\begin{aligned}
\overline{M} + \overline{N}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\left(\frac{(-1-2x)\sqrt{1+2x} + y}{\sqrt{1+2x}}\right) + \left(\sqrt{1+2x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0
\end{aligned}$$

The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial\phi}{\partial x} = \overline{M} \tag{1}$$
$$\frac{\partial\phi}{\partial y} = \overline{N} \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \overline{M}\,\mathrm{d}x$$
$$\int \frac{\partial\phi}{\partial x}\,\mathrm{d}x = \int \frac{(-1-2x)\sqrt{1+2x}+y}{\sqrt{1+2x}}\,\mathrm{d}x$$
$$\phi = \sqrt{1+2x}\,y - x - x^2 + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial\phi}{\partial y} = \sqrt{1+2x} + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial\phi}{\partial y} = \sqrt{1+2x}$. Therefore equation (4) becomes

$$\sqrt{1+2x} = \sqrt{1+2x} + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = \sqrt{1+2x}\,y - x - x^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = \sqrt{1+2x}\,y - x - x^2$$

The solution becomes

$$y = \frac{x^2 + c_1 + x}{\sqrt{1 + 2x}}$$

Summary

The solution(s) found are the following

$$y = \frac{x^2 + c_1 + x}{\sqrt{1 + 2x}} \tag{1}$$

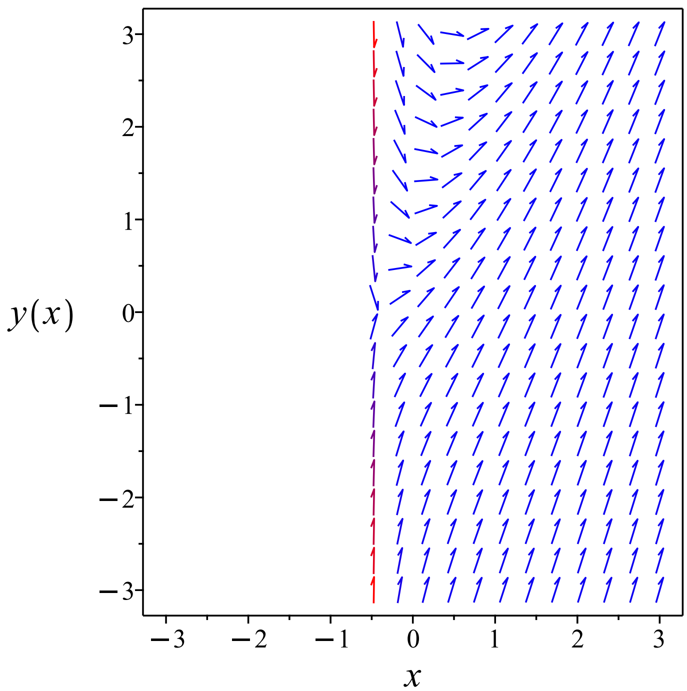


Figure 421: Slope field plot

Verification of solutions

$$y = \frac{x^2 + c_1 + x}{\sqrt{1 + 2x}}$$

Verified OK.

6.29.4 Maple step by step solution

Let's solve

$y + (1 + 2x)\, y' = (1 + 2x)^{\frac{3}{2}}$

- Highest derivative means the order of the ODE is 1

  y'

- Isolate the derivative

  $y' = -\frac{y}{1+2x} + \sqrt{1 + 2x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE

  $y' + \frac{y}{1+2x} = \sqrt{1 + 2x}$

- The ODE is linear; multiply by an integrating factor $\mu(x)$

  $\mu(x) \left(y' + \frac{y}{1+2x}\right) = \mu(x) \sqrt{1 + 2x}$

- Assume the lhs of the ODE is the total derivative $\frac{d}{dx}(\mu(x)\, y)$

  $\mu(x) \left(y' + \frac{y}{1+2x}\right) = \mu'(x)\, y + \mu(x)\, y'$

- Isolate $\mu'(x)$

  $\mu'(x) = \frac{\mu(x)}{1+2x}$

- Solve to find the integrating factor

  $\mu(x) = \sqrt{1 + 2x}$

- Integrate both sides with respect to x

  $\int \left(\frac{d}{dx}(\mu(x)\, y)\right) dx = \int \mu(x) \sqrt{1 + 2x} dx + c_1$

- Evaluate the integral on the lhs

  $\mu(x)\, y = \int \mu(x) \sqrt{1 + 2x} dx + c_1$

- Solve for y

  $y = \frac{\int \mu(x)\sqrt{1+2x} dx + c_1}{\mu(x)}$

- Substitute $\mu(x) = \sqrt{1 + 2x}$

  $y = \frac{\int (1+2x) dx + c_1}{\sqrt{1+2x}}$

- Evaluate the integrals on the rhs

  $y = \frac{x^2 + c_1 + x}{\sqrt{1+2x}}$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve(y(x)+(1+2*x)*diff(y(x),x) = (1+2*x)^(3/2),y(x), singsol=all)
```

$$y(x) = \frac{x^2 + c_1 + x}{\sqrt{1 + 2x}}$$

✓ Solution by Mathematica

Time used: 0.066 (sec). Leaf size: 43

```
DSolve[y[x]+(1+2*x)*y'[x] == (1+2*x)^(3/2),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{\frac{x\sqrt{-(2x+1)^2(x+1)}}{2x+1} + c_1}{\sqrt{-2x - 1}}$$

6.30 problem 31(a)

Internal problem ID [150]
Internal file name [`OUTPUT/150_Sunday_June_05_2022_01_36_05_AM_91460194/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 31(a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 3x^2(7 + y) = 0}$$

6.30.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x^2(21 + 3y) \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = 21 + 3y$. Integrating both sides gives

$$\begin{aligned} \frac{1}{21 + 3y}\, dy &= x^2\, dx \\ \int \frac{1}{21 + 3y}\, dy &= \int x^2\, dx \\ \frac{\ln (7 + y)}{3} &= \frac{x^3}{3} + c_1 \end{aligned}$$

Raising both side to exponential gives

$$(7 + y)^{\frac{1}{3}} = e^{\frac{x^3}{3} + c_1}$$

Which simplifies to

$$(7 + y)^{\frac{1}{3}} = c_2 e^{\frac{x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = c_2^3 e^{x^3 + 3c_1} - 7 \tag{1}$$

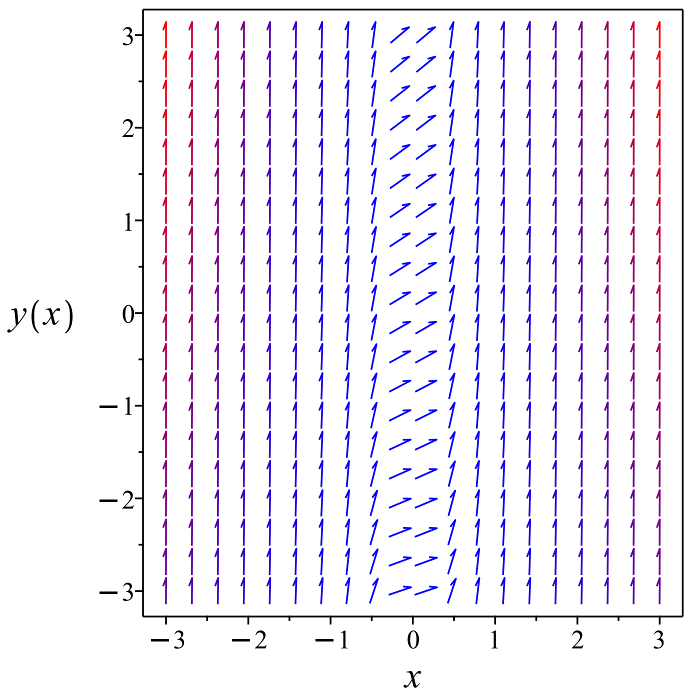


Figure 422: Slope field plot

Verification of solutions

$$y = c_2^3 e^{x^3 + 3c_1} - 7$$

Verified OK.

6.30.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3x^2 \\ q(x) &= 21x^2 \end{aligned}$$

Hence the ode is

$$y' - 3x^2y = 21x^2$$

The integrating factor μ is

$$\begin{aligned} \mu &= \mathrm{e}^{\int -3x^2 dx} \\ &= \mathrm{e}^{-x^3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(21x^2\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{-x^3}y\right) &= \left(\mathrm{e}^{-x^3}\right)\left(21x^2\right) \\ \mathrm{d}\left(\mathrm{e}^{-x^3}y\right) &= \left(21x^2\mathrm{e}^{-x^3}\right)\,\mathrm{d}x \end{aligned}$$

Integrating gives

$$\begin{aligned} \mathrm{e}^{-x^3}y &= \int 21x^2\mathrm{e}^{-x^3}\,\mathrm{d}x \\ \mathrm{e}^{-x^3}y &= -7\,\mathrm{e}^{-x^3} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \mathrm{e}^{-x^3}$ results in

$$y = -7\,\mathrm{e}^{x^3}\mathrm{e}^{-x^3} + c_1\mathrm{e}^{x^3}$$

which simplifies to

$$y = -7 + c_1\mathrm{e}^{x^3}$$

Summary

The solution(s) found are the following

$$y = -7 + c_1\mathrm{e}^{x^3} \tag{1}$$

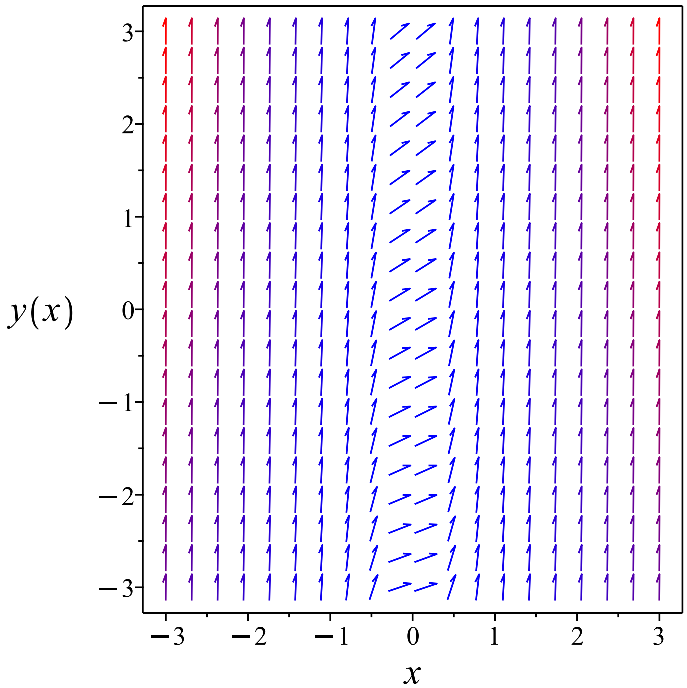


Figure 423: Slope field plot

Verification of solutions

$$y = -7 + c_1 e^{x^3}$$

Verified OK.

6.30.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x^2(7 + y)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 306: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= 0 \\
\eta(x,y) &= \mathrm{e}^{x^3} \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \qquad (1)
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^3}} dy \end{aligned}$$

Which results in

$$S = e^{-x^3} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = 3x^2(7 + y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -3x^2 e^{-x^3} y \\ S_y &= e^{-x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 21x^2 e^{-x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 21R^2 e^{-R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -7\,\mathrm{e}^{-R^3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x^3} y = -7\,\mathrm{e}^{-x^3} + c_1$$

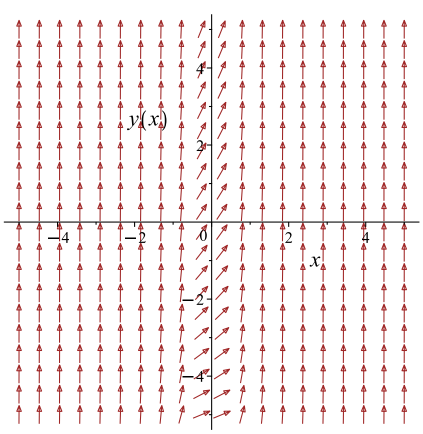
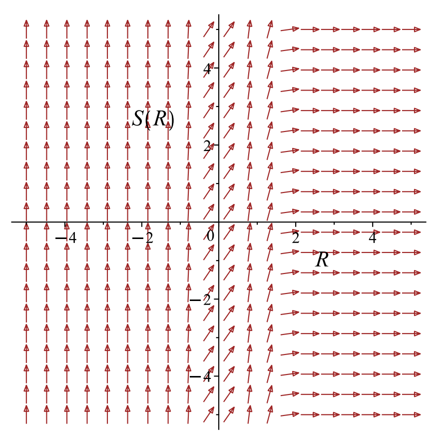
Which simplifies to

$$(7 + y)\,\mathrm{e}^{-x^3} - c_1 = 0$$

Which gives

$$y = -\left(7\,\mathrm{e}^{-x^3} - c_1\right) \mathrm{e}^{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x^2(7 + y)$ 	$R = x$ $S = \mathrm{e}^{-x^3} y$	$\frac{dS}{dR} = 21R^2\mathrm{e}^{-R^3}$ 

Summary

The solution(s) found are the following

$$y = -\left(7\,\mathrm{e}^{-x^3} - c_1\right) \mathrm{e}^{x^3} \tag{1}$$

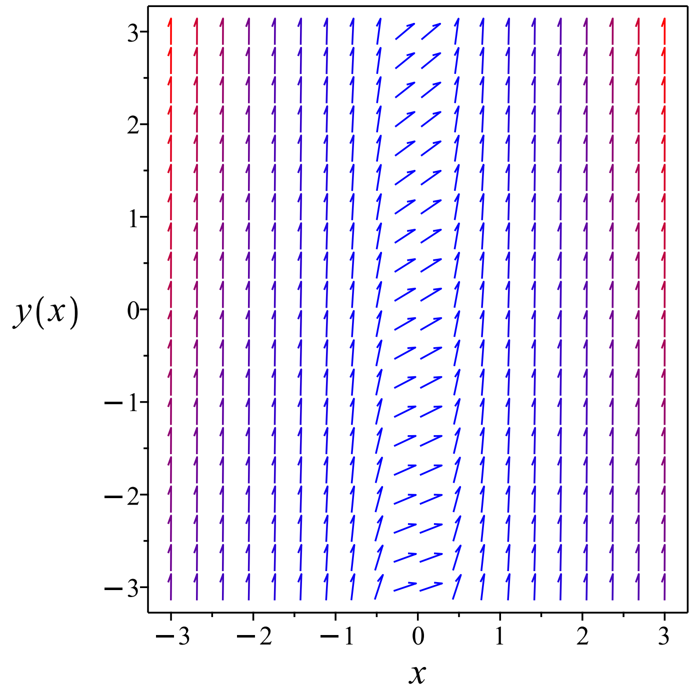


Figure 424: Slope field plot

Verification of solutions

$$y = -\left(7\,\mathrm{e}^{-x^3} - c_1\right)\mathrm{e}^{x^3}$$

Verified OK.

6.30.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= M \\
\frac{\partial \phi}{\partial y} &= N
\end{aligned}
$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$
M(x, y) \,\mathrm{d}x + N(x, y) \,\mathrm{d}y = 0 \tag{1A}
$$

Therefore

$$
\begin{aligned}
\left(\frac{1}{21 + 3y}\right) \mathrm{d}y &= \left(x^2\right) \mathrm{d}x \\
\left(-x^2\right) \mathrm{d}x + \left(\frac{1}{21 + 3y}\right) \mathrm{d}y &= 0 \qquad (2A)
\end{aligned}
$$

Comparing (1A) and (2A) shows that

$$
\begin{aligned}
M(x, y) &= -x^2 \\
N(x, y) &= \frac{1}{21 + 3y}
\end{aligned}
$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$
\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}
$$

Using result found above gives

$$
\begin{aligned}
\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y} \left(-x^2\right) \\
&= 0
\end{aligned}
$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{21+3y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x,y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -x^2\, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^3}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{21+3y}$. Therefore equation (4) becomes

$$
\frac{1}{21+3y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{21+3y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{21+3y}\right) \mathrm{d}y$$
$$f(y) = \frac{\ln(7+y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \frac{\ln(7+y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \frac{\ln(7+y)}{3}$$

The solution becomes

$$y = \mathrm{e}^{x^3+3c_1} - 7$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^3+3c_1} - 7 \tag{1}$$

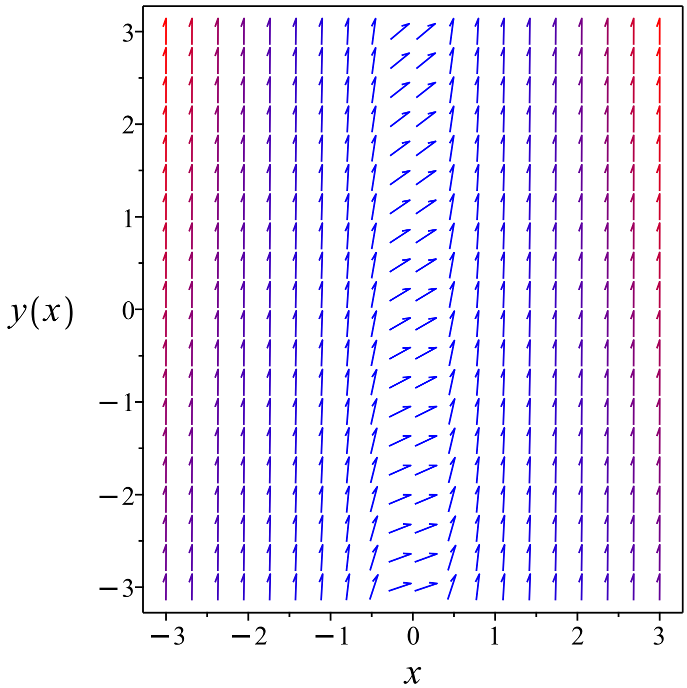


Figure 425: Slope field plot

Verification of solutions

$$y = e^{x^3+3c_1} - 7$$

Verified OK.

6.30.5 Maple step by step solution

Let's solve

$y' - 3x^2(7 + y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{7+y} = 3x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{7+y}dx = \int 3x^2dx + c_1$

- Evaluate integral

$$\ln (7+y) = x^3 + c_1$$

- Solve for y

$$y = e^{x^3+c_1} - 7$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) = 3*x^2*(7+y(x)),y(x), singsol=all)
```

$$y(x) = -7 + e^{x^3} c_1$$

✓ Solution by Mathematica

Time used: 0.045 (sec). Leaf size: 20

```
DSolve[y'[x] == 3*x^2*(7+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -7 + c_1 e^{x^3}$$
$$y(x) \to -7$$

6.31 problem 31 (b)

Internal problem ID [151]
Internal file name [`OUTPUT/151_Sunday_June_05_2022_01_36_05_AM_54149176/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 31 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' - 3x^2(7 + y) = 0}$$

6.31.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= f(x)g(y) \\ &= x^2(21 + 3y) \end{aligned}$$

Where $f(x) = x^2$ and $g(y) = 21 + 3y$. Integrating both sides gives

$$\frac{1}{21 + 3y}\, dy = x^2\, dx$$

$$\int \frac{1}{21 + 3y}\, dy = \int x^2\, dx$$

$$\frac{\ln (7 + y)}{3} = \frac{x^3}{3} + c_1$$

Raising both side to exponential gives

$$(7+y)^{\frac{1}{3}} = e^{\frac{x^3}{3}+c_1}$$

Which simplifies to

$$(7+y)^{\frac{1}{3}} = c_2 e^{\frac{x^3}{3}}$$

Summary

The solution(s) found are the following

$$y = c_2^3 e^{x^3+3c_1} - 7 \tag{1}$$

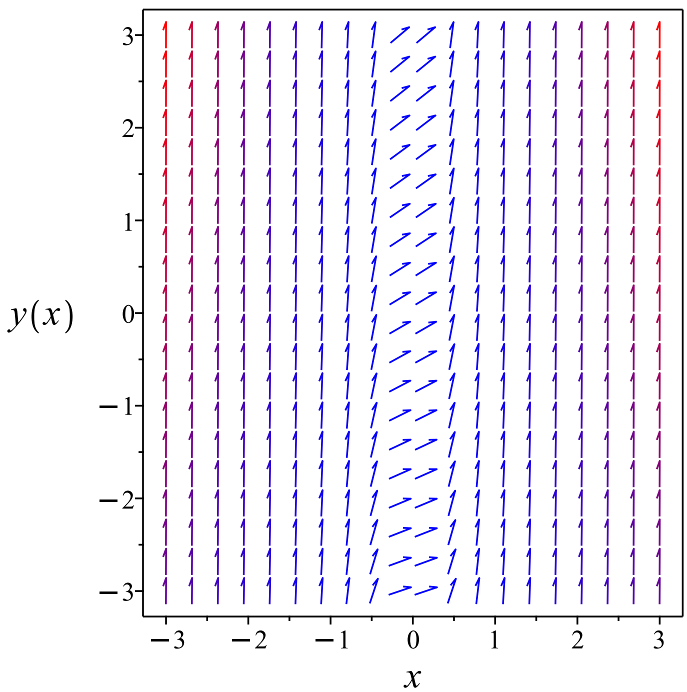


Figure 426: Slope field plot

Verification of solutions

$$y = c_2^3 e^{x^3+3c_1} - 7$$

Verified OK.

6.31.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -3x^2 \\ q(x) &= 21x^2 \end{aligned}$$

Hence the ode is

$$y' - 3x^2y = 21x^2$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -3x^2dx} \\ &= e^{-x^3} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(21x^2\right) \\ \frac{d}{dx}\left(e^{-x^3}y\right) &= \left(e^{-x^3}\right)\left(21x^2\right) \\ d\left(e^{-x^3}y\right) &= \left(21x^2e^{-x^3}\right)\, dx \end{aligned}$$

Integrating gives

$$\begin{aligned} e^{-x^3}y &= \int 21x^2e^{-x^3}\, dx \\ e^{-x^3}y &= -7\,e^{-x^3} + c_1 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = e^{-x^3}$ results in

$$y = -7\,e^{x^3}e^{-x^3} + c_1e^{x^3}$$

which simplifies to

$$y = -7 + c_1e^{x^3}$$

Summary
The solution(s) found are the following

$$y = -7 + c_1e^{x^3} \tag{1}$$

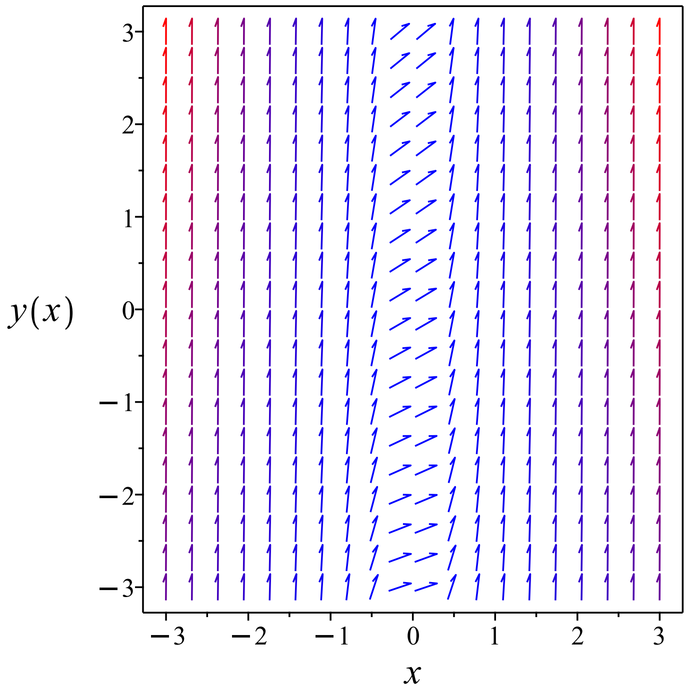


Figure 427: Slope field plot

Verification of solutions

$$y = -7 + c_1 e^{x^3}$$

Verified OK.

6.31.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = 3x^2(7 + y)$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 309: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \mathrm{e}^{x^3}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{e^{x^3}} dy \end{aligned}$$

Which results in

$$S = e^{-x^3} y$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y) S_y}{R_x + \omega(x,y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = 3x^2(7+y)$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -3x^2 e^{-x^3} y \\ S_y &= e^{-x^3} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 21x^2 e^{-x^3} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 21R^2 e^{-R^3}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by

integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = -7\,\mathrm{e}^{-R^3} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\mathrm{e}^{-x^3} y = -7\,\mathrm{e}^{-x^3} + c_1$$

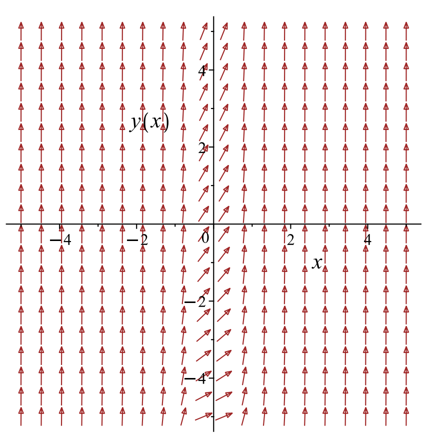
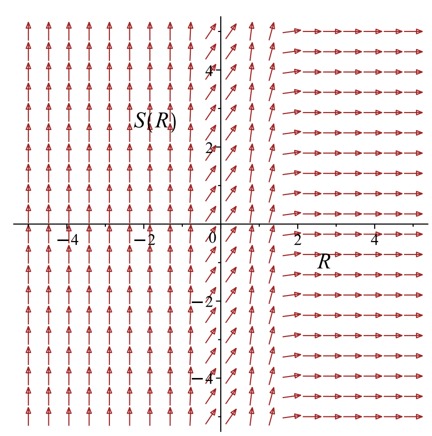
Which simplifies to

$$(7 + y)\,\mathrm{e}^{-x^3} - c_1 = 0$$

Which gives

$$y = -\left(7\,\mathrm{e}^{-x^3} - c_1\right)\mathrm{e}^{x^3}$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = 3x^2(7 + y)$ 	$R = x$ $S = \mathrm{e}^{-x^3} y$	$\frac{dS}{dR} = 21R^2\mathrm{e}^{-R^3}$ 

Summary

The solution(s) found are the following

$$y = -\left(7\,\mathrm{e}^{-x^3} - c_1\right)\mathrm{e}^{x^3} \tag{1}$$

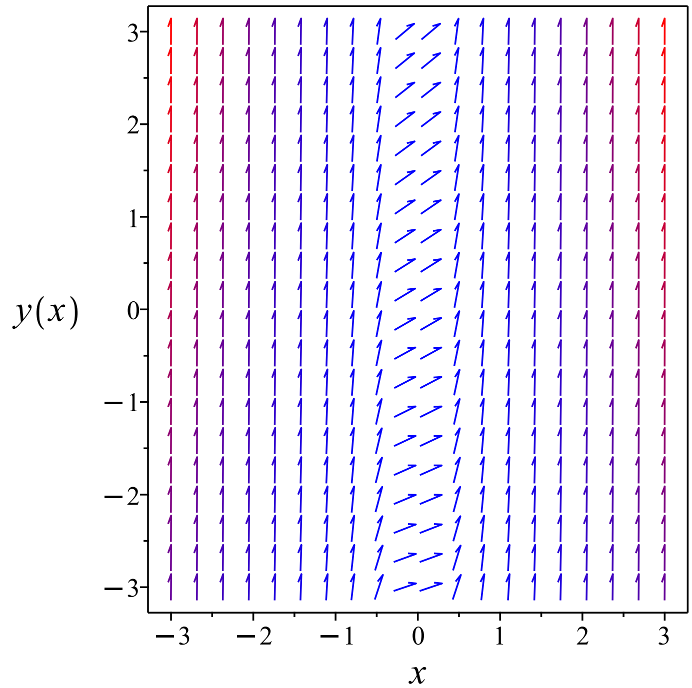


Figure 428: Slope field plot

Verification of solutions

$$y = -\left(7\,\mathrm{e}^{-x^3} - c_1\right)\mathrm{e}^{x^3}$$

Verified OK.

6.31.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\,\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\, \mathrm{d}x + N(x, y)\, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{21 + 3y}\right) \mathrm{d}y = \left(x^2\right) \mathrm{d}x$$
$$\left(-x^2\right) \mathrm{d}x + \left(\frac{1}{21 + 3y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -x^2$$
$$N(x, y) = \frac{1}{21 + 3y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-x^2\right) \\ &= 0\end{aligned}$$

And

$$
\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{21+3y}\right) \\
&= 0
\end{aligned}
$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$
\frac{\partial \phi}{\partial x} = M \tag{1}
$$

$$
\frac{\partial \phi}{\partial y} = N \tag{2}
$$

Integrating (1) w.r.t. x gives

$$
\begin{aligned}
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -x^2 \, \mathrm{d}x
\end{aligned}
$$

$$
\phi = -\frac{x^3}{3} + f(y) \tag{3}
$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$
\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}
$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{21+3y}$. Therefore equation (4) becomes

$$
\frac{1}{21+3y} = 0 + f'(y) \tag{5}
$$

Solving equation (5) for $f'(y)$ gives

$$
f'(y) = \frac{1}{21+3y}
$$

Integrating the above w.r.t y gives

$$\int f'(y)\, \mathrm{d}y = \int \left(\frac{1}{21+3y}\right) \mathrm{d}y$$

$$f(y) = \frac{\ln(7+y)}{3} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x^3}{3} + \frac{\ln(7+y)}{3} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x^3}{3} + \frac{\ln(7+y)}{3}$$

The solution becomes

$$y = \mathrm{e}^{x^3+3c_1} - 7$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{x^3+3c_1} - 7 \tag{1}$$

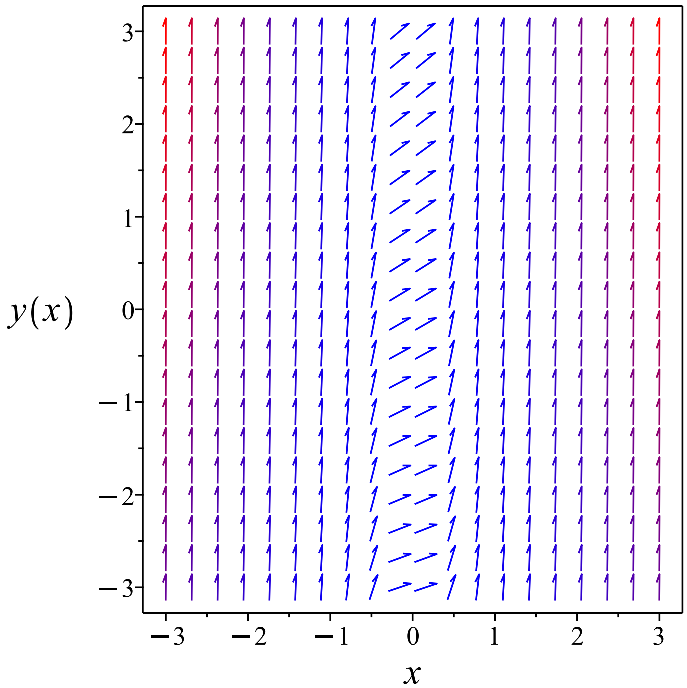


Figure 429: Slope field plot

Verification of solutions

$$y = \mathrm{e}^{x^3+3c_1} - 7$$

Verified OK.

6.31.5 Maple step by step solution

Let's solve

$y' - 3x^2(7 + y) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{7+y} = 3x^2$

- Integrate both sides with respect to x

  $\int \frac{y'}{7+y}dx = \int 3x^2dx + c_1$

- Evaluate integral

$$\ln(7+y) = x^3 + c_1$$

- Solve for y

$$y = e^{x^3+c_1} - 7$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) = 3*x^2*(7+y(x)),y(x), singsol=all)
```

$$y(x) = -7 + e^{x^3} c_1$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 20

```
DSolve[y'[x] == 3*x^2*(7+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -7 + c_1 e^{x^3}$$
$$y(x) \to -7$$

6.32 problem 32 (b)

Internal problem ID [152]
Internal file name [`OUTPUT/152_Sunday_June_05_2022_01_36_06_AM_67086637/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 32 (b).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

`[_separable]`

$$\boxed{y' + yx - xy^3 = 0}$$

6.32.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned}
y' &= F(x,y) \\
&= f(x)g(y) \\
&= x(y^3 - y)
\end{aligned}$$

Where $f(x) = x$ and $g(y) = y^3 - y$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{y^3 - y}\, dy &= x\, dx \\
\int \frac{1}{y^3 - y}\, dy &= \int x\, dx \\
-\ln(y) + \frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2} &= \frac{x^2}{2} + c_1
\end{aligned}$$

Raising both side to exponential gives

$$e^{-\ln(y)+\frac{\ln(y+1)}{2}+\frac{\ln(y-1)}{2}} = e^{\frac{x^2}{2}+c_1}$$

Which simplifies to

$$\frac{\sqrt{y+1}\,\sqrt{y-1}}{y} = c_2 e^{\frac{x^2}{2}}$$

The solution is

$$\frac{\sqrt{1+y}\,\sqrt{y-1}}{y} = c_2 e^{\frac{x^2}{2}}$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{1+y}\,\sqrt{y-1}}{y} = c_2 e^{\frac{x^2}{2}} \tag{1}$$

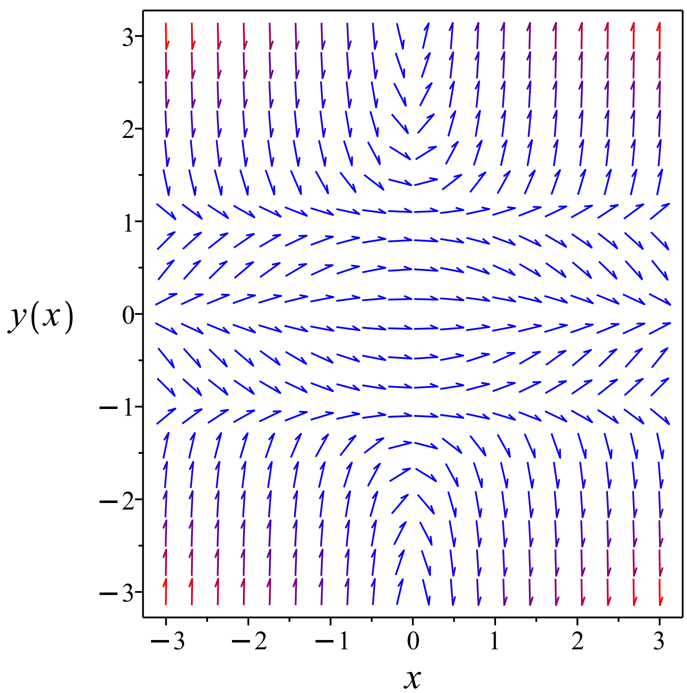


Figure 430: Slope field plot

Verification of solutions

$$\frac{\sqrt{1+y}\,\sqrt{y-1}}{y} = c_2 e^{\frac{x^2}{2}}$$

Verified OK.

6.32.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= x\,y^3 - yx \\
y' &= \omega(x, y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 312: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\,g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\,F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\,e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\,y + g(x)\,y^n$	0	$e^{-\int (n-1)f(x)dx}y^n$
Reduced Riccati	$y' = f_1(x)\,y + f_2(x)\,y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{x} \\
\eta(x,y) &= 0
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{x}} dx
\end{aligned}
$$

Which results in

$$
S = \frac{x^2}{2}
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = x\,y^3 - yx
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= x \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{y^3 - y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{R^3 - R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = -\ln(R) + \frac{\ln(R+1)}{2} + \frac{\ln(R-1)}{2} + c_1 \tag{4}
$$

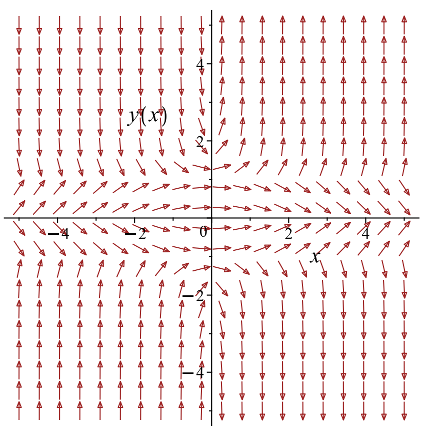
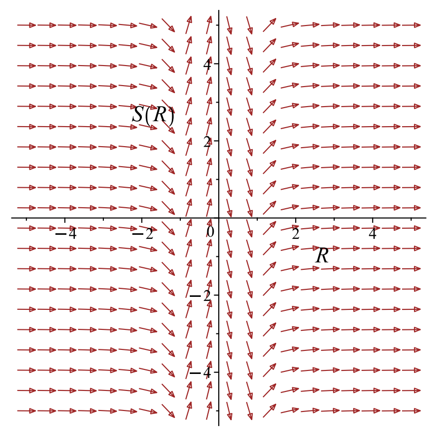
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\frac{x^2}{2} = -\ln(y) + \frac{\ln(1+y)}{2} + \frac{\ln(y-1)}{2} + c_1
$$

Which simplifies to

$$
\frac{x^2}{2} = -\ln(y) + \frac{\ln(1+y)}{2} + \frac{\ln(y-1)}{2} + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><th>Original ode in x, y coordinates</th><th>Canonical coordinates transformation</th><th>ODE in canonical coordinates (R, S)</th></tr>
<tr><td>$\frac{dy}{dx} = x\,y^3 - yx$

</td><td>$R = y$
$S = \frac{x^2}{2}$</td><td>$\frac{dS}{dR} = \frac{1}{R^3 - R}$

</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{x^2}{2} = -\ln(y) + \frac{\ln(1+y)}{2} + \frac{\ln(y-1)}{2} + c_1 \tag{1}$$

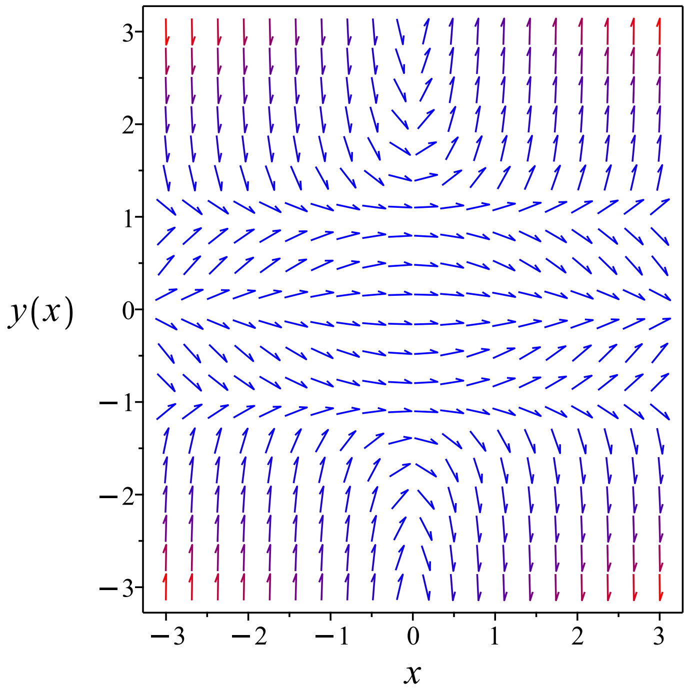


Figure 431: Slope field plot

Verification of solutions

$$\frac{x^2}{2} = -\ln(y) + \frac{\ln(1+y)}{2} + \frac{\ln(y-1)}{2} + c_1$$

Verified OK.

6.32.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= x\,y^3 - yx \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -xy + xy^3 \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -x \\ f_1(x) &= x \\ n &= 3 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = y^3$ gives

$$y' \frac{1}{y^3} = -\frac{x}{y^2} + x \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \frac{1}{y^2} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = -\frac{2}{y^3} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} -\frac{w'(x)}{2} &= -w(x)\, x + x \\ w' &= 2xw - 2x \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= -2x \\ q(x) &= -2x \end{aligned}$$

Hence the ode is

$$w'(x) - 2w(x)\, x = -2x$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -2x dx} \\
&= e^{-x^2}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)(-2x) \\
\frac{d}{dx}\left(e^{-x^2} w\right) &= \left(e^{-x^2}\right)(-2x) \\
d\left(e^{-x^2} w\right) &= \left(-2x\, e^{-x^2}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
e^{-x^2} w &= \int -2x\, e^{-x^2} dx \\
e^{-x^2} w &= e^{-x^2} + c_1
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = e^{-x^2}$ results in

$$
w(x) = e^{x^2} e^{-x^2} + c_1 e^{x^2}
$$

which simplifies to

$$
w(x) = 1 + c_1 e^{x^2}
$$

Replacing w in the above by $\frac{1}{y^2}$ using equation (5) gives the final solution.

$$
\frac{1}{y^2} = 1 + c_1 e^{x^2}
$$

Solving for y gives

$$
\begin{aligned}
y(x) &= \frac{1}{\sqrt{1 + c_1 e^{x^2}}} \\
y(x) &= -\frac{1}{\sqrt{1 + c_1 e^{x^2}}}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \frac{1}{\sqrt{1 + c_1 e^{x^2}}} \tag{1}
$$

$$
y = -\frac{1}{\sqrt{1 + c_1 e^{x^2}}} \tag{2}
$$

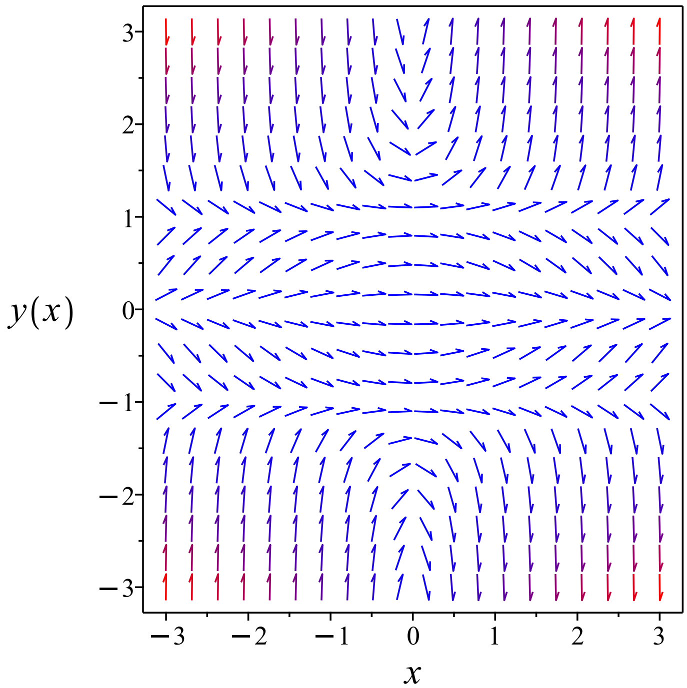


Figure 432: Slope field plot

Verification of solutions

$$y = \frac{1}{\sqrt{1 + c_1 e^{x^2}}}$$

Verified OK.

$$y = -\frac{1}{\sqrt{1 + c_1 e^{x^2}}}$$

Verified OK.

6.32.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x, y) + N(x, y) \frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x, y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x, y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial \phi}{\partial x}&=M\\ \frac{\partial \phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y}=\frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}\left(\frac{1}{y^3-y}\right)\mathrm{d}y&=(x)\,\mathrm{d}x\\ (-x)\,\mathrm{d}x+\left(\frac{1}{y^3-y}\right)\mathrm{d}y&=0 \qquad (2A)\end{aligned}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=-x\\ N(x,y)&=\frac{1}{y^3-y}\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}(-x)\\ &=0\end{aligned}$$

And

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{1}{y^3 - y}\right)$$
$$= 0$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x \, \mathrm{d}x$$
$$\phi = -\frac{x^2}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{y^3 - y}$. Therefore equation (4) becomes

$$\frac{1}{y^3 - y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{y\,(y^2 - 1)}$$
$$= \frac{1}{y^3 - y}$$

Integrating the above w.r.t y results in

$$
\begin{aligned}
\int f'(y)\,\mathrm{d}y &= \int \left(\frac{1}{y^3-y}\right)\mathrm{d}y \\
f(y) &= -\ln(y) + \frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2} + c_1
\end{aligned}
$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$
\phi = -\frac{x^2}{2} - \ln(y) + \frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2} + c_1
$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$
c_1 = -\frac{x^2}{2} - \ln(y) + \frac{\ln(y+1)}{2} + \frac{\ln(y-1)}{2}
$$

Summary

The solution(s) found are the following

$$
-\frac{x^2}{2} - \ln(y) + \frac{\ln(1+y)}{2} + \frac{\ln(y-1)}{2} = c_1 \tag{1}
$$

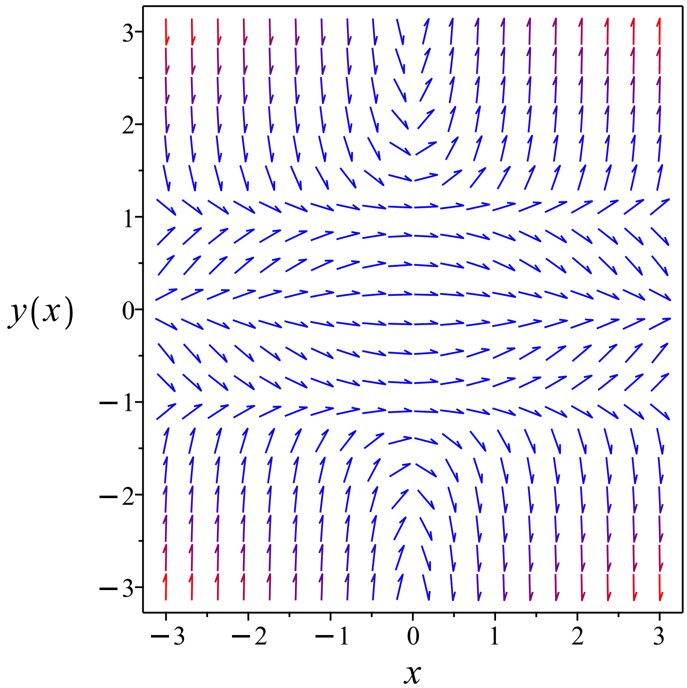


Figure 433: Slope field plot

Verification of solutions

$$-\frac{x^2}{2} - \ln(y) + \frac{\ln(1+y)}{2} + \frac{\ln(y-1)}{2} = c_1$$

Verified OK.

6.32.5 Maple step by step solution

Let's solve

$y' + yx - xy^3 = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{y(y-1)(1+y)} = x$

- Integrate both sides with respect to x

  $\int \frac{y'}{y(y-1)(1+y)} dx = \int x dx + c_1$

- Evaluate integral

$$-\ln(y)+\frac{\ln(1+y)}{2}+\frac{\ln(y-1)}{2}=\frac{x^2}{2}+c_1$$

- Solve for y

$$\left\{y=\frac{1}{\sqrt{1-e^{x^2+2c_1}}},y=-\frac{1}{\sqrt{1-e^{x^2+2c_1}}}\right\}$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x) = -x*y(x)+x*y(x)^3,y(x), singsol=all)
```

$$y(x)=\frac{1}{\sqrt{e^{x^2}c_1+1}}$$
$$y(x)=-\frac{1}{\sqrt{e^{x^2}c_1+1}}$$

✓ Solution by Mathematica

Time used: 1.917 (sec). Leaf size: 58

```
DSolve[y'[x] == -x*y[x]+x*y[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x)\to -\frac{1}{\sqrt{1+e^{x^2+2c_1}}}$$
$$y(x)\to \frac{1}{\sqrt{1+e^{x^2+2c_1}}}$$
$$y(x)\to -1$$
$$y(x)\to 0$$
$$y(x)\to 1$$

6.33 problem 33 (a)

Internal problem ID [153]
Internal file name [`OUTPUT/153_Sunday_June_05_2022_01_36_07_AM_68160363/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 33 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "homogeneousTypeD2", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, _Bernoulli]
```

$$\boxed{y' - \frac{-3x^2 - 2y^2}{4yx} = 0}$$

6.33.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{-3x^2 - 2u(x)^2\,x^2}{4u\,(x)\,x^2} = 0$$

In canonical form the ODE is

$$\begin{aligned}
u' &= F(x,u) \\
&= f(x)g(u) \\
&= -\frac{3(2u^2+1)}{4xu}
\end{aligned}$$

Where $f(x) = -\frac{3}{4x}$ and $g(u) = \frac{2u^2+1}{u}$. Integrating both sides gives

$$\frac{1}{\frac{2u^2+1}{u}}\,du = -\frac{3}{4x}\,dx$$
$$\int \frac{1}{\frac{2u^2+1}{u}}\,du = \int -\frac{3}{4x}\,dx$$
$$\frac{\ln(2u^2+1)}{4} = -\frac{3\ln(x)}{4} + c_2$$

Raising both side to exponential gives

$$\left(2u^2+1\right)^{\frac{1}{4}} = \mathrm{e}^{-\frac{3\ln(x)}{4}+c_2}$$

Which simplifies to

$$\left(2u^2+1\right)^{\frac{1}{4}} = \frac{c_3}{x^{\frac{3}{4}}}$$

Which simplifies to

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{3}{4}}}$$

The solution is

$$\left(2u(x)^2+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{3}{4}}}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\left(\frac{2y^2}{x^2}+1\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{3}{4}}}$$
$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{3}{4}}}$$

Summary

The solution(s) found are the following

$$\left(\frac{2y^2+x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 \mathrm{e}^{c_2}}{x^{\frac{3}{4}}} \tag{1}$$

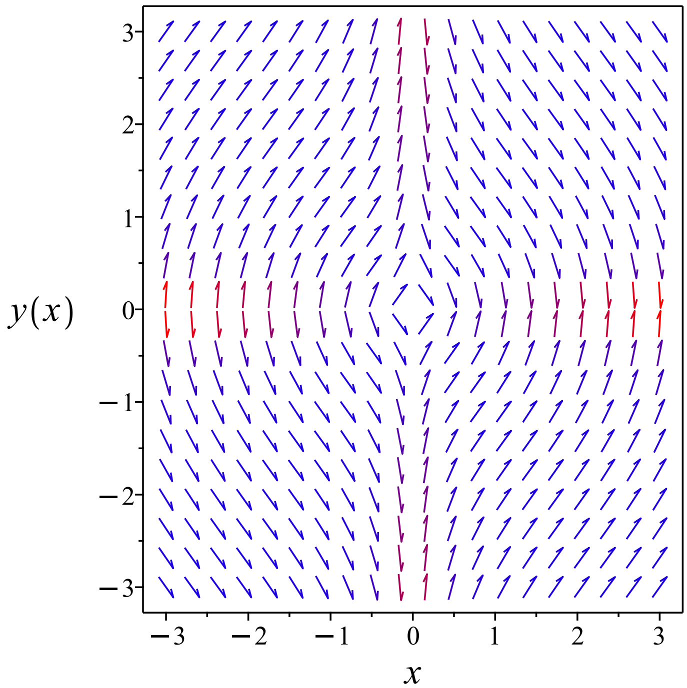


Figure 434: Slope field plot

Verification of solutions

$$\left(\frac{2y^2 + x^2}{x^2}\right)^{\frac{1}{4}} = \frac{c_3 e^{c_2}}{x^{\frac{3}{4}}}$$

Verified OK.

6.33.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = -\frac{3x^2 + 2y^2}{4yx}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `Bernoulli`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 315: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= \frac{1}{yx}
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the

canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{1}{yx}} dy \end{aligned}$$

Which results in

$$S = \frac{x\,y^2}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$\omega(x,y) = -\frac{3x^2 + 2y^2}{4yx}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= \frac{y^2}{2} \\ S_y &= yx \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = -\frac{3x^2}{4} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = -\frac{3R^2}{4}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{R^3}{4} + c_1 \tag{4}$$

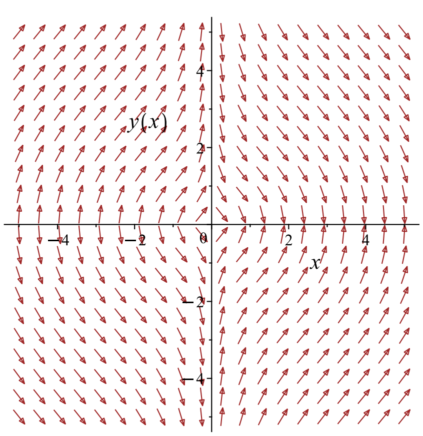
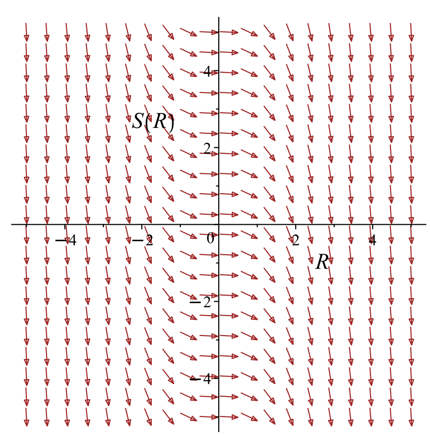
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{xy^2}{2} = -\frac{x^3}{4} + c_1$$

Which simplifies to

$$\frac{xy^2}{2} = -\frac{x^3}{4} + c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\frac{3x^2+2y^2}{4yx}$ 	$R = x$ $S = \frac{x\,y^2}{2}$	$\frac{dS}{dR} = -\frac{3R^2}{4}$ 

Summary

The solution(s) found are the following

$$\frac{xy^2}{2} = -\frac{x^3}{4} + c_1 \tag{1}$$

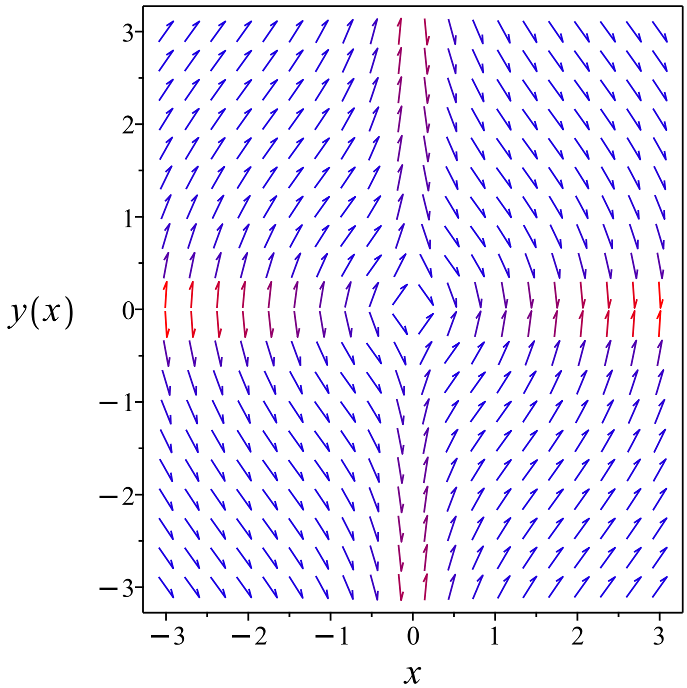


Figure 435: Slope field plot

Verification of solutions

$$\frac{xy^2}{2} = -\frac{x^3}{4} + c_1$$

Verified OK.

6.33.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x, y) \\ &= -\frac{3x^2 + 2y^2}{4yx} \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\frac{1}{2x}y - \frac{3x}{4}\frac{1}{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\frac{1}{2x} \\ f_1(x) &= -\frac{3x}{4} \\ n &= -1 \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \frac{1}{y}$ gives

$$y'y = -\frac{y^2}{2x} - \frac{3x}{4} \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= y^2 \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = 2yy' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} \frac{w'(x)}{2} &= -\frac{w(x)}{2x} - \frac{3x}{4} \\ w' &= -\frac{w}{x} - \frac{3x}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{3x}{2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{w(x)}{x} = -\frac{3x}{2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int \frac{1}{x}dx} \\ &= x\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu w) &= (\mu)\left(-\frac{3x}{2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}(xw) &= (x)\left(-\frac{3x}{2}\right) \\ \mathrm{d}(xw) &= \left(-\frac{3x^2}{2}\right)\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}xw &= \int -\frac{3x^2}{2}\,\mathrm{d}x \\ xw &= -\frac{x^3}{2} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = x$ results in

$$w(x) = -\frac{x^2}{2} + \frac{c_1}{x}$$

Replacing w in the above by y^2 using equation (5) gives the final solution.

$$y^2 = -\frac{x^2}{2} + \frac{c_1}{x}$$

Solving for y gives

$$\begin{aligned}y(x) &= \frac{\sqrt{2}\,\sqrt{-x\,(x^3 - 2c_1)}}{2x} \\ y(x) &= -\frac{\sqrt{2}\,\sqrt{-x\,(x^3 - 2c_1)}}{2x}\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{\sqrt{2}\, \sqrt{-x\,(x^3-2c_1)}}{2x} \tag{1}$$

$$y = -\frac{\sqrt{2}\, \sqrt{-x\,(x^3-2c_1)}}{2x} \tag{2}$$

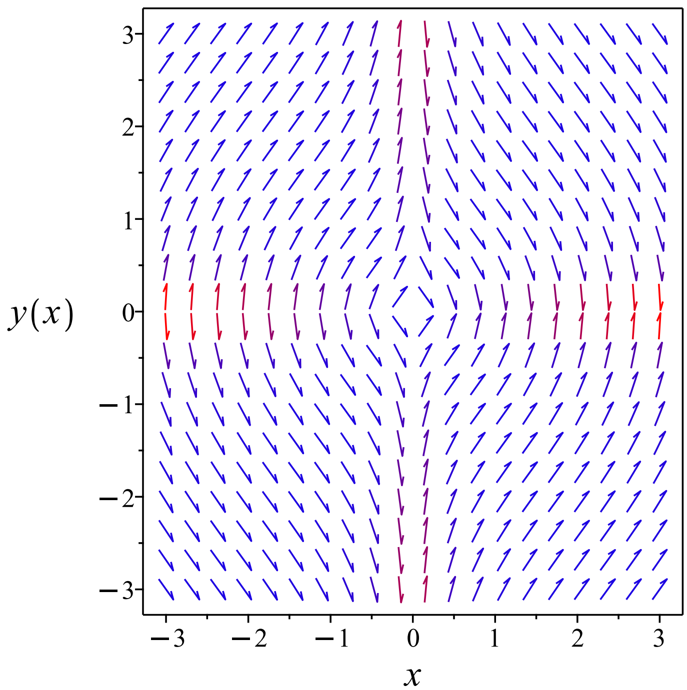


Figure 436: Slope field plot

Verification of solutions

$$y = \frac{\sqrt{2}\, \sqrt{-x\,(x^3-2c_1)}}{2x}$$

Verified OK.

$$y = -\frac{\sqrt{2}\, \sqrt{-x\,(x^3-2c_1)}}{2x}$$

Verified OK.

6.33.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y)+N(x,y)\frac{dy}{dx}=0 \tag{A}$$

We assume there exists a function $\phi(x,y)=c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y)=0$$

Hence

$$\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial y}\frac{dy}{dx}=0 \tag{B}$$

Comparing (A,B) shows that

$$\begin{aligned}\frac{\partial\phi}{\partial x}&=M\\ \frac{\partial\phi}{\partial y}&=N\end{aligned}$$

But since $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x,y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y}=\frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x,y)\,\mathrm{d}x+N(x,y)\,\mathrm{d}y=0 \tag{1A}$$

Therefore

$$\begin{aligned}(4yx)\,\mathrm{d}y&=\left(-3x^2-2y^2\right)\mathrm{d}x\\ \left(3x^2+2y^2\right)\mathrm{d}x+(4yx)\,\mathrm{d}y&=0 \end{aligned}\tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned}M(x,y)&=3x^2+2y^2\\ N(x,y)&=4yx\end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y}&=\frac{\partial}{\partial y}\left(3x^2+2y^2\right)\\&=4y\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x}&=\frac{\partial}{\partial x}(4yx)\\&=4y\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x}=M \tag{1}$$

$$\frac{\partial \phi}{\partial y}=N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int\frac{\partial \phi}{\partial x}\,\mathrm{d}x&=\int M\,\mathrm{d}x\\\int\frac{\partial \phi}{\partial x}\,\mathrm{d}x&=\int 3x^2+2y^2\,\mathrm{d}x\end{aligned}$$

$$\phi=x^3+2x\,y^2+f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y}=4yx+f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=4yx$. Therefore equation (4) becomes

$$4yx=4yx+f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = 0$$

Therefore

$$f(y) = c_1$$

Where c_1 is constant of integration. Substituting this result for $f(y)$ into equation (3) gives ϕ

$$\phi = x^3 + 2x\,y^2 + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = x^3 + 2x\,y^2$$

Summary

The solution(s) found are the following

$$x^3 + 2xy^2 = c_1 \tag{1}$$

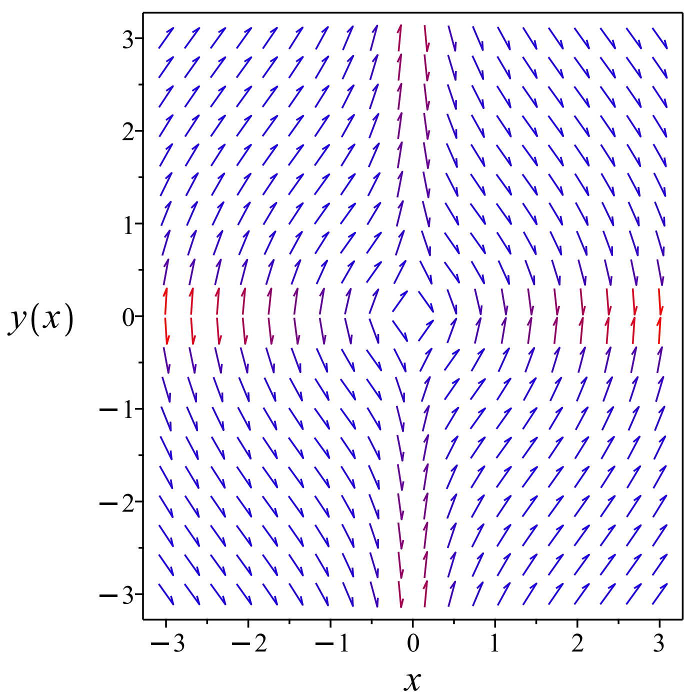


Figure 437: Slope field plot

Verification of solutions

$$x^3 + 2xy^2 = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 47

```
dsolve(diff(y(x),x) = 1/4*(-3*x^2-2*y(x)^2)/(x*y(x)),y(x), singsol=all)
```

$$y(x) = -\frac{\sqrt{2}\,\sqrt{-x\,(x^3 - 2c_1)}}{2x}$$
$$y(x) = \frac{\sqrt{2}\,\sqrt{-x\,(x^3 - 2c_1)}}{2x}$$

✓ Solution by Mathematica

Time used: 0.236 (sec). Leaf size: 60

```
DSolve[y'[x] == 1/4*(-3*x^2-2*y[x]^2)/(x*y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{\sqrt{-x^3 + 2c_1}}{\sqrt{2}\sqrt{x}}$$
$$y(x) \to \frac{\sqrt{-x^3 + 2c_1}}{\sqrt{2}\sqrt{x}}$$

6.34 problem 34 (a)

Internal problem ID [154]
Internal file name [`OUTPUT/154_Sunday_June_05_2022_01_36_08_AM_34506927/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 34 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "differentialType", "homogeneousTypeD2", "first_order_ode_lie_symmetry_calculated"**

Maple gives the following as the ode type

```
[[_homogeneous, `class A`], _rational, [_Abel, `2nd type`, `
   class A`]]
```

$$\boxed{y' - \frac{x+3y}{-3x+y} = 0}$$

6.34.1 Solving as homogeneousTypeD2 ode

Using the change of variables $y = u(x)\,x$ on the above ode results in new ode in $u(x)$

$$u'(x)\,x + u(x) - \frac{x+3u(x)\,x}{-3x+u\,(x)\,x} = 0$$

In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u^2-6u-1}{x\,(u-3)} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = \frac{u^2-6u-1}{u-3}$. Integrating both sides gives

$$\frac{1}{\frac{u^2-6u-1}{u-3}}\,du = -\frac{1}{x}\,dx$$
$$\int \frac{1}{\frac{u^2-6u-1}{u-3}}\,du = \int -\frac{1}{x}\,dx$$
$$\frac{\ln(u^2-6u-1)}{2} = -\ln(x) + c_2$$

Raising both side to exponential gives

$$\sqrt{u^2-6u-1} = e^{-\ln(x)+c_2}$$

Which simplifies to

$$\sqrt{u^2-6u-1} = \frac{c_3}{x}$$

Which simplifies to

$$\sqrt{u(x)^2-6u(x)-1} = \frac{c_3 e^{c_2}}{x}$$

The solution is

$$\sqrt{u(x)^2-6u(x)-1} = \frac{c_3 e^{c_2}}{x}$$

Replacing $u(x)$ in the above solution by $\frac{y}{x}$ results in the solution for y in implicit form

$$\sqrt{\frac{y^2}{x^2}-\frac{6y}{x}-1} = \frac{c_3 e^{c_2}}{x}$$
$$\sqrt{\frac{y^2-6yx-x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Summary

The solution(s) found are the following

$$\sqrt{\frac{y^2-6yx-x^2}{x^2}} = \frac{c_3 e^{c_2}}{x} \tag{1}$$

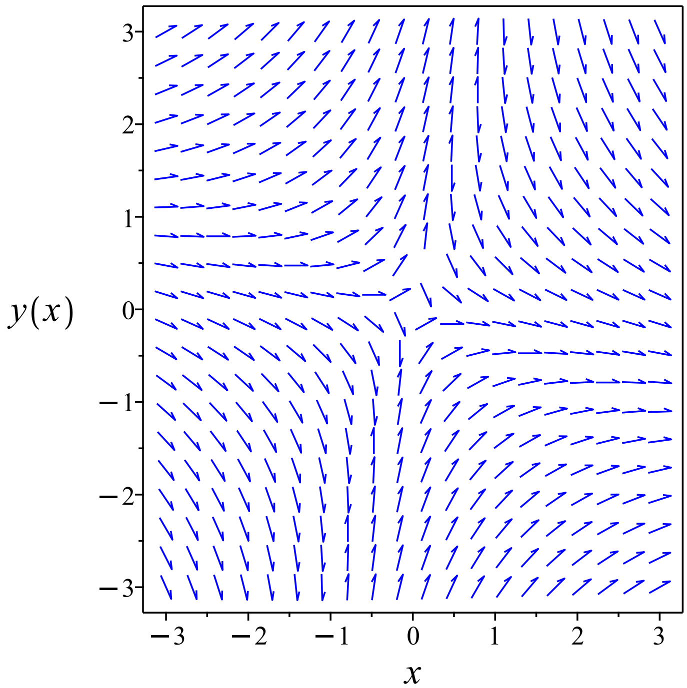


Figure 438: Slope field plot

Verification of solutions

$$\sqrt{\frac{y^2 - 6yx - x^2}{x^2}} = \frac{c_3 e^{c_2}}{x}$$

Verified OK.

6.34.2 Solving as differentialType ode

Writing the ode as

$$y' = \frac{x + 3y}{-3x + y} \tag{1}$$

Which becomes

$$(-y)\, dy = (-3x)\, dy + (-x - 3y)\, dx \tag{2}$$

But the RHS is complete differential because

$$(-3x)\, dy + (-x - 3y)\, dx = d\left(-\frac{1}{2}x^2 - 3yx\right)$$

Hence (2) becomes

$$(-y)\,dy = d\left(-\frac{1}{2}x^2 - 3yx\right)$$

Integrating both sides gives gives these solutions

$$y = 3x + \sqrt{10x^2 - 2c_1} + c_1$$
$$y = 3x - \sqrt{10x^2 - 2c_1} + c_1$$

Summary

The solution(s) found are the following

$$y = 3x + \sqrt{10x^2 - 2c_1} + c_1 \tag{1}$$
$$y = 3x - \sqrt{10x^2 - 2c_1} + c_1 \tag{2}$$

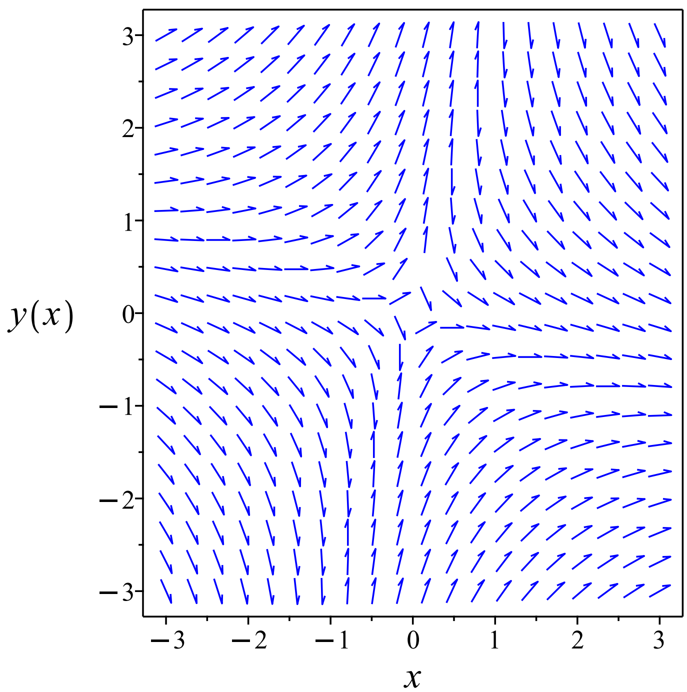


Figure 439: Slope field plot

Verification of solutions

$$y = 3x + \sqrt{10x^2 - 2c_1} + c_1$$

Verified OK.

$$y = 3x - \sqrt{10x^2 - 2c_1} + c_1$$

Verified OK.

6.34.3 Solving as first order ode lie symmetry calculated ode

Writing the ode as

$$
\begin{aligned}
y' &= \frac{x+3y}{-3x+y}\\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is not in the lookup table. To determine ξ, η then (A) is solved using ansatz. Making bivariate polynomials of degree 1 to use as anstaz gives

$$
\xi = x a_2 + y a_3 + a_1 \tag{1E}
$$

$$
\eta = x b_2 + y b_3 + b_1 \tag{2E}
$$

Where the unknown coefficients are

$$
\{a_1, a_2, a_3, b_1, b_2, b_3\}
$$

Substituting equations (1E,2E) and ω into (A) gives

$$
\begin{aligned}
&b_2 + \frac{(x+3y)(b_3-a_2)}{-3x+y} - \frac{(x+3y)^2 a_3}{(-3x+y)^2}\\
&- \left(\frac{1}{-3x+y} + \frac{3x+9y}{(-3x+y)^2}\right)(x a_2 + y a_3 + a_1)\\
&- \left(\frac{3}{-3x+y} - \frac{x+3y}{(-3x+y)^2}\right)(x b_2 + y b_3 + b_1) = 0
\end{aligned} \tag{5E}
$$

Putting the above in normal form gives

$$
\frac{3x^2a_2 - x^2a_3 + 19x^2b_2 - 3x^2b_3 - 2xya_2 - 6xya_3 - 6xyb_2 + 2xyb_3 - 3y^2a_2 - 19y^2a_3 + y^2b_2 + 3y^2b_3 + 10x}{(3x-y)^2}
$$

$$
= 0
$$

Setting the numerator to zero gives

$$
\begin{aligned}
&3x^2a_2 - x^2a_3 + 19x^2b_2 - 3x^2b_3 - 2xya_2 - 6xya_3 - 6xyb_2\\
&+ 2xyb_3 - 3y^2a_2 - 19y^2a_3 + y^2b_2 + 3y^2b_3 + 10xb_1 - 10ya_1 = 0
\end{aligned} \tag{6E}
$$

Looking at the above PDE shows the following are all the terms with $\{x, y\}$ in them.

$$\{x, y\}$$

The following substitution is now made to be able to collect on all terms with $\{x, y\}$ in them

$$\{x = v_1, y = v_2\}$$

The above PDE (6E) now becomes

$$3a_2v_1^2 - 2a_2v_1v_2 - 3a_2v_2^2 - a_3v_1^2 - 6a_3v_1v_2 - 19a_3v_2^2 + 19b_2v_1^2 - 6b_2v_1v_2 + b_2v_2^2 - 3b_3v_1^2 + 2b_3v_1v_2 + 3b_3v_2^2 - 10a_1v_2 + 10b_1v_1 = 0 \tag{7E}$$

Collecting the above on the terms v_i introduced, and these are

$$\{v_1, v_2\}$$

Equation (7E) now becomes

$$(3a_2 - a_3 + 19b_2 - 3b_3)\, v_1^2 + (-2a_2 - 6a_3 - 6b_2 + 2b_3)\, v_1v_2 + 10b_1v_1 + (-3a_2 - 19a_3 + b_2 + 3b_3)\, v_2^2 - 10a_1v_2 = 0 \tag{8E}$$

Setting each coefficients in (8E) to zero gives the following equations to solve

$$\begin{aligned}
-10a_1 &= 0 \\
10b_1 &= 0 \\
-3a_2 - 19a_3 + b_2 + 3b_3 &= 0 \\
-2a_2 - 6a_3 - 6b_2 + 2b_3 &= 0 \\
3a_2 - a_3 + 19b_2 - 3b_3 &= 0
\end{aligned}$$

Solving the above equations for the unknowns gives

$$\begin{aligned}
a_1 &= 0 \\
a_2 &= -6b_2 + b_3 \\
a_3 &= b_2 \\
b_1 &= 0 \\
b_2 &= b_2 \\
b_3 &= b_3
\end{aligned}$$

Substituting the above solution in the anstaz (1E,2E) (using 1 as arbitrary value for any unknown in the RHS) gives

$$\xi = x$$
$$\eta = y$$

Shifting is now applied to make $\xi = 0$ in order to simplify the rest of the computation

$$\begin{aligned}\eta &= \eta - \omega(x, y)\,\xi \\ &= y - \left(\frac{x + 3y}{-3x + y}\right)(x) \\ &= \frac{x^2 + 6yx - y^2}{3x - y} \\ \xi &= 0\end{aligned}$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned}S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{\frac{x^2+6yx-y^2}{3x-y}} dy\end{aligned}$$

Which results in

$$S = \frac{\ln\left(-x^2 - 6yx + y^2\right)}{2}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{x + 3y}{-3x + y}$$

Evaluating all the partial derivatives gives

$$\begin{aligned}
R_x &= 1 \\
R_y &= 0 \\
S_x &= \frac{x + 3y}{x^2 + 6yx - y^2} \\
S_y &= \frac{3x - y}{x^2 + 6yx - y^2}
\end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = 0 \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = 0$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$S(R) = c_1 \tag{4}$$

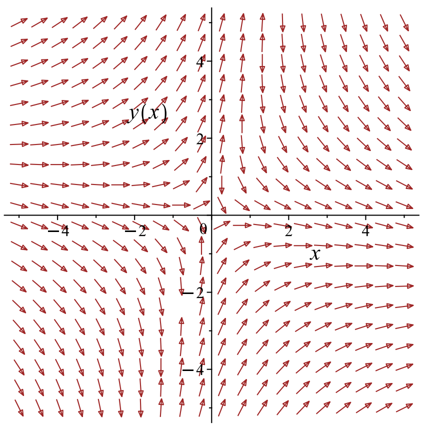
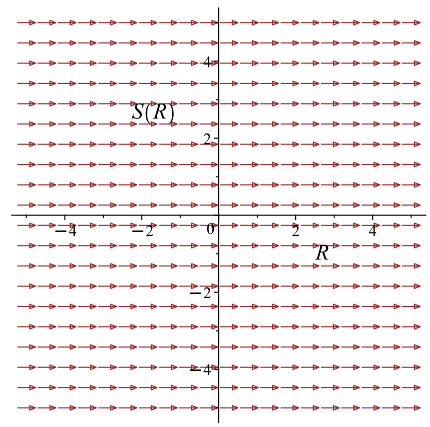
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{\ln(y^2 - 6yx - x^2)}{2} = c_1$$

Which simplifies to

$$\frac{\ln(y^2 - 6yx - x^2)}{2} = c_1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

<table>
<tr><td>Original ode in x, y coordinates</td><td>Canonical coordinates transformation</td><td>ODE in canonical coordinates (R, S)</td></tr>
<tr><td>$\frac{dy}{dx} = \frac{x+3y}{-3x+y}$
</td><td>$R = x$
$S = \frac{\ln(-x^2 - 6yx + y^2)}{2}$</td><td>$\frac{dS}{dR} = 0$
</td></tr>
</table>

Summary

The solution(s) found are the following

$$\frac{\ln(y^2 - 6yx - x^2)}{2} = c_1 \tag{1}$$

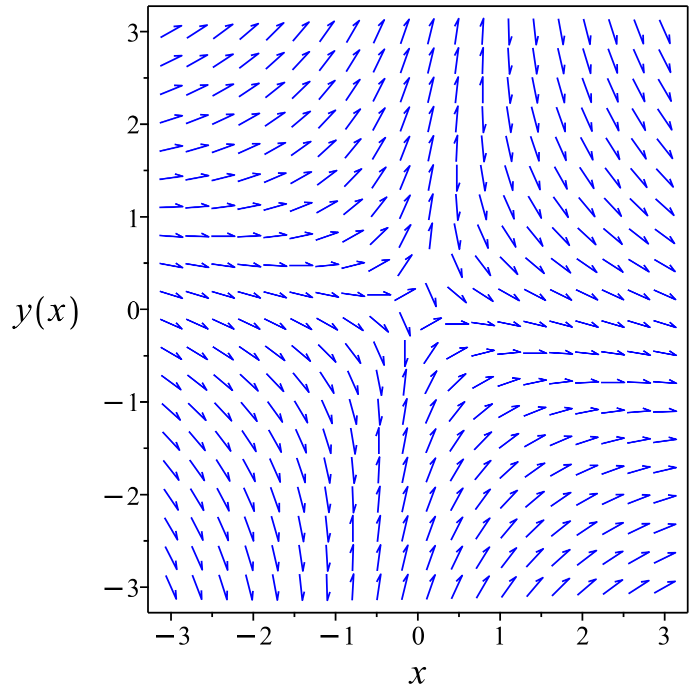


Figure 440: Slope field plot

Verification of solutions

$$\frac{\ln(y^2 - 6yx - x^2)}{2} = c_1$$

Verified OK.

6.34.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y) \, \mathrm{d}x + N(x, y) \, \mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\begin{aligned}(-3x + y) \, \mathrm{d}y &= (x + 3y) \, \mathrm{d}x \\ (-x - 3y) \, \mathrm{d}x + (-3x + y) \, \mathrm{d}y &= 0 \end{aligned} \tag{2A}$$

Comparing (1A) and (2A) shows that

$$\begin{aligned} M(x, y) &= -x - 3y \\ N(x, y) &= -3x + y \end{aligned}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned} \frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-x - 3y) \\ &= -3 \end{aligned}$$

And

$$\begin{aligned} \frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}(-3x + y) \\ &= -3 \end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$
$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int M \, \mathrm{d}x$$
$$\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x = \int -x - 3y \, \mathrm{d}x$$
$$\phi = -\frac{x(x+6y)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = -3x + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = -3x + y$. Therefore equation (4) becomes

$$-3x + y = -3x + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = y$$

Integrating the above w.r.t y gives

$$\int f'(y) \, \mathrm{d}y = \int (y) \, \mathrm{d}y$$
$$f(y) = \frac{y^2}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{x(x+6y)}{2} + \frac{y^2}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{x(x+6y)}{2} + \frac{y^2}{2}$$

Summary

The solution(s) found are the following

$$-\frac{x(x+6y)}{2} + \frac{y^2}{2} = c_1 \tag{1}$$

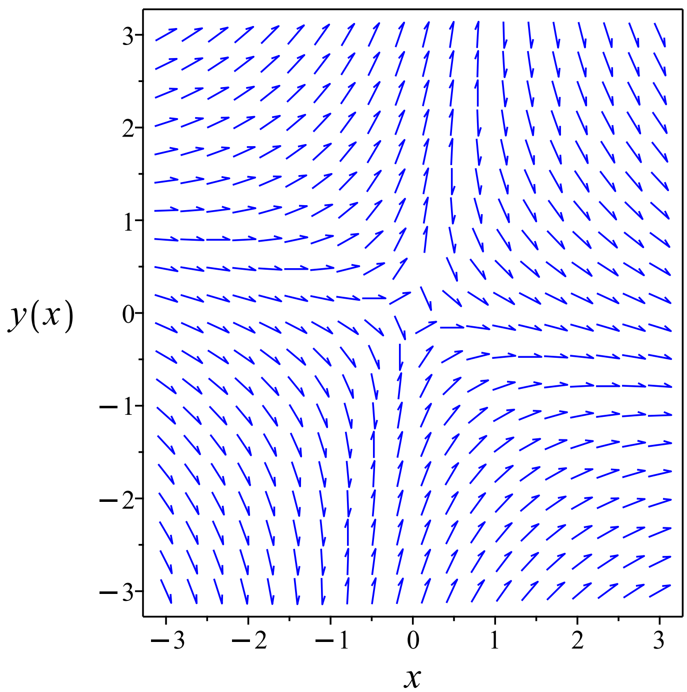


Figure 441: Slope field plot

Verification of solutions

$$-\frac{x(x+6y)}{2} + \frac{y^2}{2} = c_1$$

Verified OK.

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
trying separable
trying inverse linear
trying homogeneous types:
trying homogeneous D
<- homogeneous successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 51

```
dsolve(diff(y(x),x) = (x+3*y(x))/(-3*x+y(x)),y(x), singsol=all)
```

$$y(x) = \frac{3c_1x - \sqrt{10c_1^2x^2 + 1}}{c_1}$$
$$y(x) = \frac{3c_1x + \sqrt{10c_1^2x^2 + 1}}{c_1}$$

✓ Solution by Mathematica

Time used: 0.482 (sec). Leaf size: 94

```
DSolve[y'[x] == (x+3*y[x])/(-3*x+y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow 3x - \sqrt{10x^2 + e^{2c_1}}$$
$$y(x) \rightarrow 3x + \sqrt{10x^2 + e^{2c_1}}$$
$$y(x) \rightarrow 3x - \sqrt{10}\sqrt{x^2}$$
$$y(x) \rightarrow \sqrt{10}\sqrt{x^2} + 3x$$

6.35 problem 35 (a)

Internal problem ID [155]
Internal file name [`OUTPUT/155_Sunday_June_05_2022_01_36_09_AM_97041794/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 35 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "linear", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \frac{2x + 2yx}{x^2 + 1} = 0}$$

6.35.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \frac{x(2y+2)}{x^2+1} \end{aligned}$$

Where $f(x) = \frac{x}{x^2+1}$ and $g(y) = 2y + 2$. Integrating both sides gives

$$\frac{1}{2y+2}\,dy = \frac{x}{x^2+1}\,dx$$

$$\int \frac{1}{2y+2}\,dy = \int \frac{x}{x^2+1}\,dx$$

$$\frac{\ln(y+1)}{2} = \frac{\ln(x^2+1)}{2} + c_1$$

Raising both side to exponential gives

$$\sqrt{y+1} = e^{\frac{\ln(x^2+1)}{2}+c_1}$$

Which simplifies to

$$\sqrt{y+1} = c_2\sqrt{x^2+1}$$

Which simplifies to

$$y = c_2^2(x^2+1)\,e^{2c_1} - 1$$

Summary

The solution(s) found are the following

$$y = c_2^2(x^2+1)\,e^{2c_1} - 1 \tag{1}$$

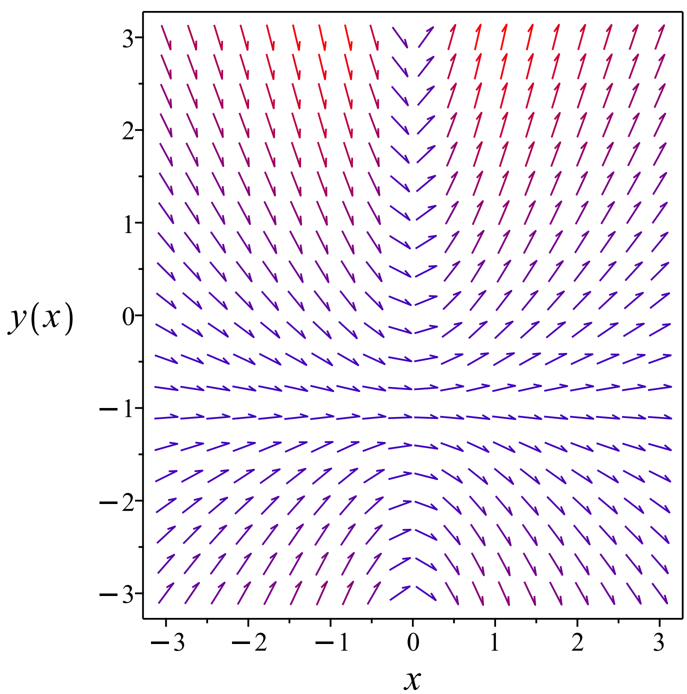


Figure 442: Slope field plot

Verification of solutions

$$y = c_2^2(x^2+1)\,e^{2c_1} - 1$$

Verified OK.

6.35.2 Solving as linear ode

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned}p(x) &= -\frac{2x}{x^2+1}\\ q(x) &= \frac{2x}{x^2+1}\end{aligned}$$

Hence the ode is

$$y' - \frac{2xy}{x^2+1} = \frac{2x}{x^2+1}$$

The integrating factor μ is

$$\begin{aligned}\mu &= e^{\int -\frac{2x}{x^2+1}dx}\\ &= \frac{1}{x^2+1}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{d}{dx}(\mu y) &= (\mu)\left(\frac{2x}{x^2+1}\right)\\ \frac{d}{dx}\left(\frac{y}{x^2+1}\right) &= \left(\frac{1}{x^2+1}\right)\left(\frac{2x}{x^2+1}\right)\\ d\left(\frac{y}{x^2+1}\right) &= \left(\frac{2x}{(x^2+1)^2}\right) dx\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x^2+1} &= \int \frac{2x}{(x^2+1)^2}\, dx\\ \frac{y}{x^2+1} &= -\frac{1}{x^2+1} + c_1\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x^2+1}$ results in

$$y = -1 + c_1(x^2+1)$$

which simplifies to

$$y = c_1x^2 + c_1 - 1$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_1 - 1 \tag{1}$$

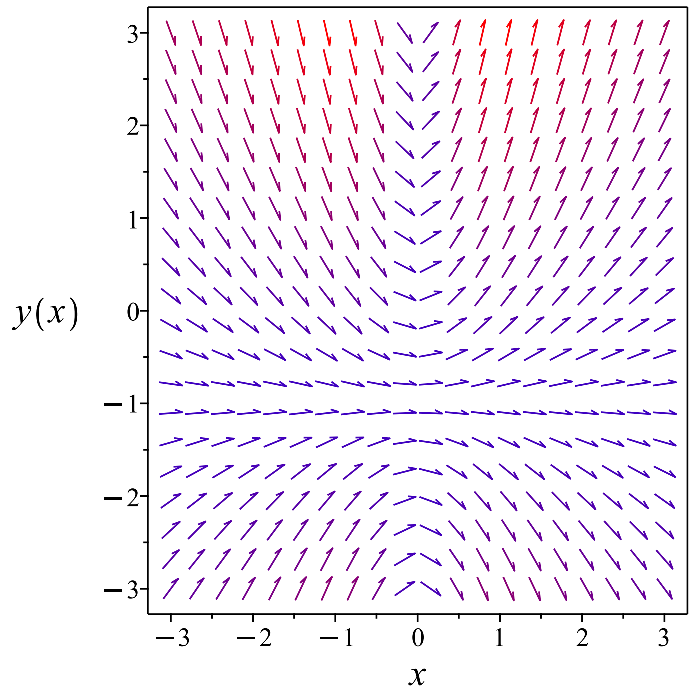


Figure 443: Slope field plot

Verification of solutions

$$y = c_1 x^2 + c_1 - 1$$

Verified OK.

6.35.3 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$y' = \frac{2x(y+1)}{x^2+1}$$
$$y' = \omega(x, y)$$

The condition of Lie symmetry is the linearized PDE given by

$$\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}$$

The type of this ode is known. It is of type `linear`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 317: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a + bx + cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx - h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx - h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x - a_2b_1x - b_1c_2 + b_2c_1}{a_1b_2 - a_2b_1}$	$\frac{a_1b_2y - a_2b_1y - a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int (n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x, y) &= 0 \\
\eta(x, y) &= x^2 + 1
\end{aligned} \tag{A1}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x, y) \to (R, S)$ where (R, S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y}\right) S(x, y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\xi = 0$ then in this special case

$$R = x$$

S is found from

$$\begin{aligned} S &= \int \frac{1}{\eta} dy \\ &= \int \frac{1}{x^2+1} dy \end{aligned}$$

Which results in

$$S = \frac{y}{x^2+1}$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$\frac{dS}{dR} = \frac{S_x + \omega(x, y) S_y}{R_x + \omega(x, y) R_y} \tag{2}$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x, y)$ is the right hand side of the original ode given by

$$\omega(x, y) = \frac{2x(y+1)}{x^2+1}$$

Evaluating all the partial derivatives gives

$$\begin{aligned} R_x &= 1 \\ R_y &= 0 \\ S_x &= -\frac{2yx}{(x^2+1)^2} \\ S_y &= \frac{1}{x^2+1} \end{aligned}$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$\frac{dS}{dR} = \frac{2x}{(x^2+1)^2} \tag{2A}$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$\frac{dS}{dR} = \frac{2R}{(R^2+1)^2}$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordinates R, S. Integrating the above gives

$$S(R) = -\frac{1}{R^2+1} + c_1 \tag{4}$$

To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$\frac{y}{x^2+1} = -\frac{1}{x^2+1} + c_1$$

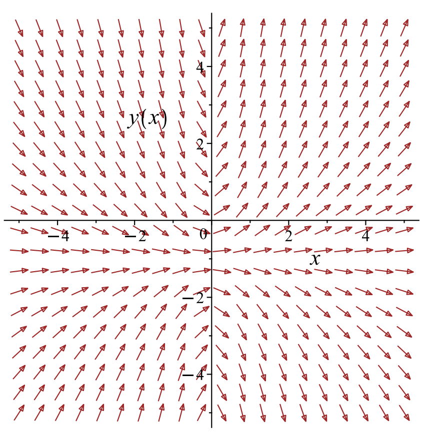
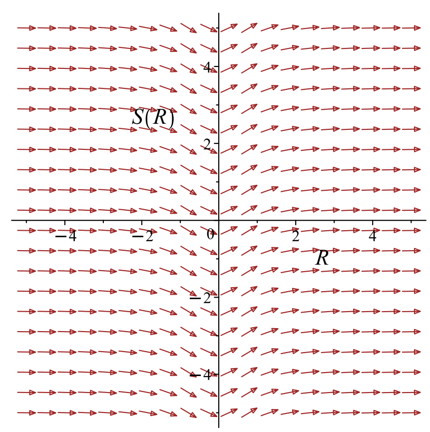
Which simplifies to

$$\frac{y}{x^2+1} = -\frac{1}{x^2+1} + c_1$$

Which gives

$$y = c_1 x^2 + c_1 - 1$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = \frac{2x(y+1)}{x^2+1}$ 	$R = x$ $S = \frac{y}{x^2+1}$	$\frac{dS}{dR} = \frac{2R}{(R^2+1)^2}$ 

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_1 - 1 \tag{1}$$

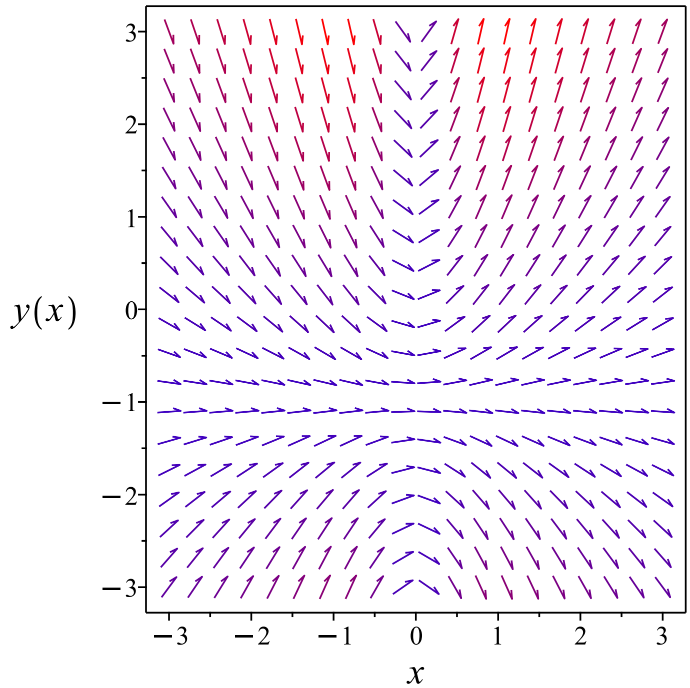


Figure 444: Slope field plot

Verification of solutions

$$y = c_1 x^2 + c_1 - 1$$

Verified OK.

6.35.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial\phi}{\partial x} = M$$
$$\frac{\partial\phi}{\partial y} = N$$

But since $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{2y+2}\right)\mathrm{d}y = \left(\frac{x}{x^2+1}\right)\mathrm{d}x$$
$$\left(-\frac{x}{x^2+1}\right)\mathrm{d}x + \left(\frac{1}{2y+2}\right)\mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\frac{x}{x^2+1}$$
$$N(x, y) = \frac{1}{2y+2}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}\left(-\frac{x}{x^2+1}\right)\\ &= 0\end{aligned}$$

And

$$\begin{aligned}\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2y+2}\right)\\ &= 0\end{aligned}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the ODE is <u>exact</u> The following equations are now set up to solve for the function $\phi(x, y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}\int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int M \, \mathrm{d}x\\ \int \frac{\partial \phi}{\partial x} \, \mathrm{d}x &= \int -\frac{x}{x^2+1} \, \mathrm{d}x\end{aligned}$$

$$\phi = -\frac{\ln(x^2+1)}{2} + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y} = \frac{1}{2y+2}$. Therefore equation (4) becomes

$$\frac{1}{2y+2} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{2y+2}$$

Integrating the above w.r.t y gives

$$\int f'(y)\,\mathrm{d}y = \int \left(\frac{1}{2y+2}\right)\mathrm{d}y$$
$$f(y) = \frac{\ln (y+1)}{2} + c_1$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\frac{\ln (x^2+1)}{2} + \frac{\ln (y+1)}{2} + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\frac{\ln (x^2+1)}{2} + \frac{\ln (y+1)}{2}$$

The solution becomes

$$y = x^2 \mathrm{e}^{2c_1} + \mathrm{e}^{2c_1} - 1$$

Summary

The solution(s) found are the following

$$y = x^2 \mathrm{e}^{2c_1} + \mathrm{e}^{2c_1} - 1 \tag{1}$$

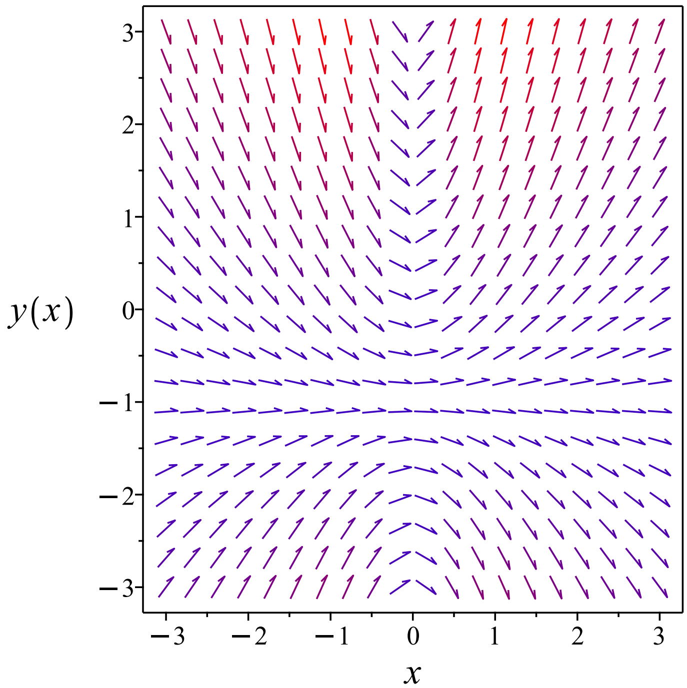


Figure 445: Slope field plot

Verification of solutions

$$y = x^2 \mathrm{e}^{2c_1} + \mathrm{e}^{2c_1} - 1$$

Verified OK.

6.35.5 Maple step by step solution

Let's solve

$y' - \frac{2x+2yx}{x^2+1} = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{1+y} = \frac{2x}{x^2+1}$

- Integrate both sides with respect to x

  $\int \frac{y'}{1+y} dx = \int \frac{2x}{x^2+1} dx + c_1$

- Evaluate integral

$$\ln(1+y) = \ln(x^2+1) + c_1$$

- Solve for y

$$y = x^2 e^{c_1} + e^{c_1} - 1$$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
<- 1st order linear successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve(diff(y(x),x) = (2*x+2*x*y(x))/(x^2+1),y(x), singsol=all)
```

$$y(x) = c_1 x^2 + c_1 - 1$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 20

```
DSolve[y'[x] == (2*x+2*x*y[x])/(x^2+1),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow -1 + c_1(x^2+1)$$
$$y(x) \rightarrow -1$$

6.36 problem 36 (a)

Internal problem ID [156]
Internal file name [`OUTPUT/156_Sunday_June_05_2022_01_36_10_AM_96305306/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Chapter 1 review problems. Page 78
Problem number: 36 (a).
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"exact", "bernoulli", "separable", "first_order_ode_lie_symmetry_lookup"**

Maple gives the following as the ode type

```
[_separable]
```

$$\boxed{y' - \cot(x)\left(\sqrt{y} - y\right) = 0}$$

6.36.1 Solving as separable ode

In canonical form the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= f(x)g(y) \\ &= \cot(x)\left(\sqrt{y} - y\right) \end{aligned}$$

Where $f(x) = \cot(x)$ and $g(y) = \sqrt{y} - y$. Integrating both sides gives

$$\frac{1}{\sqrt{y} - y}\,dy = \cot(x)\,dx$$

$$\int \frac{1}{\sqrt{y} - y}\,dy = \int \cot(x)\,dx$$

$$2\,\text{arctanh}\left(\sqrt{y}\right) - \ln(y-1) = \ln(\sin(x)) + c_1$$

The solution is

$$2 \operatorname{arctanh}\left(\sqrt{y}\right) - \ln\left(y-1\right) - \ln\left(\sin\left(x\right)\right) - c_1 = 0$$

Summary

The solution(s) found are the following

$$2 \operatorname{arctanh}\left(\sqrt{y}\right) - \ln\left(y-1\right) - \ln\left(\sin\left(x\right)\right) - c_1 = 0 \tag{1}$$

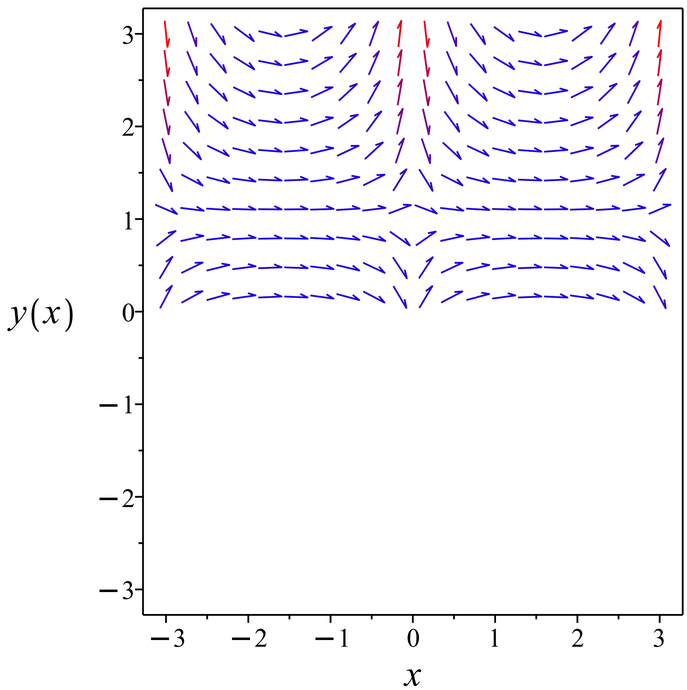


Figure 446: Slope field plot

Verification of solutions

$$2 \operatorname{arctanh}\left(\sqrt{y}\right) - \ln\left(y-1\right) - \ln\left(\sin\left(x\right)\right) - c_1 = 0$$

Verified OK.

6.36.2 Solving as first order ode lie symmetry lookup ode

Writing the ode as

$$
\begin{aligned}
y' &= -\cot(x)\left(-\sqrt{y}+y\right) \\
y' &= \omega(x,y)
\end{aligned}
$$

The condition of Lie symmetry is the linearized PDE given by

$$
\eta_x + \omega(\eta_y - \xi_x) - \omega^2 \xi_y - \omega_x \xi - \omega_y \eta = 0 \tag{A}
$$

The type of this ode is known. It is of type `separable`. Therefore we do not need to solve the PDE (A), and can just use the lookup table shown below to find ξ, η

Table 320: Lie symmetry infinitesimal lookup table for known first order ODE's

ODE class	Form	ξ	η
linear ode	$y' = f(x)y(x) + g(x)$	0	$e^{\int f dx}$
separable ode	$y' = f(x)\, g(y)$	$\frac{1}{f}$	0
quadrature ode	$y' = f(x)$	0	1
quadrature ode	$y' = g(y)$	1	0
homogeneous ODEs of Class A	$y' = f\left(\frac{y}{x}\right)$	x	y
homogeneous ODEs of Class C	$y' = (a+bx+cy)^{\frac{n}{m}}$	1	$-\frac{b}{c}$
homogeneous class D	$y' = \frac{y}{x} + g(x)\, F\left(\frac{y}{x}\right)$	x^2	xy
First order special form ID 1	$y' = g(x)\, e^{h(x)+by} + f(x)$	$\frac{e^{-\int bf(x)dx-h(x)}}{g(x)}$	$\frac{f(x)e^{-\int bf(x)dx-h(x)}}{g(x)}$
polynomial type ode	$y' = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$	$\frac{a_1b_2x-a_2b_1x-b_1c_2+b_2c_1}{a_1b_2-a_2b_1}$	$\frac{a_1b_2y-a_2b_1y-a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$
Bernoulli ode	$y' = f(x)\, y + g(x)\, y^n$	0	$e^{-\int(n-1)f(x)dx} y^n$
Reduced Riccati	$y' = f_1(x)\, y + f_2(x)\, y^2$	0	$e^{-\int f_1 dx}$

The above table shows that

$$
\begin{aligned}
\xi(x,y) &= \frac{1}{\cot (x)} \\
\eta(x,y) &= 0 \qquad \text{(A1)}
\end{aligned}
$$

The next step is to determine the canonical coordinates R, S. The canonical coordinates map $(x,y) \to (R,S)$ where (R,S) are the canonical coordinates which make the original ode become a quadrature and hence solved by integration.

The characteristic pde which is used to find the canonical coordinates is

$$
\frac{dx}{\xi} = \frac{dy}{\eta} = dS \tag{1}
$$

The above comes from the requirements that $\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) S(x,y) = 1$. Starting with the first pair of ode's in (1) gives an ode to solve for the independent variable R in the canonical coordinates, where $S(R)$. Since $\eta = 0$ then in this special case

$$
R = y
$$

S is found from

$$
\begin{aligned}
S &= \int \frac{1}{\xi} dx \\
&= \int \frac{1}{\frac{1}{\cot(x)}} dx
\end{aligned}
$$

Which results in

$$
S = \ln (\sin (x))
$$

Now that R, S are found, we need to setup the ode in these coordinates. This is done by evaluating

$$
\frac{dS}{dR} = \frac{S_x + \omega(x,y)S_y}{R_x + \omega(x,y)R_y} \tag{2}
$$

Where in the above R_x, R_y, S_x, S_y are all partial derivatives and $\omega(x,y)$ is the right hand side of the original ode given by

$$
\omega(x,y) = -\cot (x) \left(-\sqrt{y} + y\right)
$$

Evaluating all the partial derivatives gives

$$
\begin{aligned}
R_x &= 0 \\
R_y &= 1 \\
S_x &= \cot(x) \\
S_y &= 0
\end{aligned}
$$

Substituting all the above in (2) and simplifying gives the ode in canonical coordinates.

$$
\frac{dS}{dR} = \frac{1}{\sqrt{y} - y} \tag{2A}
$$

We now need to express the RHS as function of R only. This is done by solving for x, y in terms of R, S from the result obtained earlier and simplifying. This gives

$$
\frac{dS}{dR} = \frac{1}{\sqrt{R} - R}
$$

The above is a quadrature ode. This is the whole point of Lie symmetry method. It converts an ode, no matter how complicated it is, to one that can be solved by integration when the ode is in the canonical coordiates R, S. Integrating the above gives

$$
S(R) = 2 \operatorname{arctanh}\left(\sqrt{R}\right) - \ln(R-1) + c_1 \tag{4}
$$

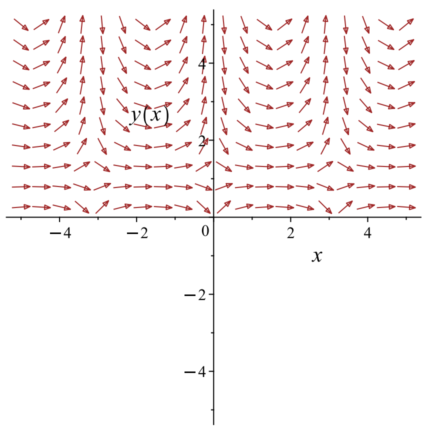
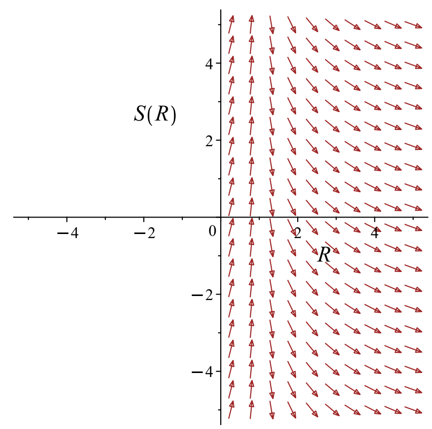
To complete the solution, we just need to transform (4) back to x, y coordinates. This results in

$$
\ln(\sin(x)) = 2 \operatorname{arctanh}(\sqrt{y}) - \ln(y-1) + c_1
$$

Which simplifies to

$$
\ln(\sin(x)) = 2 \operatorname{arctanh}(\sqrt{y}) - \ln(y-1) + c_1
$$

The following diagram shows solution curves of the original ode and how they transform in the canonical coordinates space using the mapping shown.

Original ode in x, y coordinates	Canonical coordinates transformation	ODE in canonical coordinates (R, S)
$\frac{dy}{dx} = -\cot(x)\left(-\sqrt{y}+y\right)$ 	$R = y$ $S = \ln(\sin(x))$	$\frac{dS}{dR} = \frac{1}{\sqrt{R}-R}$ 

Summary
The solution(s) found are the following

$$\ln(\sin(x)) = 2\,\mathrm{arctanh}\left(\sqrt{y}\right) - \ln(y-1) + c_1 \tag{1}$$

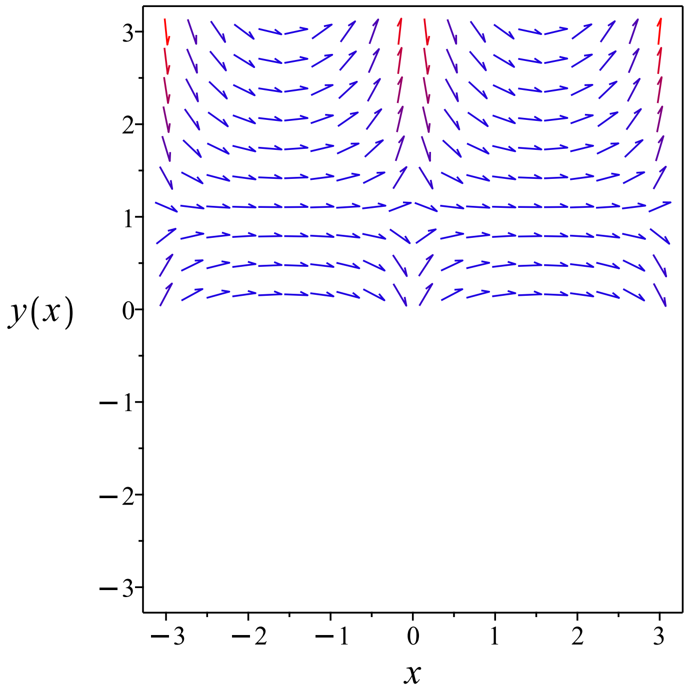


Figure 447: Slope field plot

Verification of solutions

$$\ln(\sin(x)) = 2 \operatorname{arctanh}\left(\sqrt{y}\right) - \ln(y-1) + c_1$$

Verified OK.

6.36.3 Solving as bernoulli ode

In canonical form, the ODE is

$$\begin{aligned} y' &= F(x,y) \\ &= -\cot(x)\left(-\sqrt{y}+y\right) \end{aligned}$$

This is a Bernoulli ODE.

$$y' = -\cot(x)\, y + \cot(x)\sqrt{y} \tag{1}$$

The standard Bernoulli ODE has the form

$$y' = f_0(x)y + f_1(x)y^n \tag{2}$$

The first step is to divide the above equation by y^n which gives

$$\frac{y'}{y^n} = f_0(x)y^{1-n} + f_1(x) \tag{3}$$

The next step is use the substitution $w = y^{1-n}$ in equation (3) which generates a new ODE in $w(x)$ which will be linear and can be easily solved using an integrating factor. Backsubstitution then gives the solution $y(x)$ which is what we want.

This method is now applied to the ODE at hand. Comparing the ODE (1) With (2) Shows that

$$\begin{aligned} f_0(x) &= -\cot(x) \\ f_1(x) &= \cot(x) \\ n &= \frac{1}{2} \end{aligned}$$

Dividing both sides of ODE (1) by $y^n = \sqrt{y}$ gives

$$y' \frac{1}{\sqrt{y}} = -\cot(x)\sqrt{y} + \cot(x) \tag{4}$$

Let

$$\begin{aligned} w &= y^{1-n} \\ &= \sqrt{y} \end{aligned} \tag{5}$$

Taking derivative of equation (5) w.r.t x gives

$$w' = \frac{1}{2\sqrt{y}} y' \tag{6}$$

Substituting equations (5) and (6) into equation (4) gives

$$\begin{aligned} 2w'(x) &= -\cot(x)\, w(x) + \cot(x) \\ w' &= -\frac{\cot(x)\, w}{2} + \frac{\cot(x)}{2} \end{aligned} \tag{7}$$

The above now is a linear ODE in $w(x)$ which is now solved.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$w'(x) + p(x)w(x) = q(x)$$

Where here

$$\begin{aligned} p(x) &= \frac{\cot(x)}{2} \\ q(x) &= \frac{\cot(x)}{2} \end{aligned}$$

Hence the ode is

$$w'(x) + \frac{\cot(x)\, w(x)}{2} = \frac{\cot(x)}{2}$$

The integrating factor μ is

$$\begin{aligned}
\mu &= e^{\int \frac{\cot(x)}{2}dx} \\
&= \sqrt{\sin(x)}
\end{aligned}$$

The ode becomes

$$\begin{aligned}
\frac{d}{dx}(\mu w) &= (\mu)\left(\frac{\cot(x)}{2}\right) \\
\frac{d}{dx}\left(\sqrt{\sin(x)}\, w\right) &= \left(\sqrt{\sin(x)}\right)\left(\frac{\cot(x)}{2}\right) \\
d\left(\sqrt{\sin(x)}\, w\right) &= \left(\frac{\cos(x)}{2\sqrt{\sin(x)}}\right) dx
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\sqrt{\sin(x)}\, w &= \int \frac{\cos(x)}{2\sqrt{\sin(x)}}\, dx \\
\sqrt{\sin(x)}\, w &= \sqrt{\sin(x)} + c_1
\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \sqrt{\sin(x)}$ results in

$$w(x) = 1 + \frac{c_1}{\sqrt{\sin(x)}}$$

Replacing w in the above by $\sqrt{y}$ using equation (5) gives the final solution.

$$\sqrt{y} = 1 + \frac{c_1}{\sqrt{\sin(x)}}$$

<u>Summary</u>

The solution(s) found are the following

$$\sqrt{y} = 1 + \frac{c_1}{\sqrt{\sin(x)}} \tag{1}$$

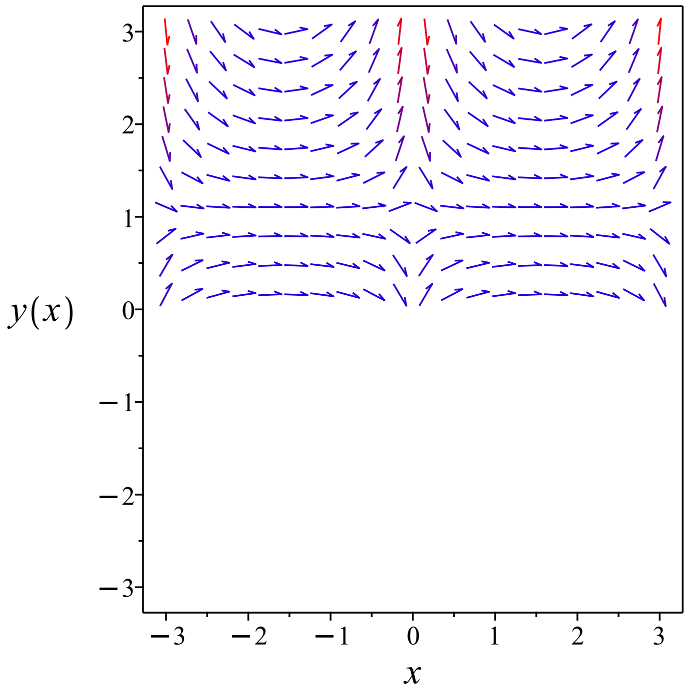


Figure 448: Slope field plot

Verification of solutions

$$\sqrt{y} = 1 + \frac{c_1}{\sqrt{\sin(x)}}$$

Verified OK.

6.36.4 Solving as exact ode

Entering Exact first order ODE solver. (Form one type)

To solve an ode of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \tag{A}$$

We assume there exists a function $\phi(x,y) = c$ where c is constant, that satisfies the ode. Taking derivative of ϕ w.r.t. x gives

$$\frac{d}{dx}\phi(x,y) = 0$$

Hence

$$\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\frac{dy}{dx} = 0 \tag{B}$$

Comparing (A,B) shows that

$$\frac{\partial \phi}{\partial x} = M$$
$$\frac{\partial \phi}{\partial y} = N$$

But since $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ then for the above to be valid, we require that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

If the above condition is satisfied, then the original ode is called exact. We still need to determine $\phi(x, y)$ but at least we know now that we can do that since the condition $\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$ is satisfied. If this condition is not satisfied then this method will not work and we have to now look for an integrating factor to force this condition, which might or might not exist. The first step is to write the ODE in standard form to check for exactness, which is

$$M(x, y)\,\mathrm{d}x + N(x, y)\,\mathrm{d}y = 0 \tag{1A}$$

Therefore

$$\left(\frac{1}{\sqrt{y} - y}\right) \mathrm{d}y = (\cot(x))\,\mathrm{d}x$$
$$(-\cot(x))\,\mathrm{d}x + \left(\frac{1}{\sqrt{y} - y}\right) \mathrm{d}y = 0 \tag{2A}$$

Comparing (1A) and (2A) shows that

$$M(x, y) = -\cot(x)$$
$$N(x, y) = \frac{1}{\sqrt{y} - y}$$

The next step is to determine if the ODE is is exact or not. The ODE is exact when the following condition is satisfied

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Using result found above gives

$$\begin{aligned}\frac{\partial M}{\partial y} &= \frac{\partial}{\partial y}(-\cot(x)) \\ &= 0\end{aligned}$$

And

$$\begin{aligned}
\frac{\partial N}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{\sqrt{y}-y}\right) \\
&= 0
\end{aligned}$$

Since $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$, then the ODE is exact The following equations are now set up to solve for the function $\phi(x,y)$

$$\frac{\partial \phi}{\partial x} = M \tag{1}$$

$$\frac{\partial \phi}{\partial y} = N \tag{2}$$

Integrating (1) w.r.t. x gives

$$\begin{aligned}
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int M\, \mathrm{d}x \\
\int \frac{\partial \phi}{\partial x}\, \mathrm{d}x &= \int -\cot(x)\, \mathrm{d}x
\end{aligned}$$

$$\phi = -\ln(\sin(x)) + f(y) \tag{3}$$

Where $f(y)$ is used for the constant of integration since ϕ is a function of both x and y. Taking derivative of equation (3) w.r.t y gives

$$\frac{\partial \phi}{\partial y} = 0 + f'(y) \tag{4}$$

But equation (2) says that $\frac{\partial \phi}{\partial y}=\frac{1}{\sqrt{y}-y}$. Therefore equation (4) becomes

$$\frac{1}{\sqrt{y}-y} = 0 + f'(y) \tag{5}$$

Solving equation (5) for $f'(y)$ gives

$$f'(y) = \frac{1}{\sqrt{y}-y}$$

Integrating the above w.r.t y gives

$$\begin{aligned}
\int f'(y)\, \mathrm{d}y &= \int \left(\frac{1}{\sqrt{y}-y}\right) \mathrm{d}y \\
f(y) &= 2\ \mathrm{arctanh}\left(\sqrt{y}\right) - \ln(y-1) + c_1
\end{aligned}$$

Where c_1 is constant of integration. Substituting result found above for $f(y)$ into equation (3) gives ϕ

$$\phi = -\ln(\sin(x)) + 2\,\text{arctanh}\left(\sqrt{y}\right) - \ln(y-1) + c_1$$

But since ϕ itself is a constant function, then let $\phi = c_2$ where c_2 is new constant and combining c_1 and c_2 constants into new constant c_1 gives the solution as

$$c_1 = -\ln(\sin(x)) + 2\,\text{arctanh}\left(\sqrt{y}\right) - \ln(y-1)$$

Summary

The solution(s) found are the following

$$2\,\text{arctanh}\left(\sqrt{y}\right) - \ln(y-1) - \ln(\sin(x)) = c_1 \tag{1}$$

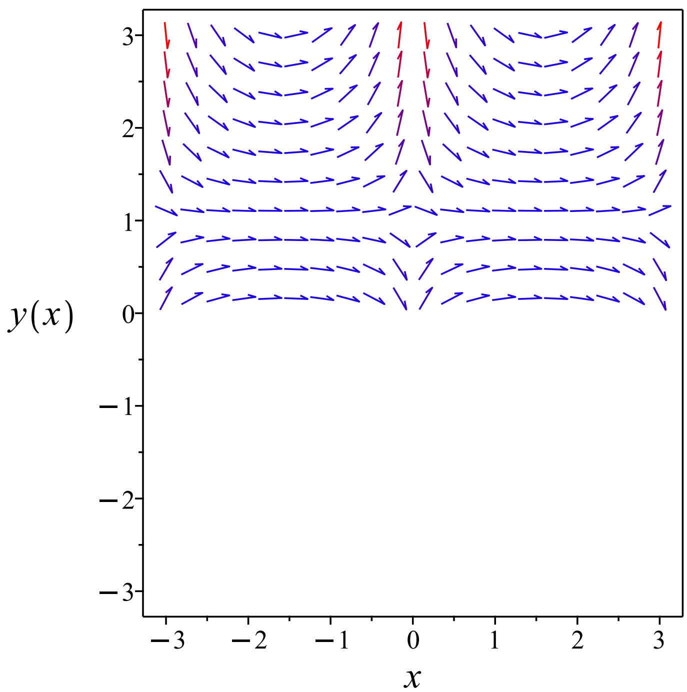


Figure 449: Slope field plot

Verification of solutions

$$2\,\text{arctanh}\left(\sqrt{y}\right) - \ln(y-1) - \ln(\sin(x)) = c_1$$

Verified OK.

6.36.5 Maple step by step solution

Let's solve

$y' - \cot(x)\left(\sqrt{y} - y\right) = 0$

- Highest derivative means the order of the ODE is 1

  y'

- Separate variables

  $\frac{y'}{\sqrt{y}-y} = \cot(x)$

- Integrate both sides with respect to x

  $\int \frac{y'}{\sqrt{y}-y}dx = \int \cot(x)\, dx + c_1$

- Evaluate integral

  $2\,\mathrm{arctanh}\left(\sqrt{y}\right) - \ln(y-1) = \ln(\sin(x)) + c_1$

Maple trace

```
`Methods for first order ODEs:
--- Trying classification methods ---
trying a quadrature
trying 1st order linear
trying Bernoulli
<- Bernoulli successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x) = cot(x)*(y(x)^(1/2)-y(x)),y(x), singsol=all)
```

$$\sqrt{y(x)} - \frac{\int \frac{\cos(x)}{\sqrt{\sin(x)}}dx + 2c_1}{2\sqrt{\sin(x)}} = 0$$

✓ Solution by Mathematica

Time used: 0.272 (sec). Leaf size: 35

```
DSolve[y'[x] == Cot[x]*(y[x]^(1/2)-y[x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \csc(x) \left(\sqrt{\sin(x)} + e^{\frac{c_1}{2}}\right)^2$$
$$y(x) \to 0$$
$$y(x) \to 1$$

7 Section 5.1, second order linear equations. Page 299

7.1 problem 1

Internal problem ID [157]
Internal file name [`OUTPUT/157_Sunday_June_05_2022_01_36_11_AM_42210530/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 5]$$

7.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\ &= \pm 1\end{aligned}$$

Hence

$$\lambda_1 = +1$$
$$\lambda_2 = -1$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(1)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 e^{x} + c_2 e^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{x} + c_2 e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{x} - c_2 e^{-x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

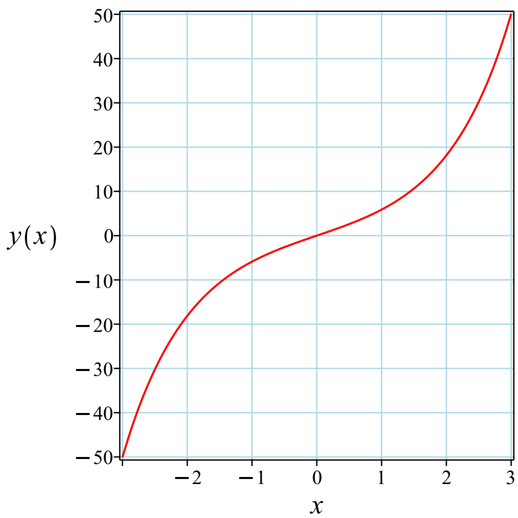
$$c_1 = \frac{5}{2}$$
$$c_2 = -\frac{5}{2}$$

Substituting these values back in above solution results in

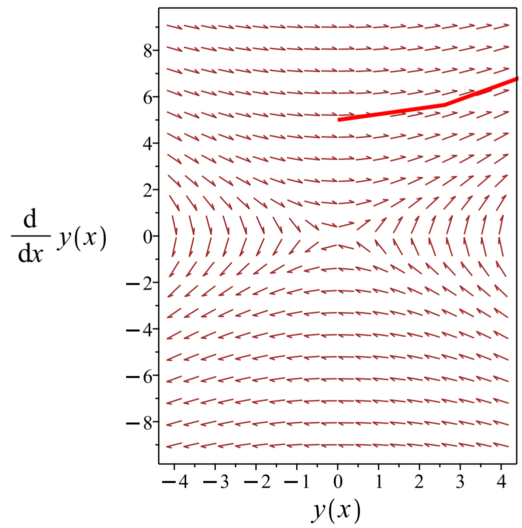
$$y = \frac{5 e^{x}}{2} - \frac{5 e^{-x}}{2}$$

Summary
The solution(s) found are the following

$$y = \frac{5\,e^{x}}{2}-\frac{5\,e^{-x}}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{5\,e^{x}}{2}-\frac{5\,e^{-x}}{2}$$

Verified OK.

7.1.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y''-yy'=0$$

Integrating the above w.r.t x gives

$$\int (y'y''-yy')\,dx = 0$$
$$\frac{y'^2}{2}-\frac{y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_2$$

Raising both side to exponential gives

$$y + \sqrt{y^2 + 2c_1} = e^{x + c_2}$$

Which simplifies to

$$y + \sqrt{y^2 + 2c_1} = c_3 e^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{y^2 + 2c_1}} dy = \int dx$$
$$-\ln\left(y + \sqrt{y^2 + 2c_1}\right) = x + c_4$$

Raising both side to exponential gives

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = e^{x + c_4}$$

Which simplifies to

$$\frac{1}{y + \sqrt{y^2 + 2c_1}} = c_5 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(e^{2x} c_3^2 - 2c_1\right) e^{-x}}{2c_3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_3^2 - 2c_1}{2c_3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(\mathrm{e}^{2x} c_3^2 - 2c_1\right) \mathrm{e}^{-x}}{2c_3} + c_3 \mathrm{e}^{x}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = \frac{c_3^2 + 2c_1}{2c_3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{25}{2} \\ c_3 &= 5 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{5\,\mathrm{e}^{-x}\left(-1 + \mathrm{e}^{2x}\right)}{2}$$

Which simplifies to

$$y = \frac{5\,\mathrm{e}^{x}}{2} - \frac{5\,\mathrm{e}^{-x}}{2}$$

Looking at the Second solution

$$y = -\frac{\left(2c_1 c_5^2 \mathrm{e}^{2x} - 1\right) \mathrm{e}^{-x}}{2c_5} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{-2c_1 c_5^2 + 1}{2c_5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_5 \mathrm{e}^{x} c_1 + \frac{\left(2c_1 c_5^2 \mathrm{e}^{2x} - 1\right) \mathrm{e}^{-x}}{2c_5}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = \frac{-2c_1 c_5^2 - 1}{2c_5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Solving for the constants gives

$$c_1 = \frac{25}{2}$$
$$c_5 = -\frac{1}{5}$$

Substituting these values back in above solution results in

$$y = \frac{5\,\mathrm{e}^{-x}(-1+\mathrm{e}^{2x})}{2}$$

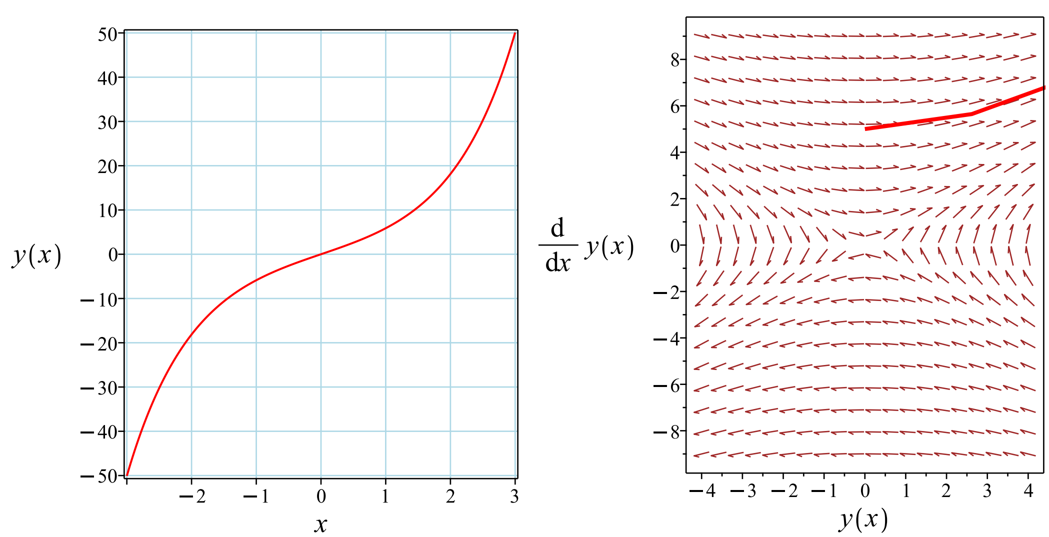
Which simplifies to

$$y = \frac{5\,\mathrm{e}^{x}}{2} - \frac{5\,\mathrm{e}^{-x}}{2}$$

<u>Summary</u>

The solution(s) found are the following

$$y = \frac{5\,\mathrm{e}^{x}}{2} - \frac{5\,\mathrm{e}^{-x}}{2} \tag{1}$$
$$y = \frac{5\,\mathrm{e}^{x}}{2} - \frac{5\,\mathrm{e}^{-x}}{2} \tag{2}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \frac{5\,e^{x}}{2} - \frac{5\,e^{-x}}{2}$$

Verified OK.

$$y = \frac{5\,e^{x}}{2} - \frac{5\,e^{-x}}{2}$$

Verified OK.

7.1.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 323: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-x} \int \frac{1}{\mathrm{e}^{-2x}}\,dx \\ &= \mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-x}\right) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{2x}}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{x}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + \frac{c_2 e^{x}}{2}$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = -c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{5}{2}$$
$$c_2 = 5$$

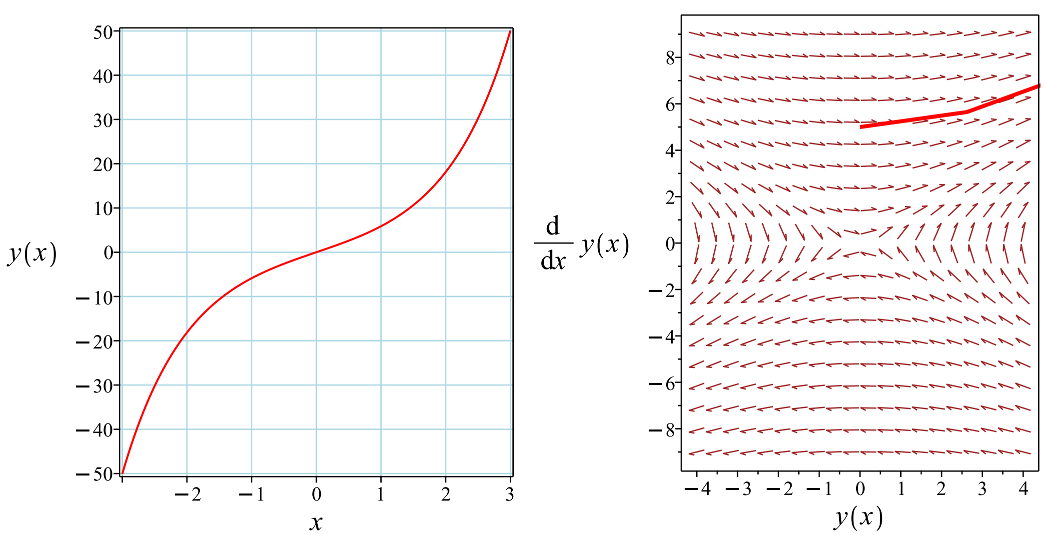
Substituting these values back in above solution results in

$$y = \frac{5\,e^{x}}{2} - \frac{5\,e^{-x}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{5\,e^{x}}{2} - \frac{5\,e^{-x}}{2} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{5\,\mathrm{e}^{x}}{2} - \frac{5\,\mathrm{e}^{-x}}{2}$$

Verified OK.

7.1.5 Maple step by step solution

Let's solve

$$\left[y'' - y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r-1)(r+1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-x}$
- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

□ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2$
  - Compute derivative of the solution

    $y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 5$

  $5 = -c_1 + c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{5}{2}, c_2 = \frac{5}{2}\right\}$
- Substitute constant values into general solution and simplify

  $y = \frac{5\,e^x}{2} - \frac{5\,e^{-x}}{2}$

- Solution to the IVP

  $y = \frac{5\,e^x}{2} - \frac{5\,e^{-x}}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-y(x)=0,y(0) = 0, D(y)(0) = 5],y(x), singsol=all)
```

$$y(x) = \frac{5\,e^x}{2} - \frac{5\,e^{-x}}{2}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 21

```
DSolve[{y''[x]-y[x]==0,{y[0]==0,y'[0]==5}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{5}{2}e^{-x}\left(e^{2x} - 1\right)$$

7.2 problem 2

Internal problem ID [158]
Internal file name [`OUTPUT/158_Sunday_June_05_2022_01_36_11_AM_42038455/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 9y = 0}$$

With initial conditions

$$[y(0) = -1, y'(0) = 15]$$

7.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -9 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 9y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-9)}\\ &= \pm 3\end{aligned}$$

Hence

$$\lambda_1 = +3$$
$$\lambda_2 = -3$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 e^{3x} + c_2 e^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + c_2 e^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} - 3c_2 e^{-3x}$$

substituting $y' = 15$ and $x = 0$ in the above gives

$$15 = 3c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

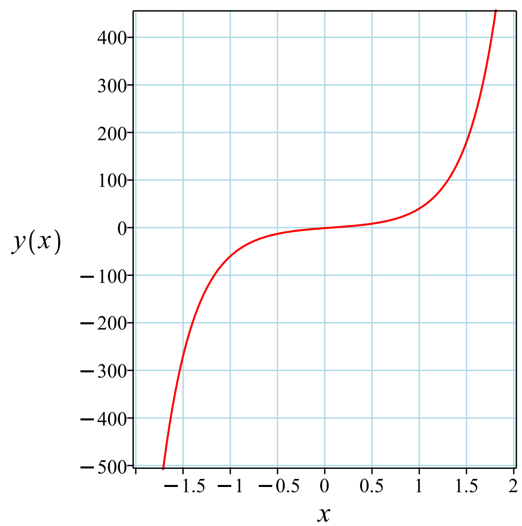
$$c_1 = 2$$
$$c_2 = -3$$

Substituting these values back in above solution results in

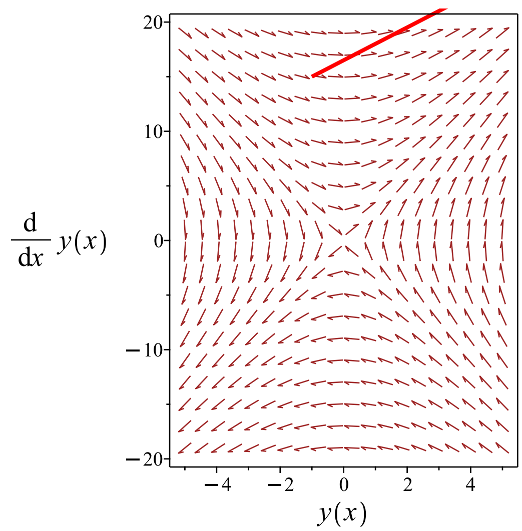
$$y = 2\,e^{3x} - 3\,e^{-3x}$$

Summary
The solution(s) found are the following

$$y = 2\,\mathrm{e}^{3x} - 3\,\mathrm{e}^{-3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,\mathrm{e}^{3x} - 3\,\mathrm{e}^{-3x}$$

Verified OK.

7.2.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 9yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 9yy')\,dx = 0$$
$$\frac{y'^2}{2} - \frac{9y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{9y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{9y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{9y^2+2c_1}}dy = \int dx$$
$$\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9} = x+c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9}} = e^{x+c_2}$$

Which simplifies to

$$\left(3y+\sqrt{9y^2+2c_1}\right)^{\frac{1}{3}} = c_3 e^x$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{9y^2+2c_1}}dy = \int dx$$
$$-\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9} = x+c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(y\sqrt{9}+\sqrt{9y^2+2c_1}\right)\sqrt{9}}{9}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{\left(3y+\sqrt{9y^2+2c_1}\right)^{\frac{1}{3}}} = c_5 e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(e^{6x}c_3^6-2c_1\right)e^{-3x}}{6c_3^3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = \frac{c_3^6 - 2c_1}{6c_3^3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(e^{6x} c_3^6 - 2c_1\right) e^{-3x}}{2c_3^3} + c_3^3 e^{3x}$$

substituting $y' = 15$ and $x = 0$ in the above gives

$$15 = \frac{c_3^6 + 2c_1}{2c_3^3} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Looking at the Second solution

$$y = -\frac{\left(2c_1 c_5^6 e^{6x} - 1\right) e^{-3x}}{6c_5^3} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = \frac{-2c_1 c_5^6 + 1}{6c_5^3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_5^3 e^{3x} c_1 + \frac{\left(2c_1 c_5^6 e^{6x} - 1\right) e^{-3x}}{2c_5^3}$$

substituting $y' = 15$ and $x = 0$ in the above gives

$$15 = \frac{-2c_1 c_5^6 - 1}{2c_5^3} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_5\}$. Warning, unable to solve for constants of integrations.

<u>Verification of solutions</u> N/A

7.2.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 325: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\&= 0 - 0\\&= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-3x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-3x} \int \frac{1}{\mathrm{e}^{-6x}}\, dx \\ &= \mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{6x}}{6}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{3x}}{6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -1$ and $x = 0$ in the above gives

$$-1 = c_1 + \frac{c_2}{6} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 e^{-3x} + \frac{c_2 e^{3x}}{2}$$

substituting $y' = 15$ and $x = 0$ in the above gives

$$15 = -3c_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

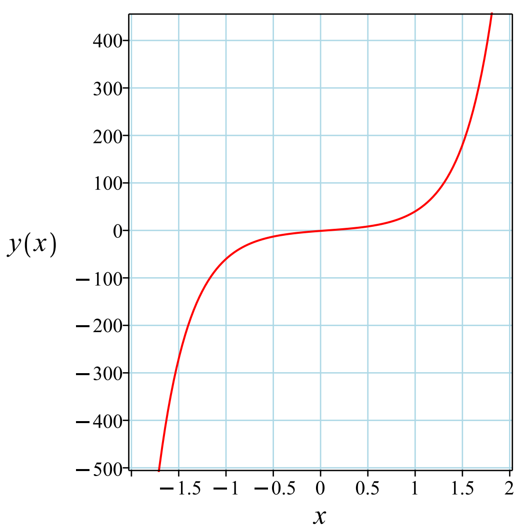
$$c_1 = -3$$
$$c_2 = 12$$

Substituting these values back in above solution results in

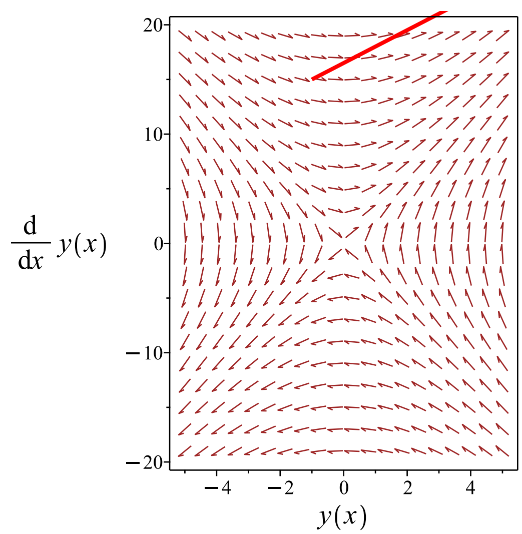
$$y = 2\,e^{3x} - 3\,e^{-3x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = 2\,e^{3x} - 3\,e^{-3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,e^{3x} - 3\,e^{-3x}$$

Verified OK.

7.2.5 Maple step by step solution

Let's solve

$$\left[y'' - 9y = 0, y(0) = -1, y'\Big|_{\{x=0\}} = 15\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 9 = 0$

- Factor the characteristic polynomial

  $(r - 3)(r + 3) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 e^{3x}$

□ Check validity of solution $y = c_1 e^{-3x} + c_2 e^{3x}$

  - Use initial condition $y(0) = -1$

    $-1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3c_1 e^{-3x} + 3c_2 e^{3x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 15$

    $15 = -3c_1 + 3c_2$

  - Solve for c_1 and c_2

    $\{c_1 = -3, c_2 = 2\}$

- Substitute constant values into general solution and simplify

  $y = 2\,\mathrm{e}^{3x} - 3\,\mathrm{e}^{-3x}$

- Solution to the IVP

  $y = 2\,\mathrm{e}^{3x} - 3\,\mathrm{e}^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)-9*y(x)=0,y(0) = -1, D(y)(0) = 15],y(x), singsol=all)
```

$$y(x) = -3\,\mathrm{e}^{-3x} + 2\,\mathrm{e}^{3x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[{y''[x]-9*y[x]==0,{y[0]==-1,y'[0]==15}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(2e^{6x} - 3\right)$$

7.3 problem 3

Internal problem ID [159]
Internal file name [`OUTPUT/159_Sunday_June_05_2022_01_36_12_AM_224056/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 4y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 8]$$

7.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 4y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.3.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + c_2 \sin(2x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 4$$

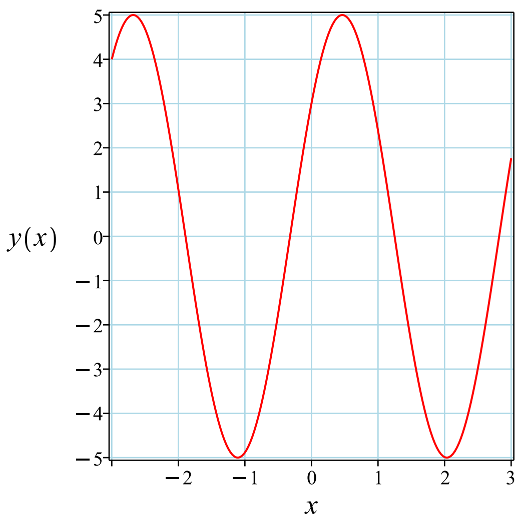
Substituting these values back in above solution results in

$$y = 3\cos(2x) + 4\sin(2x)$$

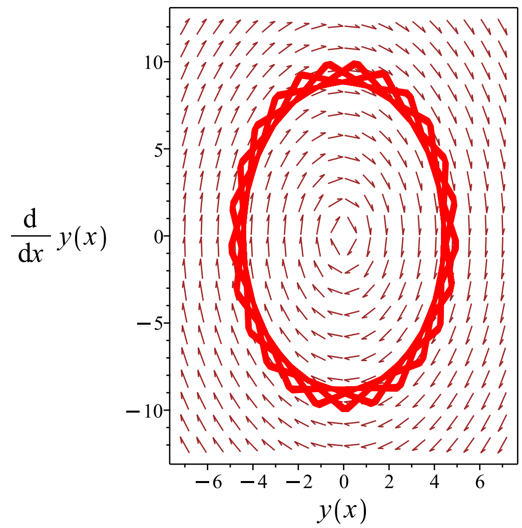
Summary

The solution(s) found are the following

$$y = 3\cos(2x) + 4\sin(2x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\cos(2x) + 4\sin(2x)$$

Verified OK.

7.3.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 4yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 4yy')\, dx = 0$$

$$\frac{y'^2}{2} + 2y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-4y^2 + 2c_1} \tag{1}$$

$$y' = -\sqrt{-4y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-4y^2 + 2c_1}} dy = \int dx$$

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-4y^2 + 2c_1}} dy = \int dx$$

$$-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$\frac{\arctan\left(\frac{6}{\sqrt{-36+2c_1}}\right)}{2} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\left(2\tan\left(2x+2c_2\right)^2+2\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(2x+2c_2)^2+1}}}{2} - \frac{\tan\left(2x+2c_2\right)^2\sqrt{2}\,c_1\left(2\tan\left(2x+2c_2\right)^2+2\right)}{2\sqrt{\frac{c_1}{\tan(2x+2c_2)^2+1}}\left(\tan\left(2x+2c_2\right)^2+1\right)^2}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = \frac{\cos\left(2c_2\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(2c_2\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 50$$
$$c_2 = \frac{\arctan\left(\frac{3}{4}\right)}{2}$$

Substituting these values back in above solution results in

$$\frac{\arctan\left(\frac{y}{\sqrt{25-y^2}}\right)}{2} = x + \frac{\arctan\left(\frac{3}{4}\right)}{2}$$

Looking at the Second solution

$$-\frac{\arctan\left(\frac{2y}{\sqrt{-4y^2+2c_1}}\right)}{2} = c_3 + x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$-\frac{\arctan\left(\frac{6}{\sqrt{-36+2c_1}}\right)}{2} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(2\tan\left(2c_3+2x\right)^2+2\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(2c_3+2x)^2+1}}}{2} + \frac{\tan\left(2c_3+2x\right)^2\sqrt{2}\,c_1\left(2\tan\left(2c_3+2x\right)^2+2\right)}{2\sqrt{\frac{c_1}{\tan(2c_3+2x)^2+1}}\left(\tan\left(2c_3+2x\right)^2+1\right)^2}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = -\frac{\cos\left(2c_3\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(2c_3\right)^2 c_1}} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. There is no solution for the constants of integrations. This solution is removed.

Summary

The solution(s) found are the following

$$\frac{\arctan\left(\frac{y}{\sqrt{25-y^2}}\right)}{2} = x + \frac{\arctan\left(\frac{3}{4}\right)}{2} \tag{1}$$

Verification of solutions

$$\frac{\arctan\left(\frac{y}{\sqrt{25-y^2}}\right)}{2} = x + \frac{\arctan\left(\frac{3}{4}\right)}{2}$$

Verified OK.

7.3.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 327: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + c_2 \cos(2x)$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 8$$

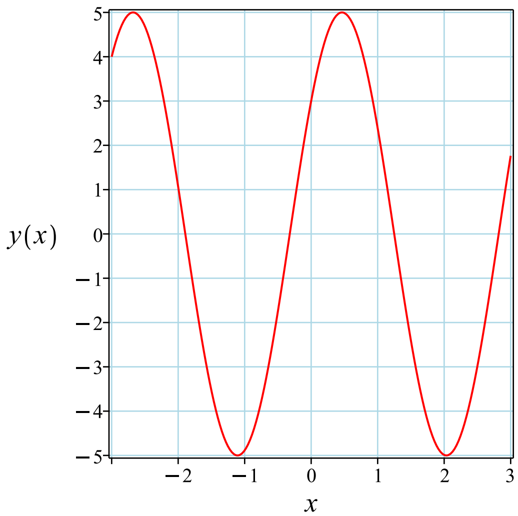
Substituting these values back in above solution results in

$$y = 3\cos(2x) + 4\sin(2x)$$

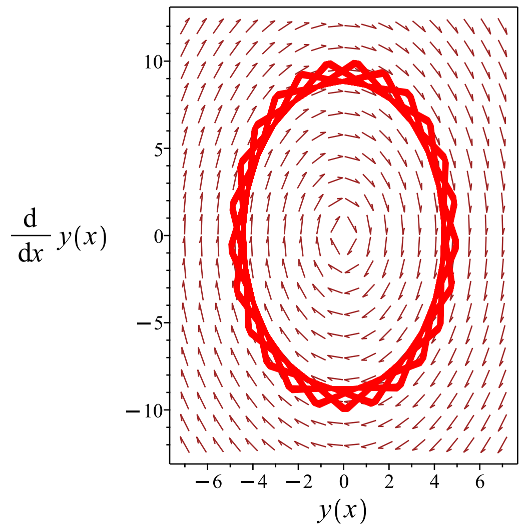
Summary

The solution(s) found are the following

$$y = 3\cos(2x) + 4\sin(2x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 3\cos(2x) + 4\sin(2x)$$

Verified OK.

7.3.5 Maple step by step solution

Let's solve

$\left[y'' + 4y = 0, y(0) = 3, y'\Big|_{\{x=0\}} = 8\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the ODE

$y_1(x) = \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(2x) + c_2 \sin(2x)$

□ Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x)$

  - Use initial condition $y(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x)$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 8$

    $8 = 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 4\}$

  - Substitute constant values into general solution and simplify

    $y = 3\cos(2x) + 4\sin(2x)$

- Solution to the IVP

  $y = 3\cos(2x) + 4\sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+4*y(x)=0,y(0) = 3, D(y)(0) = 8],y(x), singsol=all)
```

$$y(x) = 4\sin(2x) + 3\cos(2x)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[{y''[x]+4*y[x]==0,{y[0]==3,y'[0]==8}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 4\sin(2x) + 3\cos(2x)$$

7.4 problem 4

Internal problem ID [160]
Internal file name [`OUTPUT/160_Sunday_June_05_2022_01_36_13_AM_37831864/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 25y = 0}$$

With initial conditions

$$[y(0) = 10, y'(0) = -10]$$

7.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 25 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 25y = 0$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 25$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(25)} \\ &= \pm 5i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +5i \\ \lambda_2 &= -5i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 5i$$
$$\lambda_2 = -5i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(5x) + c_2 \sin(5x))$$

Or

$$y = c_1 \cos(5x) + c_2 \sin(5x)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(5x) + c_2 \sin(5x) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 0$ in the above gives

$$10 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -5c_1 \sin(5x) + 5c_2 \cos(5x)$$

substituting $y' = -10$ and $x = 0$ in the above gives

$$-10 = 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 10$$
$$c_2 = -2$$

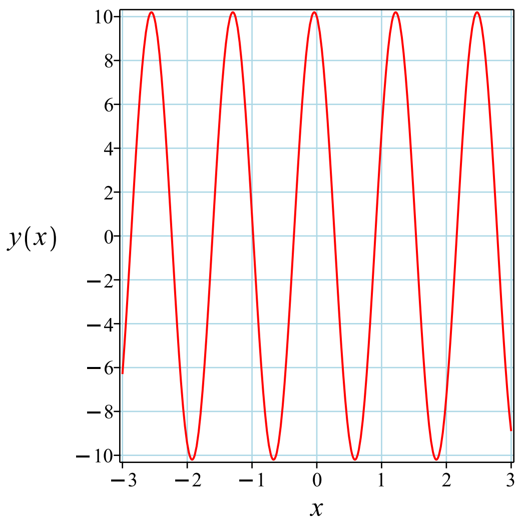
Substituting these values back in above solution results in

$$y = 10\cos(5x) - 2\sin(5x)$$

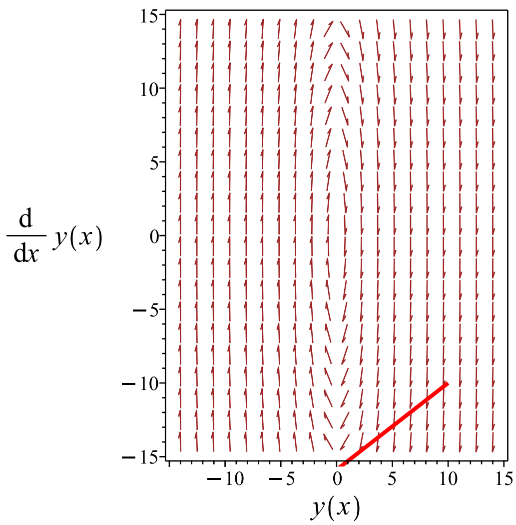
Summary

The solution(s) found are the following

$$y = 10\cos(5x) - 2\sin(5x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 10\cos(5x) - 2\sin(5x)$$

Verified OK.

7.4.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 25yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 25yy')\,dx = 0$$

$$\frac{y'^2}{2} + \frac{25y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-25y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{-25y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-25y^2 + 2c_1}} dy = \int dx$$
$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-25y^2 + 2c_1}} dy = \int dx$$
$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = c_3 + x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = x + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 0$ in the above gives

$$\frac{\arctan\left(\frac{50}{\sqrt{-2500+2c_1}}\right)}{5} = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\left(5\tan\left(5x+5c_2\right)^2+5\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(5x+5c_2)^2+1}}}{5} - \frac{\tan\left(5x+5c_2\right)^2\sqrt{2}\,c_1\left(5\tan\left(5x+5c_2\right)^2+5\right)}{5\sqrt{\frac{c_1}{\tan(5x+5c_2)^2+1}}\,\left(\tan\left(5x+5c_2\right)^2+1\right)^2}$$

substituting $y' = -10$ and $x = 0$ in the above gives

$$-10 = \frac{\cos\left(5c_2\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(5c_2\right)^2 c_1}} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_2\}$. There is no solution for the constants of integrations. This solution is removed.

Looking at the Second solution

$$-\frac{\arctan\left(\frac{5y}{\sqrt{-25y^2+2c_1}}\right)}{5} = c_3 + x \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 0$ in the above gives

$$-\frac{\arctan\left(\frac{50}{\sqrt{-2500+2c_1}}\right)}{5} = c_3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(5\tan\left(5c_3+5x\right)^2+5\right)\sqrt{2}\,\sqrt{\frac{c_1}{\tan(5c_3+5x)^2+1}}}{5} + \frac{\tan\left(5c_3+5x\right)^2\sqrt{2}\,c_1\left(5\tan\left(5c_3+5x\right)^2+5\right)}{5\sqrt{\frac{c_1}{\tan(5c_3+5x)^2+1}}\,\left(\tan\left(5c_3+5x\right)^2+1\right)^2}$$

substituting $y' = -10$ and $x = 0$ in the above gives

$$-10 = -\frac{\cos\left(5c_3\right)^2\sqrt{2}\,c_1}{\sqrt{\cos\left(5c_3\right)^2 c_1}} \tag{2A}$$

Equations $\{1A,2A\}$ are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = 1300$$
$$c_3 = -\frac{\arctan(5)}{5}$$

Substituting these values back in above solution results in

$$-\frac{\arctan\left(\frac{y}{\sqrt{-y^2+104}}\right)}{5} = -\frac{\arctan(5)}{5} + x$$

Summary

The solution(s) found are the following

$$-\frac{\arctan\left(\frac{y}{\sqrt{-y^2+104}}\right)}{5} = -\frac{\arctan(5)}{5} + x \tag{1}$$

Verification of solutions

$$-\frac{\arctan\left(\frac{y}{\sqrt{-y^2+104}}\right)}{5} = -\frac{\arctan(5)}{5} + x$$

Verified OK.

7.4.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -25z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 329: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -25$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(5x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(5x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(5x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(5x) \int \frac{1}{\cos(5x)^2}\,dx \\ &= \cos(5x)\left(\frac{\tan(5x)}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(5x)) + c_2\left(\cos(5x)\left(\frac{\tan(5x)}{5}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(5x) + \frac{c_2 \sin(5x)}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 0$ in the above gives

$$
10 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -5c_1 \sin(5x) + c_2 \cos(5x)
$$

substituting $y' = -10$ and $x = 0$ in the above gives

$$
-10 = c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 10 \\
c_2 &= -10
\end{aligned}
$$

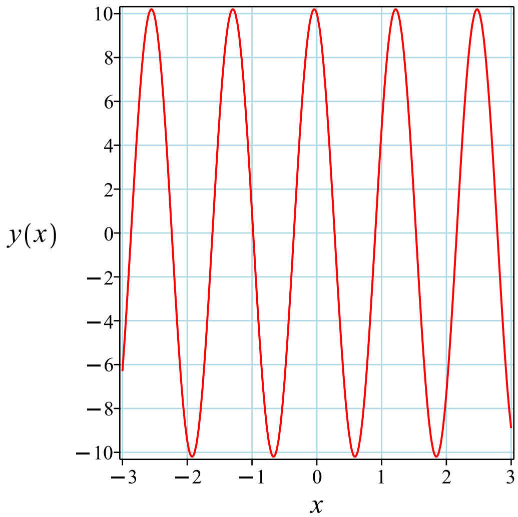
Substituting these values back in above solution results in

$$
y = 10\cos(5x) - 2\sin(5x)
$$

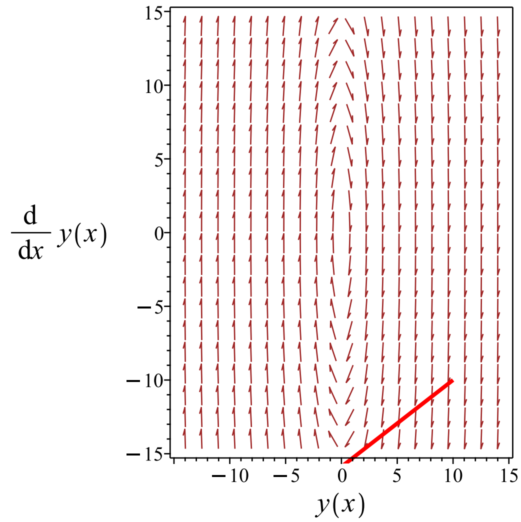
Summary

The solution(s) found are the following

$$
y = 10\cos(5x) - 2\sin(5x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 10\cos(5x) - 2\sin(5x)$$

Verified OK.

7.4.5 Maple step by step solution

Let's solve

$$\left[y'' + 25y = 0, y(0) = 10, y'\Big|_{\{x=0\}} = -10\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

  $r = (-5\,\mathrm{I}, 5\,\mathrm{I})$

- 1st solution of the ODE

$y_1(x) = \cos(5x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(5x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(5x) + c_2 \sin(5x)$

□ Check validity of solution $y = c_1 \cos(5x) + c_2 \sin(5x)$

  - Use initial condition $y(0) = 10$

    $10 = c_1$

  - Compute derivative of the solution

    $y' = -5c_1 \sin(5x) + 5c_2 \cos(5x)$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -10$

    $-10 = 5c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 10, c_2 = -2\}$

  - Substitute constant values into general solution and simplify

    $y = 10\cos(5x) - 2\sin(5x)$

- Solution to the IVP

  $y = 10\cos(5x) - 2\sin(5x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+25*y(x)=0,y(0) = 10, D(y)(0) = -10],y(x), singsol=all)
```

$$y(x) = -2\sin(5x) + 10\cos(5x)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 18

```
DSolve[{y''[x]+25*y[x]==0,{y[0]==10,y'[0]==-10}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \rightarrow 10\cos(5x) - 2\sin(5x)$$

7.5 problem 5

Internal problem ID [161]
Internal file name [`OUTPUT/161_Sunday_June_05_2022_01_36_14_AM_44083784/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 0}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

7.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 3y' + 2y = 0$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-3^2 - (4)(1)(2)} \\ &= \frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= \frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} + c_2 e^{x}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -1$$
$$c_2 = 2$$

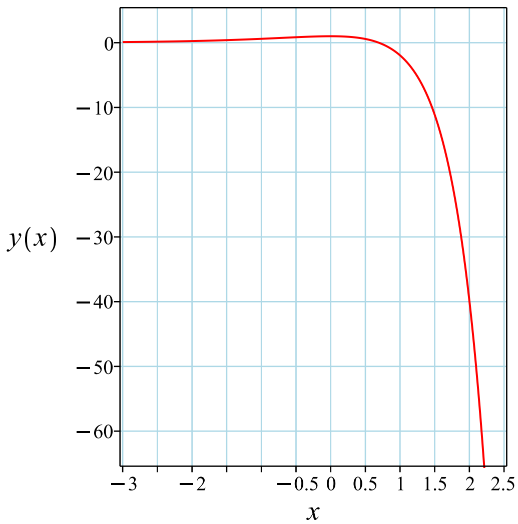
Substituting these values back in above solution results in

$$y = -e^{2x} + 2\,e^{x}$$

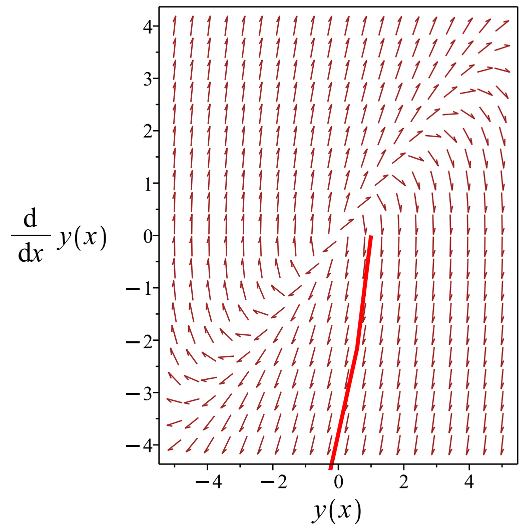
Summary

The solution(s) found are the following

$$y = -e^{2x} + 2e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{2x} + 2e^{x}$$

Verified OK.

7.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 331: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x})\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 \mathrm{e}^x + 2c_2 \mathrm{e}^{2x}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + 2c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= -1
\end{aligned}
$$

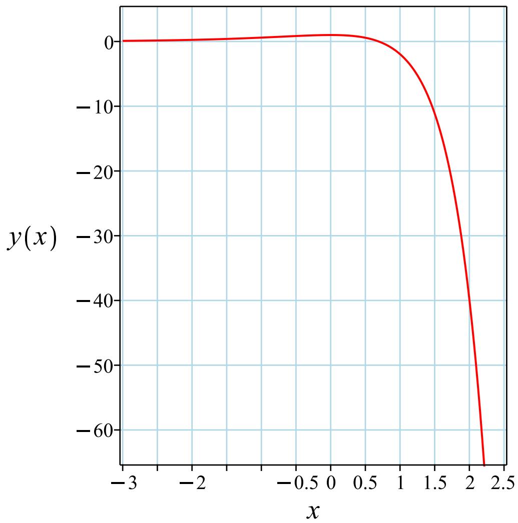
Substituting these values back in above solution results in

$$
y = -\mathrm{e}^{2x} + 2\,\mathrm{e}^x
$$

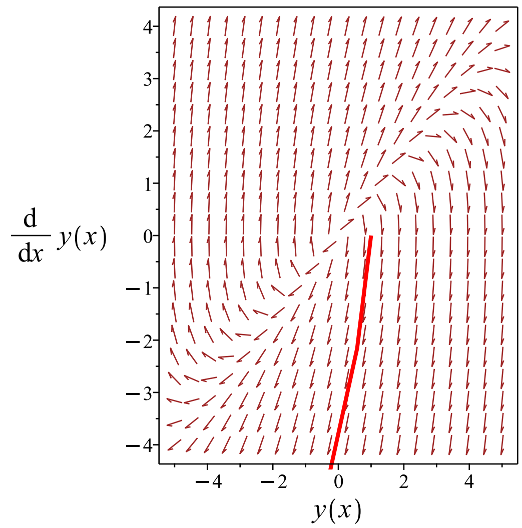
Summary

The solution(s) found are the following

$$
y = -\mathrm{e}^{2x} + 2\,\mathrm{e}^x \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -e^{2x} + 2\,e^{x}$$

Verified OK.

7.5.4 Maple step by step solution

Let's solve

$$\left[y'' - 3y' + 2y = 0, y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

$y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{2x}$

□ Check validity of solution $y = c_1 e^x + c_2 e^{2x}$

  - Use initial condition $y(0) = 1$

    $1 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = c_1 e^x + 2c_2 e^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = -1\}$

  - Substitute constant values into general solution and simplify

    $y = -e^{2x} + 2\,e^x$

- Solution to the IVP

  $y = -e^{2x} + 2\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = 2\,\mathrm{e}^{x} - \mathrm{e}^{2x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 15

```
DSolve[{y''[x]-3*y'[x]+2*y[x]==0,{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to -e^{x}(e^{x} - 2)$$

7.6 problem 6

Internal problem ID [162]
Internal file name [`OUTPUT/162_Sunday_June_05_2022_01_36_14_AM_7119039/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' - 6y = 0}$$

With initial conditions

$$[y(0) = 7, y'(0) = -1]$$

7.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -6 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' - 6y = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.6.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} - 6 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = -6$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(-6)} \\ &= -\frac{1}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{5}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{5}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = -3$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(-3)x}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-3x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-3x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 0$ in the above gives

$$7 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - 3c_2 e^{-3x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = 2c_1 - 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

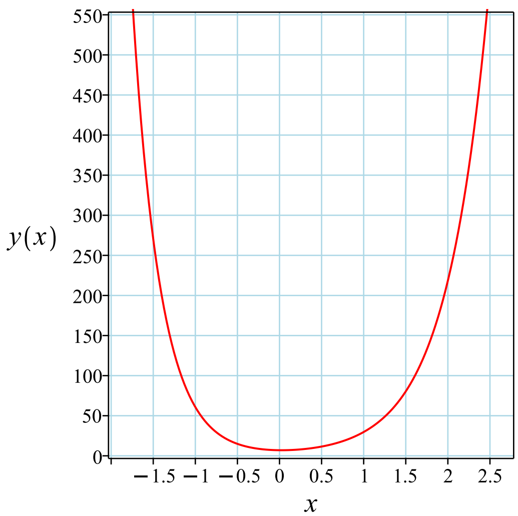
$$c_1 = 4$$
$$c_2 = 3$$

Substituting these values back in above solution results in

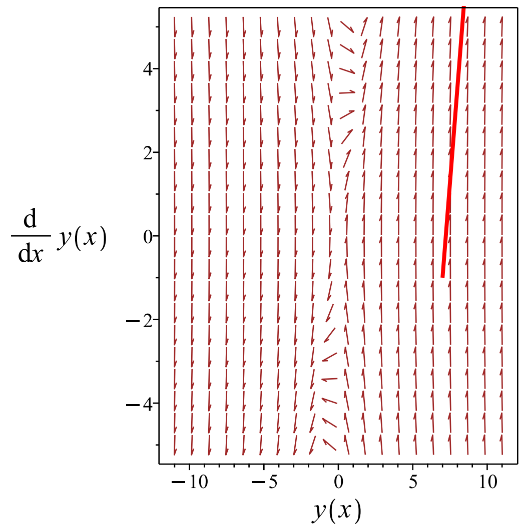
$$y = 4\,e^{2x} + 3\,e^{-3x}$$

Summary
The solution(s) found are the following

$$y = 4\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{-3x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 4\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{-3x}$$

Verified OK.

7.6.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 333: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(\mathrm{e}^{-3x}\right) + c_2\left(\mathrm{e}^{-3x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1\mathrm{e}^{-3x} + \frac{c_2\mathrm{e}^{2x}}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 0$ in the above gives

$$7 = c_1 + \frac{c_2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1\mathrm{e}^{-3x} + \frac{2c_2\mathrm{e}^{2x}}{5}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -3c_1 + \frac{2c_2}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 3 \\ c_2 &= 20 \end{aligned}$$

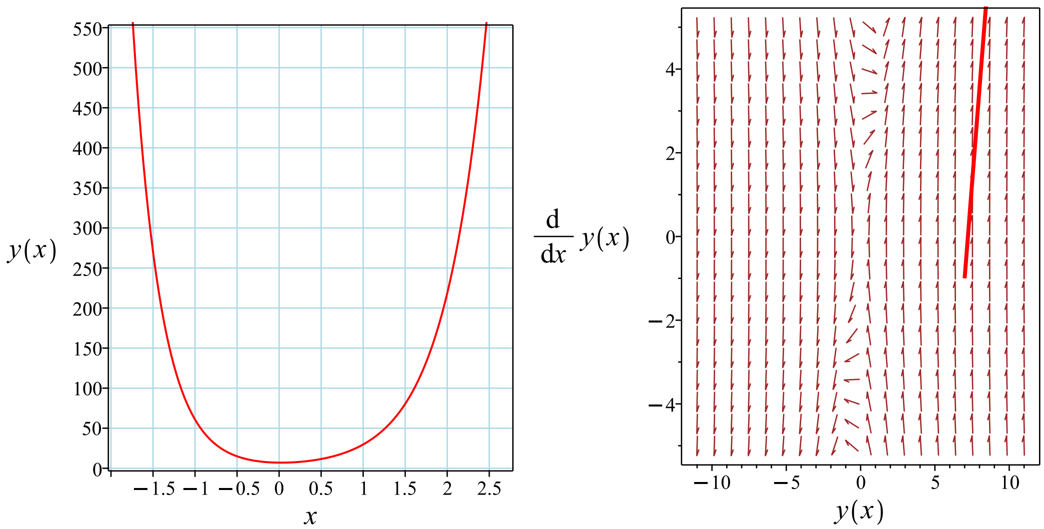
Substituting these values back in above solution results in

$$y = 4\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{-3x}$$

Summary

The solution(s) found are the following

$$y = 4\,\mathrm{e}^{2x} + 3\,\mathrm{e}^{-3x} \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$y = 4\,e^{2x} + 3\,e^{-3x}$$

Verified OK.

7.6.4 Maple step by step solution

Let's solve

$\left[y'' + y' - 6y = 0, y(0) = 7, y'\Big|_{\{x=0\}} = -1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + r - 6 = 0$

- Factor the characteristic polynomial

  $(r + 3)\,(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 2)$

- 1st solution of the ODE

$y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{2x}$

- □ Check validity of solution $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{2x}$
  - Use initial condition $y(0) = 7$

    $7 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = -3c_1 \mathrm{e}^{-3x} + 2c_2 \mathrm{e}^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = -1$

    $-1 = -3c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = 4\}$

  - Substitute constant values into general solution and simplify

    $y = (4\,\mathrm{e}^{5x} + 3)\,\mathrm{e}^{-3x}$

- Solution to the IVP

  $y = (4\,\mathrm{e}^{5x} + 3)\,\mathrm{e}^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve([diff(y(x),x$2)+diff(y(x),x)-6*y(x)=0,y(0) = 7, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = \left(4\,\mathrm{e}^{5x} + 3\right) \mathrm{e}^{-3x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[{y''[x]+y'[x]-6*y[x]==0,{y[0]==7,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}\left(4e^{5x} + 3\right)$$

7.7 problem 7

Internal problem ID [163]
Internal file name [`OUTPUT/163_Sunday_June_05_2022_01_36_15_AM_33044451/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + y' = 0}$$

With initial conditions

$$[y(0) = -2, y'(0) = 8]$$

7.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 1 \\ q(x) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + y' = 0$$

The domain of $p(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

7.7.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + \lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(0)} \\ &= -\frac{1}{2} \pm \frac{1}{2} \end{aligned}$$

Hence

$$\lambda_1 = -\frac{1}{2} + \frac{1}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{(-1)x}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + c_2 \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_2 \mathrm{e}^{-x}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = -c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_2 = -8$$

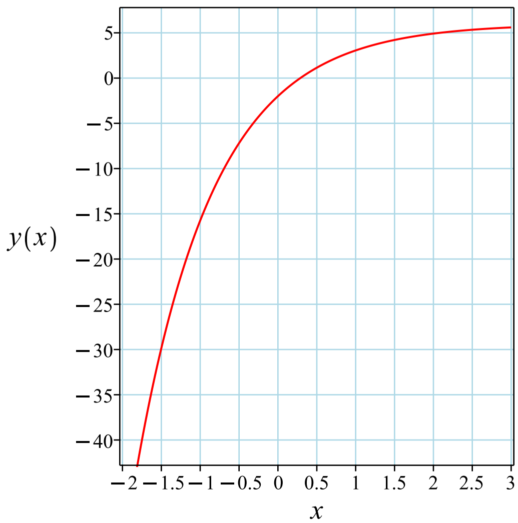
Substituting these values back in above solution results in

$$y = 6 - 8\,e^{-x}$$

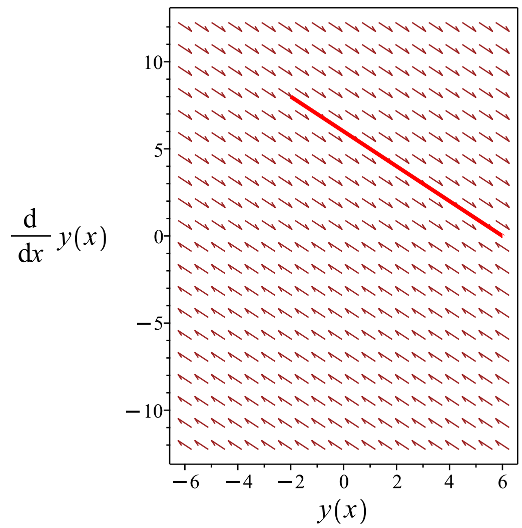
Summary

The solution(s) found are the following

$$y = 6 - 8\,e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 6 - 8\,e^{-x}$$

Verified OK.

7.7.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = 0$$
$$y' + y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-y + c_1} dy = \int dx$$
$$-\ln(-y + c_1) = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{-y+c_1} = e^{x+c_2}$$

Which simplifies to

$$\frac{1}{-y+c_1} = c_3 e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{e^{-x}}{c_3} + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = \frac{c_1 c_3 - 1}{c_3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{e^{-x}}{c_3}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = \frac{1}{c_3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_3 = \frac{1}{8}$$

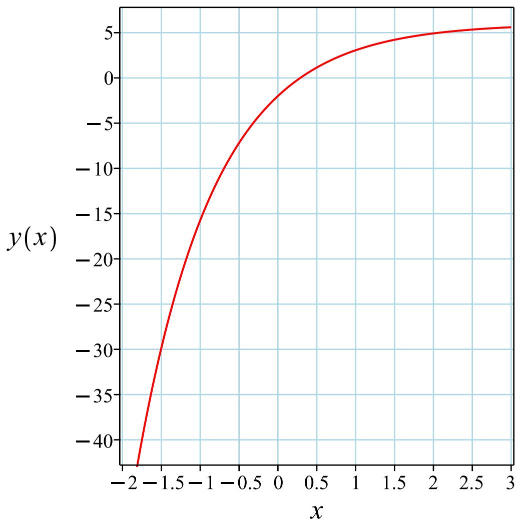
Substituting these values back in above solution results in

$$y = 6 - 8\,e^{-x}$$

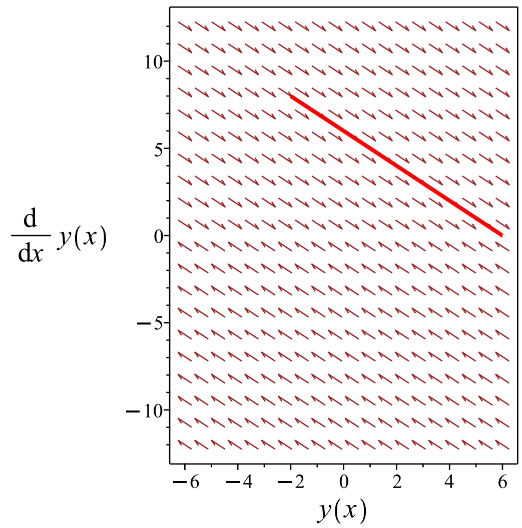
Summary

The solution(s) found are the following

$$y = 6 - 8\,e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 6 - 8\,e^{-x}$$

Verified OK.

7.7.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{p}dp = \int dx$$
$$-\ln(p) = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{p} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{p} = c_2 e^{x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $p = 8$ in the above solution gives an equation to solve for the constant of integration.

$$8 = \frac{1}{c_2}$$

$$c_2 = \frac{1}{8}$$

Substituting c_2 found above in the general solution gives

$$p(x) = 8\,e^{-x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = 8\,e^{-x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int 8\,e^{-x}\,dx \\ &= -8\,e^{-x} + c_3 \end{aligned}$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = -8 + c_3$$

$$c_3 = 6$$

Substituting c_3 found above in the general solution gives

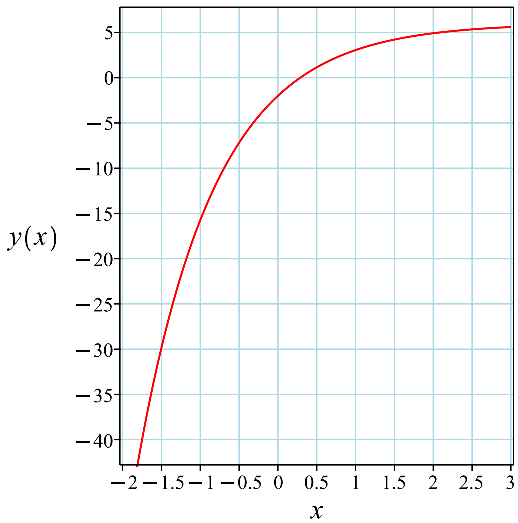
$$y = 6 - 8\,e^{-x}$$

Initial conditions are used to solve for the constants of integration.

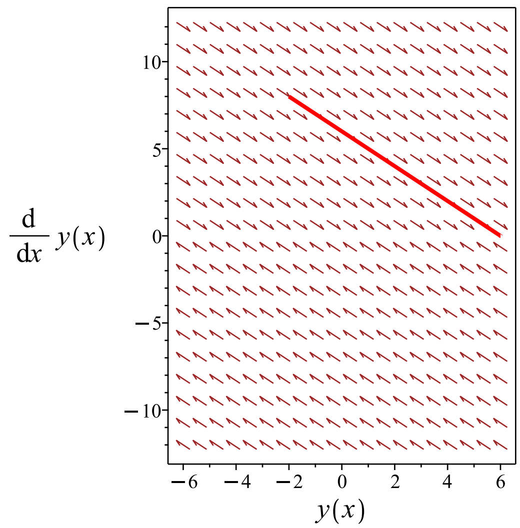
Summary

The solution(s) found are the following

$$y = 6 - 8\,\mathrm{e}^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 6 - 8\,\mathrm{e}^{-x}$$

Verified OK.

7.7.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + y')\,dx = 0$$

$$y' + y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-y + c_1} dy = \int dx$$

$$-\ln(-y + c_1) = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{-y+c_1} = e^{x+c_2}$$

Which simplifies to

$$\frac{1}{-y+c_1} = c_3 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -\frac{e^{-x}}{c_3} + c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$-2 = \frac{c_1 c_3 - 1}{c_3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{e^{-x}}{c_3}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = \frac{1}{c_3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_3 = \frac{1}{8}$$

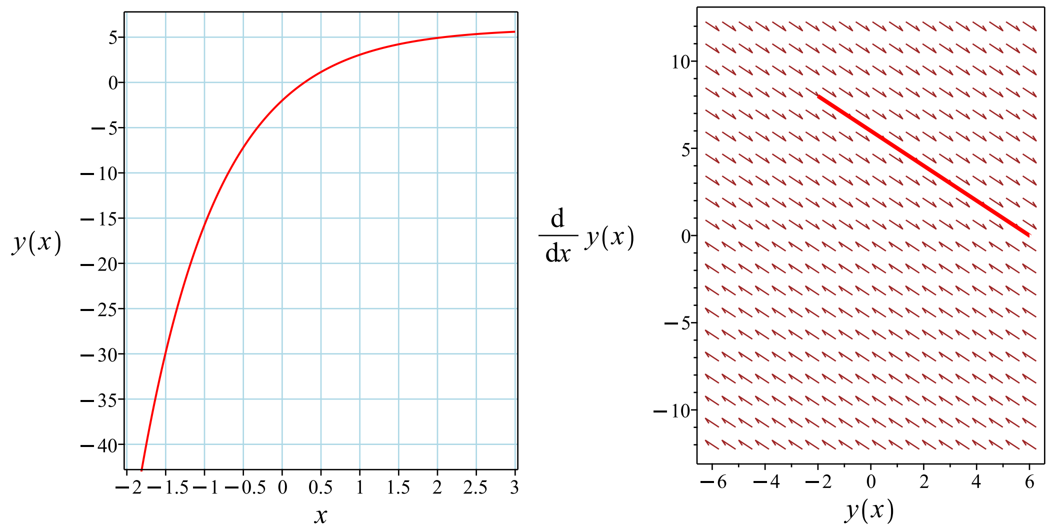
Substituting these values back in above solution results in

$$y = 6 - 8\,e^{-x}$$

Summary

The solution(s) found are the following

$$y = 6 - 8\,e^{-x} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 6 - 8\,\mathrm{e}^{-x}$$

Verified OK.

7.7.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 335: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\ &= z_1 e^{-\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x})\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} (\mathrm{e}^{x})\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = -2$ and $x = 0$ in the above gives

$$
-2 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x}
$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$
8 = -c_1 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -8 \\
c_2 &= 6
\end{aligned}
$$

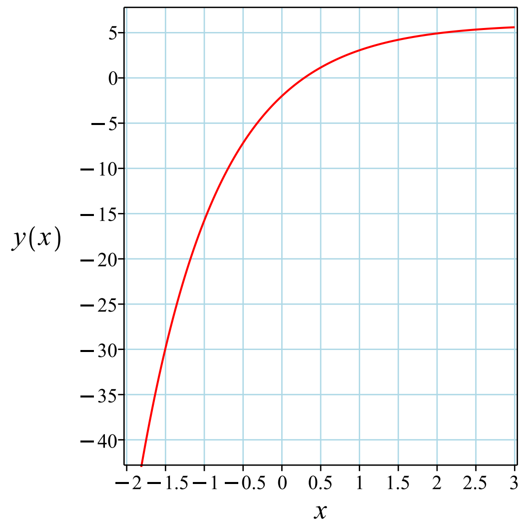
Substituting these values back in above solution results in

$$
y = 6 - 8\,\mathrm{e}^{-x}
$$

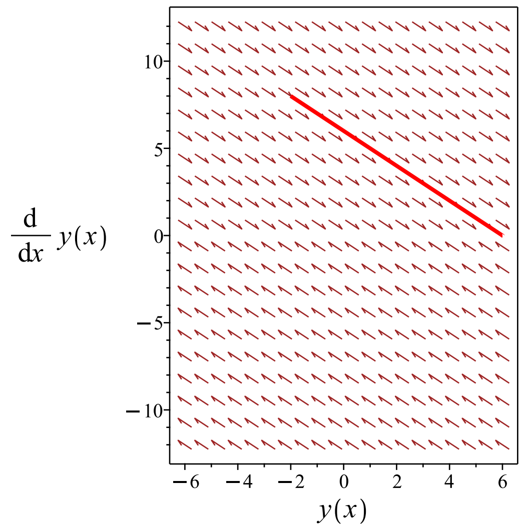
Summary

The solution(s) found are the following

$$
y = 6 - 8\,\mathrm{e}^{-x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$
y = 6 - 8\,\mathrm{e}^{-x}
$$

Verified OK.

7.7.7 Solving as exact linear second order ode ode

An ode of the form

$$
p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)
$$

is exact if

$$
p''(x) - q'(x) + r(x) = 0 \tag{1}
$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 1 \\
r(x) &= 0 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$y'+y=c_1$$

We now have a first order ode to solve which is

$$y'+y=c_1$$

Integrating both sides gives

$$\int \frac{1}{-y+c_1}dy=\int dx$$
$$-\ln\left(-y+c_1\right)=x+c_2$$

Raising both side to exponential gives

$$\frac{1}{-y+c_1}=\mathrm{e}^{x+c_2}$$

Which simplifies to

$$\frac{1}{-y+c_1}=c_3\mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y=-\frac{\mathrm{e}^{-x}}{c_3}+c_1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=-2$ and $x=0$ in the above gives

$$-2=\frac{c_1c_3-1}{c_3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{e^{-x}}{c_3}$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = \frac{1}{c_3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_3 = \frac{1}{8}$$

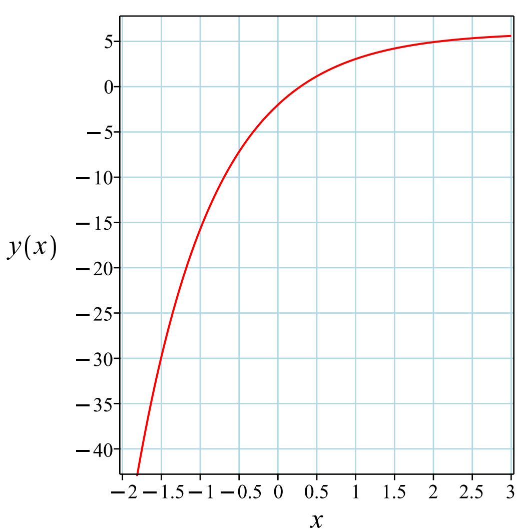
Substituting these values back in above solution results in

$$y = 6 - 8\,e^{-x}$$

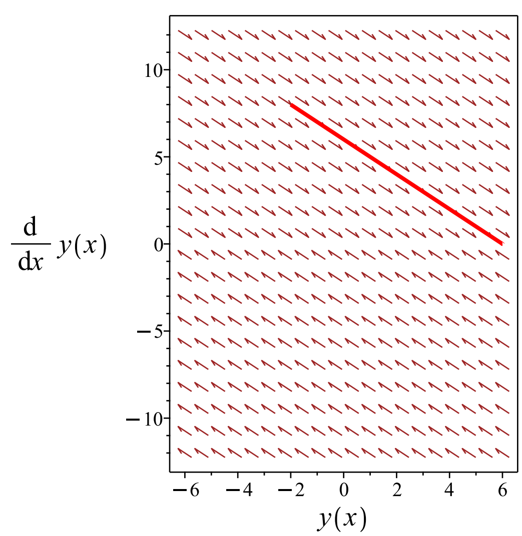
Summary

The solution(s) found are the following

$$y = 6 - 8\,e^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 6 - 8\,e^{-x}$$

Verified OK.

7.7.8 Maple step by step solution

Let's solve

$$\left[y''+y'=0, y(0)=-2, y'\Big|_{\{x=0\}}=8\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2+r=0$

- Factor the characteristic polynomial

  $r(r+1)=0$

- Roots of the characteristic polynomial

  $r=(-1,0)$

- 1st solution of the ODE

  $y_1(x)=\mathrm{e}^{-x}$

- 2nd solution of the ODE

  $y_2(x)=1$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)$

- Substitute in solutions

  $y=c_1\mathrm{e}^{-x}+c_2$

□ Check validity of solution $y=c_1\mathrm{e}^{-x}+c_2$

  - Use initial condition $y(0)=-2$

    $-2=c_1+c_2$

  - Compute derivative of the solution

    $y'=-c_1\mathrm{e}^{-x}$

  - Use the initial condition $y'\Big|_{\{x=0\}}=8$

    $8=-c_1$

  - Solve for c_1 and c_2

    $\{c_1=-8, c_2=6\}$

- Substitute constant values into general solution and simplify

  $y = 6 - 8\,e^{-x}$

- Solution to the IVP

  $y = 6 - 8\,e^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)+diff(y(x),x)=0,y(0) = -2, D(y)(0) = 8],y(x), singsol=all)
```

$$y(x) = 6 - 8\,e^{-x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 14

```
DSolve[{y''[x]+y'[x]==0,{y[0]==-2,y'[0]==8}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 6 - 8e^{-x}$$

7.8 problem 8

Internal problem ID [164]
Internal file name [`OUTPUT/164_Sunday_June_05_2022_01_36_15_AM_47605589/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = -2]$$

7.8.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -3 \\ q(x) &= 0 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 3y' = 0$$

The domain of $p(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. Hence solution exists and is unique.

7.8.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 0$ into the above gives

$$\begin{aligned} \lambda_{1,2} &= \frac{3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-3^2 - (4)\,(1)\,(0)} \\ &= \frac{3}{2} \pm \frac{3}{2} \end{aligned}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{3}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{3}{2}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 0$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(0)x}$$

Or

$$y = c_1 e^{3x} + c_2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + c_2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = 3c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{2}{3}$$
$$c_2 = \frac{14}{3}$$

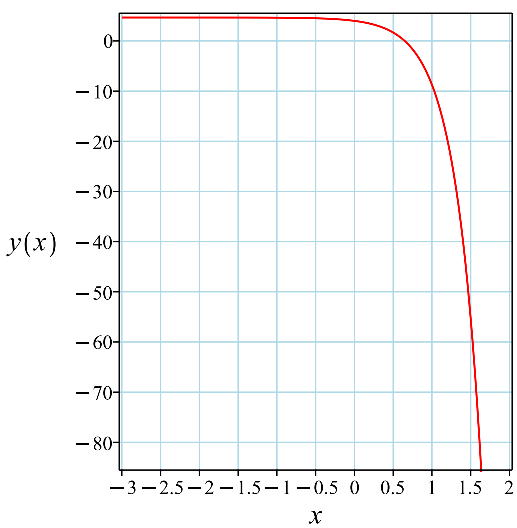
Substituting these values back in above solution results in

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$$

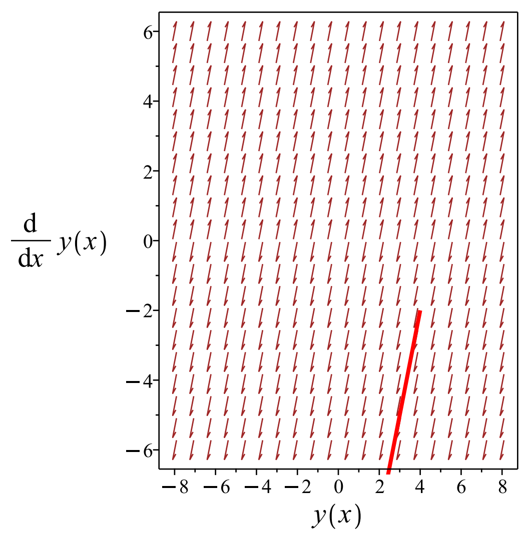
Summary

The solution(s) found are the following

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$$

Verified OK.

7.8.3 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = 0$$

$$-3y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{3y+c_1}dy = \int dx$$
$$\frac{\ln(3y+c_1)}{3} = x + c_2$$

Raising both side to exponential gives

$$(3y+c_1)^{\frac{1}{3}} = e^{x+c_2}$$

Which simplifies to

$$(3y+c_1)^{\frac{1}{3}} = c_3 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_3^3 e^{3x}}{3} - \frac{c_1}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = \frac{c_3^3}{3} - \frac{c_1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3^3 e^{3x}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = c_3^3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

<u>Verification of solutions</u> N/A

7.8.4 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) - 3p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int \frac{1}{3p} dp = \int dx$$
$$\frac{\ln(p)}{3} = x + c_1$$

Raising both side to exponential gives

$$p^{\frac{1}{3}} = e^{x+c_1}$$

Which simplifies to

$$p^{\frac{1}{3}} = c_2 e^{x}$$

Initial conditions are used to solve for c_2. Substituting $x = 0$ and $p = -2$ in the above solution gives an equation to solve for the constant of integration.

$$-2 = c_2^3$$

$$c_2 = -2^{\frac{1}{3}}$$

Substituting c_2 found above in the general solution gives

$$p(x) = -2\,e^{3x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = -2\,e^{3x}$$

Integrating both sides gives

$$y = \int -2\,e^{3x}\, dx$$
$$= -\frac{2\,e^{3x}}{3} + c_3$$

Initial conditions are used to solve for c_3. Substituting $x = 0$ and $y = 4$ in the above solution gives an equation to solve for the constant of integration.

$$4 = -\frac{2}{3} + c_3$$

$$c_3 = \frac{14}{3}$$

Substituting c_3 found above in the general solution gives

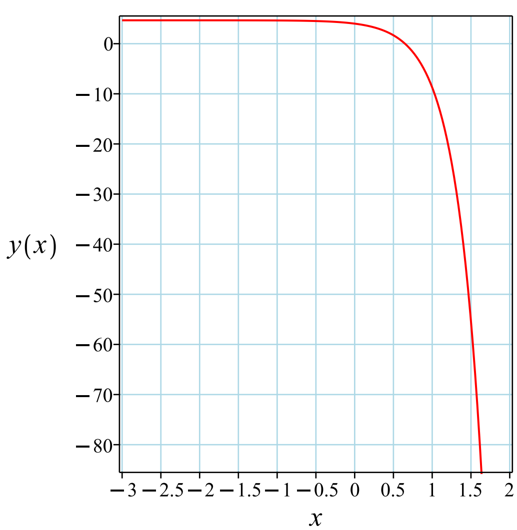
$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$$

Initial conditions are used to solve for the constants of integration.

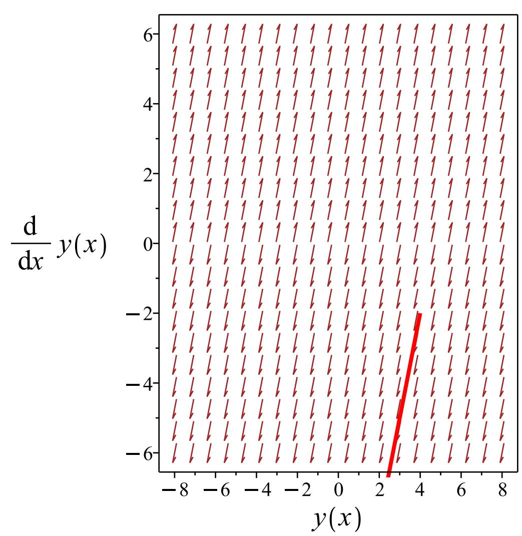
Summary

The solution(s) found are the following

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$$

Verified OK.

7.8.5 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' - 3y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' - 3y')\, dx = 0$$
$$-3y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{3y + c_1} dy = \int dx$$
$$\frac{\ln (3y + c_1)}{3} = x + c_2$$

Raising both side to exponential gives

$$(3y + c_1)^{\frac{1}{3}} = e^{x + c_2}$$

Which simplifies to

$$(3y + c_1)^{\frac{1}{3}} = c_3 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_3^3 e^{3x}}{3} - \frac{c_1}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = \frac{c_3^3}{3} - \frac{c_1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3^3 e^{3x}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = c_3^3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

7.8.6 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 9 \\
t &= 4
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = \frac{9z(x)}{4} \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 337: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2\left(1\left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 + \frac{c_2 e^{3x}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = c_1 + \frac{c_2}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_2 e^{3x}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{14}{3} \\ c_2 &= -2 \end{aligned}$$

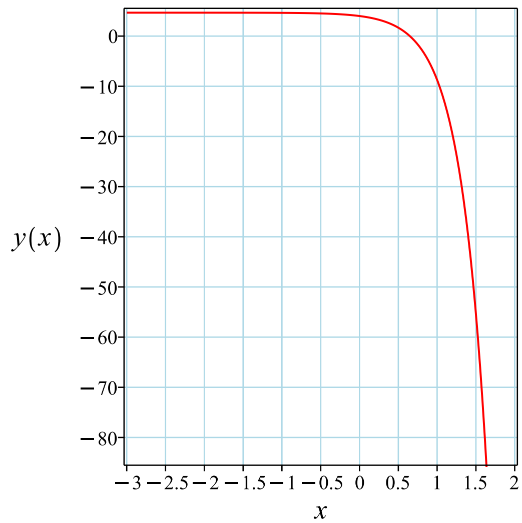
Substituting these values back in above solution results in

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$$

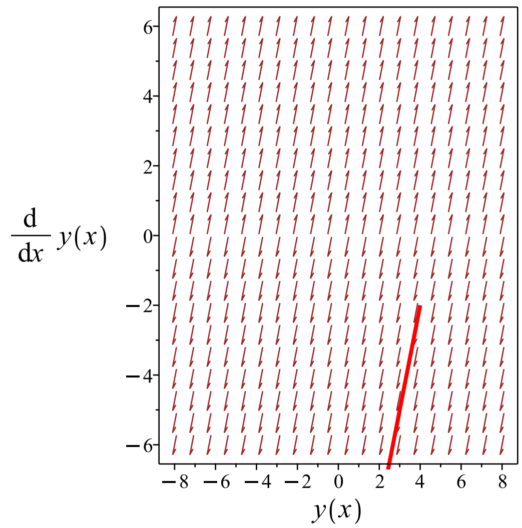
Summary

The solution(s) found are the following

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$$

Verified OK.

7.8.7 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= 1 \\ q(x) &= -3 \\ r(x) &= 0 \\ s(x) &= 0 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 0 \\ q'(x) &= 0 \end{aligned}$$

Therefore (1) becomes

$$0-(0)+(0)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p,q,r,s gives

$$-3y+y'=c_1$$

We now have a first order ode to solve which is

$$-3y+y'=c_1$$

Integrating both sides gives

$$\begin{aligned}\int \frac{1}{3y+c_1}dy&=\int dx\\ \frac{\ln(3y+c_1)}{3}&=x+c_2\end{aligned}$$

Raising both side to exponential gives

$$(3y+c_1)^{\frac{1}{3}}=\mathrm{e}^{x+c_2}$$

Which simplifies to

$$(3y+c_1)^{\frac{1}{3}}=c_3\mathrm{e}^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y=\frac{c_3^3\mathrm{e}^{3x}}{3}-\frac{c_1}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y=4$ and $x=0$ in the above gives

$$4=\frac{c_3^3}{3}-\frac{c_1}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_3^3 e^{3x}$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = c_3^3 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Warning, unable to solve for constants of integrations.

Verification of solutions N/A

7.8.8 Maple step by step solution

Let's solve

$\left[y'' - 3y' = 0, y(0) = 4, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r = 0$

- Factor the characteristic polynomial

  $r(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (0, 3)$

- 1st solution of the ODE

  $y_1(x) = 1$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 + c_2 e^{3x}$

☐ Check validity of solution $y = c_1 + c_2 e^{3x}$

- Use initial condition $y(0) = 4$

  $4 = c_1 + c_2$
- Compute derivative of the solution

  $y' = 3c_2 e^{3x}$
- Use the initial condition $y'\big|_{\{x=0\}} = -2$

  $-2 = 3c_2$
- Solve for c_1 and c_2

  $\left\{c_1 = \frac{14}{3}, c_2 = -\frac{2}{3}\right\}$
- Substitute constant values into general solution and simplify

  $y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$

- Solution to the IVP

  $y = -\frac{2\,e^{3x}}{3} + \frac{14}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)-3*diff(y(x),x)=0,y(0) = 4, D(y)(0) = -2],y(x), singsol=all)
```

$$y(x) = \frac{14}{3} - \frac{2\,e^{3x}}{3}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 16

```
DSolve[{y''[x]-3*y'[x]==0,{y[0]==4,y'[0]==-2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{2}{3}\left(e^{3x} - 7\right)$$

7.9 problem 9

Internal problem ID [165]
Internal file name [`OUTPUT/165_Sunday_June_05_2022_01_36_16_AM_4167586/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' + y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = -1]$$

7.9.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 2 \\ q(x) &= 1 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 2y' + y = 0$$

The domain of $p(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.9.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 2\lambda\, \mathrm{e}^{\lambda x} + \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(2)^2 - (4)\,(1)\,(1)}\\ &= -1\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 1$. Therefore the solution is

$$y = c_1 \mathrm{e}^{-x} + c_2 x\, \mathrm{e}^{-x} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 x e^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} + c_2 e^{-x} - c_2 x e^{-x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = -c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 1$$

Substituting these values back in above solution results in

$$y = x e^{-x} + 2 e^{-x}$$

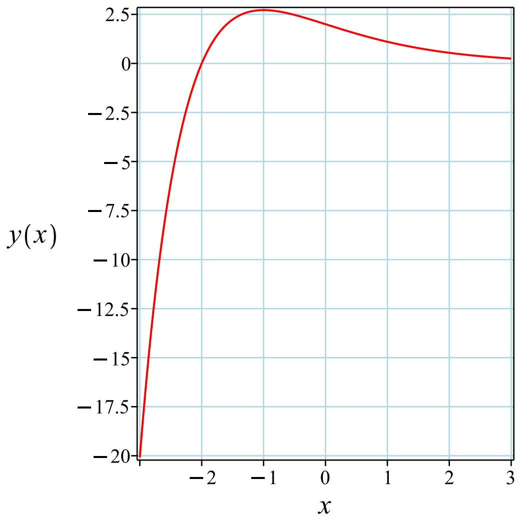
Which simplifies to

$$y = e^{-x}(2 + x)$$

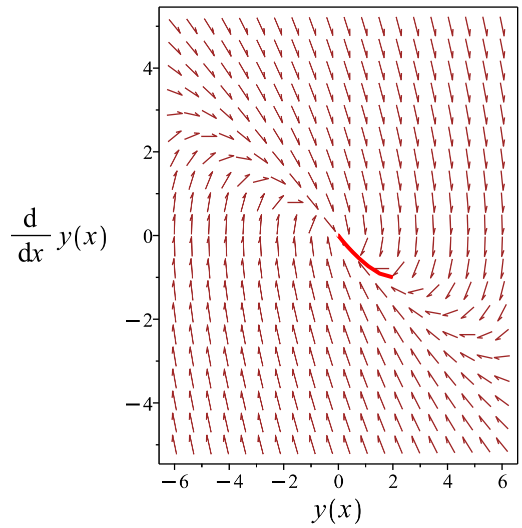
Summary

The solution(s) found are the following

$$y = e^{-x}(2 + x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-x}(2 + x)$$

Verified OK.

7.9.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 2$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 2\,dx} \\ &= e^{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ (e^{x}y)'' &= 0 \end{aligned}$$

Integrating once gives

$$(\mathrm{e}^x y)' = c_1$$

Integrating again gives

$$(\mathrm{e}^x y) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^x}$$

Or

$$y = c_1 x\,\mathrm{e}^{-x} + c_2 \mathrm{e}^{-x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\,\mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 \mathrm{e}^{-x} - c_1 x\,\mathrm{e}^{-x} - c_2 \mathrm{e}^{-x}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = x\,\mathrm{e}^{-x} + 2\,\mathrm{e}^{-x}$$

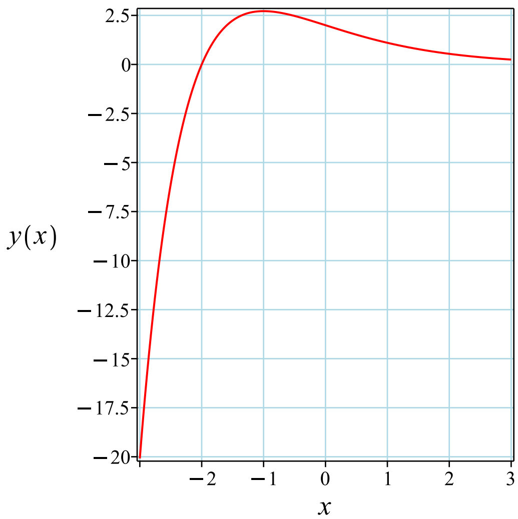
Which simplifies to

$$y = \mathrm{e}^{-x}(2 + x)$$

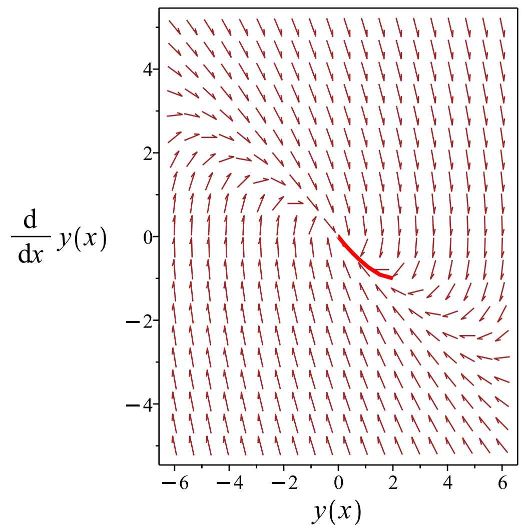
Summary

The solution(s) found are the following

$$y = e^{-x}(2 + x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{-x}(2 + x)$$

Verified OK.

7.9.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 339: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1 \left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x}(x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-x} + c_2 x \, \mathrm{e}^{-x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} - c_2 x \, \mathrm{e}^{-x}
$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$
-1 = -c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 1
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = x \, \mathrm{e}^{-x} + 2 \, \mathrm{e}^{-x}
$$

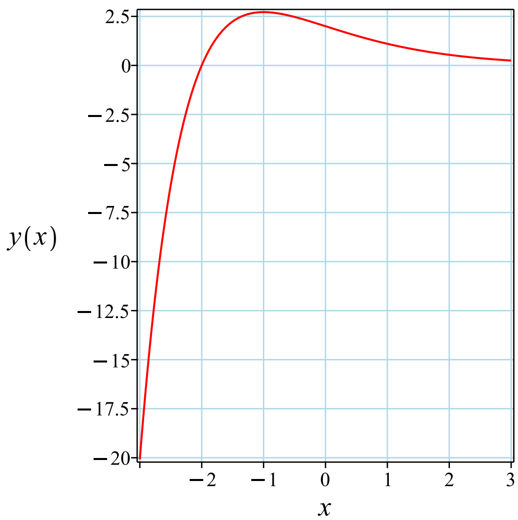
Which simplifies to

$$
y = \mathrm{e}^{-x} (2 + x)
$$

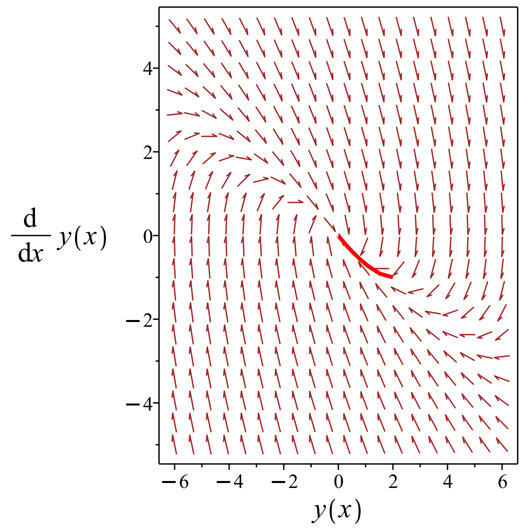
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-x} (2 + x) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{-x}(2 + x)$$

Verified OK.

7.9.5 Maple step by step solution

Let's solve

$$\left[y'' + 2y' + y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = -1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r + 1)^2 = 0$

- Root of the characteristic polynomial

  $r = -1$

- 1st solution of the ODE

$y_1(x) = \mathrm{e}^{-x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,\mathrm{e}^{-x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$
- Substitute in solutions

  $y = c_1 \mathrm{e}^{-x} + c_2 x\,\mathrm{e}^{-x}$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-x} + c_2 x \mathrm{e}^{-x}$

- Use initial condition $y(0) = 2$

  $2 = c_1$
- Compute derivative of the solution

  $y' = -c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-x} - c_2 x\,\mathrm{e}^{-x}$
- Use the initial condition $y'\Big|_{\{x=0\}} = -1$

  $-1 = -c_1 + c_2$
- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = 1\}$
- Substitute constant values into general solution and simplify

  $y = \mathrm{e}^{-x}(2 + x)$

- Solution to the IVP

  $y = \mathrm{e}^{-x}(2 + x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve([diff(y(x),x$2)+2*diff(y(x),x)+y(x)=0,y(0) = 2, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = e^{-x}(2 + x)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 14

```
DSolve[{y''[x]+2*y'[x]+y[x]==0,{y[0]==2,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x}(x + 2)$$

7.10 problem 10

Internal problem ID [166]
Internal file name [`OUTPUT/166_Sunday_June_05_2022_01_36_17_AM_76300682/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{y'' - 10y' + 25y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 13]$$

7.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -10 \\ q(x) &= 25 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 10y' + 25y = 0$$

The domain of $p(x) = -10$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 25$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.10.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -10, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 10\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 10\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -10, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{10}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(-10)^2 - (4)\,(1)\,(25)}\\ &= 5\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -5$. Therefore the solution is

$$y = c_1 e^{5x} + c_2 x\, e^{5x} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{5x} + c_2 x\, e^{5x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x\, e^{5x}$$

substituting $y' = 13$ and $x = 0$ in the above gives

$$13 = 5c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 3 \\ c_2 &= -2 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -2x\, e^{5x} + 3\, e^{5x}$$

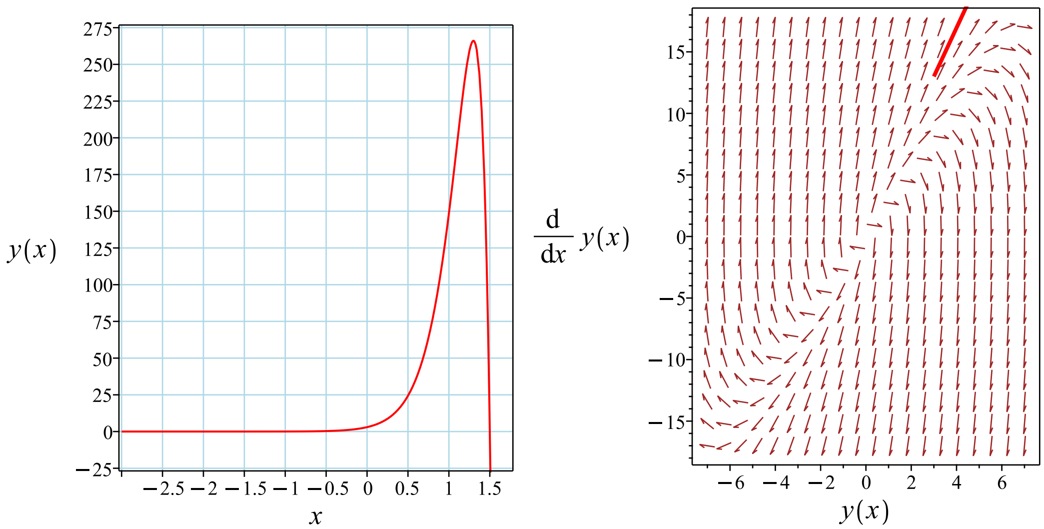
Which simplifies to

$$y = e^{5x}(-2x + 3)$$

Summary

The solution(s) found are the following

$$y = e^{5x}(-2x + 3) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$y = e^{5x}(-2x + 3)$$

Verified OK.

7.10.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x) y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -10$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -10\,dx} \\ &= e^{-5x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 0$$
$$\left(e^{-5x} y\right)'' = 0$$

Integrating once gives

$$\left(e^{-5x}y\right)' = c_1$$

Integrating again gives

$$\left(e^{-5x}y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{e^{-5x}}$$

Or

$$y = c_1 x\, e^{5x} + c_2 e^{5x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x\, e^{5x} + c_2 e^{5x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 e^{5x} + 5c_1 x\, e^{5x} + 5c_2 e^{5x}$$

substituting $y' = 13$ and $x = 0$ in the above gives

$$13 = c_1 + 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -2x\, e^{5x} + 3\, e^{5x}$$

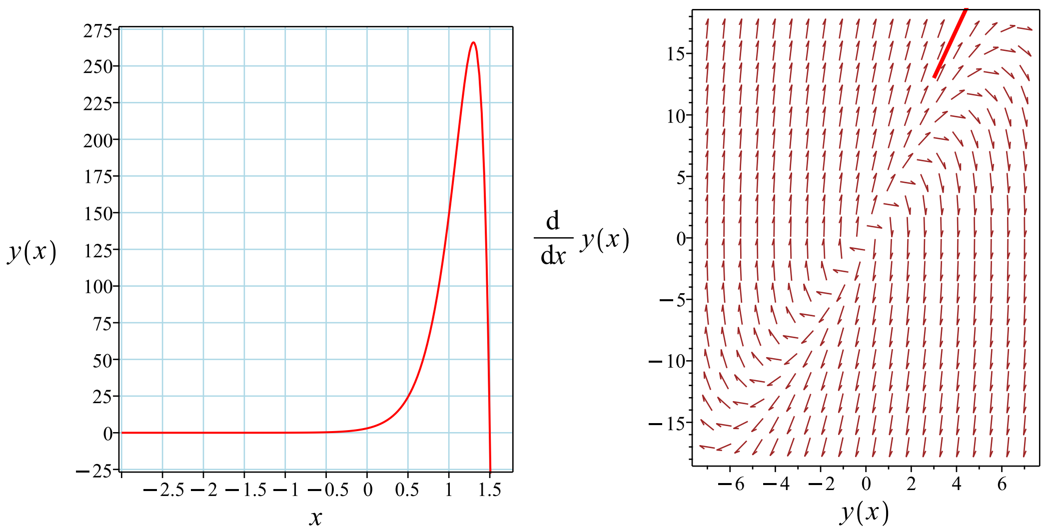
Which simplifies to

$$y = e^{5x}(-2x + 3)$$

Summary

The solution(s) found are the following

$$y = e^{5x}(-2x + 3) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{5x}(-2x + 3)$$

Verified OK.

7.10.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 10y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -10 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 341: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-10}{1}\,dx} \\
&= z_1 e^{5x} \\
&= z_1 \left(\mathrm{e}^{5x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-10}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{10x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{5x}\right) + c_2 \left(\mathrm{e}^{5x} (x)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{5x} + c_2 x \,\mathrm{e}^{5x} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 5c_1 \mathrm{e}^{5x} + c_2 \mathrm{e}^{5x} + 5c_2 x \,\mathrm{e}^{5x}
$$

substituting $y' = 13$ and $x = 0$ in the above gives

$$
13 = 5c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= -2
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -2x \,\mathrm{e}^{5x} + 3 \,\mathrm{e}^{5x}
$$

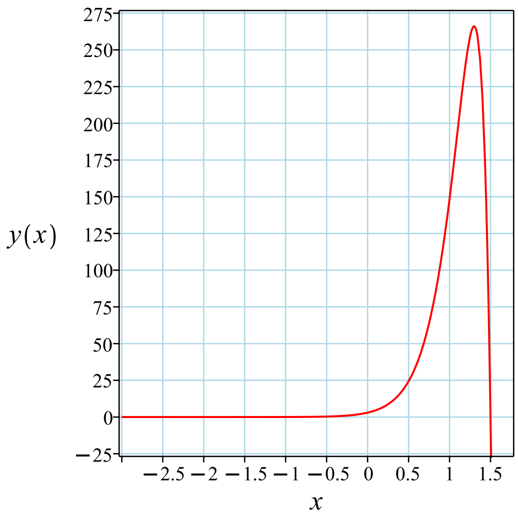
Which simplifies to

$$
y = \mathrm{e}^{5x} (-2x + 3)
$$

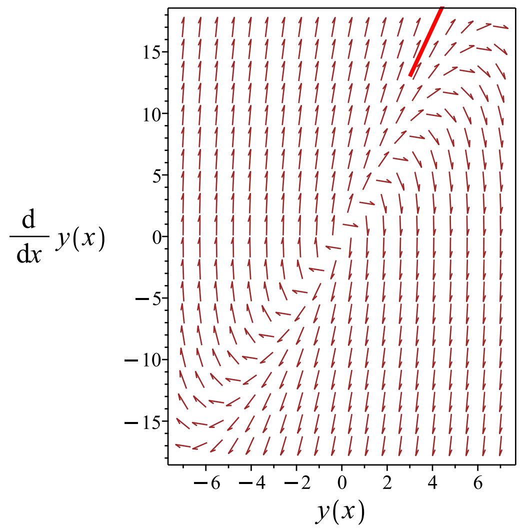
Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{5x} (-2x + 3) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = e^{5x}(-2x + 3)$$

Verified OK.

7.10.5 Maple step by step solution

Let's solve

$$\left[y'' - 10y' + 25y = 0, y(0) = 3, y'\Big|_{\{x=0\}} = 13\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 10r + 25 = 0$

- Factor the characteristic polynomial

  $(r - 5)^2 = 0$

- Root of the characteristic polynomial

  $r = 5$

- 1st solution of the ODE

$y_1(x) = e^{5x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{5x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{5x} + c_2 x\, e^{5x}$

□ Check validity of solution $y = c_1 e^{5x} + c_2 x e^{5x}$

  - Use initial condition $y(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $y' = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 x\, e^{5x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 13$

    $13 = 5c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 3, c_2 = -2\}$

  - Substitute constant values into general solution and simplify

    $y = e^{5x}(-2x + 3)$

- Solution to the IVP

  $y = e^{5x}(-2x + 3)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)-10*diff(y(x),x)+25*y(x)=0,y(0) = 3, D(y)(0) = 13],y(x), singsol=all)
```

$$y(x) = e^{5x}(3 - 2x)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 16

```
DSolve[{y''[x]-10*y'[x]+25*y[x]==0,{y[0]==3,y'[0]==13}},y[x],x,IncludeSingularSolutions -> T
```

$$y(x) \to e^{5x}(3 - 2x)$$

7.11 problem 11

Internal problem ID [167]
Internal file name [`OUTPUT/167_Sunday_June_05_2022_01_36_17_AM_60101299/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' + 2y = 0}$$

With initial conditions

$$[y(0) = 0, y'(0) = 5]$$

7.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = 0$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.11.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\, \mathrm{e}^{\lambda x} + 2\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)}\\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\lambda_1 = 1 + i$$
$$\lambda_2 = 1 - i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(-c_1 \sin(x) + c_2 \cos(x))$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$5 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

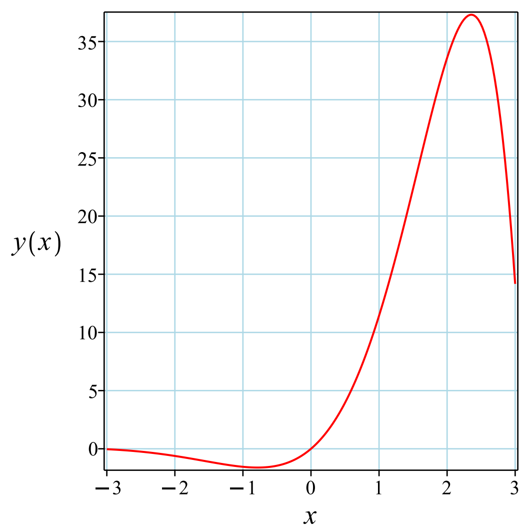
$$c_1 = 0$$
$$c_2 = 5$$

Substituting these values back in above solution results in

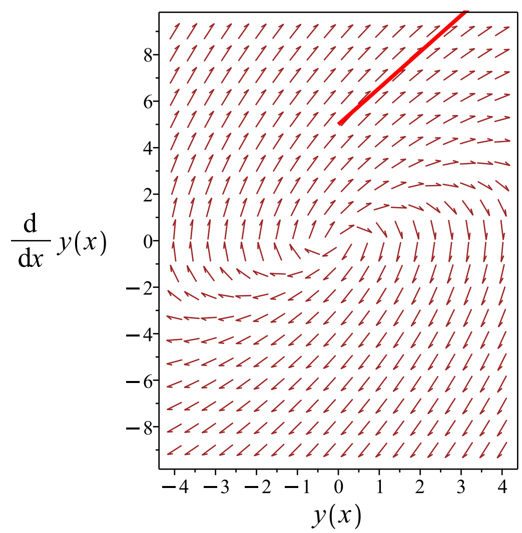
$$y = 5 \sin(x) e^{x}$$

Summary
The solution(s) found are the following

$$y = 5 \sin(x) e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 5 \sin(x) e^{x}$$

Verified OK.

7.11.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 343: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}\\ &= z_1 e^{x}\\ &= z_1(\mathrm{e}^{x})\end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx\\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx\\ &= y_1(\tan(x))\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1(\cos(x)\,\mathrm{e}^x) + c_2(\cos(x)\,\mathrm{e}^x(\tan(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(x)\,\mathrm{e}^x + \mathrm{e}^x c_2 \sin(x) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -c_1 \sin(x)\,\mathrm{e}^x + c_1 \cos(x)\,\mathrm{e}^x + \mathrm{e}^x c_2 \sin(x) + \mathrm{e}^x c_2 \cos(x)
$$

substituting $y' = 5$ and $x = 0$ in the above gives

$$
5 = c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= 5
\end{aligned}
$$

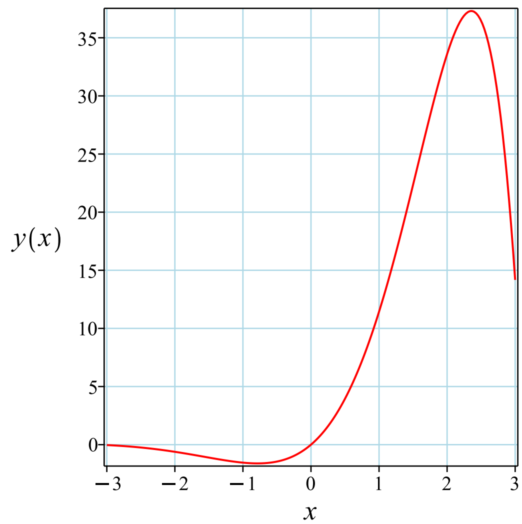
Substituting these values back in above solution results in

$$
y = 5\sin(x)\,\mathrm{e}^x
$$

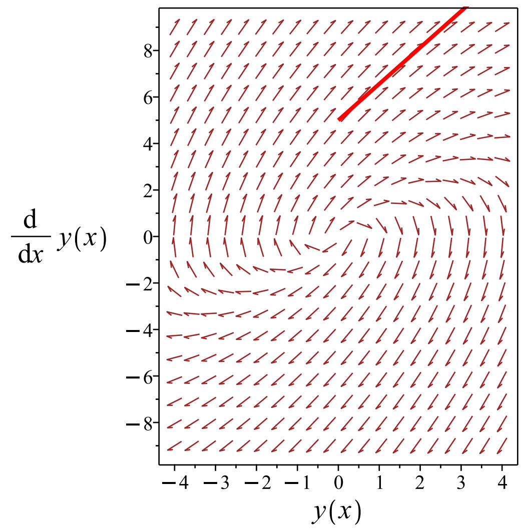
Summary

The solution(s) found are the following

$$
y = 5\sin(x)\,\mathrm{e}^x \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 5\sin(x)\,e^{x}$$

Verified OK.

7.11.4 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 2y = 0, y(0) = 0, y'\Big|_{\{x=0\}} = 5\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the ODE

$y_1(x) = \cos(x)\,e^x$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)\,e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x)\,e^x + e^x c_2 \sin(x)$

□ Check validity of solution $y = c_1 \cos(x)\,e^x + e^x c_2 \sin(x)$

  - Use initial condition $y(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $y' = -c_1 \sin(x)\,e^x + c_1 \cos(x)\,e^x + e^x c_2 \sin(x) + e^x c_2 \cos(x)$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 5$

    $5 = c_1 + c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = 5\}$

  - Substitute constant values into general solution and simplify

    $y = 5\sin(x)\,e^x$

- Solution to the IVP

  $y = 5\sin(x)\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=0,y(0) = 0, D(y)(0) = 5],y(x), singsol=all)
```

$$y(x) = 5\sin(x)\,e^{x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 12

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==0,{y[0]==0,y'[0]==5}},y[x],x,IncludeSingularSolutions -> True
```

$$y(x) \to 5e^{x}\sin(x)$$

7.12 problem 12

Internal problem ID [168]
Internal file name [`OUTPUT/168_Sunday_June_05_2022_01_36_18_AM_74995814/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 13y = 0}$$

With initial conditions

$$[y(0) = 2, y'(0) = 0]$$

7.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 6 \\ q(x) &= 13 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + 6y' + 13y = 0$$

The domain of $p(x) = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 13$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

7.12.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda\, e^{\lambda x} + 13\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{6^2 - (4)\,(1)\,(13)} \\ &= -3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -3 + 2i$$
$$\lambda_2 = -3 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3\,\mathrm{e}^{-3x}(c_1 \cos(2x) + c_2 \sin(2x)) + \mathrm{e}^{-3x}(-2c_1 \sin(2x) + 2c_2 \cos(2x))$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = -3c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 3$$

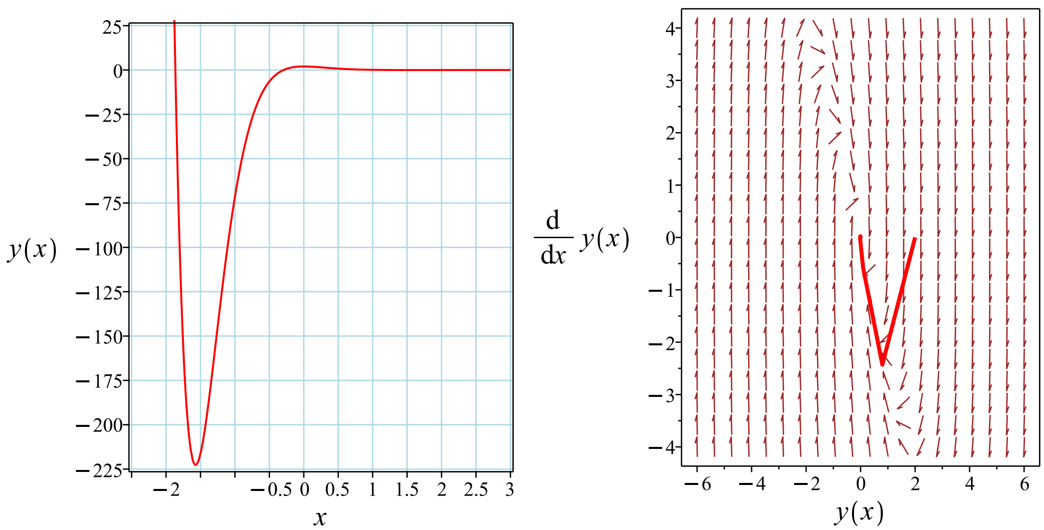
Substituting these values back in above solution results in

$$y = \mathrm{e}^{-3x}(2\cos(2x) + 3\sin(2x))$$

Summary

The solution(s) found are the following

$$y = e^{-3x}(2\cos(2x) + 3\sin(2x)) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-3x}(2\cos(2x) + 3\sin(2x))$$

Verified OK.

7.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 13y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 345: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{6}{1} \, dx} \\ &= z_1 e^{-3x} \\ &= z_1 \left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{-3x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-3x} \cos(2x) + \frac{c_2 \mathrm{e}^{-3x} \sin(2x)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 \mathrm{e}^{-3x} \cos(2x) - 2c_1 \mathrm{e}^{-3x} \sin(2x) - \frac{3c_2 \mathrm{e}^{-3x} \sin(2x)}{2} + c_2 \mathrm{e}^{-3x} \cos(2x)
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = -3c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 2 \\
c_2 &= 6
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = 2\,\mathrm{e}^{-3x} \cos(2x) + 3\,\mathrm{e}^{-3x} \sin(2x)
$$

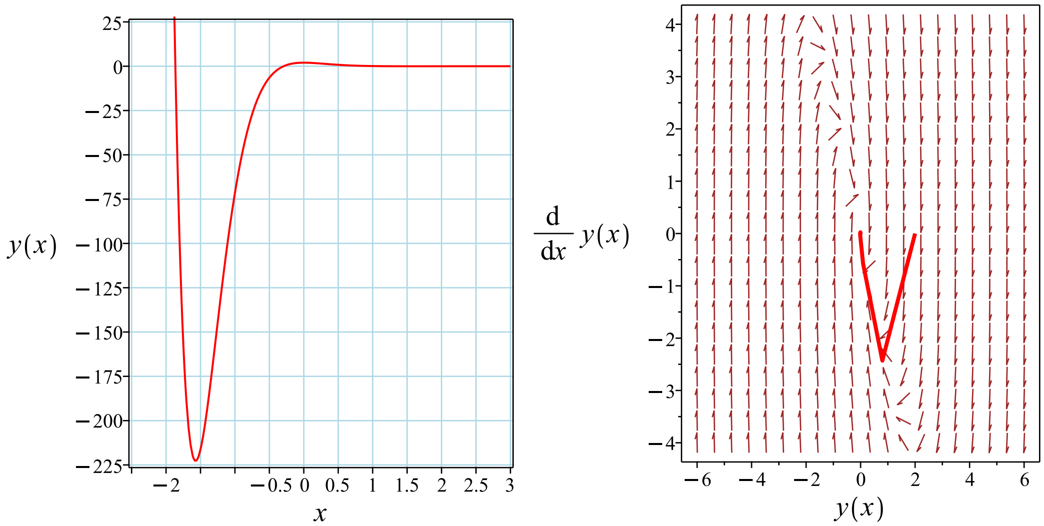
Which simplifies to

$$
y = \mathrm{e}^{-3x}(2\cos(2x) + 3\sin(2x))
$$

Summary

The solution(s) found are the following

$$
y = \mathrm{e}^{-3x}(2\cos(2x) + 3\sin(2x)) \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = e^{-3x}(2\cos(2x) + 3\sin(2x))$$

Verified OK.

7.12.4 Maple step by step solution

Let's solve

$\left[y'' + 6y' + 13y = 0, y(0) = 2, y'\Big|_{\{x=0\}} = 0\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the ODE

$y_1(x) = e^{-3x} \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = e^{-3x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} \cos(2x) + c_2 e^{-3x} \sin(2x)$

□ Check validity of solution $y = c_1 e^{-3x} \cos(2x) + c_2 e^{-3x} \sin(2x)$

  - Use initial condition $y(0) = 2$

    $2 = c_1$

  - Compute derivative of the solution

    $y' = -3c_1 e^{-3x} \cos(2x) - 2c_1 e^{-3x} \sin(2x) - 3c_2 e^{-3x} \sin(2x) + 2c_2 e^{-3x} \cos(2x)$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = -3c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = 3\}$

  - Substitute constant values into general solution and simplify

    $y = e^{-3x}(2\cos(2x) + 3\sin(2x))$

- Solution to the IVP

  $y = e^{-3x}(2\cos(2x) + 3\sin(2x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 22

```
dsolve([diff(y(x),x$2)+6*diff(y(x),x)+13*y(x)=0,y(0) = 2, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = e^{-3x}(3\sin(2x) + 2\cos(2x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 24

```
DSolve[{y''[x]+6*y'[x]+13*y[x]==0,{y[0]==2,y'[0]==0}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{-3x}(3\sin(2x) + 2\cos(2x))$$

7.13 problem 13

Internal problem ID [169]
Internal file name [`OUTPUT/169_Sunday_June_05_2022_01_36_19_AM_51144929/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2 y'' - 2y'x + 2y = 0}$$

With initial conditions

$$[y(1) = 3, y'(1) = 1]$$

7.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$$

The domain of $p(x) = -\frac{2}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{2}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.13.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 2xrx^{r-1} + 2x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 2r\,x^r + 2x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 2r + 2 = 0$$

Or

$$r^2 - 3r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 2$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = c_2 x^2 + c_1 x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_2 x^2 + c_1 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_2 x + c_1$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -2x^2 + 5x$$

Summary

The solution(s) found are the following

$$y = -2x^2 + 5x \tag{1}$$

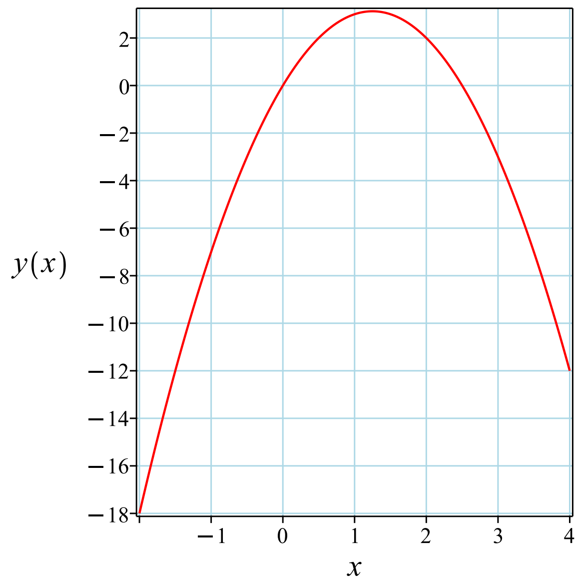


Figure 482: Solution plot

Verification of solutions

$$y = -2x^2 + 5x$$

Verified OK.

7.13.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{2}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int -\frac{2}{x}\, dx} \\ &= \frac{1}{x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\frac{y}{x}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x}\right)' = c_1$$

Integrating again gives

$$\left(\frac{y}{x}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\frac{1}{x}}$$

Or

$$y = c_1 x^2 + c_2 x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x^2 + c_2 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 x + c_2$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= 5 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -2x^2 + 5x$$

Summary

The solution(s) found are the following

$$y = -2x^2 + 5x \tag{1}$$

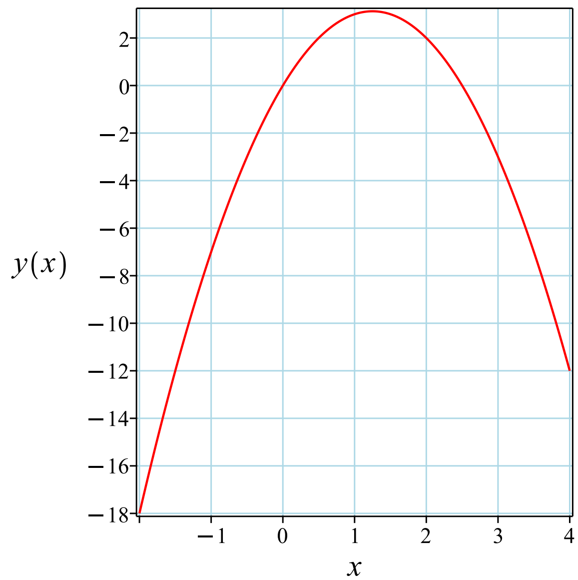


Figure 483: Solution plot

Verification of solutions

$$y = -2x^2 + 5x$$

Verified OK.

7.13.4 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{2}{x}dx\right)} dx \\
&= \int e^{2\ln(x)}\, dx \\
&= \int x^2 dx \\
&= \frac{x^3}{3} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{2}{x^2}}{x^4} \\
&= \frac{2}{x^6} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + \frac{2y(\tau)}{x^6} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{2}{x^6} = \frac{2}{9\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{2y(\tau)}{9\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$9\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 2y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$9\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 2\tau^r = 0$$

Simplifying gives

$$9r(r-1)\,\tau^r + 0\,\tau^r + 2\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$9r(r-1) + 0 + 2 = 0$$

Or

$$9r^2 - 9r + 2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{3}$$
$$r_2 = \frac{2}{3}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1\tau^{\frac{1}{3}} + c_2\tau^{\frac{2}{3}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 3^{\frac{2}{3}}(x^3)^{\frac{1}{3}}}{3} + \frac{c_2 3^{\frac{1}{3}}(x^3)^{\frac{2}{3}}}{3}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 3^{\frac{2}{3}} \left(x^3\right)^{\frac{1}{3}}}{3}+\frac{c_2 3^{\frac{1}{3}} \left(x^3\right)^{\frac{2}{3}}}{3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = \frac{3^{\frac{1}{3}} \left(c_1 3^{\frac{1}{3}}+c_2 \right)}{3} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 3^{\frac{2}{3}} x^2}{3 \left(x^3\right)^{\frac{2}{3}}}+\frac{2 c_2 3^{\frac{1}{3}} x^2}{3 \left(x^3\right)^{\frac{1}{3}}}$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = \frac{3^{\frac{1}{3}} \left(c_1 3^{\frac{1}{3}}+2 c_2 \right)}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5\,3^{\frac{1}{3}}$$
$$c_2 = -2\,3^{\frac{2}{3}}$$

Substituting these values back in above solution results in

$$y = -2\left(x^3\right)^{\frac{2}{3}}+5\left(x^3\right)^{\frac{1}{3}}$$

Summary

The solution(s) found are the following

$$y = -2\left(x^3\right)^{\frac{2}{3}}+5\left(x^3\right)^{\frac{1}{3}} \tag{1}$$

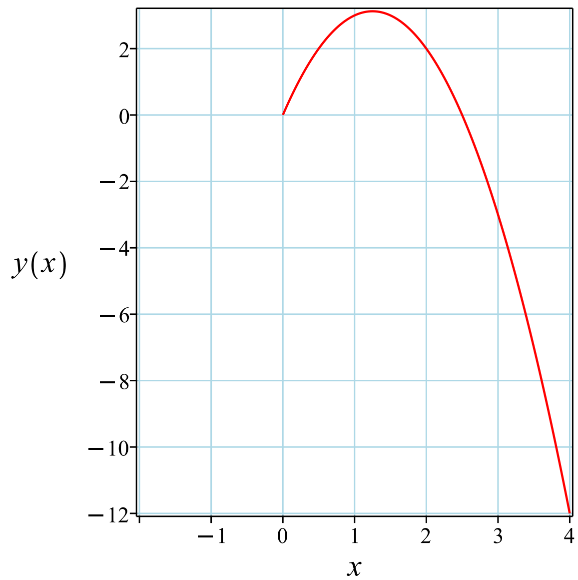


Figure 484: Solution plot

Verification of solutions

$$y = -2\left(x^3\right)^{\frac{2}{3}} + 5\left(x^3\right)^{\frac{1}{3}}$$

Verified OK.

7.13.5 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{2}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{2}{x}\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{3c\sqrt{2}}{2}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{3c\sqrt{2}\,\left(\frac{d}{d\tau}y(\tau)\right)}{2} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{3\sqrt{2}\,c\tau}{4}}\left(c_1 \cosh\left(\frac{\sqrt{2}\,c\tau}{4}\right) + ic_2 \sinh\left(\frac{\sqrt{2}\,c\tau}{4}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{2}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{2}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{3}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = x^{\frac{3}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
3 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = \frac{3\sqrt{x}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)}{2} + x^{\frac{3}{2}}\left(\frac{c_1\sinh\left(\frac{\ln(x)}{2}\right)}{2x} + \frac{ic_2\cosh\left(\frac{\ln(x)}{2}\right)}{2x}\right)
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$
1 = \frac{3c_1}{2} + \frac{ic_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 3 \\
c_2 &= 7i
\end{aligned}
$$

Substituting these values back in above solution results in

$$y = -7x^{\frac{3}{2}} \sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{3}{2}}$$

Which simplifies to

$$y = \left(-7\sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y = \left(-7\sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{3}{2}} \tag{1}$$

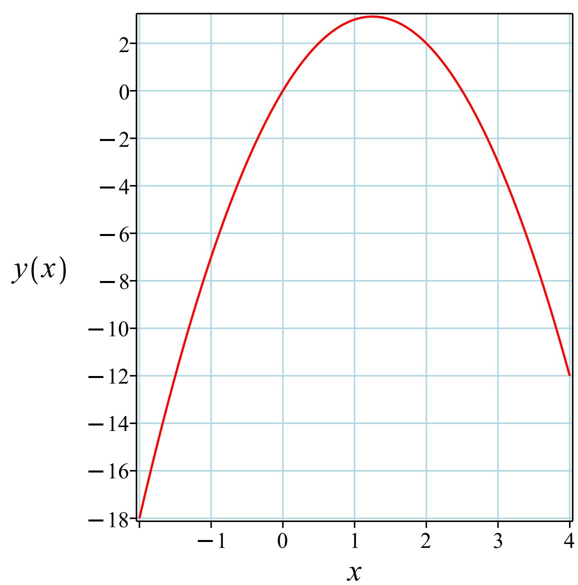


Figure 485: Solution plot

Verification of solutions

$$y = \left(-7\sinh\left(\frac{\ln(x)}{2}\right) + 3\cosh\left(\frac{\ln(x)}{2}\right)\right) x^{\frac{3}{2}}$$

Verified OK.

7.13.6 Solving as second order change of variable on y method 1 ode

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2}{x} \\ q(x) &= \frac{2}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(-\frac{2}{x}\right)'}{2} - \frac{\left(-\frac{2}{x}\right)^2}{4} \\ &= \frac{2}{x^2} - \frac{\left(\frac{2}{x^2}\right)}{2} - \frac{\left(\frac{4}{x^2}\right)}{4} \\ &= \frac{2}{x^2} - \left(\frac{1}{x^2}\right) - \frac{1}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2} dx\right)} \\ &= e^{-\int \frac{-\frac{2}{x}}{2}} \\ &= x \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x \tag{4}$$

Applying this change of variable to the original ode results in

$$x^3 v''(x) = 0$$

Which is now solved for $v(x)$ Integrating twice gives the solution

$$v(x) = c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1 x + c_2)\, (z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x$$

Hence (7) becomes

$$y = (c_1 x + c_2)\, x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 x + c_2)\, x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 x + c_2$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = 2c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -2 \\ c_2 &= 5 \end{aligned}$$

Substituting these values back in above solution results in

$$y = -x(2x - 5)$$

Summary

The solution(s) found are the following

$$y = -x(2x-5) \tag{1}$$

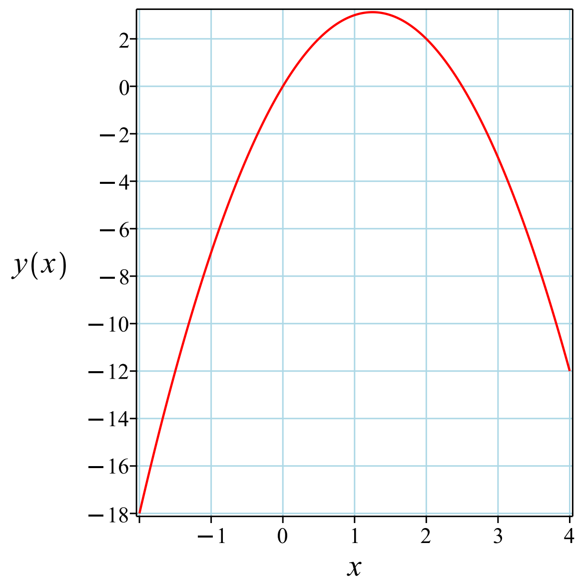


Figure 486: Solution plot

Verification of solutions

$$y = -x(2x-5)$$

Verified OK.

7.13.7 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{2}{x}$$
$$q(x) = \frac{2}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{2n}{x^2} + \frac{2}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= e^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^2 \\
&= (c_2 x - c_1)\, x
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \left(-\frac{c_1}{x} + c_2\right) x^2 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
3 = -c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 + 2\left(-\frac{c_1}{x} + c_2\right) x
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -5$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -x(2x - 5)$$

Summary

The solution(s) found are the following

$$y = -x(2x - 5) \tag{1}$$

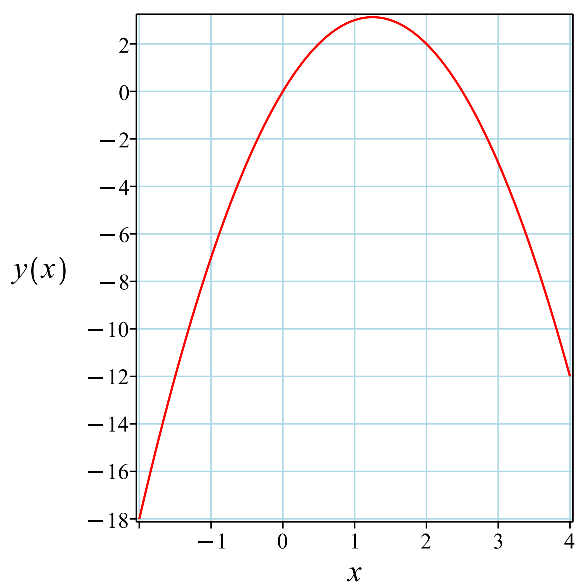


Figure 487: Solution plot

Verification of solutions

$$y = -x(2x - 5)$$

Verified OK.

7.13.8 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + (2AB' + B^2)\, v' + (AB'' + BB' + CB)\, v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + (2AB' + B^2)\, v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + (2AB' + B^2)\, u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -2x \\ C &= 2 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= (x^2)\,(0) + (-2x)\,(-2) + (2)\,(-2x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-2x^3v'' + (0)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-2x^3u'(x) = 0$$

Which is now solved for u. Integrating both sides gives

$$\begin{aligned} u(x) &= \int 0\, \mathrm{d}x \\ &= c_1 \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= c_1 \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int c_1\, \mathrm{d}x \\ &= c_1x + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-2x)\,(c_1x + c_2) \\ &= -2(c_1x + c_2)\, x \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -2(c_1x + c_2)\, x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$3 = -2c_1 - 2c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -4c_1x - 2c_2$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = -4c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -\frac{5}{2}$$

Substituting these values back in above solution results in

$$y = -x(2x - 5)$$

Summary

The solution(s) found are the following

$$y = -x(2x - 5) \tag{1}$$

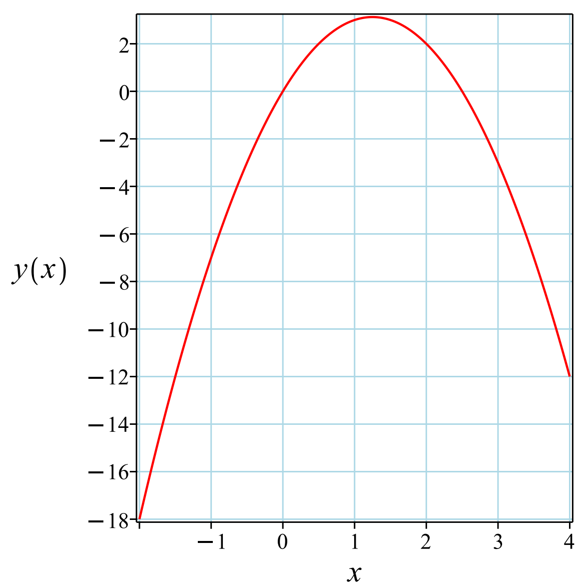


Figure 488: Solution plot

Verification of solutions

$$y = -x(2x - 5)$$

Verified OK.

7.13.9 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 2y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= x^2 \\ B &= -2x \\ C &= 2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= 0 \\ t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 347: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{\ln(x)} \\
&= z_1(x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{2\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(x))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_2 x^2 + c_1 x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 1$ in the above gives

$$
3 = c_1 + c_2 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = 2c_2 x + c_1
$$

substituting $y' = 1$ and $x = 1$ in the above gives

$$1 = c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$y = -2x^2 + 5x$$

Summary
The solution(s) found are the following

$$y = -2x^2 + 5x \tag{1}$$

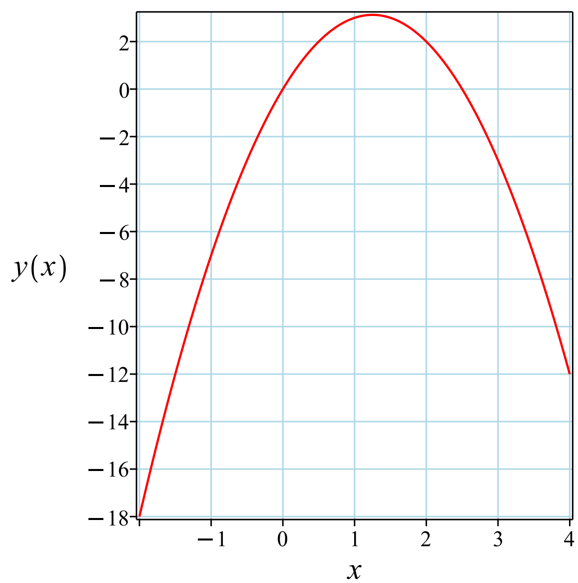


Figure 489: Solution plot

Verification of solutions

$$y = -2x^2 + 5x$$

Verified OK.

7.13.10 Maple step by step solution

Let's solve

$\left[x^2 y'' - 2y' x + 2y = 0, y(1) = 3, y'\Big|_{\{x=1\}} = 1\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{2y'}{x} - \frac{2y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{2y'}{x} + \frac{2y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - 2y' x + 2y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 2\frac{d}{dt}y(t) + 2y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 3\frac{d}{dt}y(t) + 2y(t) = 0$

- Characteristic polynomial of ODE

$r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(t) = e^t$

- 2nd solution of the ODE

  $y_2(t) = e^{2t}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 e^t + c_2 e^{2t}$

- Change variables back using $t = \ln(x)$

  $y = c_2 x^2 + c_1 x$

- Simplify

  $y = x(c_2 x + c_1)$

☐ Check validity of solution $y = x(c_2 x + c_1)$

  - Use initial condition $y(1) = 3$

    $3 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = 2c_2 x + c_1$

  - Use the initial condition $y'\Big|_{\{x=1\}} = 1$

    $1 = c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 5, c_2 = -2\}$

  - Substitute constant values into general solution and simplify

    $y = -2x^2 + 5x$

- Solution to the IVP

$$y = -2x^2 + 5x$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 13

```
dsolve([x^2*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=0,y(1) = 3, D(y)(1) = 1],y(x), singsol=al
```

$$y(x) = -2x^2 + 5x$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 12

```
DSolve[{x^2*y''[x]-2*x*y'[x]+2*y[x]==0,{y[1]==3,y'[1]==1}},y[x],x,IncludeSingularSolutions -
```

$$y(x) \to (5 - 2x)x$$

7.14 problem 14

Internal problem ID [170]
Internal file name [`OUTPUT/170_Sunday_June_05_2022_01_36_19_AM_52710539/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + 2y'x - 6y = 0}$$

With initial conditions

$$[y(2) = 10, y'(2) = 15]$$

7.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$
\begin{aligned}
p(x) &= \frac{2}{x} \\
q(x) &= -\frac{6}{x^2} \\
F &= 0
\end{aligned}
$$

Hence the ode is

$$
y'' + \frac{2y'}{x} - \frac{6y}{x^2} = 0
$$

The domain of $p(x) = \frac{2}{x}$ is

$$
\{x < 0 \lor 0 < x\}
$$

And the point $x_0 = 2$ is inside this domain. The domain of $q(x) = -\frac{6}{x^2}$ is

$$
\{x < 0 \lor 0 < x\}
$$

And the point $x_0 = 2$ is also inside this domain. Hence solution exists and is unique.

7.14.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$
x^2(r(r-1))x^{r-2} + 2xrx^{r-1} - 6x^r = 0
$$

Simplifying gives

$$
r(r-1)\,x^r + 2r\,x^r - 6x^r = 0
$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$
r(r-1) + 2r - 6 = 0
$$

Or

$$
r^2 + r - 6 = 0 \tag{1}
$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -3 \\
r_2 &= 2
\end{aligned}
$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^3} + c_2 x^2$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1}{x^3} + c_2 x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 2$ in the above gives

$$10 = \frac{c_1}{8} + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{3c_1}{x^4} + 2c_2 x$$

substituting $y' = 15$ and $x = 2$ in the above gives

$$15 = -\frac{3c_1}{16} + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -16$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = \frac{3x^5 - 16}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{3x^5 - 16}{x^3} \tag{1}$$

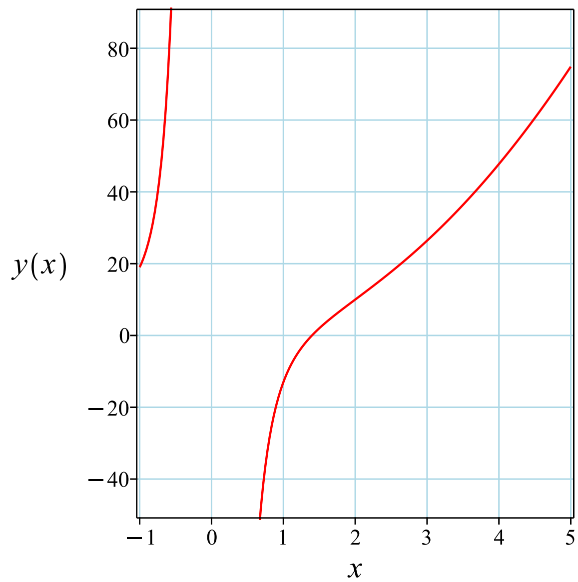


Figure 490: Solution plot

<u>Verification of solutions</u>

$$y = \frac{3x^5 - 16}{x^3}$$

Verified OK.

7.14.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 2y'x - 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{6}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)}dx \\
&= \int e^{-2\ln(x)}\,dx \\
&= \int \frac{1}{x^2}dx \\
&= -\frac{1}{x} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'\left(x\right)^2} \\
&= \frac{-\frac{6}{x^2}}{\frac{1}{x^4}} \\
&= -6x^2 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 6x^2 y(\tau) &= 0
\end{aligned}$$

But in terms of τ

$$-6x^2 = -\frac{6}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{6y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 6y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 6\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 6\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 6 = 0$$

Or

$$r^2 - r - 6 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -2 \\ r_2 &= 3 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau^2} + c_2\tau^3$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1x^5 - c_2}{x^3}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{c_1 x^5 - c_2}{x^3} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 2$ in the above gives

$$10 = 4c_1 - \frac{c_2}{8} \tag{1A}$$

Taking derivative of the solution gives

$$y' = 5c_1 x - \frac{3(c_1 x^5 - c_2)}{x^4}$$

substituting $y' = 15$ and $x = 2$ in the above gives

$$15 = 4c_1 + \frac{3c_2}{16} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 3 \\ c_2 &= 16 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{3x^5 - 16}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{3x^5 - 16}{x^3} \tag{1}$$

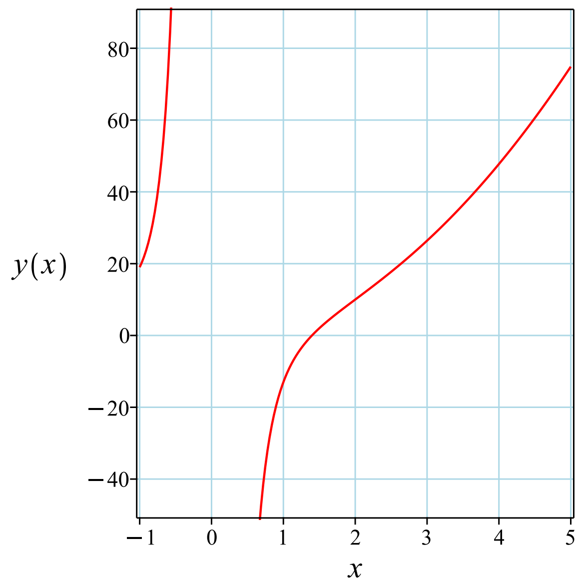


Figure 491: Solution plot

Verification of solutions

$$y = \frac{3x^5 - 16}{x^3}$$

Verified OK.

7.14.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + 2y'x - 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{6}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2} - \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{6v'(x)}{x} &= 0 \\ v''(x) + \frac{6v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{6u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{6u}{x} \end{aligned}$$

Where $f(x) = -\frac{6}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{6}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{6}{x}\, dx \\ \ln(u) &= -6\ln(x) + c_1 \\ u &= \mathrm{e}^{-6\ln(x)+c_1} \\ &= \frac{c_1}{x^6} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= -\frac{c_1}{5x^5} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\, x^n \\ &= \left(-\frac{c_1}{5x^5} + c_2\right) x^2 \\ &= \frac{5c_2x^5 - c_1}{5x^3} \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(-\frac{c_1}{5x^5} + c_2\right) x^2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 2$ in the above gives

$$10 = -\frac{c_1}{40} + 4c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1}{x^4} + 2\left(-\frac{c_1}{5x^5} + c_2\right) x$$

substituting $y' = 15$ and $x = 2$ in the above gives

$$15 = \frac{3c_1}{80} + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 80 \\ c_2 &= 3 \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{3x^5 - 16}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{3x^5 - 16}{x^3} \tag{1}$$

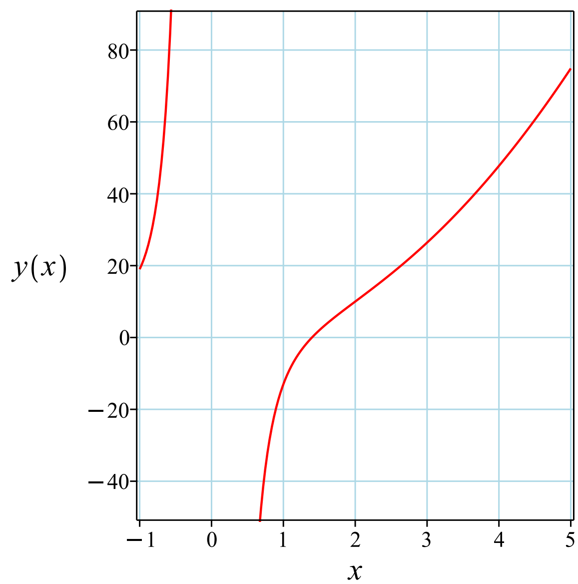


Figure 492: Solution plot

Verification of solutions

$$y = \frac{3x^5 - 16}{x^3}$$

Verified OK.

7.14.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 2y'x - 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{6}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 6 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{6}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 349: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{6}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{6}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 6$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = 3 \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = -2
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{6}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	3	-2

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	3	-2

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = -2$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= -2 - (-2) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= -\frac{2}{x} + (-)\,(0) \\
&= -\frac{2}{x} \\
&= -\frac{2}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(-\frac{2}{x}\right)(0) + \left(\left(\frac{2}{x^2}\right) + \left(-\frac{2}{x}\right)^2 - \left(\frac{6}{x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int -\frac{2}{x} dx} \\
&= \frac{1}{x^2}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2x}{x^2}\,dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1\left(\frac{1}{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x^3}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{x^5}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\frac{1}{x^3}\right) + c_2\left(\frac{1}{x^3}\left(\frac{x^5}{5}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = \frac{c_1}{x^3} + \frac{c_2 x^2}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 10$ and $x = 2$ in the above gives

$$10 = \frac{c_1}{8} + \frac{4c_2}{5} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{3c_1}{x^4} + \frac{2c_2 x}{5}$$

substituting $y' = 15$ and $x = 2$ in the above gives

$$15 = -\frac{3c_1}{16} + \frac{4c_2}{5} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -16$$
$$c_2 = 15$$

Substituting these values back in above solution results in

$$y = \frac{3x^5 - 16}{x^3}$$

Summary
The solution(s) found are the following

$$y = \frac{3x^5 - 16}{x^3} \tag{1}$$

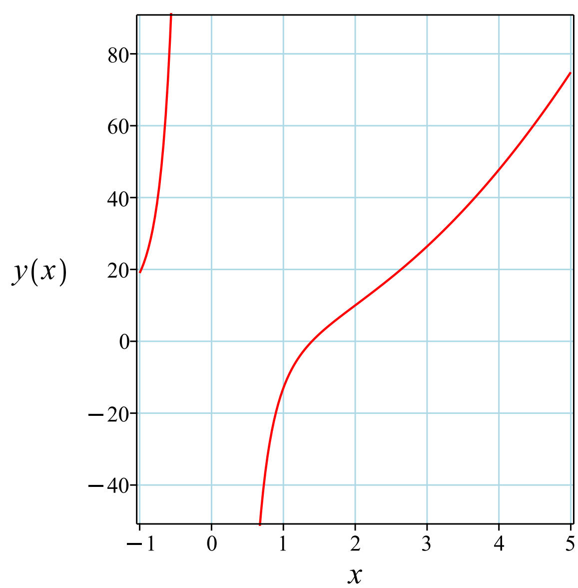


Figure 493: Solution plot

Verification of solutions

$$y = \frac{3x^5 - 16}{x^3}$$

Verified OK.

7.14.6 Maple step by step solution

Let's solve

$\left[x^2y'' + 2y'x - 6y = 0, y(2) = 10, y'\Big|_{\{x=2\}} = 15\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + \frac{6y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{6y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + 2y'x - 6y = 0$

- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  ◦ Calculate the 1st derivative of y with respect to x , using the chain rule

  $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$

  ◦ Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

  ◦ Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  ◦ Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

$$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2}-\frac{\frac{d}{dt}y(t)}{x^2}\right)+2\frac{d}{dt}y(t)-6y(t)=0$$

- Simplify
  $\frac{d^2}{dt^2}y(t)+\frac{d}{dt}y(t)-6y(t)=0$
- Characteristic polynomial of ODE
  $r^2+r-6=0$
- Factor the characteristic polynomial
  $(r+3)(r-2)=0$
- Roots of the characteristic polynomial
  $r=(-3,2)$
- 1st solution of the ODE
  $y_1(t)=e^{-3t}$
- 2nd solution of the ODE
  $y_2(t)=e^{2t}$
- General solution of the ODE
  $y(t)=c_1y_1(t)+c_2y_2(t)$
- Substitute in solutions
  $y(t)=c_1e^{-3t}+c_2e^{2t}$
- Change variables back using $t=\ln(x)$
  $y=\frac{c_1}{x^3}+c_2x^2$
- Simplify
  $y=\frac{c_1}{x^3}+c_2x^2$

☐ Check validity of solution $y=\frac{c_1}{x^3}+c_2x^2$

- Use initial condition $y(2)=10$
  $10=\frac{c_1}{8}+4c_2$
- Compute derivative of the solution
  $y'=-\frac{3c_1}{x^4}+2c_2x$
- Use the initial condition $y'\Big|_{\{x=2\}}=15$
  $15=-\frac{3c_1}{16}+4c_2$

- Solve for c_1 and c_2

  $\{c_1 = -16, c_2 = 3\}$

- Substitute constant values into general solution and simplify

  $y = -\frac{16}{x^3} + 3x^2$

- Solution to the IVP

  $y = -\frac{16}{x^3} + 3x^2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-6*y(x)=0,y(2) = 10, D(y)(2) = 15],y(x), singsol=
```

$$y(x) = -\frac{16}{x^3} + 3x^2$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 16

```
DSolve[{x^2*y''[x]+2*x*y'[x]-6*y[x]==0,{y[2]==10,y'[2]==15}},y[x],x,IncludeSingularSolutions
```

$$y(x) \to \frac{3x^5 - 16}{x^3}$$

7.15 problem 15

Internal problem ID [171]
Internal file name [`OUTPUT/171_Sunday_June_05_2022_01_36_20_AM_14166642/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' - y'x + y = 0}$$

With initial conditions

$$[y(1) = 7, y'(1) = 2]$$

7.15.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{1}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$$

The domain of $p(x) = -\frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.15.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - r + 1 = 0$$

Or

$$r^2 - 2r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 1$$
$$r_2 = 1$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x + \ln(x) c_2 x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x + \ln(x) c_2 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 1$ in the above gives

$$7 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + c_2 + c_2 \ln(x)$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 7$$
$$c_2 = -5$$

Substituting these values back in above solution results in

$$y = -5x \ln(x) + 7x$$

Which simplifies to

$$y = (-5 \ln(x) + 7) x$$

Summary
The solution(s) found are the following

$$y = (-5 \ln (x) + 7) x \tag{1}$$

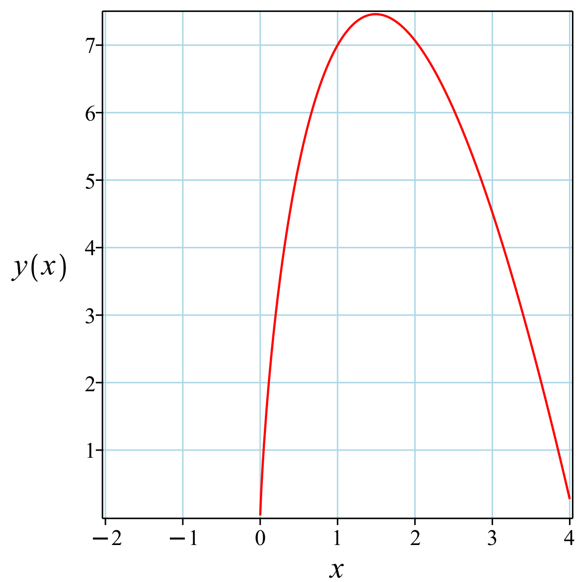


Figure 494: Solution plot

Verification of solutions

$$y = (-5 \ln (x) + 7) x$$

Verified OK.

7.15.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - y' x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{x^2} \\
&= \frac{1}{x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{x^4} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{1}{x^4} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2\ln(x) - c_2\ln(2))}{2}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \frac{x\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 1$ in the above gives

$$7 = \frac{\sqrt{2}\,(c_1 - c_2 \ln(2))}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\sqrt{2}\,(c_1 + 2c_2 \ln(x) - c_2 \ln(2))}{2} + \sqrt{2}\,c_2$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = \frac{(-c_2 \ln(2) + c_1 + 2c_2)\sqrt{2}}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -\frac{(5\ln(2) - 14)\sqrt{2}}{2}$$
$$c_2 = -\frac{5\sqrt{2}}{2}$$

Substituting these values back in above solution results in

$$y = -5x\ln(x) + 7x$$

Which simplifies to

$$y = (-5\ln(x) + 7)\,x$$

Summary

The solution(s) found are the following

$$y = (-5\ln(x) + 7)\,x \tag{1}$$

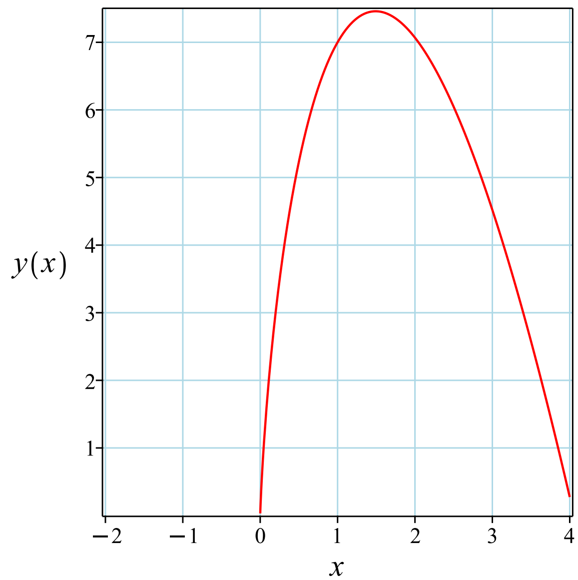


Figure 495: Solution plot

Verification of solutions

$$y = (-5 \ln (x) + 7) x$$

Verified OK.

7.15.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' - y' x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x) y' + q(x) y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -2c\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\ &= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 1$ in the above gives

$$7 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = c_1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1\}$. There is no solution for the constants of integrations. This solution is removed.

Verification of solutions N/A

7.15.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - y' x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{1}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\
\ln(u) &= -\ln(x) + c_1 \\
u &= \mathrm{e}^{-\ln(x)+c_1} \\
&= \frac{c_1}{x}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= c_1 \ln(x) + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= (c_1 \ln(x) + c_2)\,x \\
&= (c_1 \ln(x) + c_2)\,x
\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = (c_1 \ln(x) + c_2)\,x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 1$ in the above gives

$$7 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = c_1 + c_1 \ln(x) + c_2$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -5$$
$$c_2 = 7$$

Substituting these values back in above solution results in

$$y = -5x \ln(x) + 7x$$

Which simplifies to

$$y = (-5 \ln(x) + 7) x$$

Summary

The solution(s) found are the following

$$y = (-5 \ln(x) + 7) x \tag{1}$$

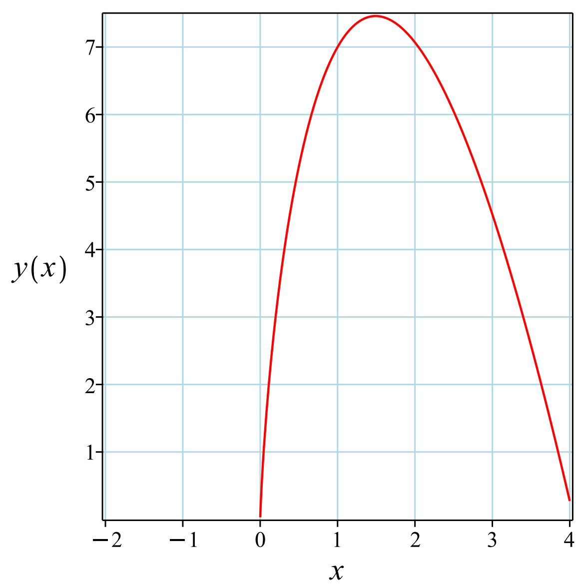


Figure 496: Solution plot

Verification of solutions

$$y = (-5 \ln(x) + 7) x$$

Verified OK.

7.15.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv = 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v = 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (-x)(-1) + (1)(-x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$-x^3v'' + (-x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$-x^2(u'(x)\, x + u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (-x)\,(c_1 \ln(x) + c_2) \\ &= -(c_1 \ln(x) + c_2)\, x \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = -(c_1 \ln(x) + c_2)\, x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 1$ in the above gives

$$7 = -c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 - c_1 \ln(x) - c_2$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$2 = -c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 5$$
$$c_2 = -7$$

Substituting these values back in above solution results in

$$y = -5x \ln(x) + 7x$$

Which simplifies to

$$y = (-5 \ln(x) + 7)\, x$$

Summary
The solution(s) found are the following

$$y = (-5 \ln(x) + 7)\, x \tag{1}$$

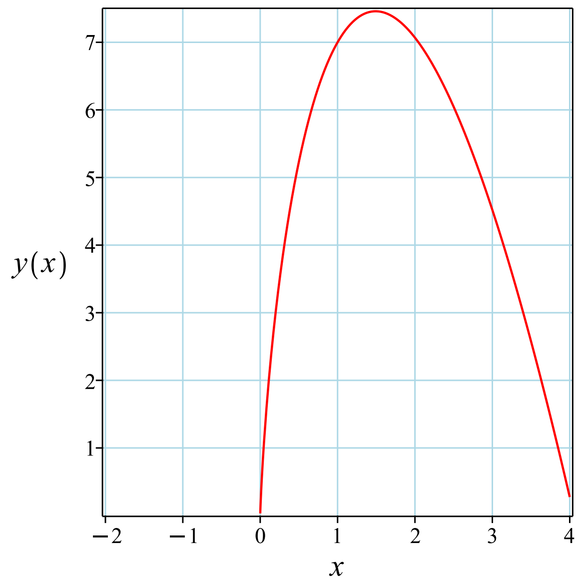


Figure 497: Solution plot

Verification of solutions

$$y = (-5\ln(x) + 7)\,x$$

Verified OK.

7.15.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - y'x + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} \tag{5}$$
$$= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 351: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0\\ 0 &= 0\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}z_1(x) &= p e^{\int \omega\, dx}\\ &= e^{\int \frac{1}{2x} dx}\\ &= \sqrt{x}\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}\\ &= z_1 e^{-\int \frac{1}{2}\frac{-x}{x^2}\, dx}\\ &= z_1 e^{\frac{\ln(x)}{2}}\\ &= z_1\left(\sqrt{x}\right)\end{aligned}$$

Which simplifies to

$$y_1 = x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(x) + c_2(x(\ln(x)))
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x + \ln(x)\, c_2 x \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 1$ in the above gives

$$
7 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 + c_2 + c_2 \ln(x)
$$

substituting $y' = 2$ and $x = 1$ in the above gives

$$
2 = c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 7 \\
c_2 &= -5
\end{aligned}
$$

Substituting these values back in above solution results in

$$
y = -5x \ln(x) + 7x
$$

Which simplifies to

$$y = (-5 \ln (x) + 7) x$$

Summary

The solution(s) found are the following

$$y = (-5 \ln (x) + 7) x \tag{1}$$

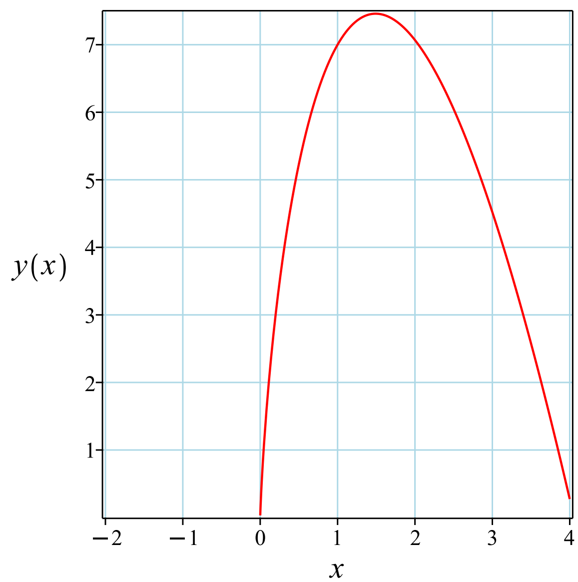


Figure 498: Solution plot

Verification of solutions

$$y = (-5 \ln (x) + 7) x$$

Verified OK.

7.15.8 Maple step by step solution

Let's solve

$$\left[x^2 y'' - y' x + y = 0, y(1) = 7, y'\Big|_{\{x=1\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

$y'' = \frac{y'}{x} - \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{x} + \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' - y' x + y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}\right) - \frac{d}{dt} y(t) + y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2} y(t) - 2 \frac{d}{dt} y(t) + y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 2r + 1 = 0$

- Factor the characteristic polynomial

  $(r - 1)^2 = 0$

- Root of the characteristic polynomial

  $r = 1$

- 1st solution of the ODE

  $y_1(t) = e^t$

- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,\mathrm{e}^t$
- General solution of the ODE

  $y(t) = c_1y_1(t) + c_2y_2(t)$
- Substitute in solutions

  $y(t) = c_1\mathrm{e}^t + c_2t\,\mathrm{e}^t$
- Change variables back using $t = \ln(x)$

  $y = c_1x + \ln(x)\,c_2x$
- Simplify

  $y = x(c_2\ln(x) + c_1)$

☐ Check validity of solution $y = x(c_2\ln(x) + c_1)$

  - Use initial condition $y(1) = 7$

    $7 = c_1$
  - Compute derivative of the solution

    $y' = c_1 + c_2 + c_2\ln(x)$
  - Use the initial condition $y'\Big|_{\{x=1\}} = 2$

    $2 = c_1 + c_2$
  - Solve for c_1 and c_2

    $\{c_1 = 7, c_2 = -5\}$
  - Substitute constant values into general solution and simplify

    $y = (-5\ln(x) + 7)\,x$

- Solution to the IVP

  $y = (-5\ln(x) + 7)\,x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 12

```
dsolve([x^2*diff(y(x),x$2)-x*diff(y(x),x)+y(x)=0,y(1) = 7, D(y)(1) = 2],y(x), singsol=all)
```

$$y(x) = x(7 - 5\ln(x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 13

```
DSolve[{x^2*y''[x]-x*y'[x]+y[x]==0,{y[1]==7,y'[1]==2}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to x(7 - 5\log(x))$$

7.16 problem 16

Internal problem ID [172]
Internal file name [`OUTPUT/172_Sunday_June_05_2022_01_36_21_AM_40336962/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x + y = 0}$$

With initial conditions

$$[y(1) = 2, y'(1) = 3]$$

7.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$$

The domain of $p(x) = \frac{1}{x}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is inside this domain. The domain of $q(x) = \frac{1}{x^2}$ is

$$\{x < 0 \lor 0 < x\}$$

And the point $x_0 = 1$ is also inside this domain. Hence solution exists and is unique.

7.16.2 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = r x^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 1 = 0$$

Or

$$r^2 + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$
\begin{aligned}
r_1 &= -i \\
r_2 &= i
\end{aligned}
$$

The roots are complex conjugate of each others. Let the roots be

$$
\begin{aligned}
r_1 &= \alpha + i\beta \\
r_2 &= \alpha - i\beta
\end{aligned}
$$

Where in this case $\alpha = 0$ and $\beta = -1$. Hence the solution becomes

$$
\begin{aligned}
y &= c_1 x^{r_1} + c_2 x^{r_2} \\
&= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\
&= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\
&= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\
&= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right)
\end{aligned}
$$

Using the values for $\alpha = 0, \beta = -1$, the above becomes

$$
y = x^0\left(c_1 e^{-i\ln(x)} + c_2 e^{i\ln(x)}\right)
$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$
y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$
2 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \sin(\ln(x))}{x} + \frac{c_2 \cos(\ln(x))}{x}$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x))$$

<u>Summary</u>
The solution(s) found are the following

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x)) \tag{1}$$

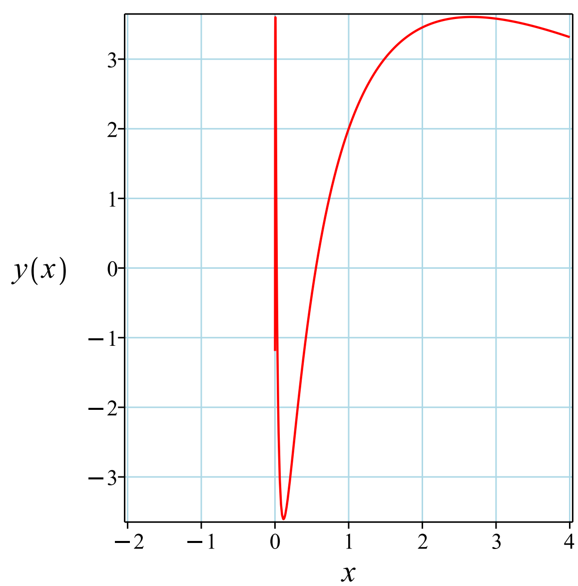


Figure 499: Solution plot

<u>Verification of solutions</u>

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x))$$

Verified OK.

7.16.3 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= 1
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} + \mathrm{e}^{\lambda\tau} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 + 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\
&= \pm i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +i \\
\lambda_2 &= -i
\end{aligned}
$$

Which simplifies to

$$\lambda_1 = i$$
$$\lambda_2 = -i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(\tau) + c_2 \sin(\tau))$$

Or

$$y(\tau) = c_1 \cos(\tau) + c_2 \sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \sin(\ln(x))}{x} + \frac{c_2 \cos(\ln(x))}{x}$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x))$$

Summary

The solution(s) found are the following

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x)) \tag{1}$$

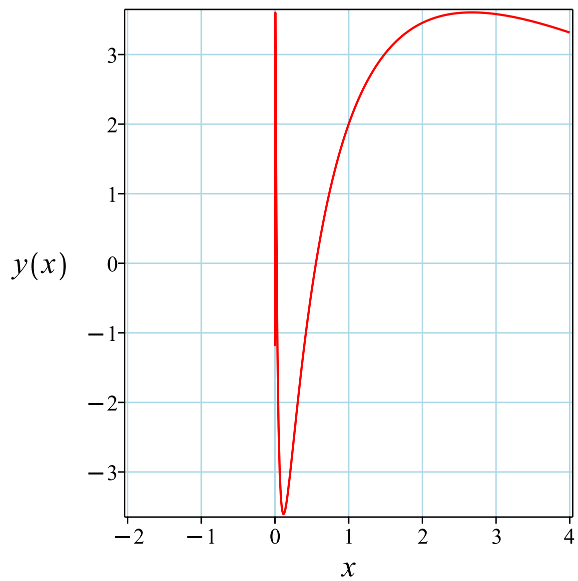


Figure 500: Solution plot

Verification of solutions

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x))$$

Verified OK.

7.16.4 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned} \tau &= \int \frac{1}{c} \sqrt{q}\, dx \\ &= \frac{\int \sqrt{\frac{1}{x^2}} dx}{c} \\ &= \frac{\sqrt{\frac{1}{x^2}}\, x \ln(x)}{c} \end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \sin(\ln(x))}{x} + \frac{c_2 \cos(\ln(x))}{x}$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 3$$

Substituting these values back in above solution results in

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x))$$

Summary
The solution(s) found are the following

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x)) \tag{1}$$

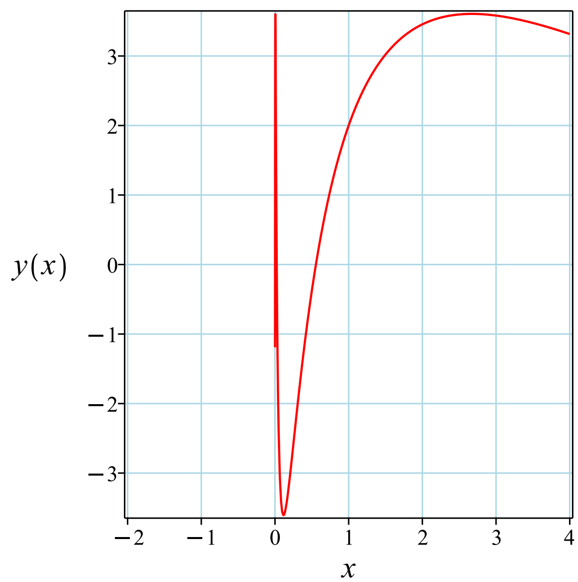


Figure 501: Solution plot

Verification of solutions

$$y = 2\cos(\ln(x)) + 3\sin(\ln(x))$$

Verified OK.

7.16.5 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+2i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(1+2i)\,u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{(-1-2i)\,u}{x}\end{aligned}$$

Where $f(x)=\frac{-1-2i}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{-1-2i}{x}\,dx\\ \int\frac{1}{u}\,du &= \int\frac{-1-2i}{x}\,dx\\ \ln(u) &= (-1-2i)\ln(x)+c_1\\ u &= \mathrm{e}^{(-1-2i)\ln(x)+c_1}\\ &= c_1\mathrm{e}^{(-1-2i)\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x)=\frac{c_1x^{-2i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x)\\ v(x) &= \int u(x)\,dx+c_2\\ &= \frac{ic_1x^{-2i}}{2}+c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n\\ &= \left(\frac{ic_1x^{-2i}}{2}+c_2\right)x^i\\ &= x^ic_2+\frac{ix^{-i}c_1}{2}\end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = \frac{i c_1}{2} + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{c_1 x^{-i}}{x} + \frac{i\left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i}{x}$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = \frac{c_1}{2} + i c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3 - 2i$$
$$c_2 = 1 - \frac{3i}{2}$$

Substituting these values back in above solution results in

$$y = \frac{3 i x^i x^{-2i}}{2} - \frac{3 i x^i}{2} + x^i x^{-2i} + x^i$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{3i}{2}\right) x^{-i} + \left(1 - \frac{3i}{2}\right) x^i \tag{1}$$

Verification of solutions

$$y = \left(1 + \frac{3i}{2}\right) x^{-i} + \left(1 - \frac{3i}{2}\right) x^i$$

Verified OK.

7.16.6 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 353: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{5}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{5}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - i$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - \left(\alpha_{c_1}^{-}\right) \\
&= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1}\right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - i}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - i}{x} \\
&= \frac{\frac{1}{2} - i}{x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right)p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{x^2}\right) + \left(\frac{\frac{1}{2} - i}{x}\right)^2 - \left(-\frac{5}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \frac{\frac{1}{2} - i}{x} dx} \\
&= x^{\frac{1}{2} - i}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-i}\right) + c_2 \left(x^{-i}\left(-\frac{i x^{2i}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 x^{-i} - \frac{i c_2 x^{i}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 1$ in the above gives

$$2 = c_1 - \frac{ic_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{ix^{-i}c_1}{x} + \frac{c_2 x^i}{2x}$$

substituting $y' = 3$ and $x = 1$ in the above gives

$$3 = -ic_1 + \frac{c_2}{2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1 + \frac{3i}{2}$$
$$c_2 = 3 + 2i$$

Substituting these values back in above solution results in

$$y = \frac{3ix^{-i}}{2} - \frac{3ix^i}{2} + x^{-i} + x^i$$

Summary

The solution(s) found are the following

$$y = \left(1 + \frac{3i}{2}\right) x^{-i} + \left(1 - \frac{3i}{2}\right) x^i \tag{1}$$

Verification of solutions

$$y = \left(1 + \frac{3i}{2}\right) x^{-i} + \left(1 - \frac{3i}{2}\right) x^i$$

Verified OK.

7.16.7 Maple step by step solution

Let's solve

$$\left[x^2 y'' + y'x + y = 0, y(1) = 2, y'\Big|_{\{x=1\}} = 3\right]$$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{y}{x^2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{y}{x^2} = 0$
- Multiply by denominators of the ODE

  $x^2 y'' + y'x + y = 0$
- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right) t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) + y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 + 1 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

$r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \cos(t)$

- 2nd solution of the ODE

  $y_2(t) = \sin(t)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \cos(t) + c_2 \sin(t)$

- Change variables back using $t = \ln(x)$

  $y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$

☐ Check validity of solution $y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$

  - Use initial condition $y(1) = 2$

    $2 = c_1$

  - Compute derivative of the solution

    $y' = -\frac{c_1 \sin(\ln(x))}{x} + \frac{c_2 \cos(\ln(x))}{x}$

  - Use the initial condition $y'\Big|_{\{x=1\}} = 3$

    $3 = c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 2, c_2 = 3\}$

  - Substitute constant values into general solution and simplify

    $y = 2\cos(\ln(x)) + 3\sin(\ln(x))$

- Solution to the IVP

  $y = 2\cos(\ln(x)) + 3\sin(\ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([x^2*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=0,y(1) = 2, D(y)(1) = 3],y(x), singsol=all)
```

$$y(x) = 3\sin(\ln(x)) + 2\cos(\ln(x))$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 16

```
DSolve[{x^2*y''[x]+x*y'[x]+y[x]==0,{y[1]==2,y'[1]==3}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to 3\sin(\log(x)) + 2\cos(\log(x))$$

7.17 problem 33

Internal problem ID [173]
Internal file name [`OUTPUT/173_Sunday_June_05_2022_01_36_22_AM_42065670/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 3y' + 2y = 0}$$

7.17.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -3, C = 2$ into the above gives

$$\lambda_{1,2} = \frac{3}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-3^2 - (4)(1)(2)}$$
$$= \frac{3}{2} \pm \frac{1}{2}$$

Hence

$$\lambda_1 = \frac{3}{2} + \frac{1}{2}$$
$$\lambda_2 = \frac{3}{2} - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = 2$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(2)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{x} \tag{1}$$

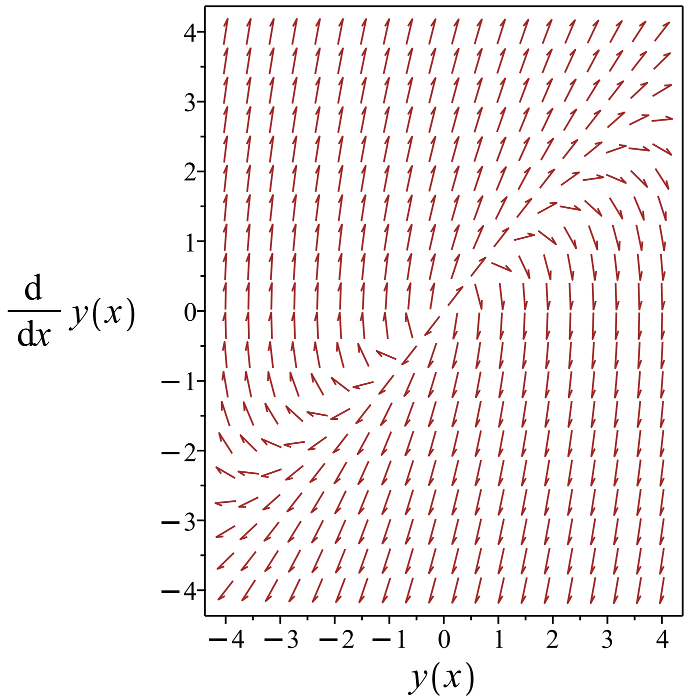


Figure 502: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{x}$$

Verified OK.

7.17.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 355: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{1}\,dx} \\ &= z_1 e^{\frac{3x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^x$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^x) + c_2(\mathrm{e}^x(\mathrm{e}^x))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x} \tag{1}
$$

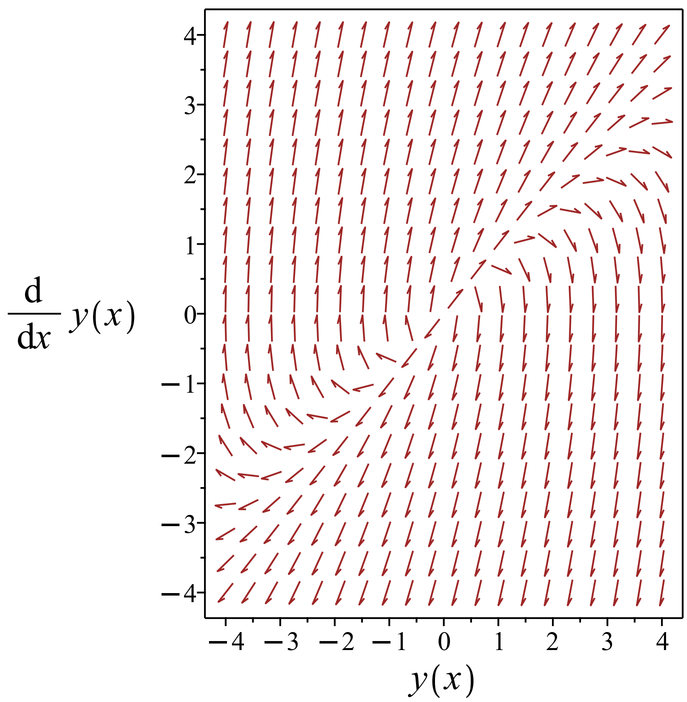


Figure 503: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^x + c_2 \mathrm{e}^{2x}
$$

Verified OK.

7.17.3 Maple step by step solution

Let's solve

$y'' - 3y' + 2y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r - 1)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (1, 2)$

- 1st solution of the ODE

  $y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(diff(y(x),x$2)-3*diff(y(x),x)+2*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^x c_1 + c_2 e^{2x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[y''[x]-3*y'[x]+2*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^x(c_2 e^x + c_1)$$

7.18 problem 34

Internal problem ID [174]
Internal file name [`OUTPUT/174_Sunday_June_05_2022_01_36_22_AM_51221852/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y' - 15y = 0}$$

7.18.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -15$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda e^{\lambda x} - 15 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 15 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -15$ into the above gives

$$\lambda_{1,2} = \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(-15)}$$
$$= -1 \pm 4$$

Hence

$$\lambda_1 = -1 + 4$$
$$\lambda_2 = -1 - 4$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = -5$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(-5)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-5x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-5x} \tag{1}$$

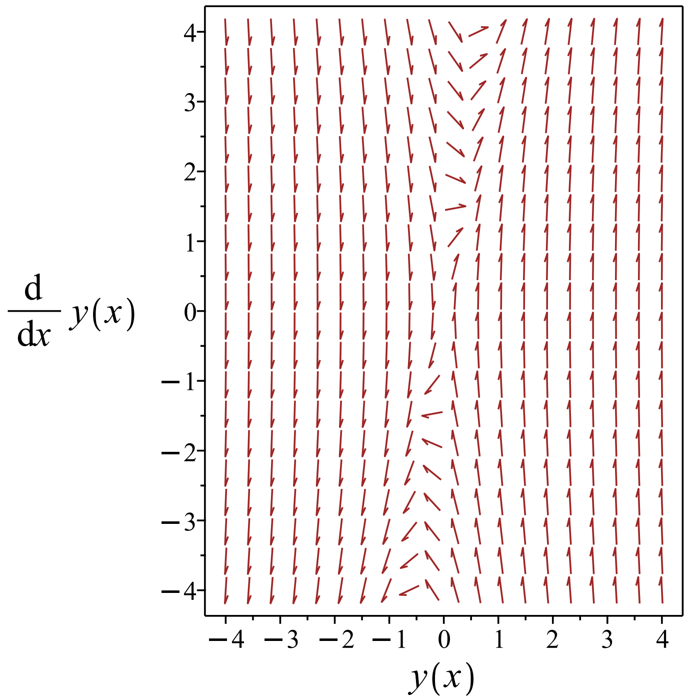


Figure 504: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{3x} + c_2 e^{-5x}$$

Verified OK.

7.18.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 15y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= -15 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 357: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-4x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{8x}}{8}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-5x}\right) + c_2\left(\mathrm{e}^{-5x}\left(\frac{\mathrm{e}^{8x}}{8}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-5x} + \frac{c_2 \mathrm{e}^{3x}}{8} \tag{1}
$$

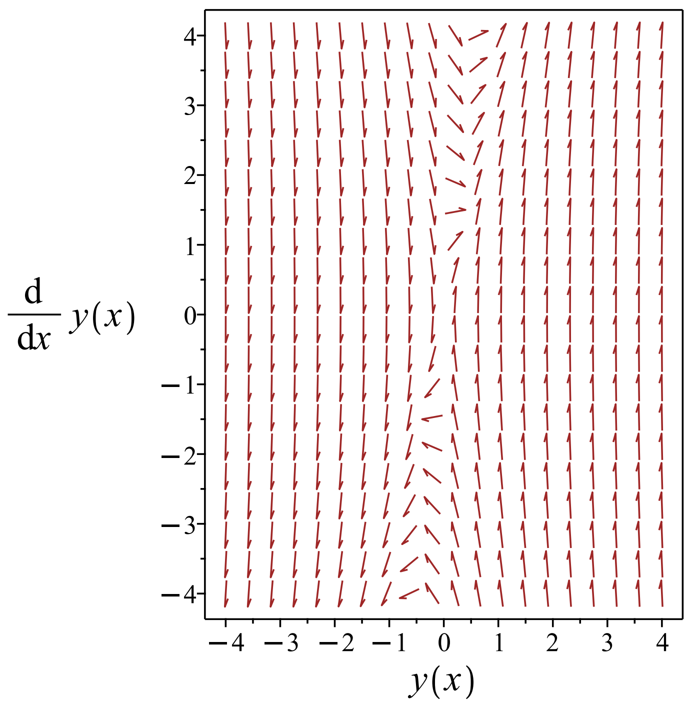


Figure 505: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-5x} + \frac{c_2 \mathrm{e}^{3x}}{8}
$$

Verified OK.

7.18.3 Maple step by step solution

Let's solve

$y'' + 2y' - 15y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 2r - 15 = 0$

- Factor the characteristic polynomial

  $(r+5)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-5, 3)$

- 1st solution of the ODE

  $y_1(x) = e^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-5x} + c_2 e^{3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-15*y(x)=0,y(x), singsol=all)
```

$$y(x) = (c_2 e^{8x} + c_1) e^{-5x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[y''[x]+2*y'[x]-15*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-5x} (c_2 e^{8x} + c_1)$$

7.19 problem 35

Internal problem ID [175]
Internal file name [`OUTPUT/175_Sunday_June_05_2022_01_36_23_AM_39130710/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' = 0}$$

7.19.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 5\lambda\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-5}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{5^2 - (4)\,(1)\,(0)}\\ &= -\frac{5}{2} \pm \frac{5}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{5}{2} + \frac{5}{2}\\ \lambda_2 &= -\frac{5}{2} - \frac{5}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(0)x} + c_2 e^{(-5)x}\end{aligned}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-5x}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-5x} \tag{1}$$

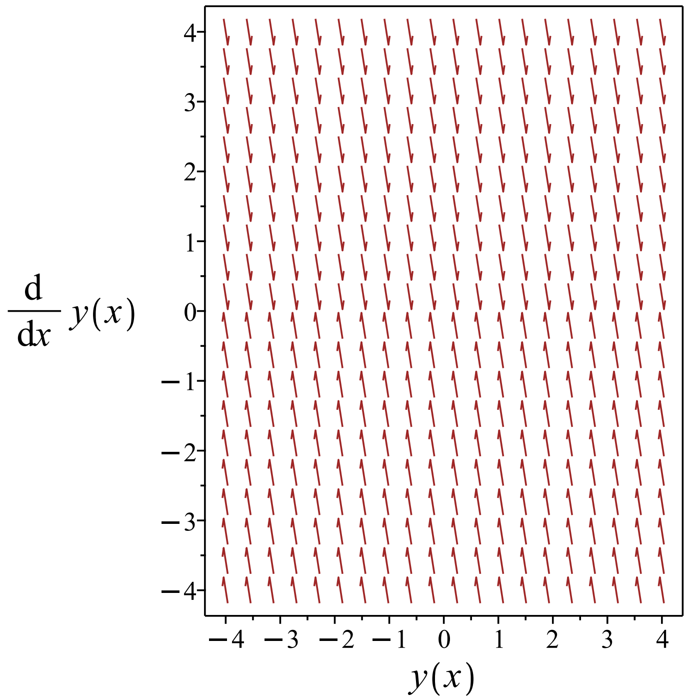


Figure 506: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 + c_2 e^{-5x}$$

Verified OK.

7.19.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 5y')\, dx = 0$$
$$5y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-5y + c_1} dy = \int dx$$
$$-\frac{\ln(-5y + c_1)}{5} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = e^{x + c_2}$$

Which simplifies to

$$\frac{1}{(-5y+c_1)^{\frac{1}{5}}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

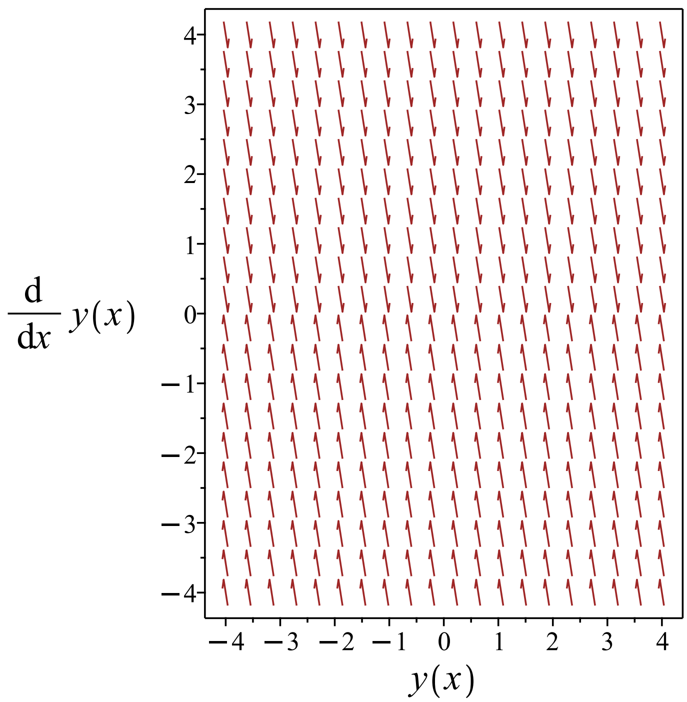


Figure 507: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

7.19.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$p'(x) + 5p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{1}{5p}dp = \int dx$$
$$-\frac{\ln(p)}{5} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{p^{\frac{1}{5}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{p^{\frac{1}{5}}} = c_2 e^{x}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{e^{-5x}}{c_2^5}$$

Integrating both sides gives

$$y = \int \frac{e^{-5x}}{c_2^5}\, dx$$
$$= -\frac{e^{-5x}}{5c_2^5} + c_3$$

Summary
The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_2^5} + c_3 \tag{1}$$

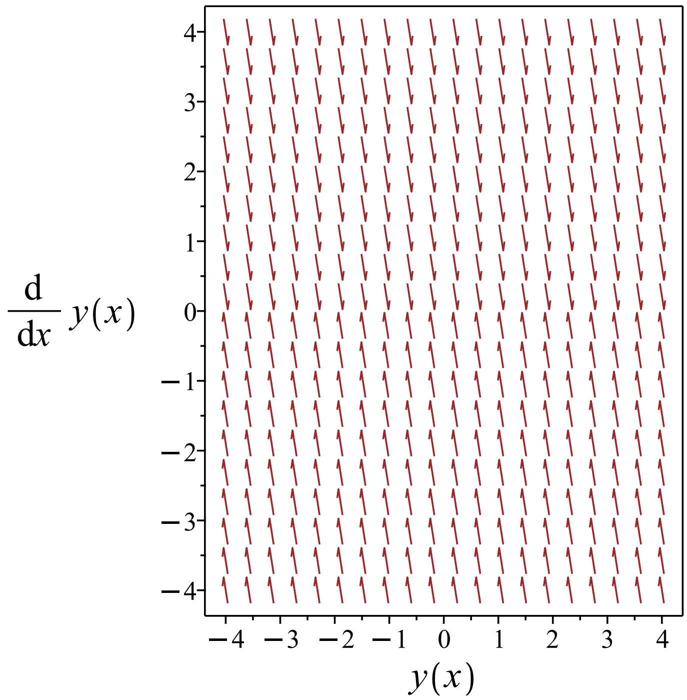


Figure 508: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_2^5} + c_3$$

Verified OK.

7.19.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$y'' + 5y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (y'' + 5y')\, dx = 0$$
$$5y + y' = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-5y + c_1} dy = \int dx$$
$$-\frac{\ln(-5y + c_1)}{5} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = e^{x+c_2}$$

Which simplifies to

$$\frac{1}{(-5y + c_1)^{\frac{1}{5}}} = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

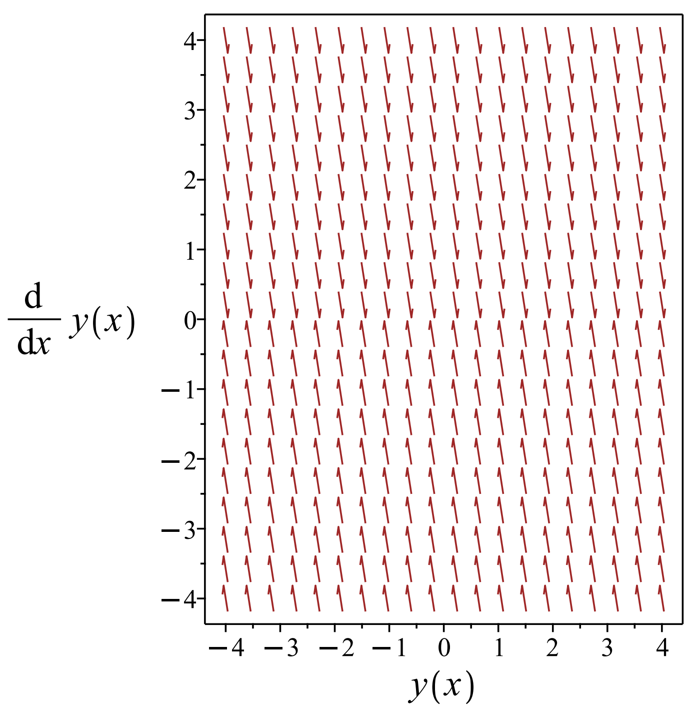


Figure 509: Slope field plot

Verification of solutions

$$y = -\frac{e^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

7.19.5 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 359: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\ &= z_1 e^{-\frac{5x}{2}} \\ &= z_1\left(\mathrm{e}^{-\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-5x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{\mathrm{e}^{5x}}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-5x}\right) + c_2\left(\mathrm{e}^{-5x}\left(\frac{\mathrm{e}^{5x}}{5}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-5x} + \frac{c_2}{5} \tag{1}$$

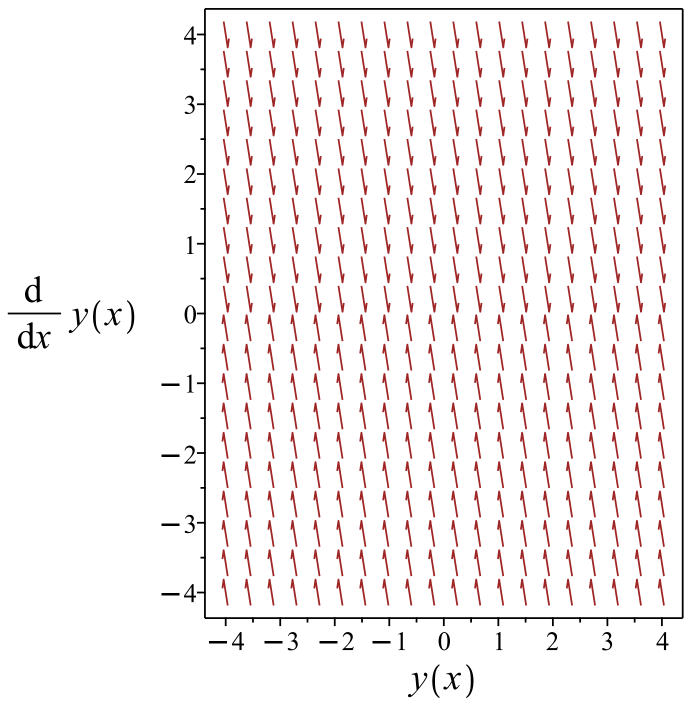


Figure 510: Slope field plot

Verification of solutions

$$y = c_1 e^{-5x} + \frac{c_2}{5}$$

Verified OK.

7.19.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 1 \\
q(x) &= 5 \\
r(x) &= 0 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\,dx
$$

Substituting the above values for p, q, r, s gives

$$
5y+y'=c_1
$$

We now have a first order ode to solve which is

$$
5y+y'=c_1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-5y+c_1}dy &= \int dx \\
-\frac{\ln(-5y+c_1)}{5} &= x+c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{(-5y+c_1)^{\frac{1}{5}}}=\mathrm{e}^{x+c_2}
$$

Which simplifies to

$$\frac{1}{\left(-5y+c_1\right)^{\frac{1}{5}}} = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{\mathrm{e}^{-5x}}{5c_3^5} + \frac{c_1}{5} \tag{1}$$

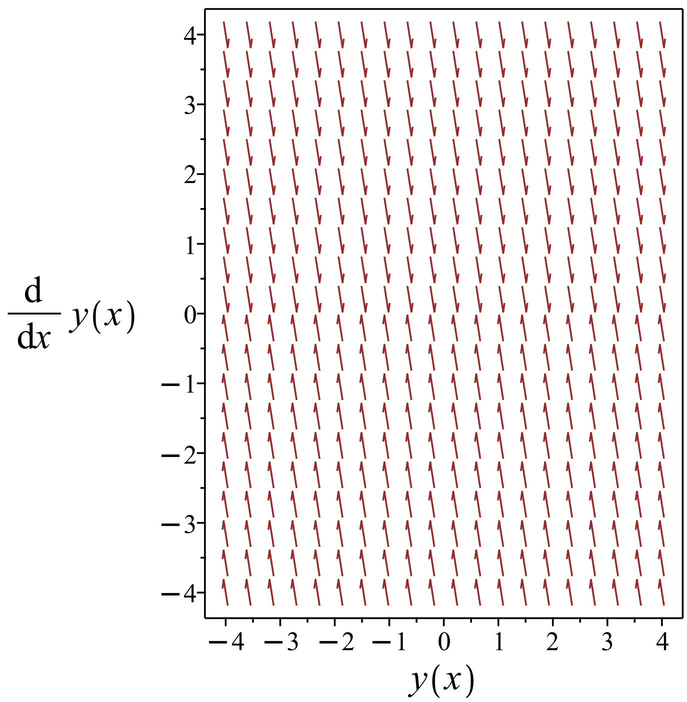


Figure 511: Slope field plot

Verification of solutions

$$y = -\frac{\mathrm{e}^{-5x}}{5c_3^5} + \frac{c_1}{5}$$

Verified OK.

7.19.7 Maple step by step solution

Let's solve

$y'' + 5y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 5r = 0$

- Factor the characteristic polynomial

  $r(r + 5) = 0$

- Roots of the characteristic polynomial

  $r = (-5, 0)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = 1$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-5x} + c_2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(diff(y(x),x$2)+5*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{-5x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 19

```
DSolve[y''[x]+5*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{1}{5} c_1 e^{-5x}$$

7.20 problem 36

Internal problem ID [176]
Internal file name [`OUTPUT/176_Sunday_June_05_2022_01_36_23_AM_33921551/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 36.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' + 3y' = 0}$$

7.20.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 3, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 3, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{3^2 - (4)\,(2)\,(0)}\\ &= -\frac{3}{4} \pm \frac{3}{4}\end{aligned}$$

Hence

$$\lambda_1 = -\frac{3}{4} + \frac{3}{4}$$
$$\lambda_2 = -\frac{3}{4} - \frac{3}{4}$$

Which simplifies to

$$\lambda_1 = 0$$
$$\lambda_2 = -\frac{3}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(0)x} + c_2 e^{\left(-\frac{3}{2}\right)x}$$

Or

$$y = c_1 + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 + c_2 \mathrm{e}^{-\frac{3x}{2}} \tag{1}$$

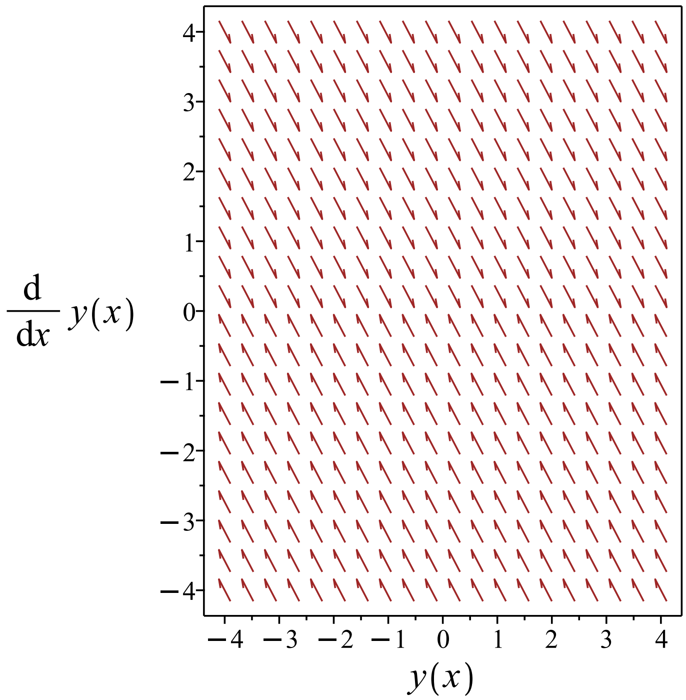


Figure 512: Slope field plot

Verification of solutions

$$y = c_1 + c_2 e^{-\frac{3x}{2}}$$

Verified OK.

7.20.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (2y'' + 3y')\, dx = 0$$

$$2y' + 3y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{3y}{2} + \frac{c_1}{2}} dy = \int dx$$

$$-\frac{2\ln(-3y + c_1)}{3} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{(-3y + c_1)^{\frac{2}{3}}} = e^{x + c_2}$$

Which simplifies to

$$\frac{1}{\left(-3y+c_1\right)^{\frac{2}{3}}} = c_3 \mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{3\left(c_3 \mathrm{e}^{x}\right)^{\frac{3}{2}}}+\frac{c_1}{3} \tag{1}$$

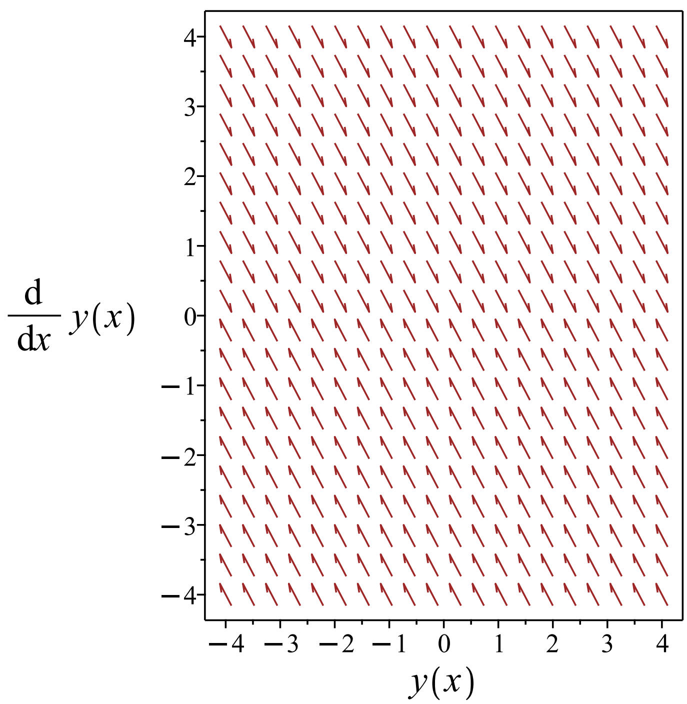


Figure 513: Slope field plot

Verification of solutions

$$y = -\frac{1}{3\left(c_3 \mathrm{e}^{x}\right)^{\frac{3}{2}}}+\frac{c_1}{3}$$

Verified OK.

7.20.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$2p'(x) + 3p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\int -\frac{2}{3p} dp = \int dx$$
$$-\frac{2\ln(p)}{3} = x + c_1$$

Raising both side to exponential gives

$$\frac{1}{p^{\frac{2}{3}}} = e^{x+c_1}$$

Which simplifies to

$$\frac{1}{p^{\frac{2}{3}}} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{1}{(c_2 e^x)^{\frac{3}{2}}}$$

Integrating both sides gives

$$y = \int \frac{1}{(c_2 e^x)^{\frac{3}{2}}} dx$$
$$= -\frac{2}{3 (c_2 e^x)^{\frac{3}{2}}} + c_3$$

Summary

The solution(s) found are the following

$$y = -\frac{2}{3 (c_2 e^x)^{\frac{3}{2}}} + c_3 \tag{1}$$

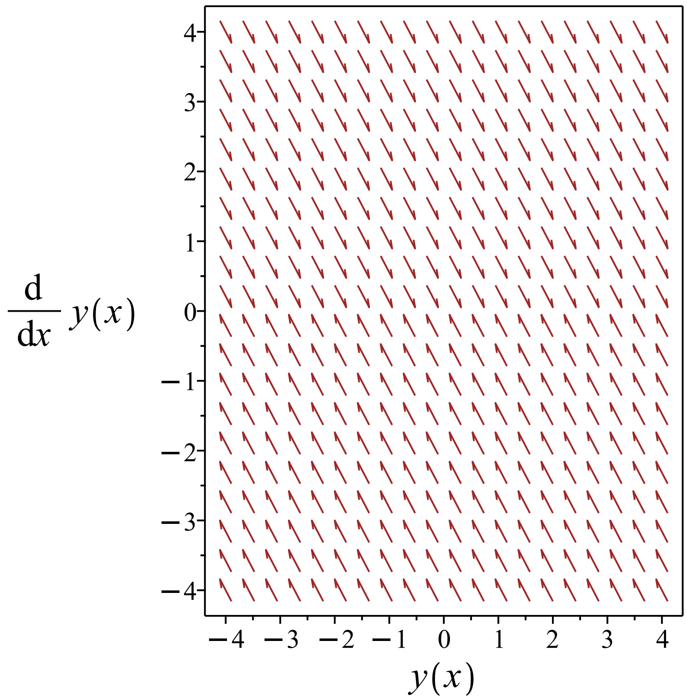


Figure 514: Slope field plot

Verification of solutions

$$y = -\frac{2}{3\left(c_2 e^{x}\right)^{\frac{3}{2}}} + c_3$$

Verified OK.

7.20.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$2y'' + 3y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (2y'' + 3y')\, dx = 0$$
$$2y' + 3y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{-\frac{3y}{2}+\frac{c_1}{2}}dy = \int dx$$
$$-\frac{2\ln\left(-3y+c_1\right)}{3} = x + c_2$$

Raising both side to exponential gives

$$\frac{1}{\left(-3y+c_1\right)^{\frac{2}{3}}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\frac{1}{\left(-3y+c_1\right)^{\frac{2}{3}}} = c_3\mathrm{e}^{x}$$

<u>Summary</u>

The solution(s) found are the following

$$y = -\frac{1}{3\left(c_3\mathrm{e}^{x}\right)^{\frac{3}{2}}} + \frac{c_1}{3} \tag{1}$$

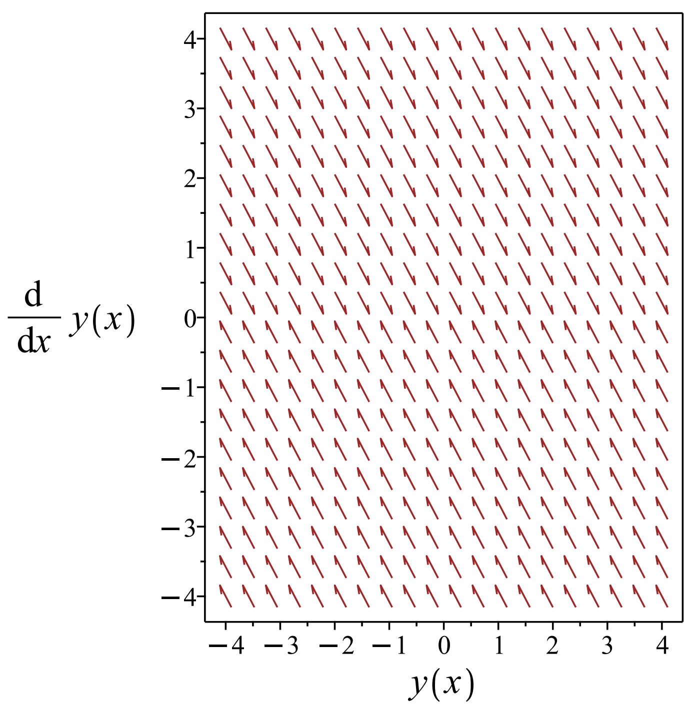


Figure 515: Slope field plot

Verification of solutions

$$y = -\frac{1}{3 \left(c_3 e^x\right)^{\frac{3}{2}}} + \frac{c_1}{3}$$

Verified OK.

7.20.5 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 361: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{2}\,dx} \\
&= z_1 e^{-\frac{3x}{4}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{4}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{3x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{\mathrm{e}^{-\frac{3x}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) + c_2 \left(\mathrm{e}^{-\frac{3x}{2}} \left(\frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3}\right)\right)
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-\frac{3x}{2}} + \frac{2c_2}{3} \tag{1}$$

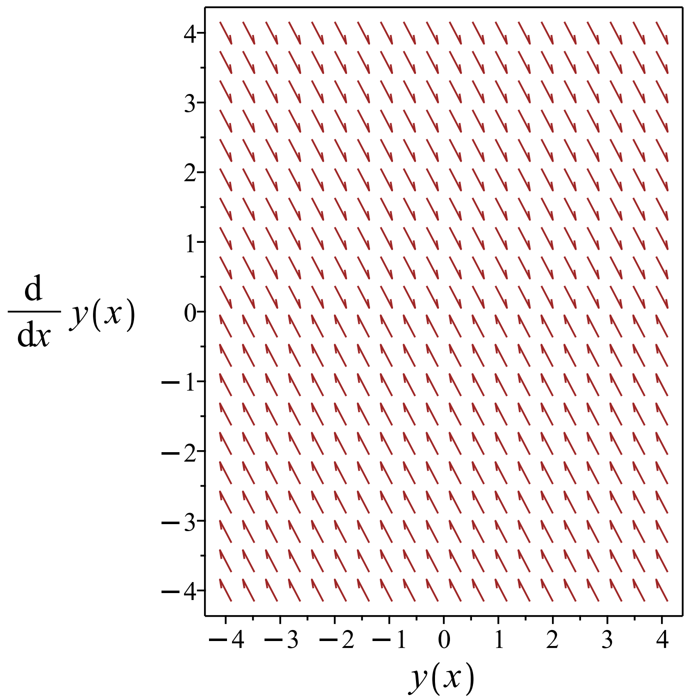


Figure 516: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{3x}{2}} + \frac{2c_2}{3}$$

Verified OK.

7.20.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x) y'' + q(x) y' + r(x) y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 2\\
q(x) &= 3\\
r(x) &= 0\\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0\\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
\left(p(x)\, y' + (q(x) - p'(x))\, y\right)' = s(x)
$$

Integrating gives

$$
p(x)\, y' + (q(x) - p'(x))\, y = \int s(x)\, dx
$$

Substituting the above values for p, q, r, s gives

$$
2y' + 3y = c_1
$$

We now have a first order ode to solve which is

$$
2y' + 3y = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{-\frac{3y}{2}+\frac{c_1}{2}} dy &= \int dx\\
-\frac{2\ln\left(-3y+c_1\right)}{3} &= x + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\frac{1}{\left(-3y+c_1\right)^{\frac{2}{3}}} = \mathrm{e}^{x+c_2}
$$

Which simplifies to

$$\frac{1}{(-3y+c_1)^{\frac{2}{3}}} = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = -\frac{1}{3(c_3 e^x)^{\frac{3}{2}}} + \frac{c_1}{3} \tag{1}$$

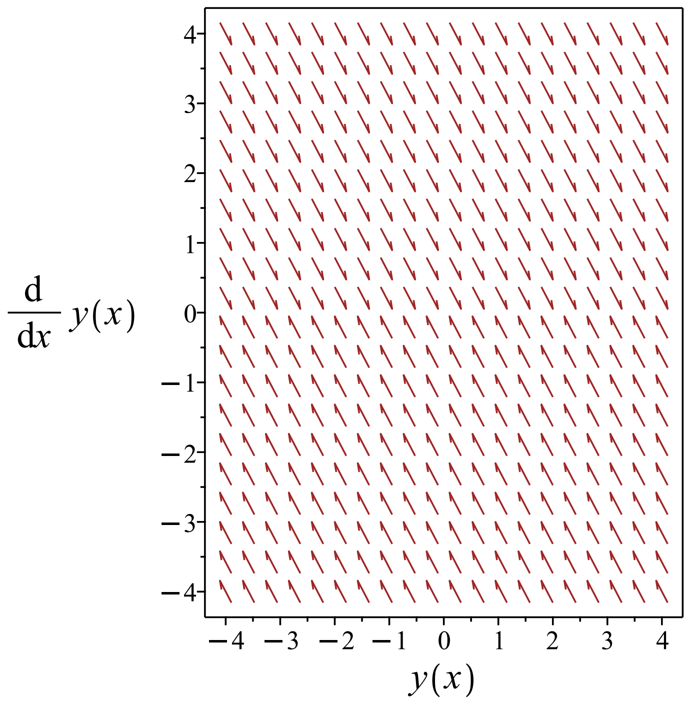


Figure 517: Slope field plot

Verification of solutions

$$y = -\frac{1}{3(c_3 e^x)^{\frac{3}{2}}} + \frac{c_1}{3}$$

Verified OK.

7.20.7 Maple step by step solution

Let's solve

$2y'' + 3y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{3y'}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{3y'}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 + \frac{3}{2}r = 0$

- Factor the characteristic polynomial

  $\frac{r(2r+3)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(0, -\frac{3}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = 1$

- 2nd solution of the ODE

  $y_2(x) = e^{-\frac{3x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 + c_2 e^{-\frac{3x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(2*diff(y(x),x$2)+3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 21

```
DSolve[2*y''[x]+3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_2 - \frac{2}{3} c_1 e^{-3x/2}$$

7.21 problem 37

Internal problem ID [177]
Internal file name [`OUTPUT/177_Sunday_June_05_2022_01_36_24_AM_23455174/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 37.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' - y' - y = 0}$$

7.21.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = -1, C = -1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} - \lambda\, e^{\lambda x} - e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - \lambda - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -1, C = -1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(2)} \pm \frac{1}{(2)\,(2)} \sqrt{-1^2 - (4)\,(2)\,(-1)} \\
&= \frac{1}{4} \pm \frac{3}{4}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= \frac{1}{4} + \frac{3}{4} \\
\lambda_2 &= \frac{1}{4} - \frac{3}{4}
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 1 \\
\lambda_2 &= -\frac{1}{2}
\end{aligned}
$$

Since roots are real and distinct, then the solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(1)x} + c_2 e^{\left(-\frac{1}{2}\right)x}
\end{aligned}
$$

Or

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-\frac{x}{2}}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}
$$

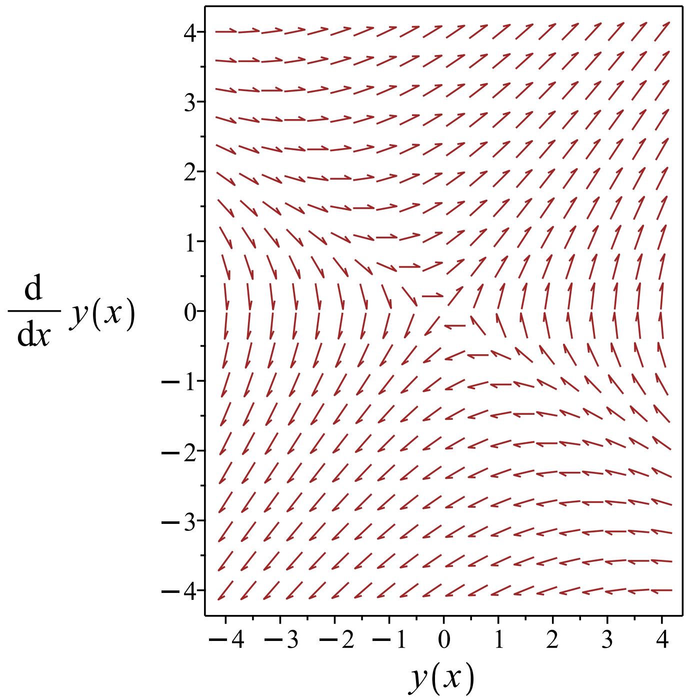


Figure 518: Slope field plot

Verification of solutions

$$y = c_1 e^{x} + c_2 e^{-\frac{x}{2}}$$

Verified OK.

7.21.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - y' - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -1 \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 363: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{9}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{3x}{4}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-1}{2}\,dx} \\
&= z_1 e^{\frac{x}{4}} \\
&= z_1 \left(\mathrm{e}^{\frac{x}{4}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{x}{2}}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$\begin{aligned} y &= c_1y_1 + c_2y_2 \\ &= c_1\left(e^{-\frac{x}{2}}\right) + c_2\left(e^{-\frac{x}{2}}\left(\frac{2\,e^{\frac{3x}{2}}}{3}\right)\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1e^{-\frac{x}{2}} + \frac{2c_2e^{x}}{3} \tag{1}$$

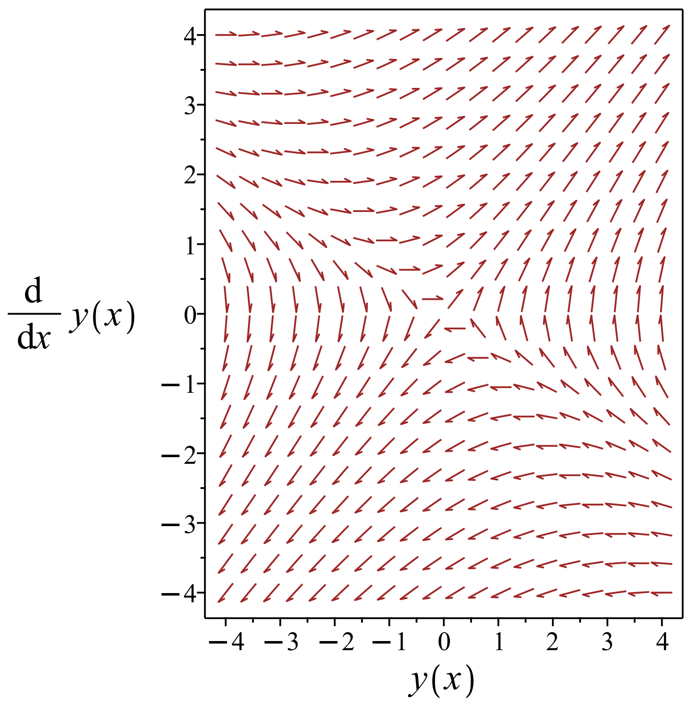


Figure 519: Slope field plot

<u>Verification of solutions</u>

$$y = c_1e^{-\frac{x}{2}} + \frac{2c_2e^{x}}{3}$$

Verified OK.

7.21.3 Maple step by step solution

Let's solve

$2y'' - y' - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{2} + \frac{y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{2} - \frac{y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{1}{2}r - \frac{1}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)(r-1)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(1, -\frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{x} + c_2 \mathrm{e}^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(2*diff(y(x),x$2)-diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{x} c_1 + c_2 \mathrm{e}^{-\frac{x}{2}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[2*y''[x]-y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-x/2} + c_2 e^{x}$$

7.22 problem 38

Internal problem ID [178]
Internal file name [`OUTPUT/178_Sunday_June_05_2022_01_36_24_AM_82526729/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 38.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 8y' + 3y = 0}$$

7.22.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 8, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 8\lambda e^{\lambda x} + 3 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 8\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 8, C = 3$ into the above gives

$$\lambda_{1,2} = \frac{-8}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{8^2 - (4)(4)(3)}$$
$$= -1 \pm \frac{1}{2}$$

Hence

$$\lambda_1 = -1 + \frac{1}{2}$$
$$\lambda_2 = -1 - \frac{1}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2}$$
$$\lambda_2 = -\frac{3}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(-\frac{1}{2}\right)x} + c_2 e^{\left(-\frac{3}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

Summary
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}} \tag{1}$$

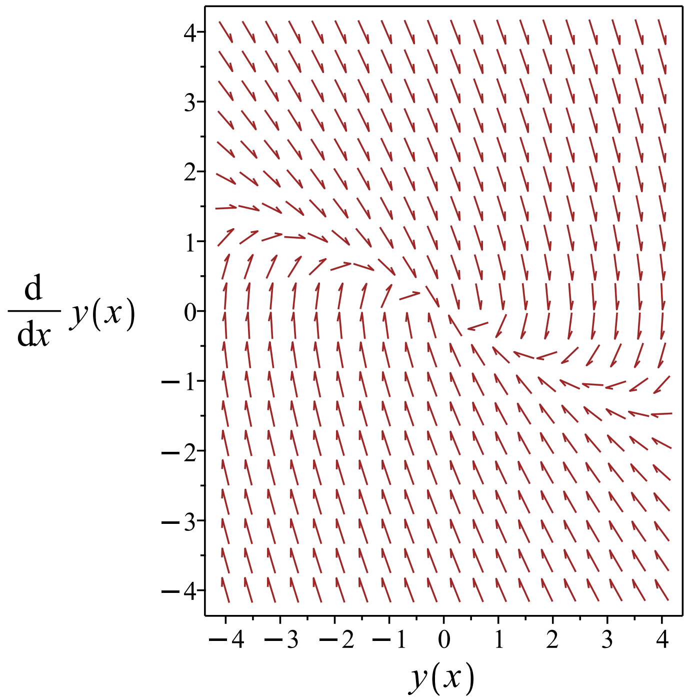


Figure 520: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + c_2 e^{-\frac{3x}{2}}$$

Verified OK.

7.22.2 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 8y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 8 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 365: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8}{4}\,dx} \\ &= z_1 e^{-x} \\ &= z_1\left(\mathrm{e}^{-x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{3x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8}{4}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-\frac{3x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{3x}{2}}\left(\mathrm{e}^{x}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}$$

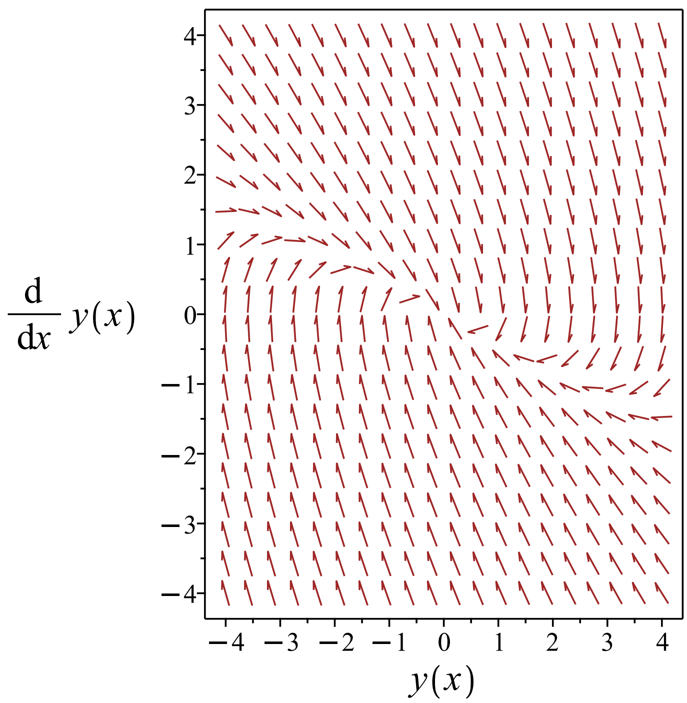


Figure 521: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Verified OK.

7.22.3 Maple step by step solution

Let's solve

$4y'' + 8y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -2y' - \frac{3y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + \frac{3y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + 2r + \frac{3}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+3)(2r+1)}{4} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2}, -\frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{3x}{2}}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{3x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(4*diff(y(x),x$2)+8*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[4*y''[x]+8*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x/2}(c_2 e^x + c_1)$$

7.23 problem 39

Internal problem ID [179]
Internal file name [`OUTPUT/179_Sunday_June_05_2022_01_36_25_AM_83565190/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 39.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' + 4y' + y = 0}$$

7.23.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 4, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 4\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 4, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{(4)^2 - (4)(4)(1)} \\ &= -\frac{1}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = \frac{1}{2}$. Therefore the solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 x\, e^{-\frac{x}{2}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-\frac{x}{2}} + c_2 x\, e^{-\frac{x}{2}} \tag{1}$$

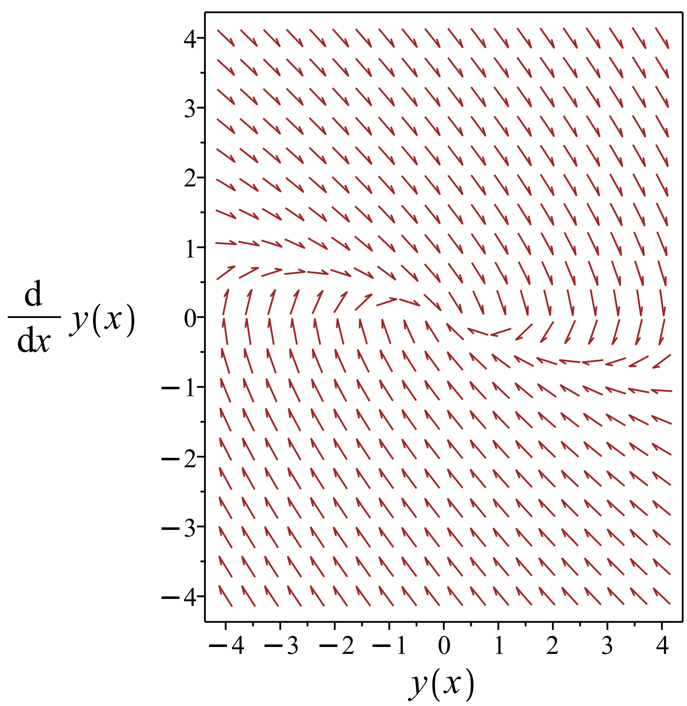


Figure 522: Slope field plot

Verification of solutions

$$y = c_1 e^{-\frac{x}{2}} + c_2 x\, e^{-\frac{x}{2}}$$

Verified OK.

7.23.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 1$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\, dx} \\ &= e^{\int 1\, dx} \\ &= \mathrm{e}^{\frac{x}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{\frac{x}{2}} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{\frac{x}{2}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{\frac{x}{2}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{\frac{x}{2}}}$$

Or

$$y = c_1 x\, \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}} \tag{1}$$

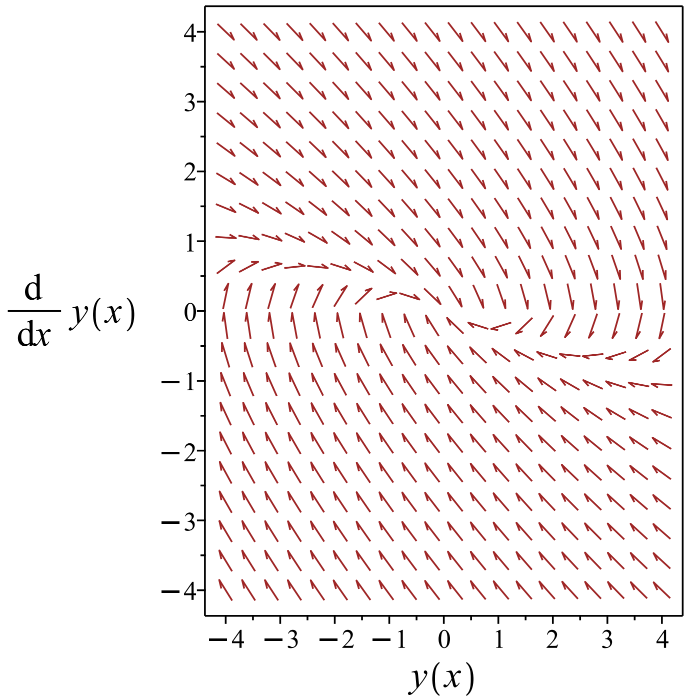


Figure 523: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-\frac{x}{2}} + c_2 \mathrm{e}^{-\frac{x}{2}}$$

Verified OK.

7.23.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 4y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 4 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 367: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{4}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 x \,\mathrm{e}^{-\frac{x}{2}} \tag{1}
$$

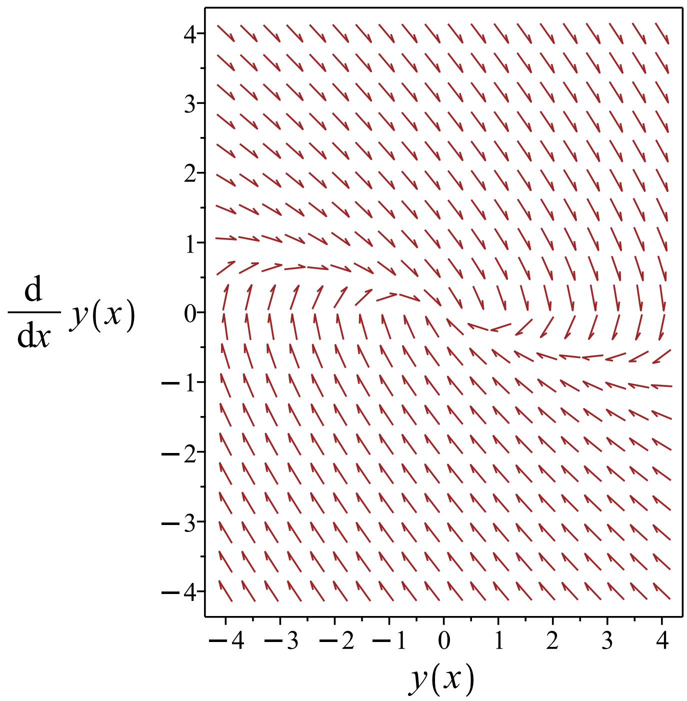


Figure 524: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 x \,\mathrm{e}^{-\frac{x}{2}}
$$

Verified OK.

7.23.4 Maple step by step solution

Let's solve

$4y'' + 4y' + y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 + r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = -\frac{1}{2}$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{x}{2}} + c_2 x\,e^{-\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(4*diff(y(x),x$2)+4*diff(y(x),x)+y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-\frac{x}{2}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 20

```
DSolve[4*y''[x]+4*y'[x]+y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-x/2}(c_2x + c_1)$$

7.24 problem 40

Internal problem ID [180]
Internal file name [`OUTPUT/180_Sunday_June_05_2022_01_36_25_AM_6498307/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 40.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{9y'' - 12y' + 4y = 0}$$

7.24.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 9, B = -12, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$9\lambda^2 \mathrm{e}^{\lambda x} - 12\lambda\,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$9\lambda^2 - 12\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = -12, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{12}{(2)(9)} \pm \frac{1}{(2)(9)}\sqrt{(-12)^2 - (4)(9)(4)} \\ &= \frac{2}{3}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{2}{3}$. Therefore the solution is

$$y = c_1 e^{\frac{2x}{3}} + c_2 x\, e^{\frac{2x}{3}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{2x}{3}} + c_2 x\, e^{\frac{2x}{3}} \tag{1}$$

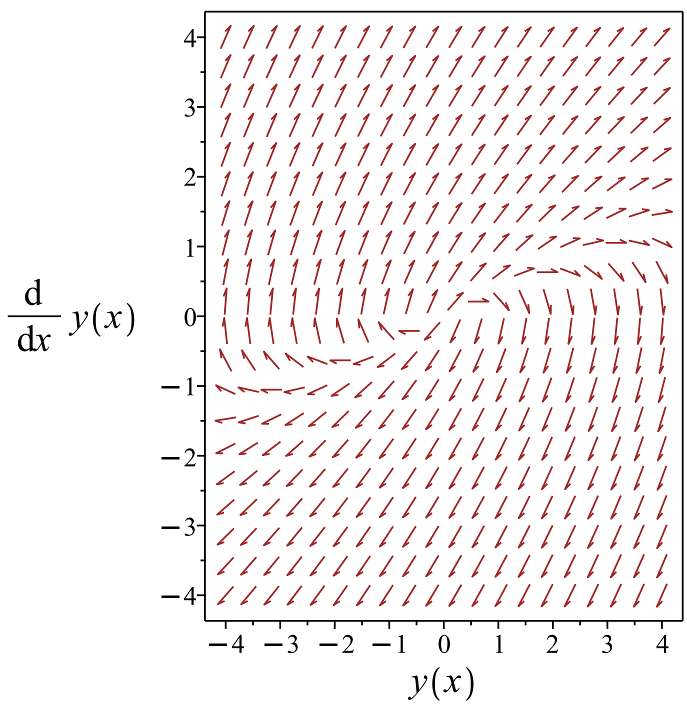


Figure 525: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{2x}{3}} + c_2 x\, e^{\frac{2x}{3}}$$

Verified OK.

7.24.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{3}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -\frac{4}{3}\,dx} \\ &= \mathrm{e}^{-\frac{2x}{3}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(\mathrm{e}^{-\frac{2x}{3}} y\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-\frac{2x}{3}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-\frac{2x}{3}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-\frac{2x}{3}}}$$

Or

$$y = c_1 x\,\mathrm{e}^{\frac{2x}{3}} + \mathrm{e}^{\frac{2x}{3}} c_2$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{\frac{2x}{3}} + \mathrm{e}^{\frac{2x}{3}} c_2 \tag{1}$$

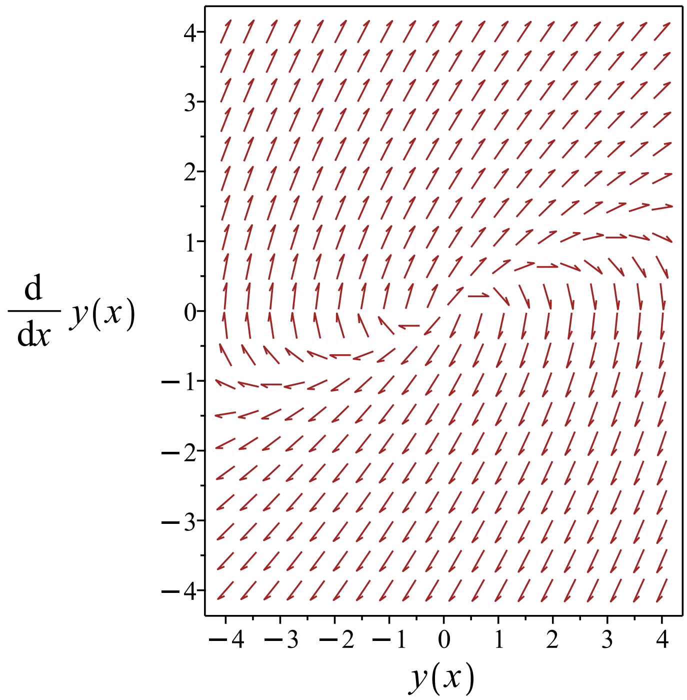


Figure 526: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{\frac{2x}{3}} + \mathrm{e}^{\frac{2x}{3}} c_2$$

Verified OK.

7.24.3 Solving using Kovacic algorithm

Writing the ode as

$$9y'' - 12y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= -12 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 369: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-12}{9}\,dx} \\
&= z_1 e^{\frac{2x}{3}} \\
&= z_1 \left(\mathrm{e}^{\frac{2x}{3}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{2x}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-12}{9}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\frac{4x}{3}}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(e^{\frac{2x}{3}}\right) + c_2\left(e^{\frac{2x}{3}}(x)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{2x}{3}} + c_2 x\, e^{\frac{2x}{3}} \tag{1}$$

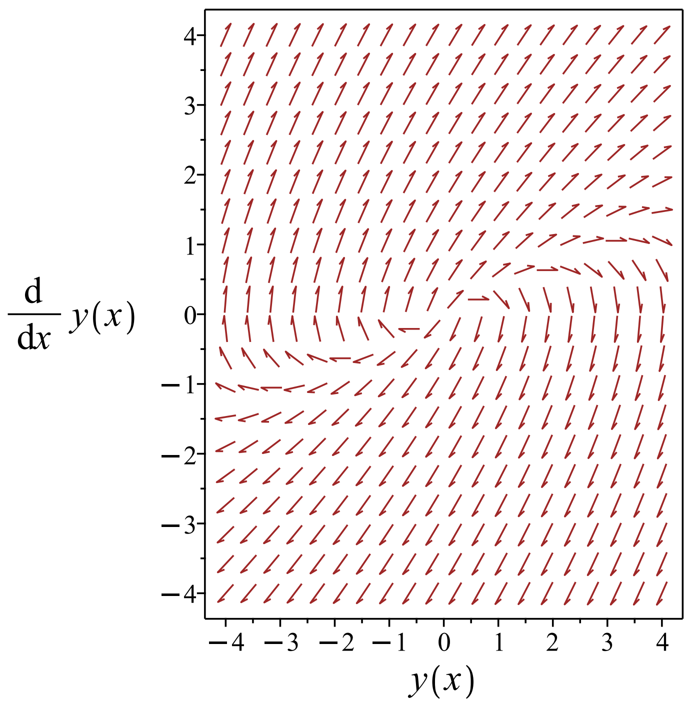


Figure 527: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{2x}{3}} + c_2 x\, e^{\frac{2x}{3}}$$

Verified OK.

7.24.4 Maple step by step solution

Let's solve

$9y'' - 12y' + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{4y'}{3} - \frac{4y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{4y'}{3} + \frac{4y}{9} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{4}{3}r + \frac{4}{9} = 0$

- Factor the characteristic polynomial

  $\frac{(3r-2)^2}{9} = 0$

- Root of the characteristic polynomial

  $r = \frac{2}{3}$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{2x}{3}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{\frac{2x}{3}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{2x}{3}} + c_2 x\, e^{\frac{2x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(9*diff(y(x),x$2)-12*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{\frac{2x}{3}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[9*y''[x]-12*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x/3}(c_2x + c_1)$$

7.25 problem 41

Internal problem ID [181]
Internal file name [`OUTPUT/181_Sunday_June_05_2022_01_36_26_AM_38496875/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 41.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{6y'' - 7y' - 20y = 0}$$

7.25.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 6, B = -7, C = -20$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$6\lambda^2 e^{\lambda x} - 7\lambda e^{\lambda x} - 20 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$6\lambda^2 - 7\lambda - 20 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 6, B = -7, C = -20$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)(6)} \pm \frac{1}{(2)(6)}\sqrt{-7^2 - (4)(6)(-20)} \\ &= \frac{7}{12} \pm \frac{23}{12}\end{aligned}$$

Hence

$$\lambda_1 = \frac{7}{12} + \frac{23}{12}$$
$$\lambda_2 = \frac{7}{12} - \frac{23}{12}$$

Which simplifies to

$$\lambda_1 = \frac{5}{2}$$
$$\lambda_2 = -\frac{4}{3}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{5}{2}\right)x} + c_2 e^{\left(-\frac{4}{3}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{5x}{2}} + c_2 \mathrm{e}^{-\frac{4x}{3}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{5x}{2}} + c_2 \mathrm{e}^{-\frac{4x}{3}} \tag{1}$$

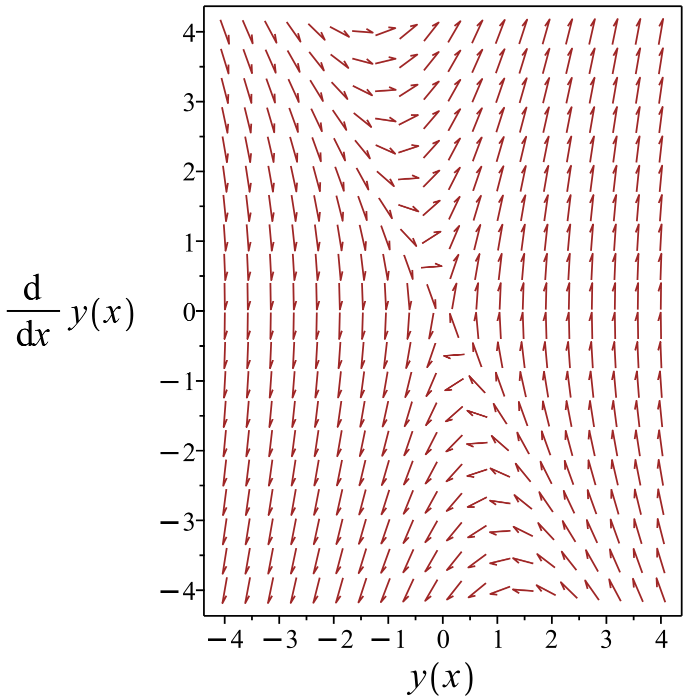


Figure 528: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{5x}{2}} + c_2 e^{-\frac{4x}{3}}$$

Verified OK.

7.25.2 Solving using Kovacic algorithm

Writing the ode as

$$6y'' - 7y' - 20y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 6 \\ B &= -7 \\ C &= -20 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{529}{144} \tag{6}$$

Comparing the above to (5) shows that

$$s = 529$$
$$t = 144$$

Therefore eq. (4) becomes

$$z''(x) = \frac{529z(x)}{144} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 371: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{529}{144}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{23x}{12}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-7}{6}\,dx} \\ &= z_1 e^{\frac{7x}{12}} \\ &= z_1 \left(\mathrm{e}^{\frac{7x}{12}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{4x}{3}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{6}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{7x}{6}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{6\,\mathrm{e}^{\frac{23x}{6}}}{23}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{4x}{3}}\right) + c_2 \left(\mathrm{e}^{-\frac{4x}{3}} \left(\frac{6\,\mathrm{e}^{\frac{23x}{6}}}{23}\right)\right)
\end{aligned}$$

<u>Summary</u>

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{4x}{3}} + \frac{6 c_2 \mathrm{e}^{\frac{5x}{2}}}{23} \tag{1}$$

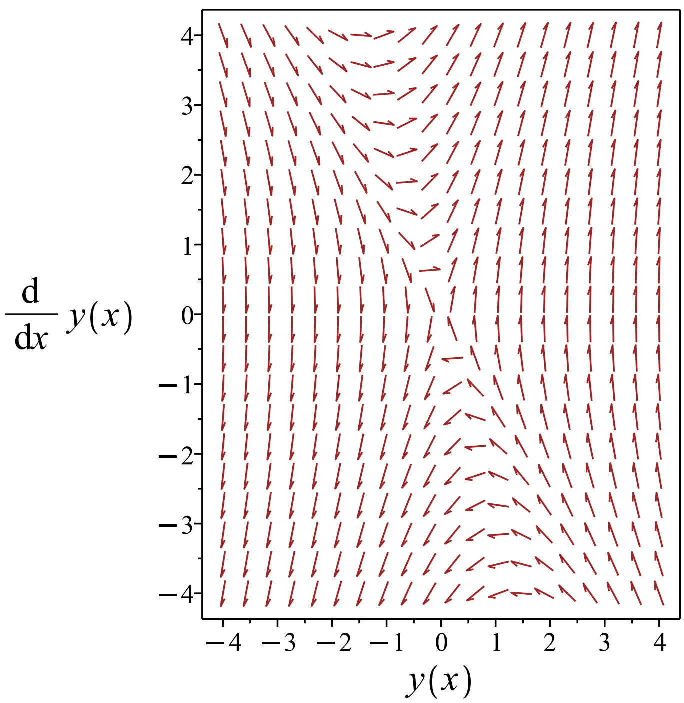


Figure 529: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{-\frac{4x}{3}} + \frac{6 c_2 \mathrm{e}^{\frac{5x}{2}}}{23}$$

Verified OK.

7.25.3 Maple step by step solution

Let's solve

$6y'' - 7y' - 20y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{7y'}{6} + \frac{10y}{3}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{7y'}{6} - \frac{10y}{3} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{7}{6}r - \frac{10}{3} = 0$

- Factor the characteristic polynomial

  $\frac{(3r+4)(2r-5)}{6} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{4}{3}, \frac{5}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{-\frac{4x}{3}}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{5x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-\frac{4x}{3}} + c_2 e^{\frac{5x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(6*diff(y(x),x$2)-7*diff(y(x),x)-20*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{\frac{23x}{6}} + c_2\right) e^{-\frac{4x}{3}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[6*y''[x]-7*y'[x]-20*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-4x/3} + c_2 e^{5x/2}$$

7.26 problem 42

Internal problem ID [182]
Internal file name [`OUTPUT/182_Sunday_June_05_2022_01_36_26_AM_13739999/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 42.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{35y'' - y' - 12y = 0}$$

7.26.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 35, B = -1, C = -12$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$35\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 12 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$35\lambda^2 - \lambda - 12 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 35, B = -1, C = -12$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(35)} \pm \frac{1}{(2)(35)}\sqrt{-1^2 - (4)(35)(-12)}\\ &= \frac{1}{70} \pm \frac{41}{70}\end{aligned}$$

Hence

$$\lambda_1 = \frac{1}{70} + \frac{41}{70}$$
$$\lambda_2 = \frac{1}{70} - \frac{41}{70}$$

Which simplifies to

$$\lambda_1 = \frac{3}{5}$$
$$\lambda_2 = -\frac{4}{7}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(\frac{3}{5}\right)x} + c_2 e^{\left(-\frac{4}{7}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\frac{3x}{5}} + c_2 \mathrm{e}^{-\frac{4x}{7}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{3x}{5}} + c_2 \mathrm{e}^{-\frac{4x}{7}} \tag{1}$$

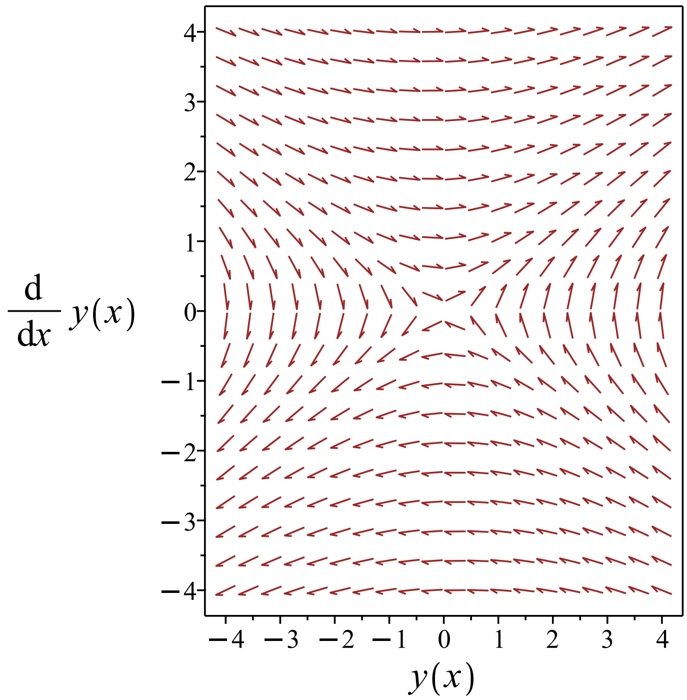


Figure 530: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{3x}{5}} + c_2 e^{-\frac{4x}{7}}$$

Verified OK.

7.26.2 Solving using Kovacic algorithm

Writing the ode as

$$35y'' - y' - 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 35 \\ B &= -1 \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1681}{4900} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1681$$
$$t = 4900$$

Therefore eq. (4) becomes

$$z''(x) = \frac{1681z(x)}{4900} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 373: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1681}{4900}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{41x}{70}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{35}\,dx} \\ &= z_1 e^{\frac{x}{70}} \\ &= z_1 \left(\mathrm{e}^{\frac{x}{70}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{4x}{7}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{35}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{x}{35}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{35\,\mathrm{e}^{\frac{41x}{35}}}{41}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{4x}{7}}\right) + c_2 \left(\mathrm{e}^{-\frac{4x}{7}} \left(\frac{35\,\mathrm{e}^{\frac{41x}{35}}}{41}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-\frac{4x}{7}} + \frac{35 c_2 \mathrm{e}^{\frac{3x}{5}}}{41} \tag{1}$$

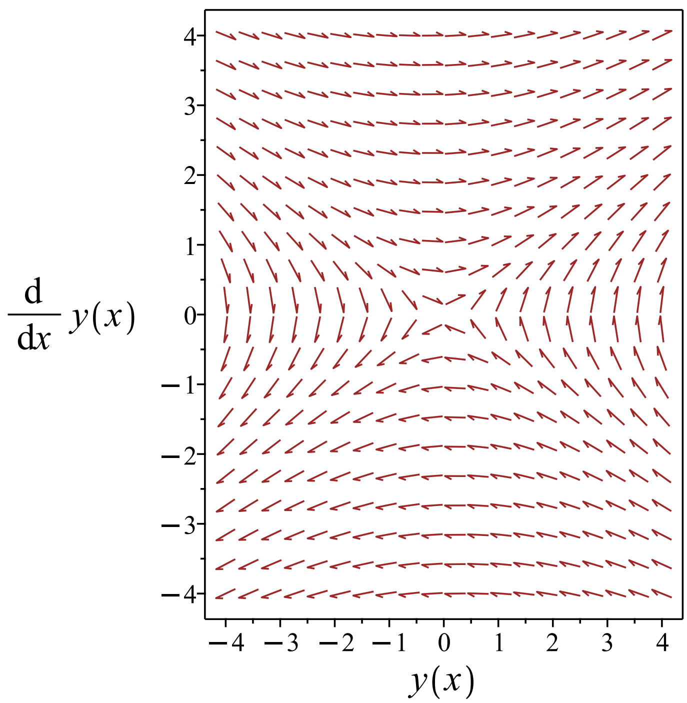


Figure 531: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-\frac{4x}{7}} + \frac{35 c_2 \mathrm{e}^{\frac{3x}{5}}}{41}$$

Verified OK.

7.26.3 Maple step by step solution

Let's solve

$35y'' - y' - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{y'}{35} + \frac{12y}{35}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{y'}{35} - \frac{12y}{35} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{1}{35}r - \frac{12}{35} = 0$

- Factor the characteristic polynomial

  $\frac{(7r+4)(5r-3)}{35} = 0$

- Roots of the characteristic polynomial

  $r = \left(-\frac{4}{7}, \frac{3}{5}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{4x}{7}}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{3x}{5}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{4x}{7}} + c_2 \mathrm{e}^{\frac{3x}{5}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(35*diff(y(x),x$2)-diff(y(x),x)-12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_1 e^{\frac{41x}{35}} + c_2\right) e^{-\frac{4x}{7}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[35*y''[x]-y'[x]-12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{-4x/7} + c_2 e^{3x/5}$$

7.27 problem 52

Internal problem ID [183]
Internal file name [`OUTPUT/183_Sunday_June_05_2022_01_36_27_AM_77948201/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 52.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _homogeneous]]
```

$$\boxed{x^2y'' + y'x - y = 0}$$

7.27.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -1 \\ r_2 &= 1 \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + c_2 x$$

Verified OK.

7.27.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\ &= \int e^{-\ln(x)}\, dx \\ &= \int \frac{1}{x} dx \\ &= \ln(x) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= -1
\end{aligned} \tag{7}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - y(\tau) &= 0
\end{aligned}
$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$
Ay''(\tau) + By'(\tau) + Cy(\tau) = 0
$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$
\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}
$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$
\lambda^2 - 1 = 0 \tag{2}
$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$
\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}
$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)} \\
&= \pm 1
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= +1 \\
\lambda_2 &= -1
\end{aligned}
$$

Which simplifies to

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$

Since roots are real and distinct, then the solution is

$$y(\tau) = c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}$$
$$y(\tau) = c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x}$$

Verified OK.

7.27.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2 y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q}\\ &= \frac{\sqrt{-\frac{1}{x^2}}}{c}\\ \tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2}\\ &= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2}\\ &= 0\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0\\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1\cos(c\tau) + c_2\sin(c\tau)$$

Now from (6)

$$\begin{aligned}\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\ &= \frac{\int \sqrt{-\frac{1}{x^2}}dx}{c} \\ &= \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Verified OK.

7.27.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2}\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= e^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{2x^2} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x \\
&= \left(-\frac{c_1}{2x^2} + c_2\right) x
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}
$$

Verification of solutions

$$
y = \left(-\frac{c_1}{2x^2} + c_2\right) x
$$

Verified OK.

7.27.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(x^2 y'' + y'x - y\right) dx &= 0 \\
y'x^2 - yx &= c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x)y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y) &= (\mu)\left(\frac{c_1}{x^2}\right) \\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right) \\ \mathrm{d}\left(\frac{y}{x}\right) &= \left(\frac{c_1}{x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\begin{aligned}\frac{y}{x} &= \int \frac{c_1}{x^3}\,\mathrm{d}x \\ \frac{y}{x} &= -\frac{c_1}{2x^2} + c_2\end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

7.27.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right) v' + \left(AB'' + BB' + CB\right) v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right) v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right) u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \\ F &= 0 \end{aligned}$$

The above shows that for this ode

$$\begin{aligned} AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\ &= 0 \end{aligned}$$

Hence the ode in v given in (1) now simplifies to

$$x^3v'' + (3x^2)\, v' = 0$$

Now by applying $v' = u$ the above becomes

$$x^2(u'(x)\, x + 3u(x)) = 0$$

Which is now solved for u. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{3}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^3}\, \mathrm{d}x \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

7.27.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2 y'' + y'x - y = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (x^2 y'' + y'x - y)\, dx = 0$$
$$y'x^2 - yx = c_1$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{c_1}{x^2}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{c_1}{x^2}$$

The integrating factor μ is

$$\mu = e^{\int -\frac{1}{x} dx}$$
$$= \frac{1}{x}$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{c_1}{x^3}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{c_1}{x^3} dx$$
$$\frac{y}{x} = -\frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

7.27.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + y' x - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 375: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1 + 4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1 + 4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{x}{x^2} \, dx} \\ &= z_1 e^{-\frac{\ln(x)}{2}} \\ &= z_1 \left(\frac{1}{\sqrt{x}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{x^2}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2}$$

Verified OK.

7.27.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= x \\ r(x) &= -1 \\ s(x) &= 0 \end{aligned}$$

Hence

$$
\begin{aligned}
p''(x) &= 2 \\
q'(x) &= 1
\end{aligned}
$$

Therefore (1) becomes

$$
2-(1)+(-1)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)
$$

Integrating gives

$$
p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx
$$

Substituting the above values for p, q, r, s gives

$$
y'x^2-yx=c_1
$$

We now have a first order ode to solve which is

$$
y'x^2-yx=c_1
$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y'+p(x)y=q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y'-\frac{y}{x}=\frac{c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= \mathrm{e}^{\int -\frac{1}{x}dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$\frac{d}{dx}(\mu y) = (\mu)\left(\frac{c_1}{x^2}\right)$$
$$\frac{d}{dx}\left(\frac{y}{x}\right) = \left(\frac{1}{x}\right)\left(\frac{c_1}{x^2}\right)$$
$$d\left(\frac{y}{x}\right) = \left(\frac{c_1}{x^3}\right) dx$$

Integrating gives

$$\frac{y}{x} = \int \frac{c_1}{x^3} dx$$
$$\frac{y}{x} = -\frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = -\frac{c_1}{2x} + c_2 x$$

Summary

The solution(s) found are the following

$$y = -\frac{c_1}{2x} + c_2 x \tag{1}$$

Verification of solutions

$$y = -\frac{c_1}{2x} + c_2 x$$

Verified OK.

7.27.10 Maple step by step solution

Let's solve

$x^2 y'' + y'x - y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} + \frac{y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + y'x - y = 0$
- Make a change of variables

  $t = \ln(x)$

☐ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) - y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 - 1 = 0$
- Factor the characteristic polynomial

  $(r - 1)(r + 1) = 0$
- Roots of the characteristic polynomial

  $r = (-1, 1)$
- 1st solution of the ODE

  $y_1(t) = e^{-t}$
- 2nd solution of the ODE

  $y_2(t) = e^{t}$
- General solution of the ODE

$y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x} + c_2 x$

- Simplify

  $y = \frac{c_1}{x} + c_2 x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 + c_2}{x}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 16

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_1}{x} + c_2 x$$

7.28 problem 53

Internal problem ID [184]
Internal file name [`OUTPUT/184_Sunday_June_05_2022_01_36_27_AM_20255227/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 53.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 2y'x - 12y = 0}$$

7.28.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 2xrx^{r-1} - 12x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 2r\,x^r - 12x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 2r - 12 = 0$$

Or

$$r^2 + r - 12 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -4$$
$$r_2 = 3$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^4} + c_2 x^3$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + c_2 x^3 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + c_2 x^3$$

Verified OK.

7.28.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' + 2y'x - 12y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{12}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau) \right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2}{x} dx\right)} dx \\
&= \int e^{-2\ln(x)}\, dx \\
&= \int \frac{1}{x^2} dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{12}{x^2}}{\frac{1}{x^4}} \\
&= -12x^2
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2} y(\tau) - 12x^2 y(\tau) = 0$$

But in terms of τ

$$-12x^2 = -\frac{12}{\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{12y(\tau)}{\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 12y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 12\tau^r = 0$$

Simplifying gives

$$r(r-1)\,\tau^r + 0\,\tau^r - 12\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$r(r-1) + 0 - 12 = 0$$

Or

$$r^2 - r - 12 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -3$$
$$r_2 = 4$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\tau^3} + c_2\tau^4$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{-c_1x^7 + c_2}{x^4}$$

Summary

The solution(s) found are the following

$$y = \frac{-c_1x^7 + c_2}{x^4} \tag{1}$$

Verification of solutions

$$y = \frac{-c_1 x^7 + c_2}{x^4}$$

Verified OK.

7.28.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' + 2y'x - 12y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{12}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2} - \frac{12}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{8v'(x)}{x} = 0$$

$$v''(x) + \frac{8v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{8u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}
u' &= F(x, u) \\
&= f(x)g(u) \\
&= -\frac{8u}{x}
\end{aligned}$$

Where $f(x) = -\frac{8}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{8}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{8}{x}\, dx \\
\ln(u) &= -8\ln(x) + c_1 \\
u &= \mathrm{e}^{-8\ln(x)+c_1} \\
&= \frac{c_1}{x^8}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{7x^7} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{7x^7} + c_2\right) x^3 \\
&= \frac{7c_2x^7 - c_1}{7x^4}
\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(-\frac{c_1}{7x^7} + c_2\right) x^3 \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{7x^7} + c_2\right) x^3$$

Verified OK.

7.28.4 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + 2y'x - 12y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 2x \\ C &= -12 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{12}{x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 12 \\ t &= x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{12}{x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 377: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{12}{x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{12}{x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = 12$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = 4 \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -3
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{12}{x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	4	-3

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	4	-3

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_{\infty}^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_{\infty}^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_{\infty}^{-} = -3$ then

$$\begin{aligned} d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\ &= -3 - (-3) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\ &= -\frac{3}{x} + (-)\,(0) \\ &= -\frac{3}{x} \\ &= -\frac{3}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(-\frac{3}{x}\right)(0) + \left(\left(\frac{3}{x^2}\right) + \left(-\frac{3}{x}\right)^2 - \left(\frac{12}{x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= \mathrm{e}^{\int -\frac{3}{x} dx} \\ &= \frac{1}{x^3} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{2x}{x^2} \, dx} \\ &= z_1 e^{-\ln(x)} \\ &= z_1 \left(\frac{1}{x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^4}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2x}{x^2} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{x^7}{7}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\frac{1}{x^4}\right) + c_2 \left(\frac{1}{x^4} \left(\frac{x^7}{7}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^4} + \frac{c_2 x^3}{7} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^4} + \frac{c_2 x^3}{7}$$

Verified OK.

7.28.5 Maple step by step solution

Let's solve

$x^2 y'' + 2y'x - 12y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + \frac{12y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{12y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 2y'x - 12y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

$y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

Substitute the change of variables back into the ODE

$x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 2\frac{d}{dt}y(t) - 12y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - 12y(t) = 0$
- Characteristic polynomial of ODE

  $r^2 + r - 12 = 0$
- Factor the characteristic polynomial

  $(r + 4)(r - 3) = 0$
- Roots of the characteristic polynomial

  $r = (-4, 3)$
- 1st solution of the ODE

  $y_1(t) = e^{-4t}$
- 2nd solution of the ODE

  $y_2(t) = e^{3t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{-4t} + c_2 e^{3t}$
- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^4} + c_2 x^3$
- Simplify

  $y = \frac{c_1}{x^4} + c_2 x^3$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(x^2*diff(y(x),x$2)+2*x*diff(y(x),x)-12*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_2 x^7 + c_1}{x^4}$$

✓ Solution by Mathematica

Time used: 0.01 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]+2*x*y'[x]-12*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^7 + c_1}{x^4}$$

7.29 problem 54

Internal problem ID [185]
Internal file name [`OUTPUT/185_Sunday_June_05_2022_01_36_28_AM_48273024/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 54.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{4x^2y'' + 8y'x - 3y = 0}$$

7.29.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$4x^2(r(r-1))x^{r-2} + 8xrx^{r-1} - 3x^r = 0$$

Simplifying gives

$$4r(r-1)\,x^r + 8r\,x^r - 3x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$4r(r-1) + 8r - 3 = 0$$

Or

$$4r^2 + 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{3}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are real and distinct, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x^{\frac{3}{2}}} + c_2\sqrt{x}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} + c_2\sqrt{x} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}} + c_2\sqrt{x}$$

Verified OK.

7.29.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$4x^2y'' + 8y'x - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{2}{x}$$
$$q(x) = -\frac{3}{4x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int \frac{2}{x}dx\right)}dx \\
&= \int e^{-2\ln(x)}\,dx \\
&= \int \frac{1}{x^2}dx \\
&= -\frac{1}{x}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{4x^2}}{\frac{1}{x^4}} \\
&= -\frac{3x^2}{4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3x^2 y(\tau)}{4} = 0$$

But in terms of τ

$$-\frac{3x^2}{4} = -\frac{3}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) - \frac{3y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 - 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} - 3\tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r - 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 - 3 = 0$$

Or

$$4r^2 - 4r - 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -\frac{1}{2}$$
$$r_2 = \frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = \frac{c_1}{\sqrt{\tau}} + c_2\tau^{\frac{3}{2}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x^2 \sqrt{-\frac{1}{x}}}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x^2 \sqrt{-\frac{1}{x}}} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 x^2 + c_2}{x^2 \sqrt{-\frac{1}{x}}}$$

Verified OK.

7.29.3 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$4x^2 y'' + 8y'x - 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{2}{x} \\ q(x) &= -\frac{3}{4x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{2n}{x^2} - \frac{3}{4x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{1}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{3}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{3}{x}\,dx \\ \ln(u) &= -3\ln(x) + c_1 \\ u &= \mathrm{e}^{-3\ln(x)+c_1} \\ &= \frac{c_1}{x^3} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(-\frac{c_1}{2x^2} + c_2\right)\sqrt{x} \\ &= \frac{2c_2x^2 - c_1}{2x^{\frac{3}{2}}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \left(-\frac{c_1}{2x^2} + c_2\right)\sqrt{x} \tag{1}$$

Verification of solutions

$$y = \left(-\frac{c_1}{2x^2} + c_2\right)\sqrt{x}$$

Verified OK.

7.29.4 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' + 8y'x - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= 8x \\ C &= -3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 379: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned} [\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2} \end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned}
d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\
&= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\
&= -\frac{1}{2x} + (-)\,(0) \\
&= -\frac{1}{2x} \\
&= -\frac{1}{2x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d=0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$
$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int -\frac{1}{2x} dx} \\
&= \frac{1}{\sqrt{x}}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8x}{4x^2} \, dx} \\
&= z_1 e^{-\ln(x)} \\
&= z_1 \left(\frac{1}{x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \frac{1}{x^{\frac{3}{2}}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$y_2 = y_1 \int \frac{e^{\int -\frac{8x}{4x^2} dx}}{(y_1)^2} dx$$
$$= y_1 \int \frac{e^{-2\ln(x)}}{(y_1)^2} dx$$
$$= y_1 \left(\frac{x^2}{2}\right)$$

Therefore the solution is

$$y = c_1 y_1 + c_2 y_2$$
$$= c_1 \left(\frac{1}{x^{\frac{3}{2}}}\right) + c_2 \left(\frac{1}{x^{\frac{3}{2}}} \left(\frac{x^2}{2}\right)\right)$$

Summary
The solution(s) found are the following

$$y = \frac{c_1}{x^{\frac{3}{2}}} + \frac{c_2\sqrt{x}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x^{\frac{3}{2}}} + \frac{c_2\sqrt{x}}{2}$$

Verified OK.

7.29.5 Maple step by step solution

Let's solve

$4x^2y'' + 8y'x - 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{2y'}{x} + \frac{3y}{4x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{2y'}{x} - \frac{3y}{4x^2} = 0$

- Multiply by denominators of the ODE

  $4x^2y'' + 8y'x - 3y = 0$
- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$
  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$
  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$
  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $4x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 8\frac{d}{dt}y(t) - 3y(t) = 0$

- Simplify

  $4\frac{d^2}{dt^2}y(t) + 4\frac{d}{dt}y(t) - 3y(t) = 0$
- Isolate 2nd derivative

  $\frac{d^2}{dt^2}y(t) = -\frac{d}{dt}y(t) + \frac{3y(t)}{4}$
- Group terms with $y(t)$ on the lhs of the ODE and the rest on the rhs of the ODE; ODE is lin

  $\frac{d^2}{dt^2}y(t) + \frac{d}{dt}y(t) - \frac{3y(t)}{4} = 0$
- Characteristic polynomial of ODE

  $r^2 + r - \frac{3}{4} = 0$
- Factor the characteristic polynomial

  $\frac{(2r+3)(2r-1)}{4} = 0$
- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2}, \frac{1}{2}\right)$
- 1st solution of the ODE

$y_1(t) = \mathrm{e}^{-\frac{3t}{2}}$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{\frac{t}{2}}$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-\frac{3t}{2}} + c_2 \mathrm{e}^{\frac{t}{2}}$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1}{x^{\frac{3}{2}}} + c_2 \sqrt{x}$

- Simplify

  $y = \frac{c_1}{x^{\frac{3}{2}}} + c_2 \sqrt{x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve(4*x^2*diff(y(x),x$2)+8*x*diff(y(x),x)-3*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 x^2 + c_2}{x^{\frac{3}{2}}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[4*x^2*y''[x]+8*x*y'[x]-3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 x^2 + c_1}{x^{3/2}}$$

7.30 problem 55

Internal problem ID [186]
Internal file name [OUTPUT/186_Sunday_June_05_2022_01_36_28_AM_14840970/index.tex]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 55.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_ode_missing_y"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_y]]
```

$$\boxed{x^2 y'' + y'x = 0}$$

The ODE is

$$x^2 y'' + y'x = 0$$

Or

$$x(y''x + y') = 0$$

For $x \neq 0$ the above simplifies to

$$y''x + y' = 0$$

7.30.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 0 = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 0 = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 0 = 0$$

Or

$$r^2 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 0$$
$$r_2 = 0$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_2 \ln(x) + c_1$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_2 \ln(x) + c_1 \tag{1}$$

<u>Verification of solutions</u>

$$y = c_2 \ln(x) + c_1$$

Verified OK. `{x <> 0}`

7.30.2 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$x^2 p'(x) + p(x)\, x = 0$$

Which is now solve for $p(x)$ as first order ode. In canonical form the ODE is

$$\begin{aligned} p' &= F(x,p) \\ &= f(x)g(p) \\ &= -\frac{p}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(p) = p$. Integrating both sides gives

$$\begin{aligned} \frac{1}{p}\, dp &= -\frac{1}{x}\, dx \\ \int \frac{1}{p}\, dp &= \int -\frac{1}{x}\, dx \\ \ln(p) &= -\ln(x) + c_1 \\ p &= e^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Since $p = y'$ then the new first order ode to solve is

$$y' = \frac{c_1}{x}$$

Integrating both sides gives

$$\begin{aligned} y &= \int \frac{c_1}{x}\, \mathrm{d}x \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \ln(x) + c_2 \tag{1}$$

Verification of solutions

$$y = c_1 \ln(x) + c_2$$

Verified OK. `{x <> 0}`

7.30.3 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 381: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{1}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2}$ then

$$
\begin{aligned}
d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\
&= \frac{1}{2} - \left(\frac{1}{2}\right) \\
&= 0
\end{aligned}
$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$
\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}
$$

The above gives

$$
\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{1}{2x} + (-)\,(0) \\
&= \frac{1}{2x} \\
&= \frac{1}{2x}
\end{aligned}
$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$
p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}
$$

Let

$$
p(x) = 1 \tag{2A}
$$

Substituting the above in eq. (1A) gives

$$
\begin{aligned}
(0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\
0 &= 0
\end{aligned}
$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{1}{2x} dx} \\
&= \sqrt{x}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = 1
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(1) + c_2(1(\ln(x)))
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_2 \ln(x) + c_1 \tag{1}
$$

Verification of solutions

$$
y = c_2 \ln(x) + c_1
$$

Verified OK. `{x <> 0}`

7.30.4 Maple step by step solution

Let's solve

$x^2 y'' + y' x = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} = 0$

- Multiply by denominators of the ODE

  $y'' x + y' = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt} y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2} y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt} y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $\left(\frac{\frac{d^2}{dt^2} y(t)}{x^2} - \frac{\frac{d}{dt} y(t)}{x^2}\right) x + \frac{\frac{d}{dt} y(t)}{x} = 0$

- Simplify

  $\frac{\frac{d^2}{dt^2} y(t)}{x} = 0$

- Isolate 2nd derivative

$\frac{d^2}{dt^2}y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 = 0$
- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{0})}{2}$
- Roots of the characteristic polynomial

  $r = 0$
- 1st solution of the ODE

  $y_1(t) = 1$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_2 t + c_1$
- Change variables back using $t = \ln(x)$

  $y = c_2 \ln(x) + c_1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 10

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_2 \ln(x) + c_1$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 13

```
DSolve[x^2*y''[x]+x*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \log(x) + c_2$$

7.31 problem 56

Internal problem ID [187]
Internal file name [`OUTPUT/187_Sunday_June_05_2022_01_36_29_AM_37289031/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.1, second order linear equations. Page 299
Problem number: 56.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' - 3y'x + 4y = 0}$$

7.31.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1 y_1 + c_2 y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

7.31.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2 y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)}dx \\
&= \int e^{3\ln(x)}\,dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} = 0$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^r + 0\,\tau^r + \tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$4r(r-1) + 0 + 1 = 0$$

Or

$$4r^2 - 4r + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2}$$
$$r_2 = \frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau) = c_1y_1 + c_2y_2$$

Where $y_1 = \tau^r$ and $y_2 = \tau^r \ln(\tau)$. Hence

$$y(\tau) = c_1\sqrt{\tau} + c_2\sqrt{\tau}\,\ln(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{(-2c_2\ln(2) + c_2\ln(x^4) + c_1)\sqrt{x^4}}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1)\sqrt{x^4}}{2} \tag{1}$$

Verification of solutions

$$y = \frac{(-2c_2 \ln(2) + c_2 \ln(x^4) + c_1)\sqrt{x^4}}{2}$$

Verified OK.

7.31.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{2\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\, x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - 2c\left(\frac{d}{d\tau}y(\tau)\right) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$
y(\tau) = \mathrm{e}^{c\tau} c_1
$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 x^2
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^2 \tag{1}
$$

Verification of solutions

$$
y = c_1 x^2
$$

Verified OK.

7.31.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2 y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{3}{x}$$
$$q(x) = \frac{4}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$v''(x) + \frac{v'(x)}{x} = 0$$
$$v''(x) + \frac{v'(x)}{x} = 0 \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\,du &= -\frac{1}{x}\,dx \\ \int \frac{1}{u}\,du &= \int -\frac{1}{x}\,dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= \mathrm{e}^{-\ln(x)+c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= (c_1 \ln(x) + c_2)\,x^2 \\ &= (c_1 \ln(x) + c_2)\,x^2 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = (c_1 \ln(x) + c_2)\,x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = (c_1 \ln(x) + c_2)\,x^2$$

Verified OK.

7.31.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 383: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{1}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{1}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)\left(0\right) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^2$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

7.31.6 Maple step by step solution

Let's solve

$x^2y'' - 3y'x + 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{x} - \frac{4y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{x} + \frac{4y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' - 3y'x + 4y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) - 3\frac{d}{dt}y(t) + 4y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) - 4\frac{d}{dt}y(t) + 4y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$
- Root of the characteristic polynomial

  $r = 2$
- 1st solution of the ODE

  $y_1(t) = e^{2t}$
- Repeated root, multiply $y_1(t)$ by t to ensure linear independence

  $y_2(t) = t\,e^{2t}$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 e^{2t} + c_2 t\,e^{2t}$
- Change variables back using $t = \ln(x)$

  $y = c_1 x^2 + c_2 x^2 \ln(x)$
- Simplify

  $y = x^2(c_2 \ln(x) + c_1)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=0,y(x), singsol=all)
```

$$y(x) = x^2(c_2 \ln(x) + c_1)$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 18

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(2c_2 \log(x) + c_1)$$

8 Section 5.2, second order linear equations. Page 311

8.1 problem 21

Internal problem ID [188]
Internal file name [`OUTPUT/188_Sunday_June_05_2022_01_36_29_AM_38650824/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.2, second order linear equations. Page 311
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + y = 3x}$$

With initial conditions

$$[y(0) = 2, y'(0) = -2]$$

8.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= 3x \end{aligned}$$

Hence the ode is

$$y'' + y = 3x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 3x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = 3x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2 x + A_1 = 3x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (3x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + 3x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) + 3$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = 3 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -5$$

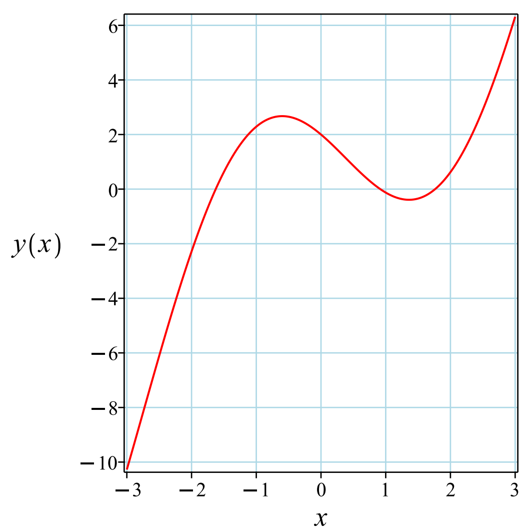
Substituting these values back in above solution results in

$$y = 2\cos(x) - 5\sin(x) + 3x$$

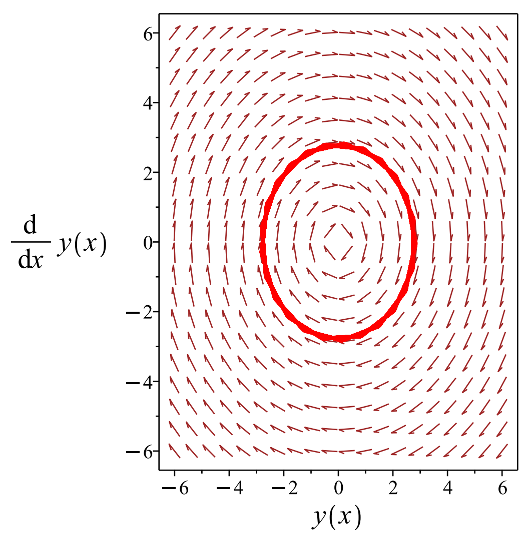
Summary

The solution(s) found are the following

$$y = 2\cos(x) - 5\sin(x) + 3x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\cos(x) - 5\sin(x) + 3x$$

Verified OK.

8.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 385: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} dx}}{y_1^2} dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2} dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2} dx \\ &= \cos(x) (\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x) (\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_2x + A_1 = 3x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + (3x) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + 3x \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 2$ and $x = 0$ in the above gives

$$2 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) + 3$$

substituting $y' = -2$ and $x = 0$ in the above gives

$$-2 = 3 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -5$$

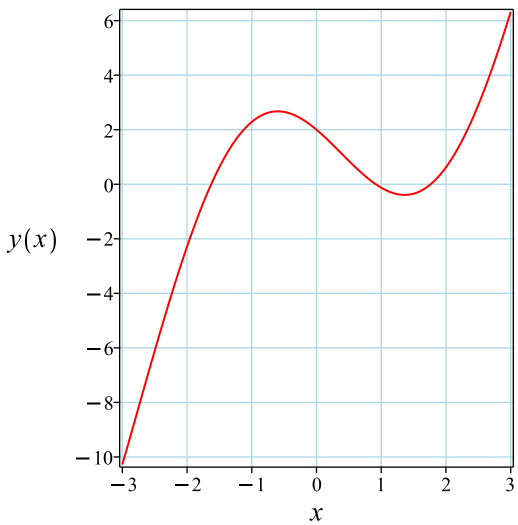
Substituting these values back in above solution results in

$$y = 2\cos(x) - 5\sin(x) + 3x$$

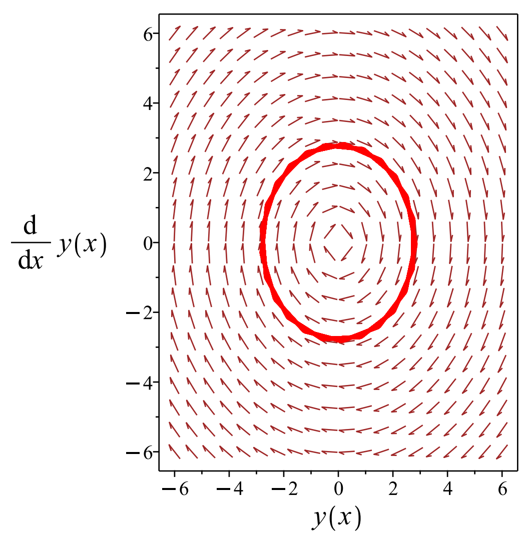
Summary

The solution(s) found are the following

$$y = 2\cos(x) - 5\sin(x) + 3x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\cos(x) - 5\sin(x) + 3x$$

Verified OK.

8.1.4 Maple step by step solution

Let's solve

$\left[y'' + y = 3x, y(0) = 2, y'\Big|_{\{x=0\}} = -2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -3\cos(x)\left(\int \sin(x)\, x dx\right) + 3\sin(x)\left(\int x\cos(x)\, dx\right)$

- Compute integrals

  $y_p(x) = 3x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + 3x$

☐ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x) + 3x$

- Use initial condition $y(0) = 2$

  $2 = c_1$

- Compute derivative of the solution

  $y' = -c_1 \sin(x) + c_2 \cos(x) + 3$

- Use the initial condition $y'\big|_{\{x=0\}} = -2$

  $-2 = 3 + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -5\}$

- Substitute constant values into general solution and simplify

  $y = 2\cos(x) - 5\sin(x) + 3x$

- Solution to the IVP

  $y = 2\cos(x) - 5\sin(x) + 3x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)+y(x)=3*x,y(0) = 2, D(y)(0) = -2],y(x), singsol=all)
```

$$y(x) = -5\sin(x) + 2\cos(x) + 3x$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 17

```
DSolve[{y''[x]+y[x]==3*x,{y[0]==2,y'[0]==-2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x - 5\sin(x) + 2\cos(x)$$

8.2 problem 22

Internal problem ID [189]
Internal file name [`OUTPUT/189_Sunday_June_05_2022_01_36_30_AM_70119664/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.2, second order linear equations. Page 311
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y = 12}$$

With initial conditions

$$[y(0) = 0, y'(0) = 10]$$

8.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= -4 \\ F &= 12 \end{aligned}$$

Hence the ode is

$$y'' - 4y = 12$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 12$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.2.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = 12$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 12$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-2x}\right) + (-3) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{2x} + c_2 e^{-2x} - 3 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 - 3 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 2c_1 e^{2x} - 2c_2 e^{-2x}$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$10 = 2c_1 - 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = -1$$

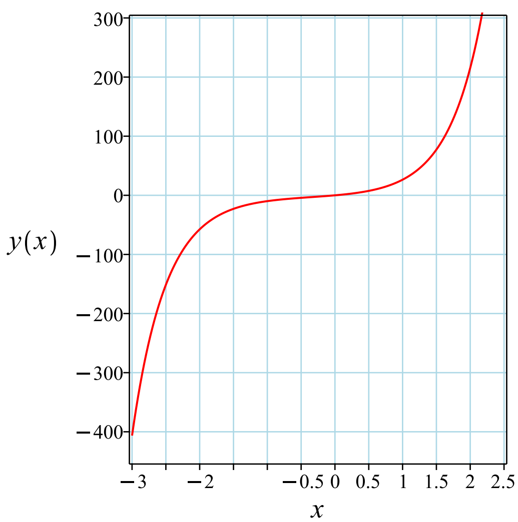
Substituting these values back in above solution results in

$$y = -3 + 4\,\mathrm{e}^{2x} - \mathrm{e}^{-2x}$$

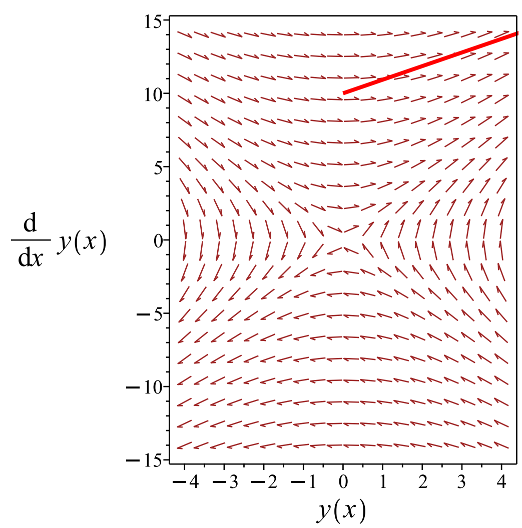
Summary

The solution(s) found are the following

$$y = -3 + 4\,\mathrm{e}^{2x} - \mathrm{e}^{-2x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = -3 + 4\,\mathrm{e}^{2x} - \mathrm{e}^{-2x}$$

Verified OK.

8.2.3 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 4yy' - 12y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 4yy' - 12y')\, dx = 0$$
$$\frac{y'^2}{2} - 2y^2 - 12y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{4y^2 + 24y + 2c_1} \tag{1}$$
$$y' = -\sqrt{4y^2 + 24y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{4y^2 + 2c_1 + 24y}} dy = \int dx$$
$$\frac{\ln\left(\frac{(4y+12)\sqrt{4}}{4} + \sqrt{4y^2 + 2c_1 + 24y}\right)\sqrt{4}}{4} = x + c_2$$

Raising both side to exponential gives

$$e^{\frac{\ln\left(\frac{(4y+12)\sqrt{4}}{4} + \sqrt{4y^2 + 2c_1 + 24y}\right)\sqrt{4}}{4}} = e^{x+c_2}$$

Which simplifies to

$$\sqrt{2y + 6 + \sqrt{4y^2 + 2c_1 + 24y}} = c_3 e^{x}$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{4y^2 + 2c_1 + 24y}} dy = \int dx$$
$$-\frac{\ln\left(\frac{(4y+12)\sqrt{4}}{4} + \sqrt{4y^2 + 2c_1 + 24y}\right)\sqrt{4}}{4} = x + c_4$$

Raising both side to exponential gives

$$e^{-\frac{\ln\left(\frac{(4y+12)\sqrt{4}}{4}+\sqrt{4y^2+2c_1+24y}\right)\sqrt{4}}{4}} = e^{x+c_4}$$

Which simplifies to

$$\frac{1}{\sqrt{2y+6+\sqrt{4y^2+2c_1+24y}}} = c_5 e^x$$

Initial conditions are used to solve for the constants of integration.

Looking at the First solution

$$y = \frac{\left(e^{4x}c_3^4 - 12\,e^{2x}c_3^2 - 2c_1 + 36\right)e^{-2x}}{4c_3^2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{c_3^4 - 12c_3^2 - 2c_1 + 36}{4c_3^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = \frac{\left(4\,e^{4x}c_3^4 - 24\,e^{2x}c_3^2\right)e^{-2x}}{4c_3^2} - \frac{\left(e^{4x}c_3^4 - 12\,e^{2x}c_3^2 - 2c_1 + 36\right)e^{-2x}}{2c_3^2}$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$10 = \frac{c_3^4 + 2c_1 - 36}{2c_3^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_3\}$. Solving for the constants gives

$$c_1 = 50$$
$$c_3 = 4$$

Substituting these values back in above solution results in

$$y = e^{-2x}\left(4\,e^{4x} - 3\,e^{2x} - 1\right)$$

Which simplifies to

$$y = -3 + 4\,e^{2x} - e^{-2x}$$

Looking at the Second solution

$$y = -\frac{\left(2c_1 c_5^4 \mathrm{e}^{4x} - 36c_5^4 \mathrm{e}^{4x} + 12c_5^2 \mathrm{e}^{2x} - 1\right) \mathrm{e}^{-2x}}{4c_5^2} \tag{2}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = \frac{1 + \left(-2c_1 + 36\right) c_5^4 - 12c_5^2}{4c_5^2} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\left(8c_1 c_5^4 \mathrm{e}^{4x} - 144c_5^4 \mathrm{e}^{4x} + 24c_5^2 \mathrm{e}^{2x}\right) \mathrm{e}^{-2x}}{4c_5^2} + \frac{\left(2c_1 c_5^4 \mathrm{e}^{4x} - 36c_5^4 \mathrm{e}^{4x} + 12c_5^2 \mathrm{e}^{2x} - 1\right) \mathrm{e}^{-2x}}{2c_5^2}$$

substituting $y' = 10$ and $x = 0$ in the above gives

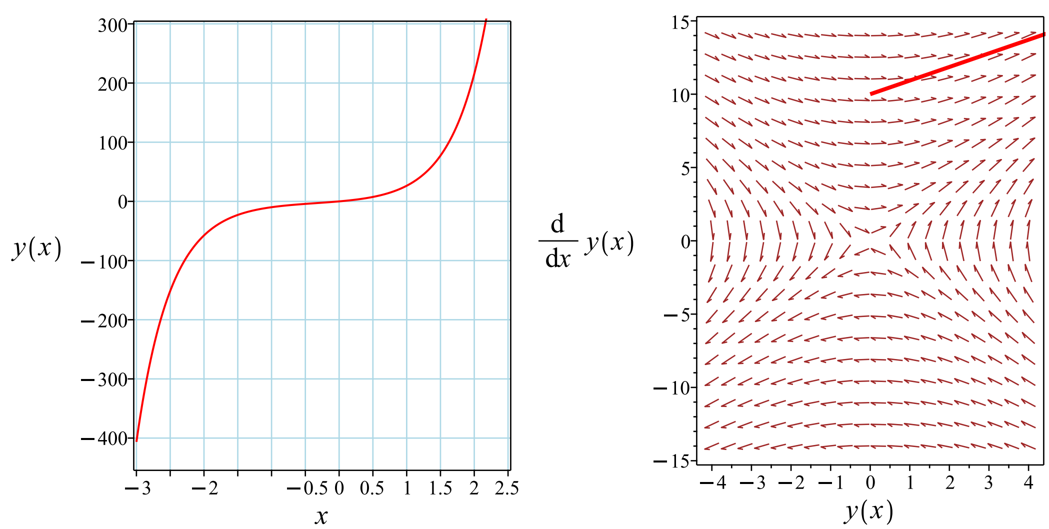
$$10 = \frac{-1 + \left(-2c_1 + 36\right) c_5^4}{2c_5^2} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_5\}$. Warning, unable to solve for constants of integrations.

Summary

The solution(s) found are the following

$$y = -3 + 4\,\mathrm{e}^{2x} - \mathrm{e}^{-2x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -3 + 4\,\mathrm{e}^{2x} - \mathrm{e}^{-2x}$$

Verified OK.

8.2.4 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 387: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1 \\ &= \mathrm{e}^{-2x}\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\,dx \\ &= \mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{2x}}{4}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1 = 12$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -3$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}\right) + (-3)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4} - 3 \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$
0 = c_1 + \frac{c_2}{4} - 3 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -2c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{2}
$$

substituting $y' = 10$ and $x = 0$ in the above gives

$$
10 = -2c_1 + \frac{c_2}{2} \tag{2A}
$$

Equations $\{1A, 2A\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -1 \\
c_2 &= 16
\end{aligned}
$$

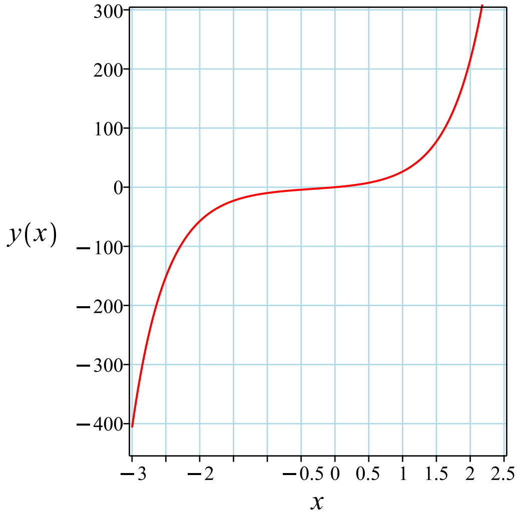
Substituting these values back in above solution results in

$$
y = -3 + 4\,\mathrm{e}^{2x} - \mathrm{e}^{-2x}
$$

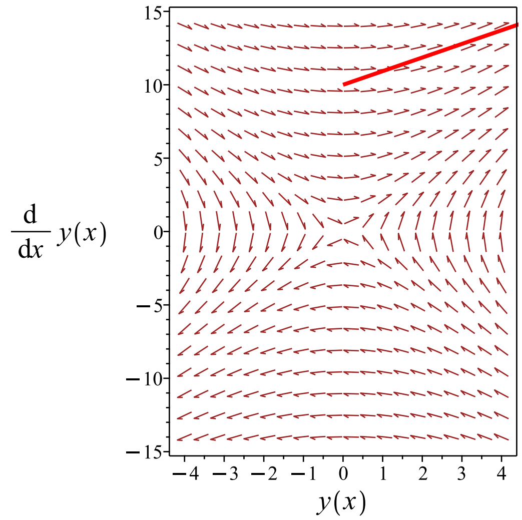
Summary

The solution(s) found are the following

$$
y = -3 + 4\,\mathrm{e}^{2x} - \mathrm{e}^{-2x} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -3 + 4\,e^{2x} - e^{-2x}$$

Verified OK.

8.2.5 Maple step by step solution

Let's solve

$$\left[y'' - 4y = 12, y(0) = 0, y'\Big|_{\{x=0\}} = 10\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r-2)(r+2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 12\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{2x} \\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -3\,\mathrm{e}^{-2x}\left(\int \mathrm{e}^{2x} dx\right) + 3\,\mathrm{e}^{2x}\left(\int \mathrm{e}^{-2x} dx\right)$

  - Compute integrals

    $y_p(x) = -3$

- Substitute particular solution into general solution to ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} - 3$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{2x} - 3$

  - Use initial condition $y(0) = 0$

    $0 = c_1 + c_2 - 3$

  - Compute derivative of the solution

    $y' = -2c_1 \mathrm{e}^{-2x} + 2c_2 \mathrm{e}^{2x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 10$

    $10 = -2c_1 + 2c_2$

  - Solve for c_1 and c_2

$\{c_1 = -1, c_2 = 4\}$

- Substitute constant values into general solution and simplify

  $y = -3 + 4\,e^{2x} - e^{-2x}$

- Solution to the IVP

  $y = -3 + 4\,e^{2x} - e^{-2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$2)-4*y(x)=12,y(0) = 0, D(y)(0) = 10],y(x), singsol=all)
```

$$y(x) = 4\,e^{2x} - e^{-2x} - 3$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 21

```
DSolve[{y''[x]-4*y[x]==12,{y[0]==0,y'[0]==10}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -e^{-2x} + 4e^{2x} - 3$$

8.3 problem 23

Internal problem ID [190]
Internal file name [`OUTPUT/190_Sunday_June_05_2022_01_36_31_AM_52050187/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.2, second order linear equations. Page 311
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2y' - 3y = 6}$$

With initial conditions

$$[y(0) = 3, y'(0) = 11]$$

8.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= -3 \\ F &= 6 \end{aligned}$$

Hence the ode is

$$y'' - 2y' - 3y = 6$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = -3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.3.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -3, f(x) = 6$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} - 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(-3)} \\ &= 1 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 2 \\ \lambda_2 &= 1 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(3)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 = 6$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 e^{-x}\right) + (-2) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + c_2 e^{-x} - 2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + c_2 - 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} - c_2 e^{-x}$$

substituting $y' = 11$ and $x = 0$ in the above gives

$$11 = 3c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 1$$

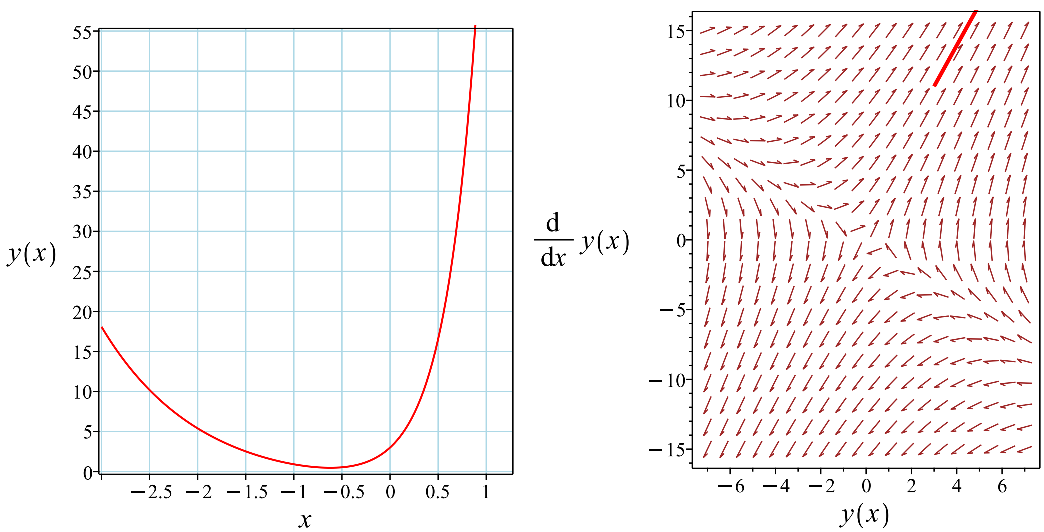
Substituting these values back in above solution results in

$$y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x}$$

Summary

The solution(s) found are the following

$$y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x}$$

Verified OK.

8.3.3 Solving using Kovacic algorithm

Writing the ode as

$$
\begin{align*}
y'' - 2y' - 3y &= 0 \tag{1}\\
Ay'' + By' + Cy &= 0 \tag{2}
\end{align*}
$$

Comparing (1) and (2) shows that

$$
\begin{align*}
A &= 1\\
B &= -2 \tag{3}\\
C &= -3
\end{align*}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{align*}
r &= \frac{s}{t} \tag{5}\\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{align*}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{align*}
s &= 4\\
t &= 1
\end{align*}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\,e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 389: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\mathrm{e}^{-x}) + c_2\left(\mathrm{e}^{-x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' - 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-x} + \frac{c_2\mathrm{e}^{3x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{4}, e^{-x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 = 6$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = -2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + \frac{c_2 e^{3x}}{4}\right) + (-2) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{3x}}{4} - 2 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 + \frac{c_2}{4} - 2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \mathrm{e}^{-x} + \frac{3c_2 \mathrm{e}^{3x}}{4}$$

substituting $y' = 11$ and $x = 0$ in the above gives

$$11 = -c_1 + \frac{3c_2}{4} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 1 \\ c_2 &= 16 \end{aligned}$$

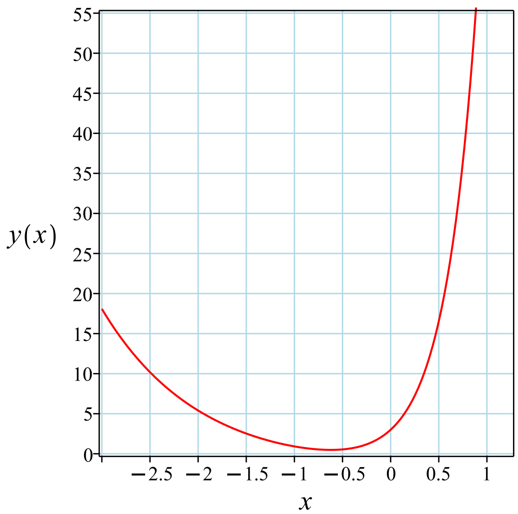
Substituting these values back in above solution results in

$$y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x}$$

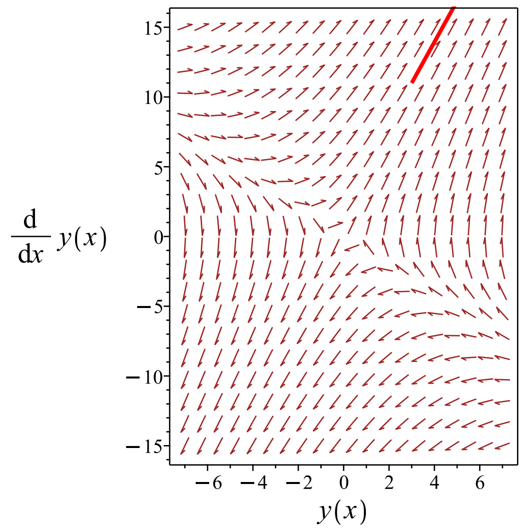
Summary

The solution(s) found are the following

$$y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x}$$

Verified OK.

8.3.4 Maple step by step solution

Let's solve

$$\left[y'' - 2y' - 3y = 6, y(0) = 3, y'\Big|_{\{x=0\}} = 11\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (-1, 3)$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1y_1(x) + c_2y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1e^{-x} + c_2e^{3x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 6\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 4e^{2x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{3e^{-x}\left(\int e^{x}dx\right)}{2} + \frac{3e^{3x}\left(\int e^{-3x}dx\right)}{2}$

- Compute integrals

  $y_p(x) = -2$

- Substitute particular solution into general solution to ODE

  $y = c_1e^{-x} + c_2e^{3x} - 2$

□ Check validity of solution $y = c_1e^{-x} + c_2e^{3x} - 2$

- Use initial condition $y(0) = 3$

  $3 = c_1 + c_2 - 2$

- Compute derivative of the solution

  $y' = -c_1e^{-x} + 3c_2e^{3x}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 11$

  $11 = -c_1 + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = 4\}$

- Substitute constant values into general solution and simplify

  $y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x}$

- Solution to the IVP

  $y = -2 + 4\,\mathrm{e}^{3x} + \mathrm{e}^{-x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 16

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)-3*y(x)=6,y(0) = 3, D(y)(0) = 11],y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-x} + 4\,\mathrm{e}^{3x} - 2$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 19

```
DSolve[{y''[x]-2*y'[x]-3*y[x]==6,{y[0]==3,y'[0]==11}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \rightarrow e^{-x} + 4e^{3x} - 2$$

8.4 problem 24

Internal problem ID [191]
Internal file name [`OUTPUT/191_Sunday_June_05_2022_01_36_32_AM_11390539/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.2, second order linear equations. Page 311
Problem number: 24.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2y' + 2y = 2x}$$

With initial conditions

$$[y(0) = 4, y'(0) = 8]$$

8.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= 2x \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = 2x$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

8.4.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(2)}\\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i\\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,e^x, \sin(x)\,e^x\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 - 2A_2 = 2x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^x(c_1 \cos(x) + c_2 \sin(x))) + (x + 1) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^x(c_1 \cos(x) + c_2 \sin(x)) + x + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(-c_1 \sin(x) + c_2 \cos(x)) + 1$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = c_1 + c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

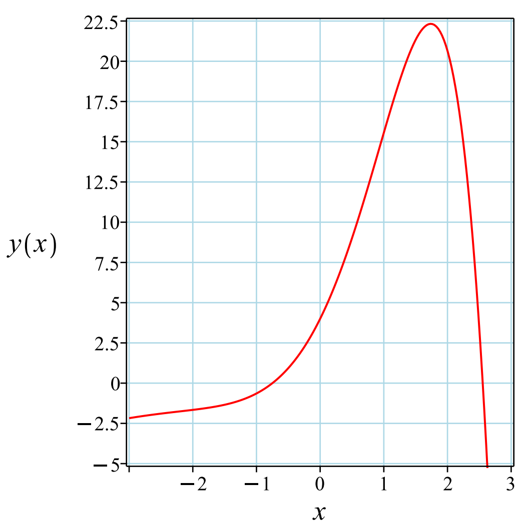
$$c_1 = 3$$
$$c_2 = 4$$

Substituting these values back in above solution results in

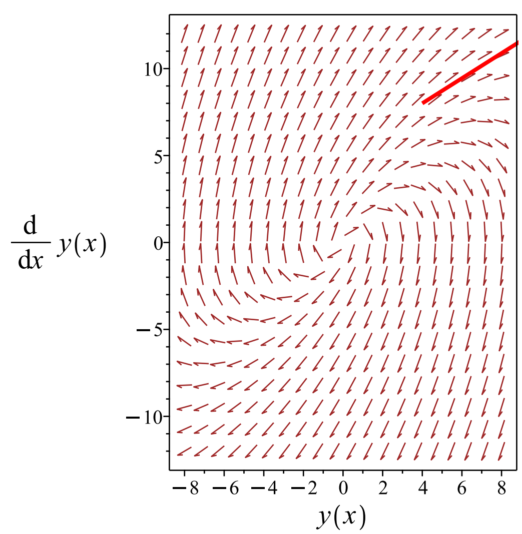
$$y = 1 + 3\cos(x)\,e^{x} + 4\sin(x)\,e^{x} + x$$

Summary
The solution(s) found are the following

$$y = 1 + 3\cos(x)\,e^{x} + 4\sin(x)\,e^{x} + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + 3\cos(x)\,e^{x} + 4\sin(x)\,e^{x} + x$$

Verified OK.

8.4.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 391: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)\,\mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)\,\mathrm{e}^x) + c_2(\cos(x)\,\mathrm{e}^x(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x)\,\mathrm{e}^x + \mathrm{e}^x \sin(x)\,c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,\mathrm{e}^{x}, \sin(x)\,\mathrm{e}^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 - 2A_2 = 2x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x)\,\mathrm{e}^{x} + \mathrm{e}^{x} \sin(x)\, c_2) + (x + 1) \end{aligned}$$

Which simplifies to

$$y = \mathrm{e}^{x}(c_1 \cos(x) + c_2 \sin(x)) + x + 1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + x + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + e^{x}(-c_1 \sin(x) + c_2 \cos(x)) + 1$$

substituting $y' = 8$ and $x = 0$ in the above gives

$$8 = c_1 + c_2 + 1 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

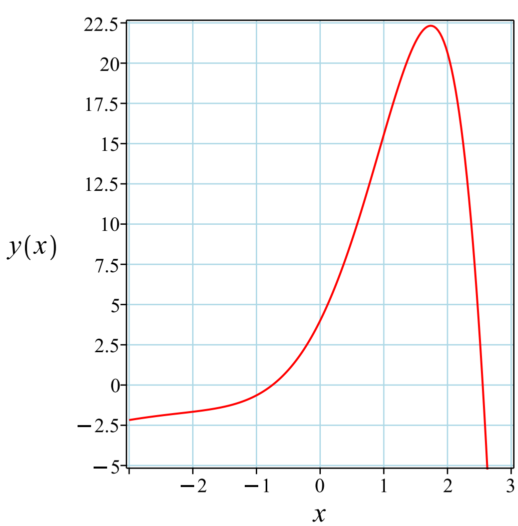
$$c_1 = 3$$
$$c_2 = 4$$

Substituting these values back in above solution results in

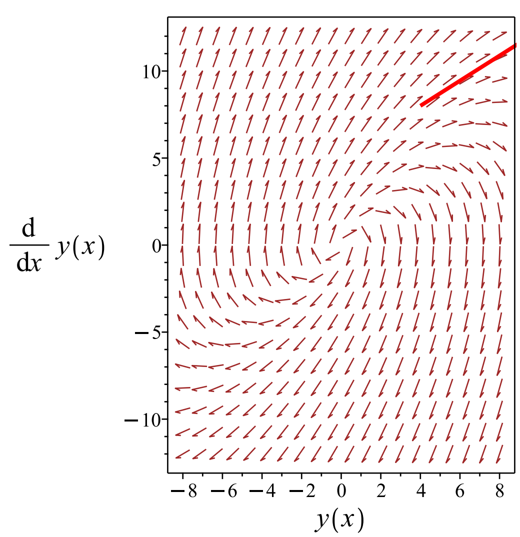
$$y = 1 + 3\cos(x) e^{x} + 4\sin(x) e^{x} + x$$

Summary
The solution(s) found are the following

$$y = 1 + 3\cos(x) e^{x} + 4\sin(x) e^{x} + x \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 + 3\cos(x)\,\mathrm{e}^{x} + 4\sin(x)\,\mathrm{e}^{x} + x$$

Verified OK.

8.4.4 Maple step by step solution

Let's solve

$\left[y'' - 2y' + 2y = 2x, y(0) = 4, y'\Big|_{\{x=0\}} = 8\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)\,\mathrm{e}^{x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\,\mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos(x)\,\mathrm{e}^{x} + \mathrm{e}^{x}\sin(x)\,c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2x\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\,e^x & \sin(x)\,e^x \\ -\sin(x)\,e^x + \cos(x)\,e^x & \cos(x)\,e^x + \sin(x)\,e^x \end{bmatrix}$$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -2\,e^x \left(\cos(x)\left(\int \sin(x)\,x\,e^{-x}dx\right) - \sin(x)\left(\int x\cos(x)\,e^{-x}dx\right)\right)$

- ○ Compute integrals

  $y_p(x) = x + 1$

- ● Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x)\,e^x + e^x \sin(x)\,c_2 + x + 1$

- □ Check validity of solution $y = c_1 \cos(x)\,e^x + e^x \sin(x)\,c_2 + x + 1$

- ○ Use initial condition $y(0) = 4$

  $4 = 1 + c_1$

- ○ Compute derivative of the solution

  $y' = -c_1 \sin(x)\,e^x + c_1 \cos(x)\,e^x + e^x \sin(x)\,c_2 + e^x \cos(x)\,c_2 + 1$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = 8$

  $8 = c_1 + c_2 + 1$

- ○ Solve for c_1 and c_2

  $\{c_1 = 3, c_2 = 4\}$

- ○ Substitute constant values into general solution and simplify

  $y = x + 1 + (3\cos(x) + 4\sin(x))\,e^x$

- ● Solution to the IVP

  $y = x + 1 + (3\cos(x) + 4\sin(x))\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 19

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=2*x,y(0) = 4, D(y)(0) = 8],y(x), singsol=all)
```

$$y(x) = x + 1 + (4\sin(x) + 3\cos(x))\,\mathrm{e}^{x}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 22

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==2*x,{y[0]==4,y'[0]==8}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to x + 4e^{x}\sin(x) + 3e^{x}\cos(x) + 1$$

8.5 problem 26(a.1)

Internal problem ID [192]
Internal file name [`OUTPUT/192_Sunday_June_05_2022_01_36_33_AM_62275496/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.2, second order linear equations. Page 311
Problem number: 26(a.1).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 2y = 4}$$

8.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Or

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\, x\right), \sin\left(\sqrt{2}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) + (2) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 2 \tag{1}$$

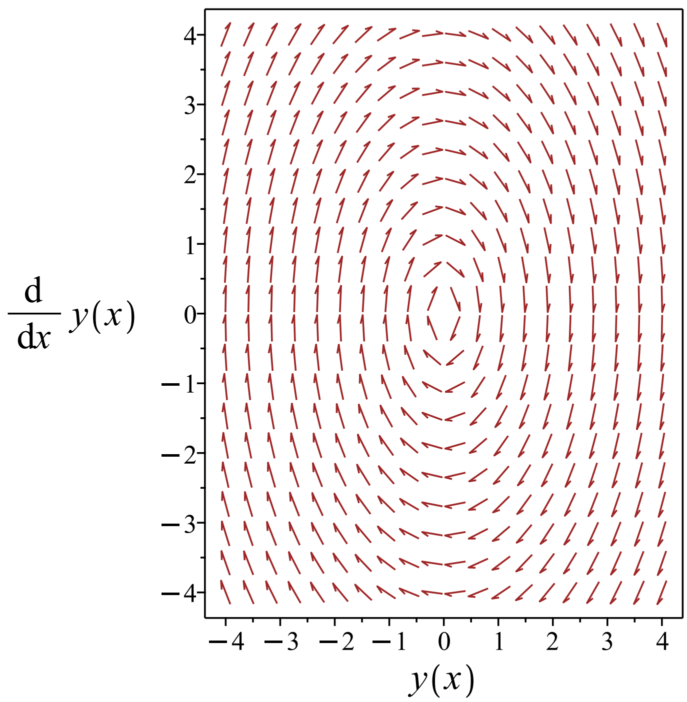


Figure 541: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 2$$

Verified OK.

8.5.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' + 2yy' - 4y' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' + 2yy' - 4y')\, dx = 0$$
$$\frac{y'^2}{2} + y^2 - 4y = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{-2y^2 + 8y + 2c_1} \tag{1}$$
$$y' = -\sqrt{-2y^2 + 8y + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{1}{\sqrt{-2y^2 + 2c_1 + 8y}} dy = \int dx$$
$$\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y-2)}{\sqrt{-2y^2+8y+2c_1}}\right)}{2} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{1}{\sqrt{-2y^2 + 2c_1 + 8y}} dy = \int dx$$
$$-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y-2)}{\sqrt{-2y^2+8y+2c_1}}\right)}{2} = c_3 + x$$

Summary

The solution(s) found are the following

$$\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y-2)}{\sqrt{-2y^2+8y+2c_1}}\right)}{2} = x + c_2 \tag{1}$$
$$-\frac{\sqrt{2}\,\arctan\left(\frac{\sqrt{2}\,(y-2)}{\sqrt{-2y^2+8y+2c_1}}\right)}{2} = c_3 + x \tag{2}$$

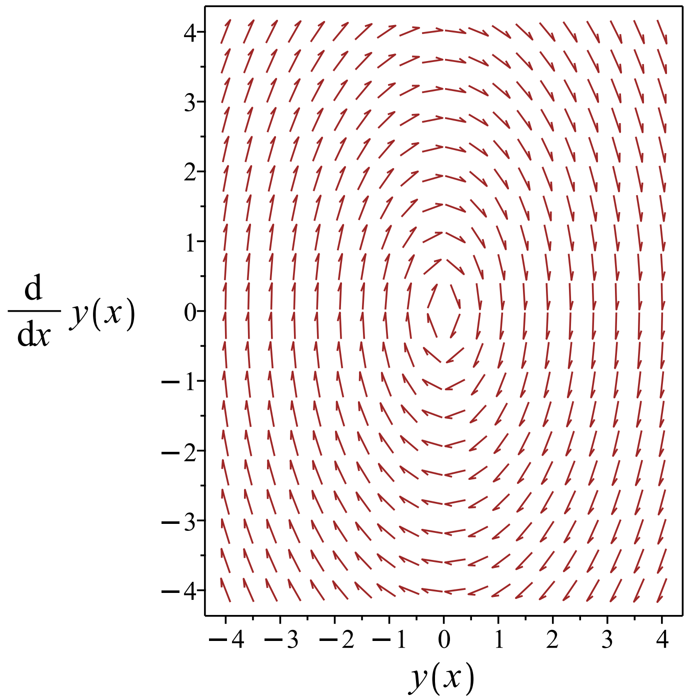


Figure 542: Slope field plot

Verification of solutions

$$\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\,(y-2)}{\sqrt{-2y^2+8y+2c_1}}\right)}{2} = x + c_2$$

Verified OK.

$$-\frac{\sqrt{2}\, \arctan\left(\frac{\sqrt{2}\,(y-2)}{\sqrt{-2y^2+8y+2c_1}}\right)}{2} = c_3 + x$$

Verified OK.

8.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 393: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{2}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos\left(\sqrt{2}\, x\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \int \frac{1}{\cos\left(\sqrt{2}\, x\right)^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, x\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}, \cos\left(\sqrt{2}\,x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 = 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}\right) + (2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} + 2 \tag{1}$$

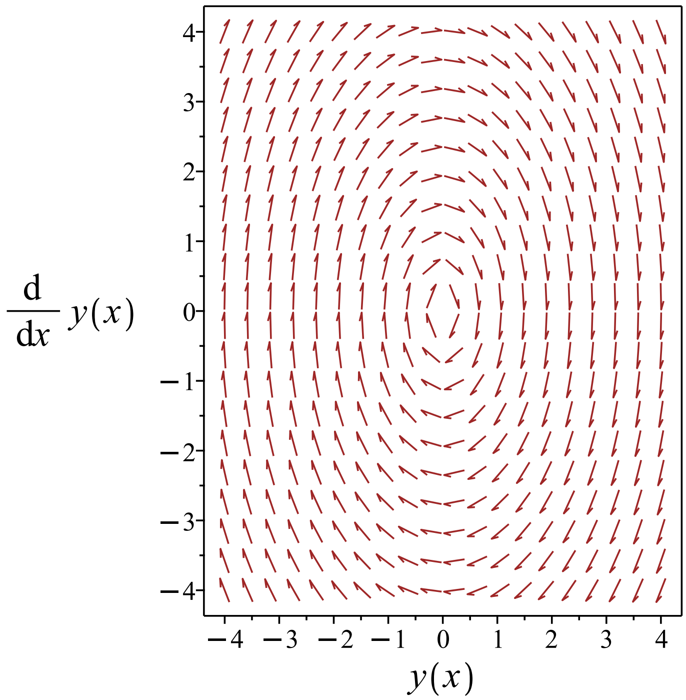


Figure 543: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} + 2$$

Verified OK.

8.5.4 Maple step by step solution

Let's solve

$y'' + 2y = 4$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

$r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos\left(\sqrt{2}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin\left(\sqrt{2}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 4\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos\left(\sqrt{2}\, x\right) & \sin\left(\sqrt{2}\, x\right) \\ -\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) & \sqrt{2}\, \cos\left(\sqrt{2}\, x\right) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \sqrt{2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\sqrt{2}\left(\cos\left(\sqrt{2}\, x\right)\left(\int \sin\left(\sqrt{2}\, x\right) dx\right) - \sin\left(\sqrt{2}\, x\right)\left(\int \cos\left(\sqrt{2}\, x\right) dx\right)\right)$

  - Compute integrals

    $y_p(x) = 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+2*y(x)=4,y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{2}\,x\right) c_2 + \cos\left(\sqrt{2}\,x\right) c_1 + 2$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 29

```
DSolve[y''[x]+2*y[x]==4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right) + 2$$

8.6 problem 26(a.2)

Internal problem ID [193]
Internal file name [`OUTPUT/193_Sunday_June_05_2022_01_36_34_AM_12734694/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.2, second order linear equations. Page 311
Problem number: 26(a.2).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y = 6x}$$

8.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = 6x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A=1, B=0, C=2$. Let the solution be $y=e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x}+2\,\mathrm{e}^{\lambda x}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2+2=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1, B=0, C=2$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{0^2-(4)(1)(2)}\\&=\pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=+i\sqrt{2}\\\lambda_2&=-i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=i\sqrt{2}\\\lambda_2&=-i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=0$ and $\beta=\sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y=e^{\alpha x}(c_1\cos(\beta x)+c_2\sin(\beta x))$$

Which becomes

$$y=e^0\left(c_1\cos\left(\sqrt{2}\,x\right)+c_2\sin\left(\sqrt{2}\,x\right)\right)$$

Or

$$y=c_1\cos\left(\sqrt{2}\,x\right)+c_2\sin\left(\sqrt{2}\,x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\, x\right), \sin\left(\sqrt{2}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 = 6x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) + (3x) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 3x \tag{1}$$

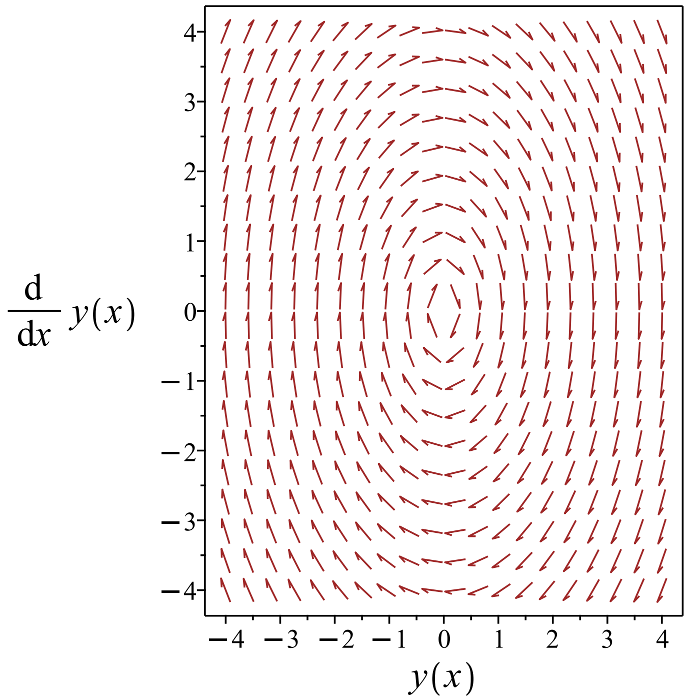


Figure 544: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 3x$$

Verified OK.

8.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned}A &= 1\\B &= 0 \\C &= 2\end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned}r &= \frac{s}{t}\\&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}\end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned}s &= -2\\t &= 1\end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 395: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\sqrt{2}\, x\right)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos\left(\sqrt{2}\, x\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \int \frac{1}{\cos\left(\sqrt{2}\, x\right)^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, x\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}, \cos\left(\sqrt{2}\,x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 = 6x$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}\right) + (3x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} + 3x \tag{1}$$

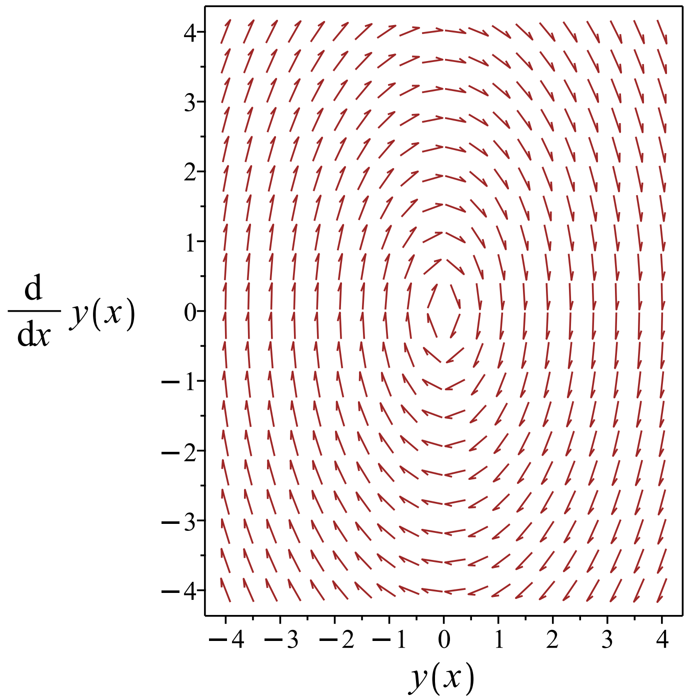


Figure 545: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2\sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2} + 3x$$

Verified OK.

8.6.3 Maple step by step solution

Let's solve

$y'' + 2y = 6x$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

$r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos\left(\sqrt{2}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin\left(\sqrt{2}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 6x\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \left[\begin{array}{cc} \cos\left(\sqrt{2}\, x\right) & \sin\left(\sqrt{2}\, x\right) \\ -\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) & \sqrt{2}\, \cos\left(\sqrt{2}\, x\right) \end{array}\right]$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \sqrt{2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -3\sqrt{2}\left(\cos\left(\sqrt{2}\, x\right)\left(\int \sin\left(\sqrt{2}\, x\right) x dx\right) - \sin\left(\sqrt{2}\, x\right)\left(\int \cos\left(\sqrt{2}\, x\right) x dx\right)\right)$

  - Compute integrals

    $y_p(x) = 3x$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 3x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+2*y(x)=6*x,y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{2}\,x\right) c_2 + \cos\left(\sqrt{2}\,x\right) c_1 + 3x$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 31

```
DSolve[y''[x]+2*y[x]==6*x,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x + c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right)$$

8.7 problem 26(b)

Internal problem ID [194]
Internal file name [OUTPUT/194_Sunday_June_05_2022_01_36_35_AM_18444330/index.tex]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.2, second order linear equations. Page 311
Problem number: 26(b).
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 2y = 6x + 4}$$

8.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 2, f(x) = 6x + 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(2)} \\ &= \pm i\sqrt{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\sqrt{2} \\ \lambda_2 &= -i\sqrt{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = \sqrt{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0\left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right)$$

Or

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\cos\left(\sqrt{2}\, x\right), \sin\left(\sqrt{2}\, x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 = 6x + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x + 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right)\right) + (3x + 2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 3x + 2 \tag{1}$$

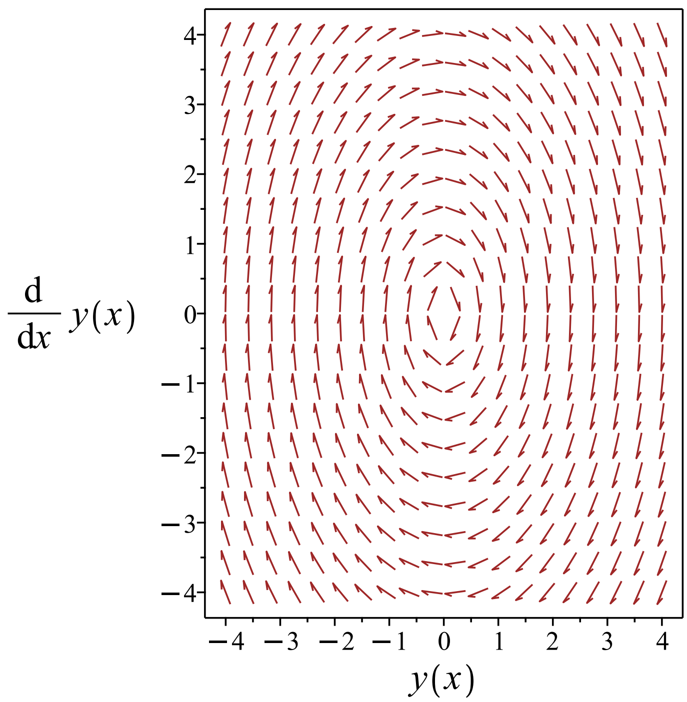


Figure 546: Slope field plot

Verification of solutions

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 3x + 2$$

Verified OK.

8.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-2}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -2 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -2z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 397: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -2$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\sqrt{2}\, x\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos\left(\sqrt{2}\, x\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\sqrt{2}\, x\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \int \frac{1}{\cos\left(\sqrt{2}\, x\right)^2}\, dx \\
&= \cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\sqrt{2}\, x\right)\right) + c_2 \left(\cos\left(\sqrt{2}\, x\right) \left(\frac{\sqrt{2}\, \tan\left(\sqrt{2}\, x\right)}{2}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}, \cos\left(\sqrt{2}\,x\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 = 6x + 4$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 3]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = 3x + 2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2}\right) + (3x + 2) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos\left(\sqrt{2}\,x\right) + \frac{c_2\sqrt{2}\,\sin\left(\sqrt{2}\,x\right)}{2} + 3x + 2 \tag{1}$$

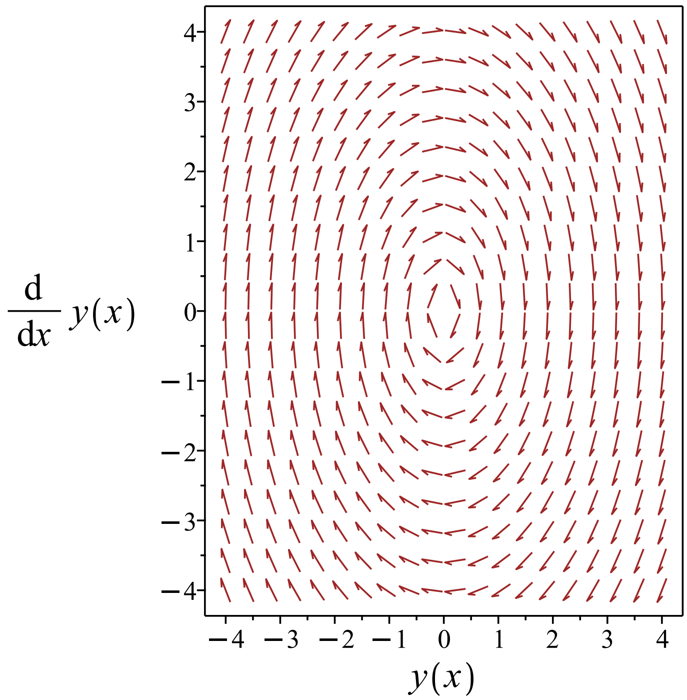


Figure 547: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos\left(\sqrt{2}\, x\right) + \frac{c_2 \sqrt{2}\, \sin\left(\sqrt{2}\, x\right)}{2} + 3x + 2$$

Verified OK.

8.7.3 Maple step by step solution

Let's solve

$y'' + 2y = 6x + 4$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-8})}{2}$

- Roots of the characteristic polynomial

$r = \left(-\mathrm{I}\sqrt{2}, \mathrm{I}\sqrt{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos\left(\sqrt{2}\, x\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin\left(\sqrt{2}\, x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 6x + 4\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos\left(\sqrt{2}\, x\right) & \sin\left(\sqrt{2}\, x\right) \\ -\sqrt{2}\, \sin\left(\sqrt{2}\, x\right) & \sqrt{2}\, \cos\left(\sqrt{2}\, x\right) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \sqrt{2}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \sqrt{2}\left(-\cos\left(\sqrt{2}\, x\right)\left(\int \sin\left(\sqrt{2}\, x\right)(3x+2)\, dx\right) + \sin\left(\sqrt{2}\, x\right)\left(\int \cos\left(\sqrt{2}\, x\right)(3x+2)\, dx\right)\right)$

  - Compute integrals

    $y_p(x) = 3x + 2$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos\left(\sqrt{2}\, x\right) + c_2 \sin\left(\sqrt{2}\, x\right) + 3x + 2$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)+2*y(x)=6*x+4,y(x), singsol=all)
```

$$y(x) = \sin\left(\sqrt{2}\,x\right) c_2 + \cos\left(\sqrt{2}\,x\right) c_1 + 3x + 2$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 32

```
DSolve[y''[x]+2*y[x]==6*x+4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x + c_1 \cos\left(\sqrt{2}x\right) + c_2 \sin\left(\sqrt{2}x\right) + 2$$

9 Section 5.3, second order linear equations. Page 323

9.1 problem 1

Internal problem ID [195]
Internal file name [`OUTPUT/195_Sunday_June_05_2022_01_36_36_AM_14768880/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y = 0}$$

9.1.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x} \tag{1}$$

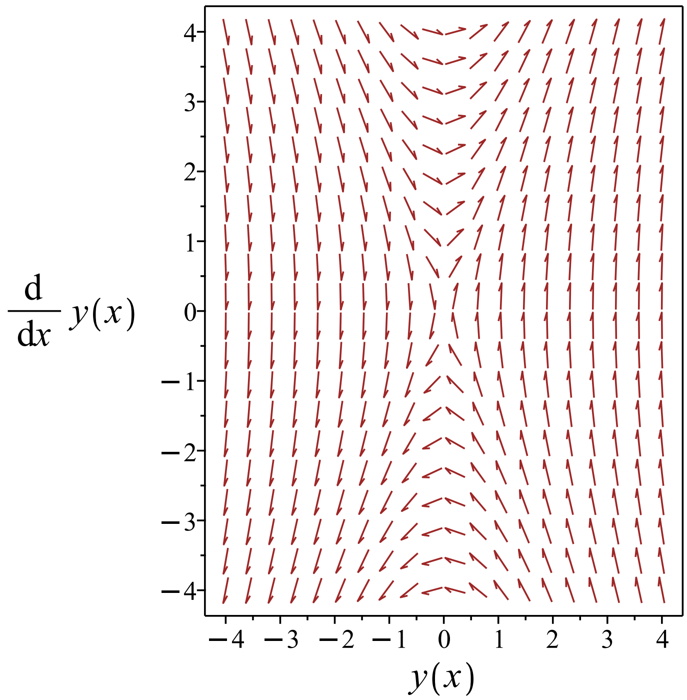


Figure 548: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

Verified OK.

9.1.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$y'y'' - 4yy' = 0$$

Integrating the above w.r.t x gives

$$\int (y'y'' - 4yy')\, dx = 0$$
$$\frac{y'^2}{2} - 2y^2 = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = \sqrt{4y^2 + 2c_1} \tag{1}$$
$$y' = -\sqrt{4y^2 + 2c_1} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$
\int \frac{1}{\sqrt{4y^2+2c_1}}dy = \int dx
$$
$$
\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4} = x+c_2
$$

Raising both side to exponential gives

$$
e^{\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4}} = e^{x+c_2}
$$

Which simplifies to

$$
\sqrt{2y+\sqrt{4y^2+2c_1}} = c_3 e^x
$$

Solving equation (2)

Integrating both sides gives

$$
\int -\frac{1}{\sqrt{4y^2+2c_1}}dy = \int dx
$$
$$
-\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4} = x+c_4
$$

Raising both side to exponential gives

$$
e^{-\frac{\ln\left(y\sqrt{4}+\sqrt{4y^2+2c_1}\right)\sqrt{4}}{4}} = e^{x+c_4}
$$

Which simplifies to

$$
\frac{1}{\sqrt{2y+\sqrt{4y^2+2c_1}}} = c_5 e^x
$$

Summary

The solution(s) found are the following

$$
y = \frac{\left(e^{4x}c_3^4-2c_1\right)e^{-2x}}{4c_3^2} \tag{1}
$$
$$
y = -\frac{\left(2c_1c_5^4e^{4x}-1\right)e^{-2x}}{4c_5^2} \tag{2}
$$

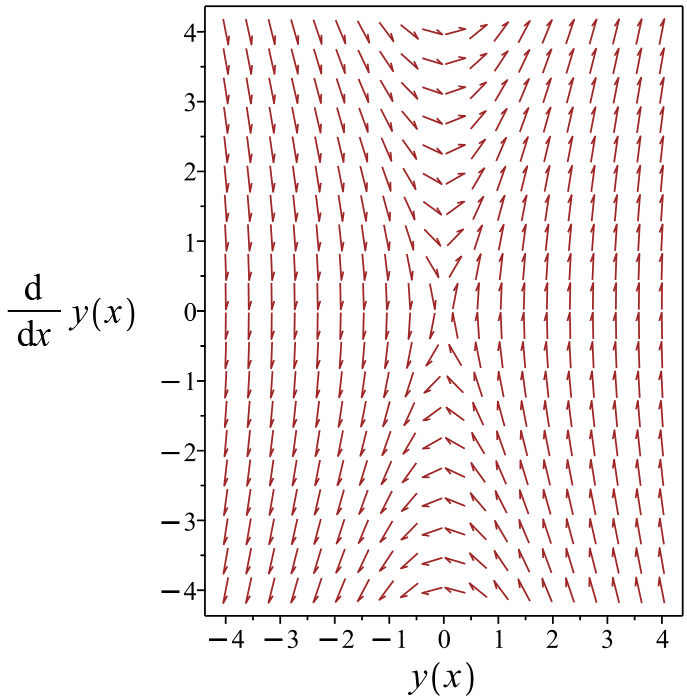


Figure 549: Slope field plot

Verification of solutions

$$y = \frac{(e^{4x}c_3^4 - 2c_1)\,e^{-2x}}{4c_3^2}$$

Verified OK.

$$y = -\frac{(2c_1c_5^4e^{4x} - 1)\,e^{-2x}}{4c_5^2}$$

Verified OK.

9.1.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= -4
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 399: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = e^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= e^{-2x}
\end{aligned}
$$

Which simplifies to

$$y_1 = e^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= e^{-2x} \int \frac{1}{e^{-4x}}\, dx \\ &= e^{-2x} \left(\frac{e^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x} \left(\frac{e^{4x}}{4}\right)\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} \tag{1}$$

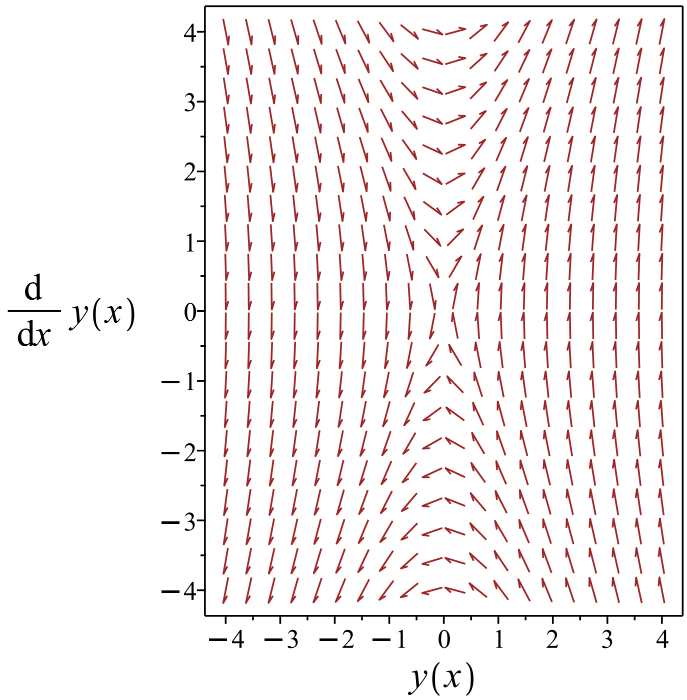


Figure 550: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}$$

Verified OK.

9.1.4 Maple step by step solution

Let's solve

$y'' - 4y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r + 2) = 0$

- Roots of the characteristic polynomial

$$r = (-2, 2)$$

- 1st solution of the ODE

$$y_1(x) = e^{-2x}$$

- 2nd solution of the ODE

$$y_2(x) = e^{2x}$$

- General solution of the ODE

$$y = c_1 y_1(x) + c_2 y_2(x)$$

- Substitute in solutions

$$y = c_1 e^{-2x} + c_2 e^{2x}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-4*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{2x} c_1 + c_2 e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 22

```
DSolve[y''[x]-4*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2x}\left(c_1 e^{4x} + c_2\right)$$

9.2 problem 2

Internal problem ID [196]
Internal file name [`OUTPUT/196_Sunday_June_05_2022_01_36_36_AM_2377207/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "exact linear second order ode", "second_order_integrable_as_is", "second_order_ode_missing_y", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' - 3y' = 0}$$

9.2.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = -3, C = 0$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} - 3\lambda e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - 3\lambda = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -3, C = 0$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{3}{(2)(2)} \pm \frac{1}{(2)(2)}\sqrt{-3^2 - (4)(2)(0)}\\ &= \frac{3}{4} \pm \frac{3}{4}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{3}{4} + \frac{3}{4}\\ \lambda_2 &= \frac{3}{4} - \frac{3}{4}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{3}{2}\\ \lambda_2 &= 0\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{\left(\frac{3}{2}\right)x} + c_2 e^{(0)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{\frac{3x}{2}} + c_2$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\frac{3x}{2}} + c_2 \tag{1}$$

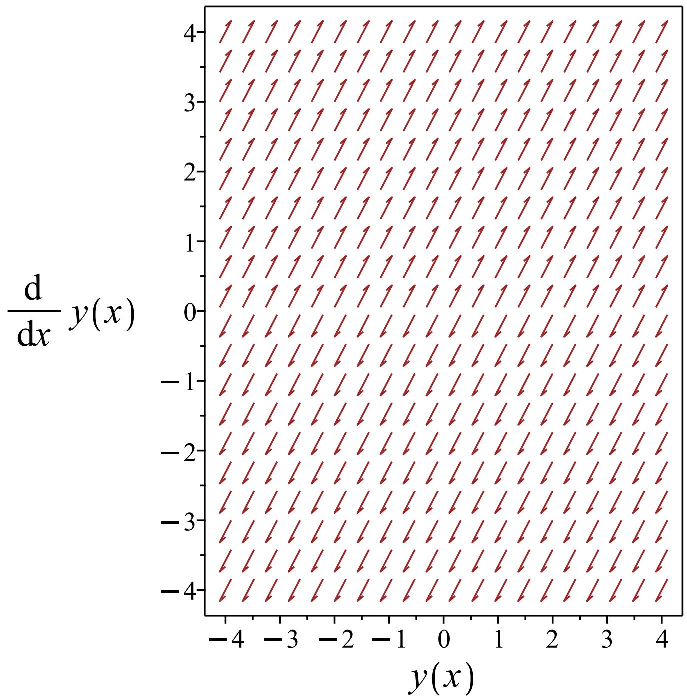


Figure 551: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{\frac{3x}{2}} + c_2$$

Verified OK.

9.2.2 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$\int (2y'' - 3y')\, dx = 0$$
$$2y' - 3y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{\frac{3y}{2} + \frac{c_1}{2}} dy = \int dx$$
$$\frac{2\ln(3y + c_1)}{3} = x + c_2$$

Raising both side to exponential gives

$$(3y + c_1)^{\frac{2}{3}} = e^{x+c_2}$$

Which simplifies to

$$(3y + c_1)^{\frac{2}{3}} = c_3 e^x$$

Summary

The solution(s) found are the following

$$y = \frac{(c_3 e^x)^{\frac{3}{2}}}{3} - \frac{c_1}{3} \tag{1}$$

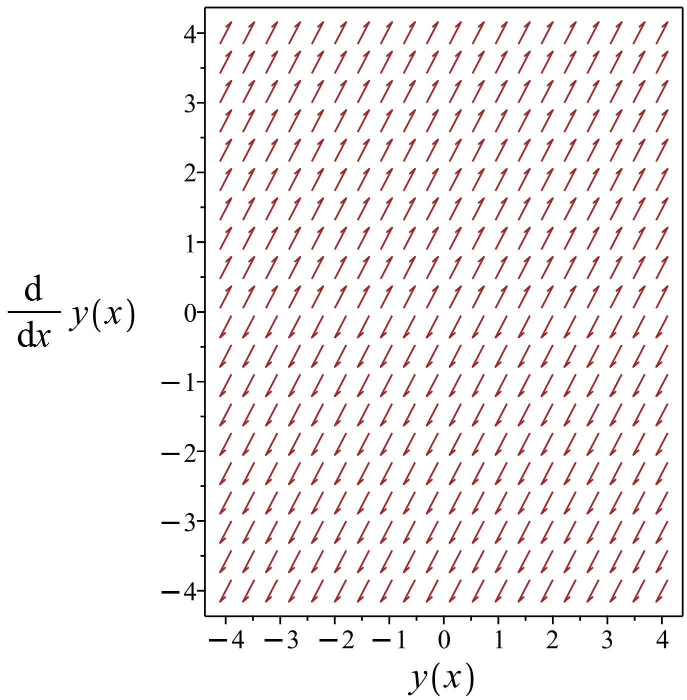


Figure 552: Slope field plot

Verification of solutions

$$y = \frac{(c_3 e^x)^{\frac{3}{2}}}{3} - \frac{c_1}{3}$$

Verified OK.

9.2.3 Solving as second order ode missing y ode

This is second order ode with missing dependent variable y. Let

$$p(x) = y'$$

Then

$$p'(x) = y''$$

Hence the ode becomes

$$2p'(x) - 3p(x) = 0$$

Which is now solve for $p(x)$ as first order ode. Integrating both sides gives

$$\begin{aligned}\int \frac{2}{3p} dp &= \int dx \\ \frac{2\ln(p)}{3} &= x + c_1\end{aligned}$$

Raising both side to exponential gives

$$p^{\frac{2}{3}} = e^{x+c_1}$$

Which simplifies to

$$p^{\frac{2}{3}} = c_2 e^x$$

Since $p = y'$ then the new first order ode to solve is

$$y' = (c_2 e^x)^{\frac{3}{2}}$$

Integrating both sides gives

$$\begin{aligned}y &= \int (c_2 e^x)^{\frac{3}{2}} \, dx \\ &= \frac{2(c_2 e^x)^{\frac{3}{2}}}{3} + c_3\end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{2(c_2 e^x)^{\frac{3}{2}}}{3} + c_3 \tag{1}$$

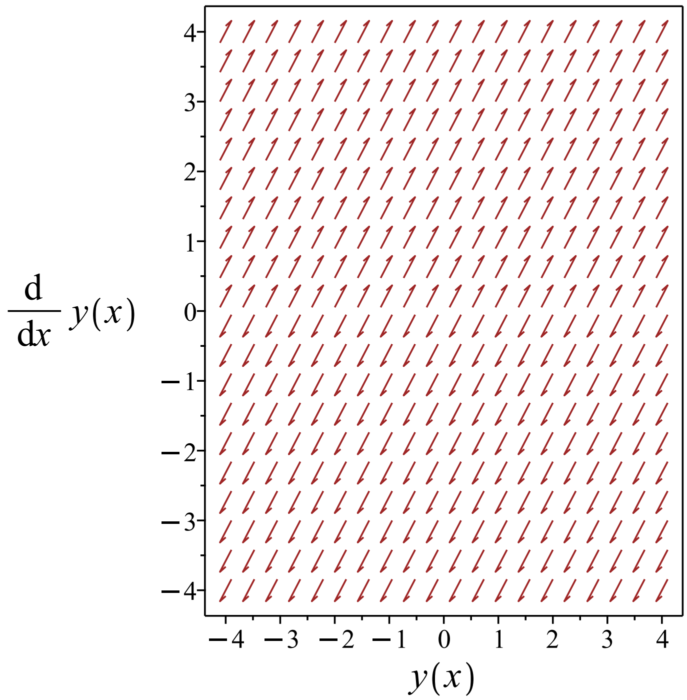


Figure 553: Slope field plot

Verification of solutions

$$y = \frac{2(c_2 e^x)^{\frac{3}{2}}}{3} + c_3$$

Verified OK.

9.2.4 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$2y'' - 3y' = 0$$

Integrating both sides of the ODE w.r.t x gives

$$\int (2y'' - 3y')\,dx = 0$$
$$2y' - 3y = c_1$$

Which is now solved for y. Integrating both sides gives

$$\int \frac{1}{\frac{3y}{2}+\frac{c_1}{2}}dy = \int dx$$
$$\frac{2\ln\left(3y+c_1\right)}{3} = x + c_2$$

Raising both side to exponential gives

$$\left(3y+c_1\right)^{\frac{2}{3}} = \mathrm{e}^{x+c_2}$$

Which simplifies to

$$\left(3y+c_1\right)^{\frac{2}{3}} = c_3\mathrm{e}^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_3\mathrm{e}^{x}\right)^{\frac{3}{2}}}{3} - \frac{c_1}{3} \tag{1}$$

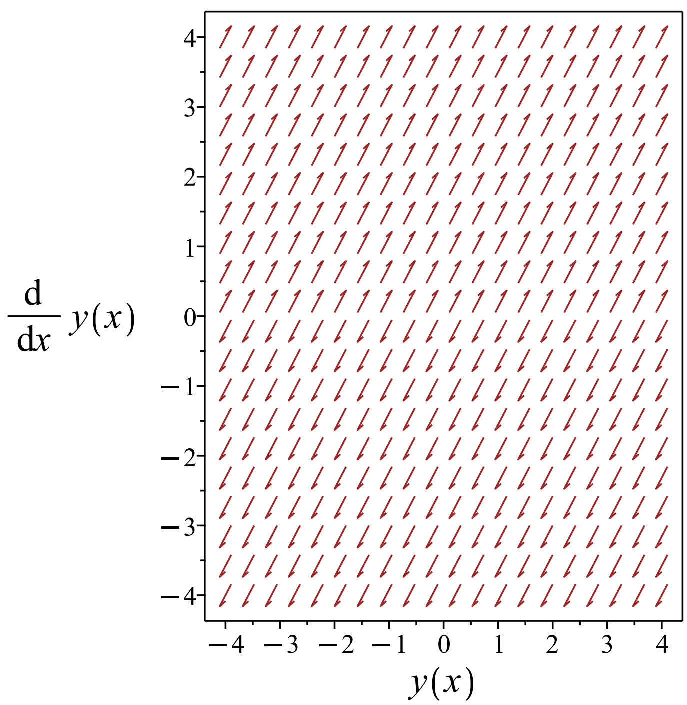


Figure 554: Slope field plot

Verification of solutions

$$y = \frac{(c_3 e^x)^{\frac{3}{2}}}{3} - \frac{c_1}{3}$$

Verified OK.

9.2.5 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - 3y' = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -3 \\ C &= 0 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 401: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-3}{2}\,dx} \\ &= z_1 e^{\frac{3x}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{3x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = 1$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-3}{2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{3x}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(1) + c_2\left(1\left(\frac{2\,\mathrm{e}^{\frac{3x}{2}}}{3}\right)\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 + \frac{2c_2 e^{\frac{3x}{2}}}{3} \tag{1}$$

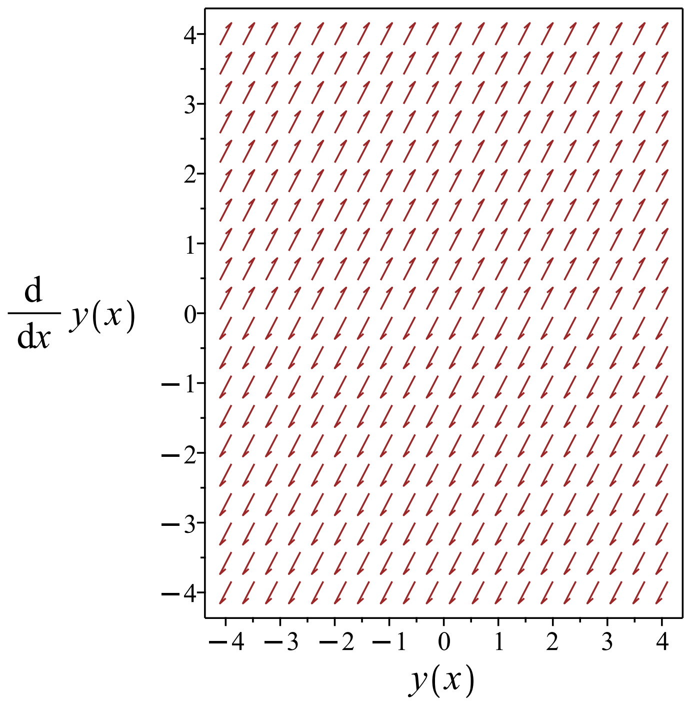


Figure 555: Slope field plot

Verification of solutions

$$y = c_1 + \frac{2c_2 e^{\frac{3x}{2}}}{3}$$

Verified OK.

9.2.6 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\, y'' + q(x)\, y' + r(x)\, y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$
\begin{aligned}
p(x) &= 2 \\
q(x) &= -3 \\
r(x) &= 0 \\
s(x) &= 0
\end{aligned}
$$

Hence

$$
\begin{aligned}
p''(x) &= 0 \\
q'(x) &= 0
\end{aligned}
$$

Therefore (1) becomes

$$
0-(0)+(0)=0
$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$
(p(x)\,y' + (q(x) - p'(x))\,y)' = s(x)
$$

Integrating gives

$$
p(x)\,y' + (q(x) - p'(x))\,y = \int s(x)\,dx
$$

Substituting the above values for p, q, r, s gives

$$
2y' - 3y = c_1
$$

We now have a first order ode to solve which is

$$
2y' - 3y = c_1
$$

Integrating both sides gives

$$
\begin{aligned}
\int \frac{1}{\frac{3y}{2}+\frac{c_1}{2}}dy &= \int dx \\
\frac{2\ln\left(3y+c_1\right)}{3} &= x + c_2
\end{aligned}
$$

Raising both side to exponential gives

$$
\left(3y+c_1\right)^{\frac{2}{3}} = \mathrm{e}^{x+c_2}
$$

Which simplifies to

$$(3y + c_1)^{\frac{2}{3}} = c_3 e^{x}$$

Summary

The solution(s) found are the following

$$y = \frac{(c_3 e^{x})^{\frac{3}{2}}}{3} - \frac{c_1}{3} \tag{1}$$

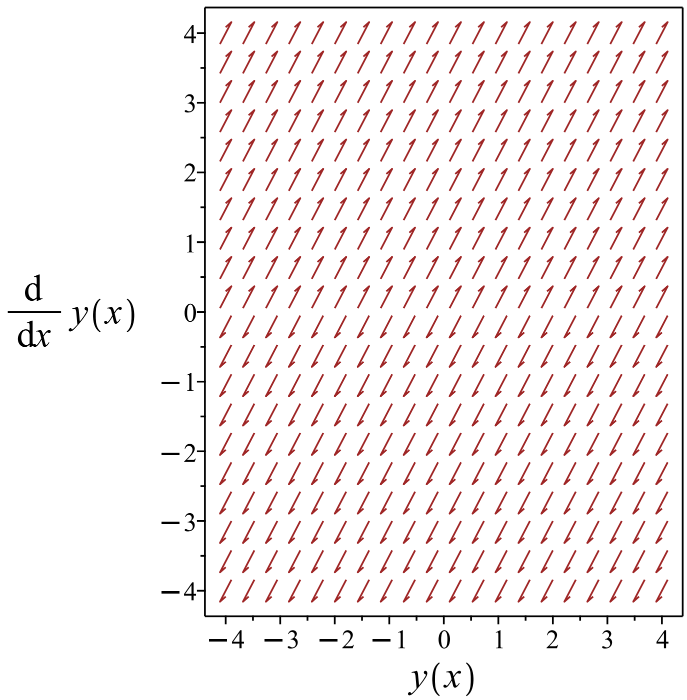


Figure 556: Slope field plot

Verification of solutions

$$y = \frac{(c_3 e^{x})^{\frac{3}{2}}}{3} - \frac{c_1}{3}$$

Verified OK.

9.2.7 Maple step by step solution

Let's solve

$2y'' - 3y' = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{3y'}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{3y'}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{3}{2}r = 0$

- Factor the characteristic polynomial

  $\frac{r(2r-3)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(0, \frac{3}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = 1$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\frac{3x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 + c_2 \mathrm{e}^{\frac{3x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 12

```
dsolve(2*diff(y(x),x$2)-3*diff(y(x),x)=0,y(x), singsol=all)
```

$$y(x) = c_1 + c_2 e^{\frac{3x}{2}}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 21

```
DSolve[2*y''[x]-3*y'[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2}{3}c_1 e^{3x/2} + c_2$$

9.3 problem 3

Internal problem ID [197]
Internal file name [`OUTPUT/197_Sunday_June_05_2022_01_36_37_AM_65436008/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 3y' - 10y = 0}$$

9.3.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = -10$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda e^{\lambda x} - 10 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda - 10 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = -10$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(-10)} \\ &= -\frac{3}{2} \pm \frac{7}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{7}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{7}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -5\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-5)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-5x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-5x} \tag{1}$$

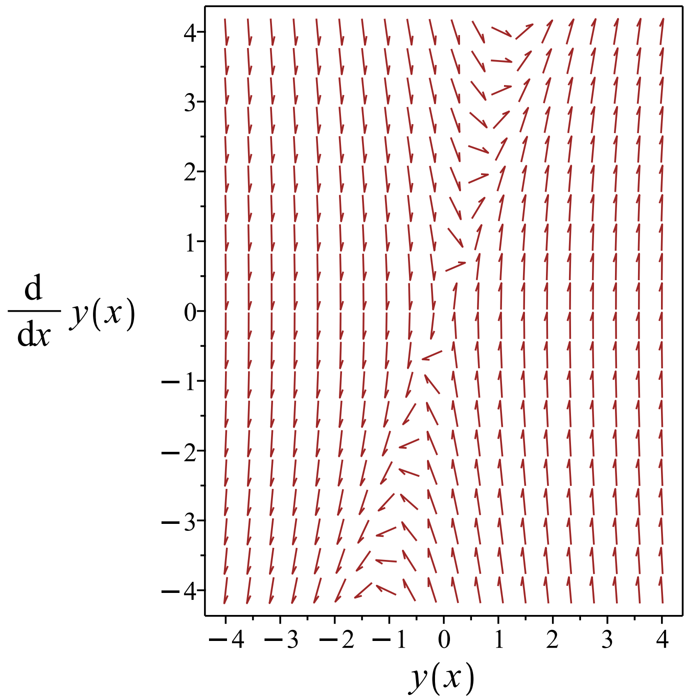


Figure 557: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-5x}$$

Verified OK.

9.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' - 10y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= -10 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{49}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 49$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{49z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 403: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = \frac{49}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-\frac{7x}{2}}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-5x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\mathrm{e}^{7x}}{7}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1y_1 + c_2y_2 \\
&= c_1\left(\mathrm{e}^{-5x}\right) + c_2\left(\mathrm{e}^{-5x}\left(\frac{\mathrm{e}^{7x}}{7}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{-5x} + \frac{c_2\mathrm{e}^{2x}}{7} \tag{1}
$$

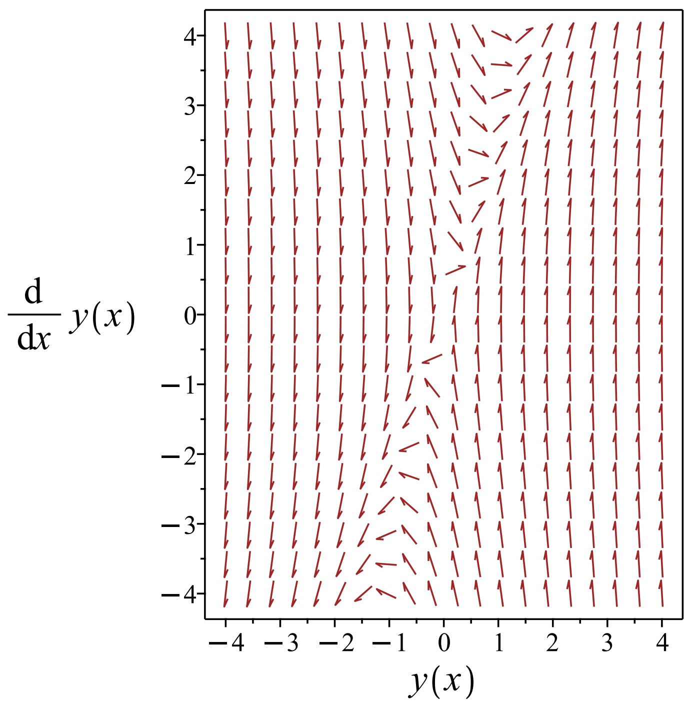


Figure 558: Slope field plot

Verification of solutions

$$
y = c_1\mathrm{e}^{-5x} + \frac{c_2\mathrm{e}^{2x}}{7}
$$

Verified OK.

9.3.3 Maple step by step solution

Let's solve

$y'' + 3y' - 10y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 3r - 10 = 0$

- Factor the characteristic polynomial

  $(r + 5)(r - 2) = 0$

- Roots of the characteristic polynomial

  $r = (-5, 2)$

- 1st solution of the ODE

  $y_1(x) = e^{-5x}$

- 2nd solution of the ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-5x} + c_2 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)-10*y(x)=0,y(x), singsol=all)
```

$$y(x) = \left(c_2 e^{7x} + c_1\right) e^{-5x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[y''[x]+3*y'[x]-10*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-5x}\left(c_2 e^{7x} + c_1\right)$$

9.4 problem 4

Internal problem ID [198]
Internal file name [`OUTPUT/198_Sunday_June_05_2022_01_36_37_AM_84432007/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2y'' - 7y' + 3y = 0}$$

9.4.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = -7, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} - 7\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 - 7\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = -7, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{7}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{-7^2 - (4)\,(2)\,(3)}\\ &= \frac{7}{4} \pm \frac{5}{4}\end{aligned}$$

Hence

$$\lambda_1 = \frac{7}{4} + \frac{5}{4}$$
$$\lambda_2 = \frac{7}{4} - \frac{5}{4}$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = \frac{1}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{\left(\frac{1}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{\frac{x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{\frac{x}{2}} \tag{1}$$

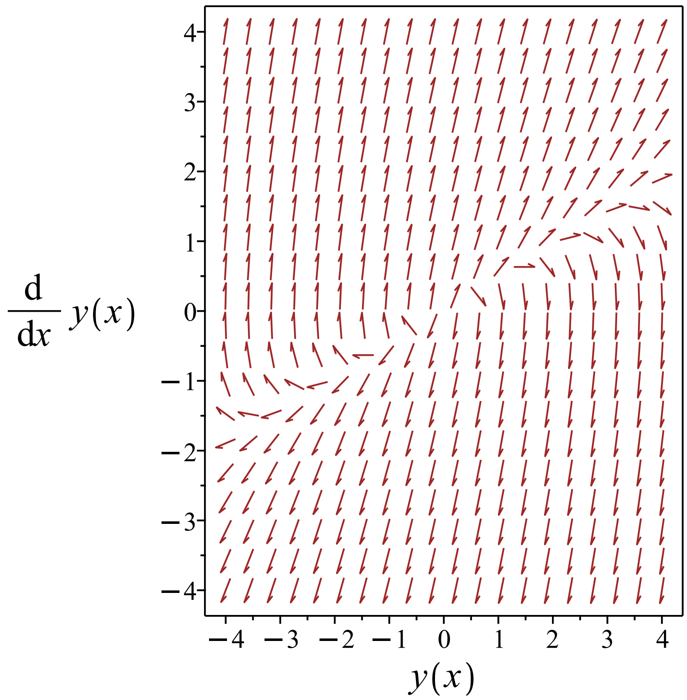


Figure 559: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{\frac{x}{2}}$$

Verified OK.

9.4.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' - 7y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= -7 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{16} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 16 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{16} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 405: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{16}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{4}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-7}{2}\,dx} \\ &= z_1 e^{\frac{7x}{4}} \\ &= z_1 \left(\mathrm{e}^{\frac{7x}{4}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-7}{2}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{\frac{7x}{2}}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{\frac{x}{2}}\left(\frac{2\,\mathrm{e}^{\frac{5x}{2}}}{5}\right)\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1\mathrm{e}^{\frac{x}{2}} + \frac{2c_2\mathrm{e}^{3x}}{5} \tag{1}
$$

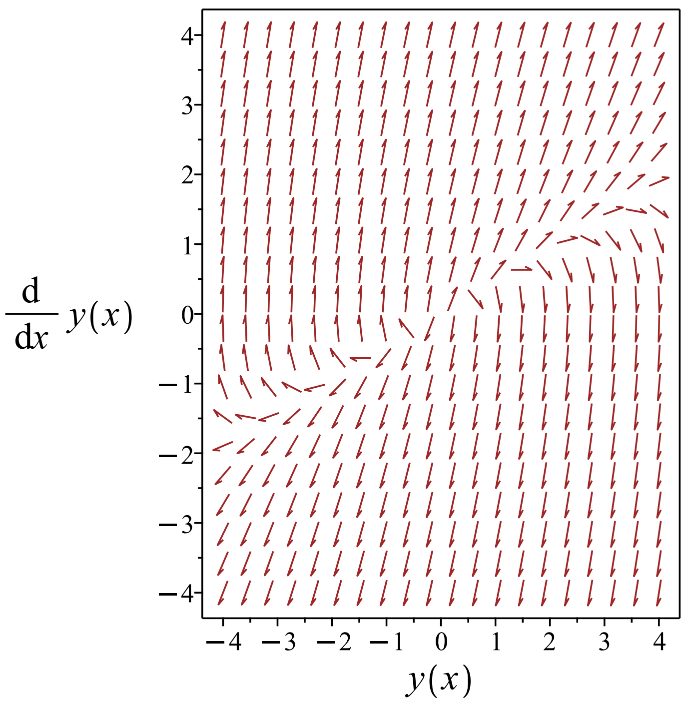


Figure 560: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1\mathrm{e}^{\frac{x}{2}} + \frac{2c_2\mathrm{e}^{3x}}{5}
$$

Verified OK.

9.4.3 Maple step by step solution

Let's solve

$2y'' - 7y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = \frac{7y'}{2} - \frac{3y}{2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - \frac{7y'}{2} + \frac{3y}{2} = 0$

- Characteristic polynomial of ODE

  $r^2 - \frac{7}{2}r + \frac{3}{2} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-1)(r-3)}{2} = 0$

- Roots of the characteristic polynomial

  $r = \left(3, \frac{1}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = e^{3x}$

- 2nd solution of the ODE

  $y_2(x) = e^{\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{3x} + c_2 e^{\frac{x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(2*diff(y(x),x$2)-7*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{x}{2}} + c_2 e^{3x}$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 24

```
DSolve[2*y''[x]-7*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x/2} + c_2 e^{3x}$$

9.5 problem 5

Internal problem ID [199]
Internal file name [`OUTPUT/199_Sunday_June_05_2022_01_36_38_AM_61218353/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 6y' + 9y = 0}$$

9.5.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 6, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 6\lambda e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 6\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(6)^2 - (4)(1)(9)} \\ &= -3\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 3$. Therefore the solution is

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + c_2 x e^{-3x} \tag{1}$$

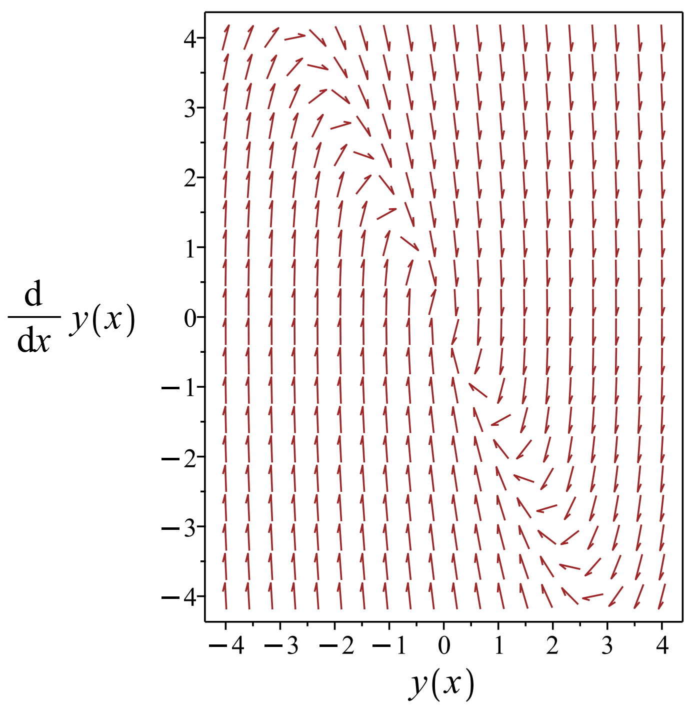


Figure 561: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + c_2 x e^{-3x}$$

Verified OK.

9.5.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = 6$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 6\,dx} \\ &= \mathrm{e}^{3x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= 0 \\ \left(y\,\mathrm{e}^{3x}\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(y\,\mathrm{e}^{3x}\right)' = c_1$$

Integrating again gives

$$\left(y\,\mathrm{e}^{3x}\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{3x}}$$

Or

$$y = c_1 x\,\mathrm{e}^{-3x} + \mathrm{e}^{-3x} c_2$$

Summary

The solution(s) found are the following

$$y = c_1 x\,\mathrm{e}^{-3x} + \mathrm{e}^{-3x} c_2 \tag{1}$$

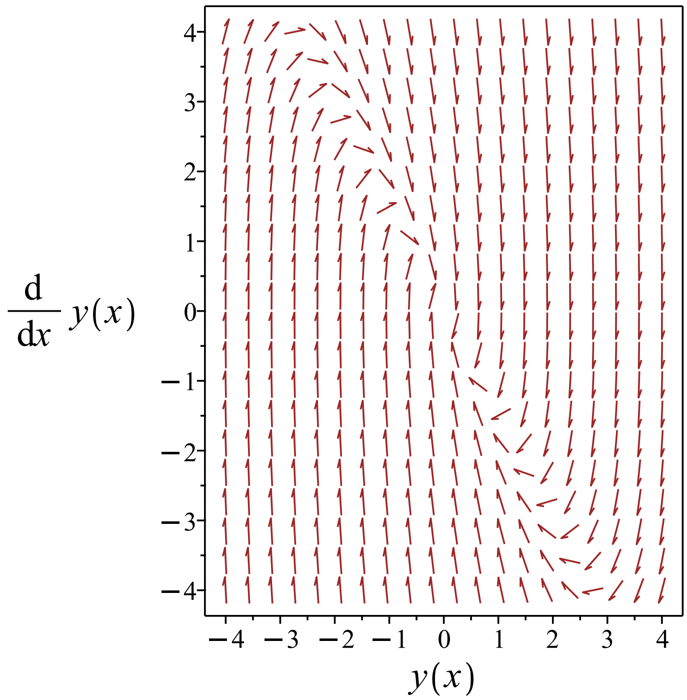


Figure 562: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-3x} + \mathrm{e}^{-3x} c_2$$

Verified OK.

9.5.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 6y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 407: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dx} \\ &= z_1 e^{-3x} \\ &= z_1 \left(\mathrm{e}^{-3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-6x}}{(y_1)^2}\,dx \\ &= y_1(x) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-3x} + c_2 x \, \mathrm{e}^{-3x} \tag{1}
$$

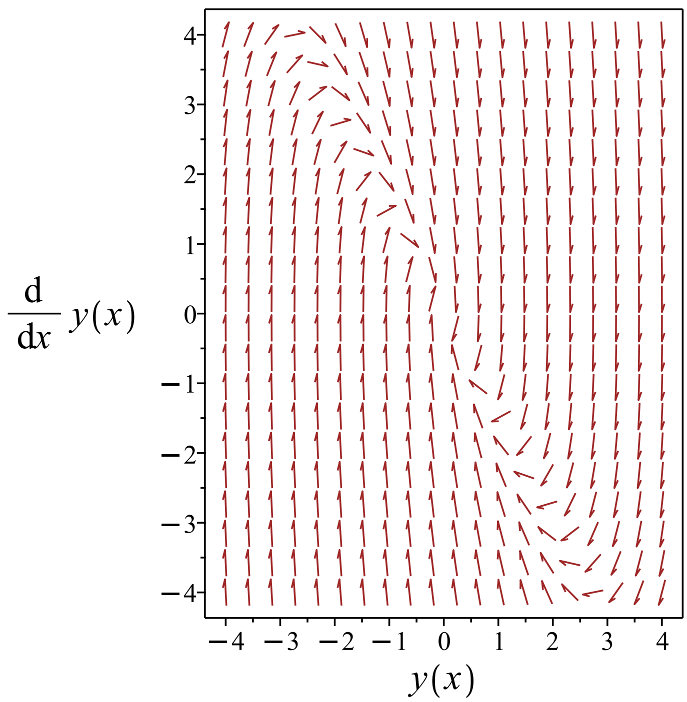


Figure 563: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-3x} + c_2 x \, \mathrm{e}^{-3x}
$$

Verified OK.

9.5.4 Maple step by step solution

Let's solve

$y'' + 6y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 6r + 9 = 0$

- Factor the characteristic polynomial

  $(r + 3)^2 = 0$

- Root of the characteristic polynomial

  $r = -3$

- 1st solution of the ODE

  $y_1(x) = e^{-3x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{-3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{-3x} + c_2 x\, e^{-3x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(diff(y(x),x$2)+6*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{-3x}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 18

```
DSolve[y''[x]+6*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-3x}(c_2x + c_1)$$

9.6 problem 6

Internal problem ID [200]
Internal file name [`OUTPUT/200_Sunday_June_05_2022_01_36_39_AM_34186377/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 5y' + 5y = 0}$$

9.6.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 5, C = 5$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 5\lambda e^{\lambda x} + 5 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 5\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 5, C = 5$ into the above gives

$$\lambda_{1,2} = \frac{-5}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{5^2 - (4)(1)(5)}$$
$$= -\frac{5}{2} \pm \frac{\sqrt{5}}{2}$$

Hence

$$\lambda_1 = -\frac{5}{2} + \frac{\sqrt{5}}{2}$$
$$\lambda_2 = -\frac{5}{2} - \frac{\sqrt{5}}{2}$$

Which simplifies to

$$\lambda_1 = -\frac{5}{2} + \frac{\sqrt{5}}{2}$$
$$\lambda_2 = -\frac{5}{2} - \frac{\sqrt{5}}{2}$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{\left(-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)x} + c_2 e^{\left(-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)x}$$

Or

$$y = c_1 \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)x}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)x} \tag{1}$$

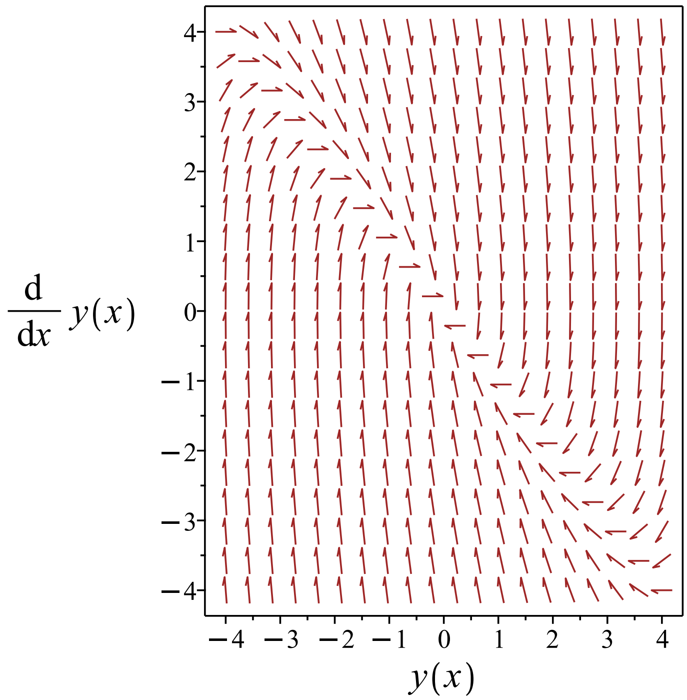


Figure 564: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)x} + c_2 e^{\left(-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)x}$$

Verified OK.

9.6.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 5y' + 5y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 5 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{5}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 5$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{5z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 409: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{5}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x\sqrt{5}}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{5}{1}\,dx} \\ &= z_1 e^{-\frac{5x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{5x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{\left(5+\sqrt{5}\right)x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{5}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-5x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\sqrt{5}\,\mathrm{e}^{x\sqrt{5}}}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-\frac{(5+\sqrt{5})x}{2}} \right) + c_2 \left(e^{-\frac{(5+\sqrt{5})x}{2}} \left(\frac{\sqrt{5}\, e^{x\sqrt{5}}}{5} \right) \right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{-\frac{(5+\sqrt{5})x}{2}} + \frac{c_2 \sqrt{5}\, e^{\frac{(-5+\sqrt{5})x}{2}}}{5} \tag{1}
$$

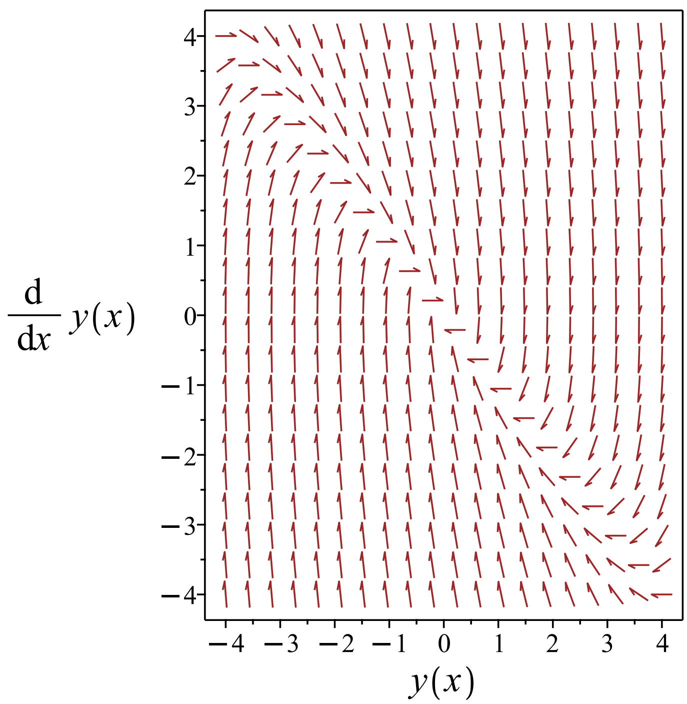


Figure 565: Slope field plot

Verification of solutions

$$
y = c_1 e^{-\frac{(5+\sqrt{5})x}{2}} + \frac{c_2 \sqrt{5}\, e^{\frac{(-5+\sqrt{5})x}{2}}}{5}
$$

Verified OK.

9.6.3 Maple step by step solution

Let's solve

$y'' + 5y' + 5y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 5r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-5)\pm\left(\sqrt{5}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{5}{2} - \frac{\sqrt{5}}{2}, -\frac{5}{2} + \frac{\sqrt{5}}{2}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)x}$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{\left(-\frac{5}{2}-\frac{\sqrt{5}}{2}\right)x} + c_2 \mathrm{e}^{\left(-\frac{5}{2}+\frac{\sqrt{5}}{2}\right)x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(y(x),x$2)+5*diff(y(x),x)+5*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{\frac{\left(-5+\sqrt{5}\right)x}{2}} + c_2 e^{-\frac{\left(5+\sqrt{5}\right)x}{2}}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 35

```
DSolve[y''[x]+5*y'[x]+5*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-\frac{1}{2}\left(5+\sqrt{5}\right)x}\left(c_2 e^{\sqrt{5}x} + c_1\right)$$

9.7 problem 7

Internal problem ID [201]
Internal file name [`OUTPUT/201_Sunday_June_05_2022_01_36_39_AM_83462194/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{4y'' - 12y' + 9y = 0}$$

9.7.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = -12, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 \mathrm{e}^{\lambda x} - 12\lambda\,\mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 - 12\lambda + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = -12, C = 9$ into the above gives

$$\lambda_{1,2} = \frac{12}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{(-12)^2 - (4)(4)(9)}$$
$$= \frac{3}{2}$$

Hence this is the case of a double root $\lambda_{1,2} = -\frac{3}{2}$. Therefore the solution is

$$y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}} \tag{1}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}} \tag{1}$$

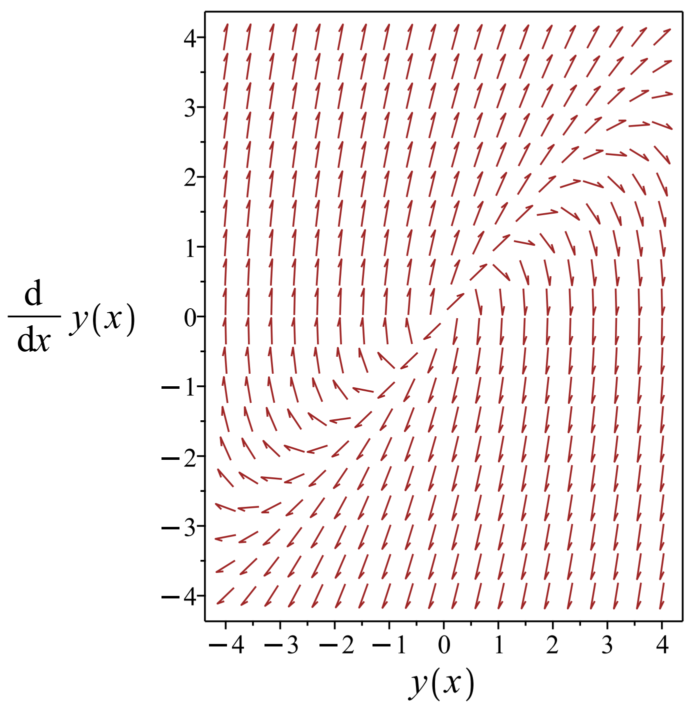


Figure 566: Slope field plot

Verification of solutions

$$y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}}$$

Verified OK.

9.7.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^{2} + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -3$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -3\,dx} \\
&= \mathrm{e}^{-\frac{3x}{2}}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= 0 \\
\left(\mathrm{e}^{-\frac{3x}{2}} y\right)'' &= 0
\end{aligned}$$

Integrating once gives

$$\left(\mathrm{e}^{-\frac{3x}{2}} y\right)' = c_1$$

Integrating again gives

$$\left(\mathrm{e}^{-\frac{3x}{2}} y\right) = c_1 x + c_2$$

Hence the solution is

$$y = \frac{c_1 x + c_2}{\mathrm{e}^{-\frac{3x}{2}}}$$

Or

$$y = c_1 x\, \mathrm{e}^{\frac{3x}{2}} + c_2 \mathrm{e}^{\frac{3x}{2}}$$

Summary

The solution(s) found are the following

$$y = c_1 x\, \mathrm{e}^{\frac{3x}{2}} + c_2 \mathrm{e}^{\frac{3x}{2}} \tag{1}$$

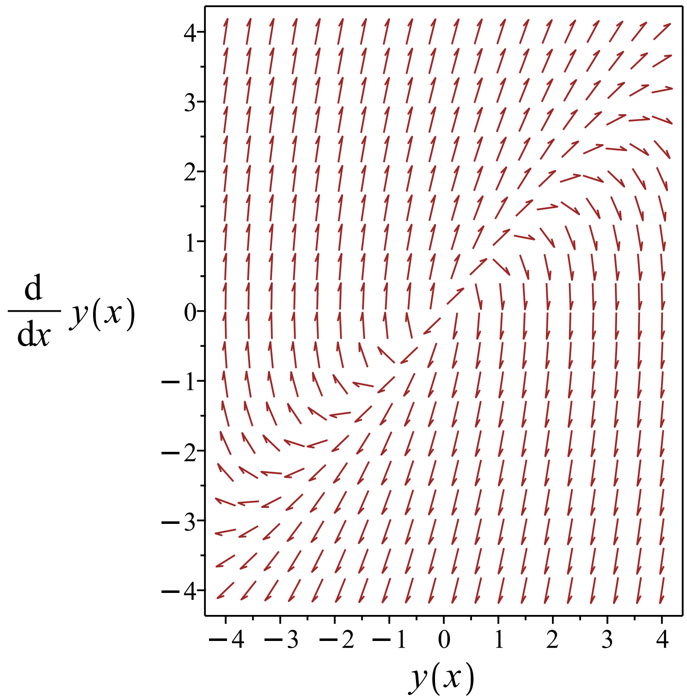


Figure 567: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{\frac{3x}{2}} + c_2 \mathrm{e}^{\frac{3x}{2}}$$

Verified OK.

9.7.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' - 12y' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= -12 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 411: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-12}{4}\,dx} \\
&= z_1 e^{\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{\frac{3x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\frac{3x}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-12}{4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(e^{\frac{3x}{2}}\right) + c_2\left(e^{\frac{3x}{2}}(x)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}} \tag{1}
$$

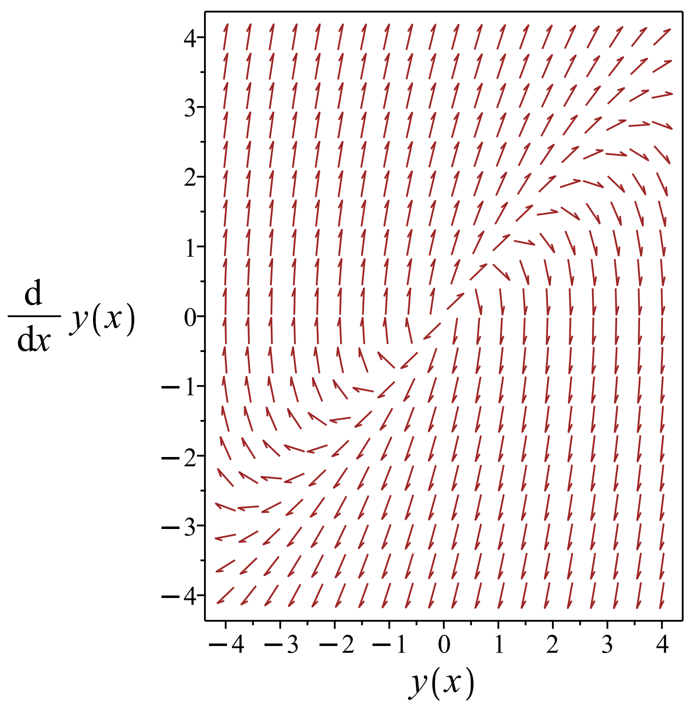


Figure 568: Slope field plot

Verification of solutions

$$
y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}}
$$

Verified OK.

9.7.4 Maple step by step solution

Let's solve

$4y'' - 12y' + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 3y' - \frac{9y}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 3y' + \frac{9y}{4} = 0$

- Characteristic polynomial of ODE

  $r^2 - 3r + \frac{9}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r-3)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = \frac{3}{2}$

- 1st solution of the ODE

  $y_1(x) = e^{\frac{3x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\, e^{\frac{3x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^{\frac{3x}{2}} + c_2 x\, e^{\frac{3x}{2}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve(4*diff(y(x),x$2)-12*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{\frac{3x}{2}}(c_2x + c_1)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[4*y''[x]-12*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x/2}(c_2x + c_1)$$

9.8 problem 8

Internal problem ID [202]
Internal file name [`OUTPUT/202_Sunday_June_05_2022_01_36_40_AM_18678395/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 13y = 0}$$

9.8.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 6\lambda\, \mathrm{e}^{\lambda x} + 13\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-6^2 - (4)(1)(13)}\\ &= 3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 3 + 2i\\ \lambda_2 &= 3 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3 + 2i\\ \lambda_2 &= 3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) \tag{1}$$

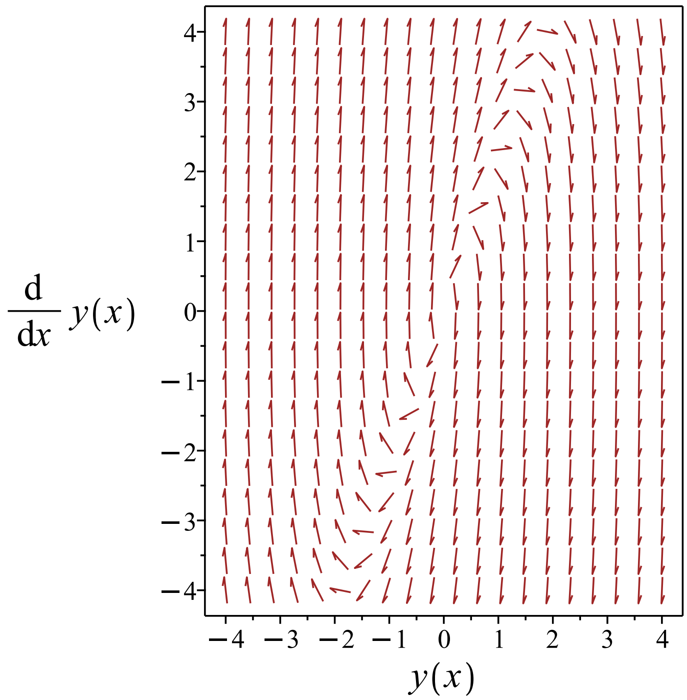


Figure 569: Slope field plot

Verification of solutions

$$y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Verified OK.

9.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 13y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 413: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\
&= z_1 e^{3x} \\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{3x} \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{3x} \cos(2x)\right) + c_2 \left(e^{3x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} \cos(2x) + \frac{c_2 e^{3x} \sin(2x)}{2} \tag{1}$$

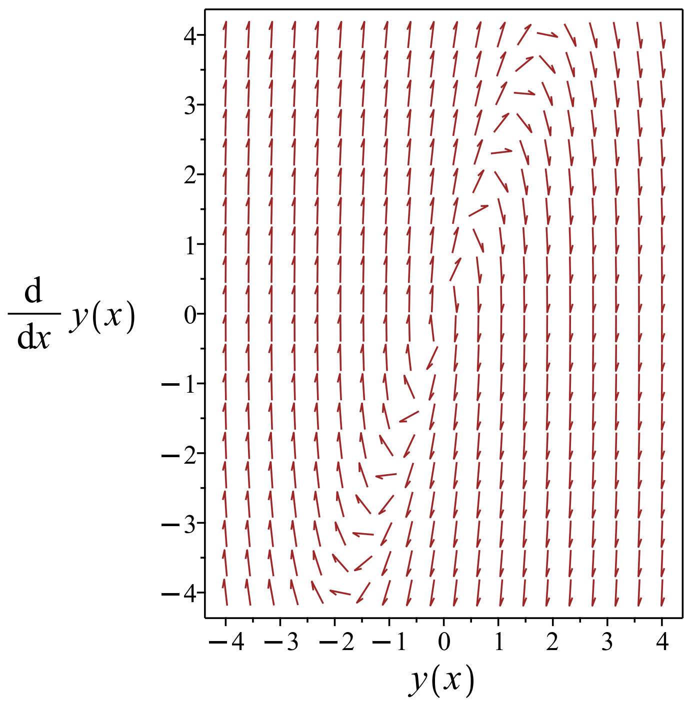


Figure 570: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} \cos(2x) + \frac{c_2 e^{3x} \sin(2x)}{2}$$

Verified OK.

9.8.3 Maple step by step solution

Let's solve

$y'' - 6y' + 13y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{6 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (3 - 2\,\mathrm{I}, 3 + 2\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{3x} \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{3x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{3x} \cos(2x) + c_2 \mathrm{e}^{3x} \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+13*y(x)=0,y(x), singsol=all)
```

$$y(x) = e^{3x}(\sin(2x) c_1 + c_2 \cos(2x))$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 26

```
DSolve[y''[x]-6*y'[x]+13*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{3x}(c_2 \cos(2x) + c_1 \sin(2x))$$

9.9 problem 9

Internal problem ID [203]
Internal file name [`OUTPUT/203_Sunday_June_05_2022_01_36_40_AM_20550739/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' + 8y' + 25y = 0}$$

9.9.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 8, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 8\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 8\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{8^2 - (4)(1)(25)} \\ &= -4 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -4 + 3i \\ \lambda_2 &= -4 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -4 + 3i \\ \lambda_2 &= -4 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -4$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-4x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Summary

The solution(s) found are the following

$$y = \mathrm{e}^{-4x}(c_1 \cos(3x) + c_2 \sin(3x)) \tag{1}$$

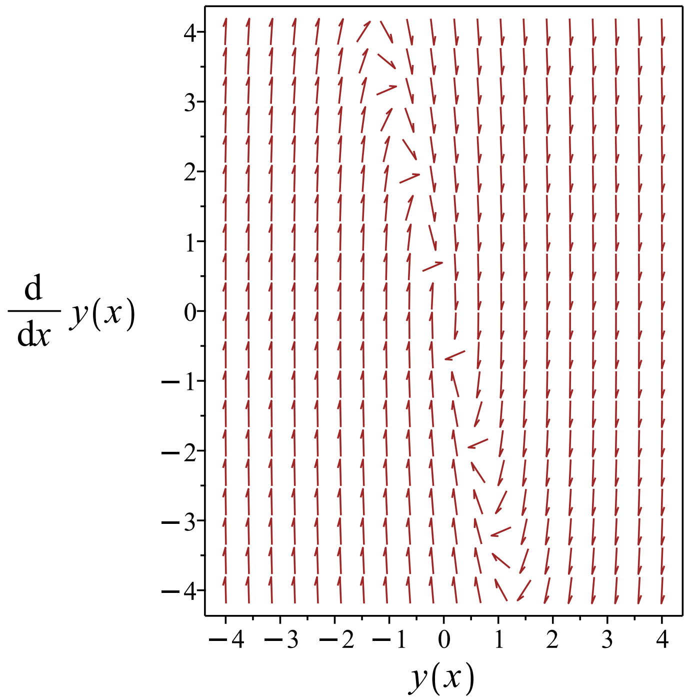


Figure 571: Slope field plot

Verification of solutions

$$y = e^{-4x}(c_1 \cos(3x) + c_2 \sin(3x))$$

Verified OK.

9.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 8y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 415: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\,dx} \\ &= z_1 e^{-4x} \\ &= z_1 \left(\mathrm{e}^{-4x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-4x}\cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{8}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-8x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-4x} \cos(3x)\right) + c_2 \left(\mathrm{e}^{-4x} \cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-4x} \cos(3x) + \frac{c_2 \mathrm{e}^{-4x} \sin(3x)}{3} \tag{1}
$$

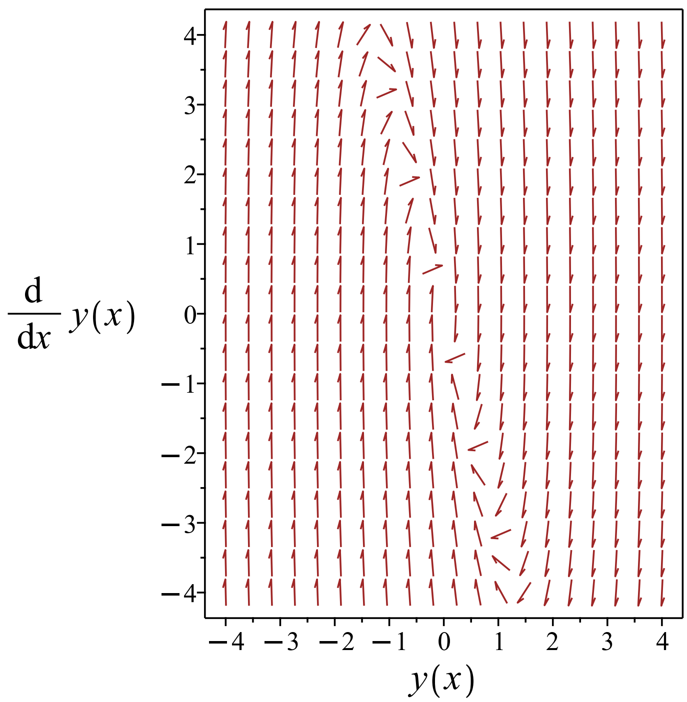


Figure 572: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{-4x} \cos(3x) + \frac{c_2 \mathrm{e}^{-4x} \sin(3x)}{3}
$$

Verified OK.

9.9.3 Maple step by step solution

Let's solve

$y'' + 8y' + 25y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 + 8r + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-8)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-4 - 3\,\mathrm{I}, -4 + 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-4x} \cos(3x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-4x} \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-4x} \cos(3x) + c_2 \mathrm{e}^{-4x} \sin(3x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 22

```
dsolve(diff(y(x),x$2)+8*diff(y(x),x)+25*y(x)=0,y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-4x}(c_1 \sin(3x) + c_2 \cos(3x))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 26

```
DSolve[y''[x]+8*y'[x]+25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-4x}(c_2 \cos(3x) + c_1 \sin(3x))$$

9.10 problem 21

Internal problem ID [204]
Internal file name [`OUTPUT/204_Sunday_June_05_2022_01_36_41_AM_14205893/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 4y' + 3y = 0}$$

With initial conditions

$$[y(0) = 7, y'(0) = 11]$$

9.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -4 \\ q(x) &= 3 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 4y' + 3y = 0$$

The domain of $p(x) = -4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.10.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-4^2 - (4)\,(1)\,(3)}\\ &= 2 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = 2 + 1$$
$$\lambda_2 = 2 - 1$$

Which simplifies to

$$\lambda_1 = 3$$
$$\lambda_2 = 1$$

Since roots are real and distinct, then the solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$y = c_1 e^{(3)x} + c_2 e^{(1)x}$$

Or

$$y = c_1 e^{3x} + c_2 e^{x}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} + c_2 e^{x} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 0$ in the above gives

$$7 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} + c_2 e^{x}$$

substituting $y' = 11$ and $x = 0$ in the above gives

$$11 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = 5$$

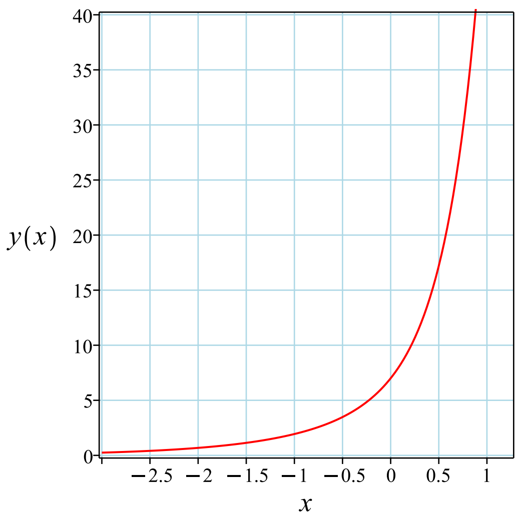
Substituting these values back in above solution results in

$$y = 2\,e^{3x} + 5\,e^{x}$$

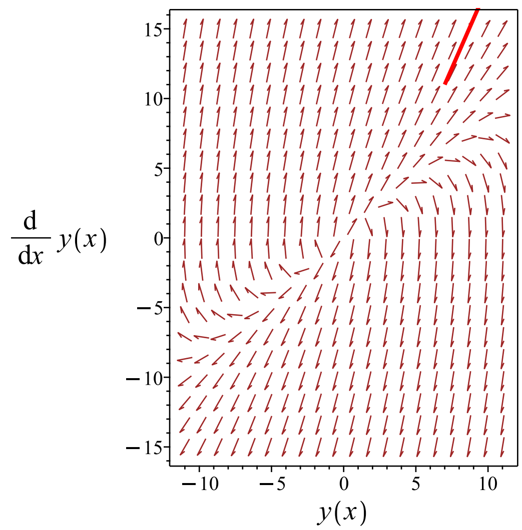
Summary

The solution(s) found are the following

$$y = 2\,e^{3x} + 5\,e^{x} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 2\,e^{3x} + 5\,e^{x}$$

Verified OK.

9.10.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 417: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-4}{1} \, dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2} \, dx \\
&= y_1 \left(\frac{\mathrm{e}^{2x}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(e^x) + c_2\left(e^x\left(\frac{e^{2x}}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 e^x + \frac{c_2 e^{3x}}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 7$ and $x = 0$ in the above gives

$$
7 = c_1 + \frac{c_2}{2} \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = c_1 e^x + \frac{3c_2 e^{3x}}{2}
$$

substituting $y' = 11$ and $x = 0$ in the above gives

$$
11 = c_1 + \frac{3c_2}{2} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 5 \\
c_2 &= 4
\end{aligned}
$$

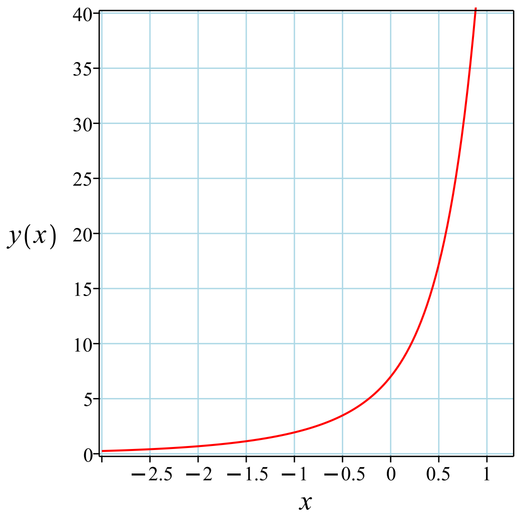
Substituting these values back in above solution results in

$$
y = 2\,e^{3x} + 5\,e^x
$$

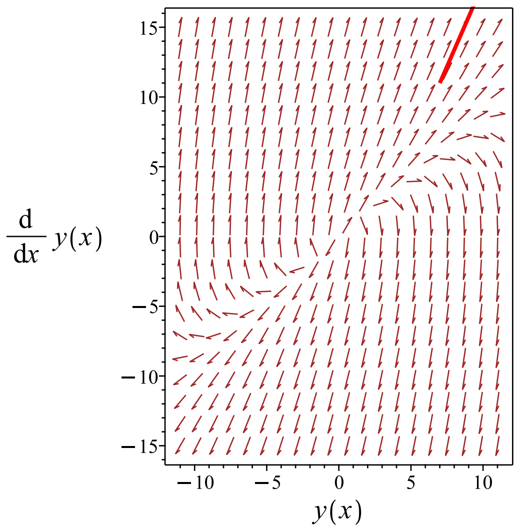
Summary

The solution(s) found are the following

$$
y = 2\,e^{3x} + 5\,e^x \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = 2\,e^{3x} + 5\,e^{x}$$

Verified OK.

9.10.4 Maple step by step solution

Let's solve

$$\left[y'' - 4y' + 3y = 0, y(0) = 7, y'\Big|_{\{x=0\}} = 11\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 4r + 3 = 0$

- Factor the characteristic polynomial

  $(r-1)(r-3) = 0$

- Roots of the characteristic polynomial

  $r = (1, 3)$

- 1st solution of the ODE

$y_1(x) = e^x$

- 2nd solution of the ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 e^x + c_2 e^{3x}$

□ Check validity of solution $y = c_1 e^x + c_2 e^{3x}$

  - Use initial condition $y(0) = 7$

    $7 = c_1 + c_2$

  - Compute derivative of the solution

    $y' = c_1 e^x + 3c_2 e^{3x}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 11$

    $11 = c_1 + 3c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 5, c_2 = 2\}$

  - Substitute constant values into general solution and simplify

    $y = 2\,e^{3x} + 5\,e^x$

- Solution to the IVP

  $y = 2\,e^{3x} + 5\,e^x$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 15

```
dsolve([diff(y(x),x$2)-4*diff(y(x),x)+3*y(x)=0,y(0) = 7, D(y)(0) = 11],y(x), singsol=all)
```

$$y(x) = 5\,e^{x} + 2\,e^{3x}$$

✓ Solution by Mathematica

Time used: 0.011 (sec). Leaf size: 18

```
DSolve[{y''[x]-4*y'[x]+3*y[x]==0,{y[0]==7,y'[0]==11}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to e^{x}\left(2e^{2x} + 5\right)$$

9.11 problem 22

Internal problem ID [205]
Internal file name [`OUTPUT/205_Sunday_June_05_2022_01_36_42_AM_62818003/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 22.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{9y'' + 6y' + 4y = 0}$$

With initial conditions

$$[y(0) = 3, y'(0) = 4]$$

9.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= \frac{2}{3} \\ q(x) &= \frac{4}{9} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' + \frac{2y'}{3} + \frac{4y}{9} = 0$$

The domain of $p(x) = \frac{2}{3}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = \frac{4}{9}$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.11.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 9, B = 6, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$9\lambda^2 \mathrm{e}^{\lambda x} + 6\lambda \,\mathrm{e}^{\lambda x} + 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$9\lambda^2 + 6\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 9, B = 6, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)\,(9)} \pm \frac{1}{(2)\,(9)}\sqrt{6^2 - (4)\,(9)\,(4)}\\ &= -\frac{1}{3} \pm \frac{i\sqrt{3}}{3}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{3} + \frac{i\sqrt{3}}{3}\\ \lambda_2 &= -\frac{1}{3} - \frac{i\sqrt{3}}{3}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{3} + \frac{i\sqrt{3}}{3}$$
$$\lambda_2 = -\frac{1}{3} - \frac{i\sqrt{3}}{3}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{3}$ and $\beta = \frac{\sqrt{3}}{3}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{3}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{3}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{3}\right)\right)$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{-\frac{x}{3}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{3}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{3}\right)\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{\mathrm{e}^{-\frac{x}{3}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{3}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{3}\right)\right)}{3} + \mathrm{e}^{-\frac{x}{3}}\left(-\frac{c_1\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{3}\right)}{3} + \frac{c_2\sqrt{3}\,\cos\left(\frac{\sqrt{3}\,x}{3}\right)}{3}\right)$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = -\frac{c_1}{3} + \frac{\sqrt{3}\,c_2}{3} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 5\sqrt{3}$$

Substituting these values back in above solution results in

$$y = 5\,\mathrm{e}^{-\frac{x}{3}}\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{3}\right) + 3\,\mathrm{e}^{-\frac{x}{3}}\cos\left(\frac{\sqrt{3}\,x}{3}\right)$$

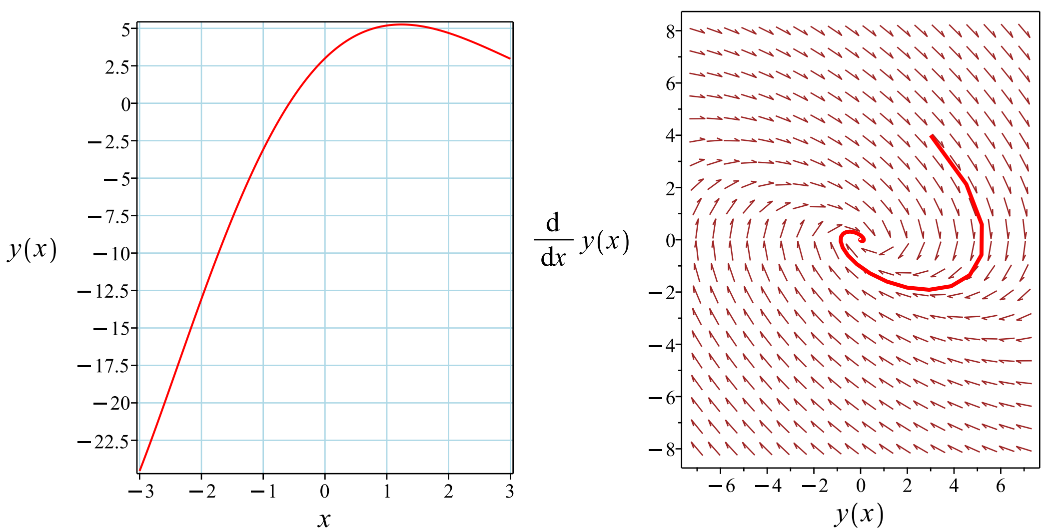
Which simplifies to

$$y = \left(5\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\,x}{3}\right)\right)\mathrm{e}^{-\frac{x}{3}}$$

Summary

The solution(s) found are the following

$$y = \left(5\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\,x}{3}\right)\right)\mathrm{e}^{-\frac{x}{3}} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \left(5\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\,x}{3}\right)\right)\mathrm{e}^{-\frac{x}{3}}$$

Verified OK.

9.11.3 Solving using Kovacic algorithm

Writing the ode as

$$9y'' + 6y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 9 \\ B &= 6 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{3} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 3 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{z(x)}{3} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 419: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{1}{3}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos\left(\frac{\sqrt{3}\, x}{3}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{9}\,dx} \\ &= z_1 e^{-\frac{x}{3}} \\ &= z_1 \left(\mathrm{e}^{-\frac{x}{3}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{6}{9}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-\frac{2x}{3}}}{(y_1)^2}\,dx \\ &= y_1 \left(\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{3}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right) \left(\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{3}\right)\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right) + c_2 \mathrm{e}^{-\frac{x}{3}} \sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{3}\right) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\, x}{3}\right)}{3} - \frac{c_1 \mathrm{e}^{-\frac{x}{3}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{3}\right)}{3} - \frac{c_2 \mathrm{e}^{-\frac{x}{3}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{3}\right)}{3} + c_2 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\, x}{3}\right)$$

substituting $y' = 4$ and $x = 0$ in the above gives

$$4 = -\frac{c_1}{3} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 3$$
$$c_2 = 5$$

Substituting these values back in above solution results in

$$y = 5\,\mathrm{e}^{-\frac{x}{3}} \sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{3}\right) + 3\,\mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\, x}{3}\right)$$

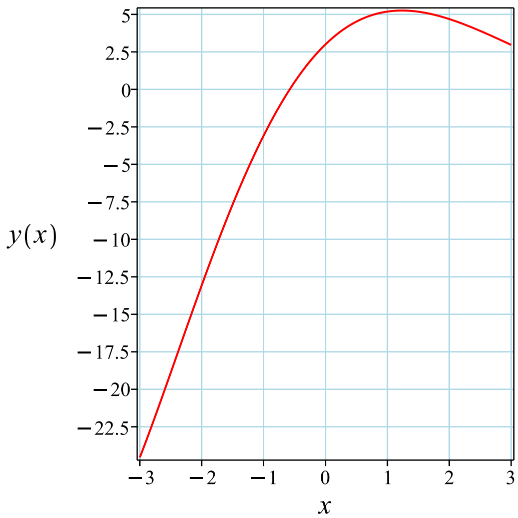
Which simplifies to

$$y = \left(5\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\, x}{3}\right)\right) \mathrm{e}^{-\frac{x}{3}}$$

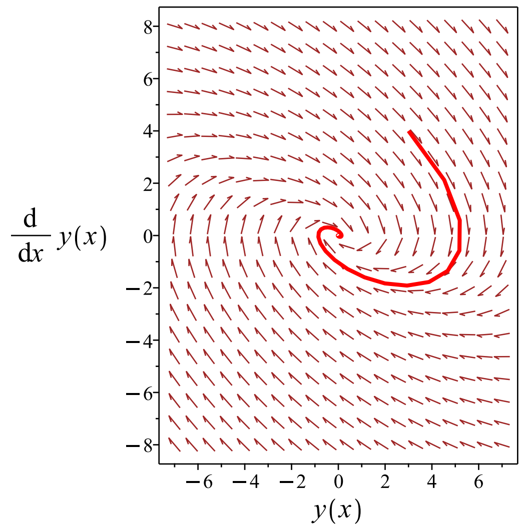
Summary

The solution(s) found are the following

$$y = \left(5\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\, x}{3}\right)\right) \mathrm{e}^{-\frac{x}{3}} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \left(5\sqrt{3}\,\sin\left(\frac{\sqrt{3}\,x}{3}\right)+3\cos\left(\frac{\sqrt{3}\,x}{3}\right)\right)\mathrm{e}^{-\frac{x}{3}}$$

Verified OK.

9.11.4 Maple step by step solution

Let's solve

$\left[9y''+6y'+4y=0, y(0)=3, y'\Big|_{\{x=0\}}=4\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y''=-\frac{2y'}{3}-\frac{4y}{9}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y''+\frac{2y'}{3}+\frac{4y}{9}=0$

- Characteristic polynomial of ODE

  $r^2+\frac{2}{3}r+\frac{4}{9}=0$

- Use quadratic formula to solve for r

  $r = \frac{\left(-\frac{2}{3}\right)\pm\left(\sqrt{-\frac{4}{3}}\right)}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{3} - \frac{I\sqrt{3}}{3}, -\frac{1}{3} + \frac{I\sqrt{3}}{3}\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{\sqrt{3}\,x}{3}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right) + \mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{\sqrt{3}\,x}{3}\right) c_2$

☐ Check validity of solution $y = c_1 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right) + \mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{\sqrt{3}\,x}{3}\right) c_2$

  - Use initial condition $y(0) = 3$

    $3 = c_1$

  - Compute derivative of the solution

    $y' = -\frac{c_1 \mathrm{e}^{-\frac{x}{3}} \cos\left(\frac{\sqrt{3}\,x}{3}\right)}{3} - \frac{c_1 \mathrm{e}^{-\frac{x}{3}} \sqrt{3} \sin\left(\frac{\sqrt{3}\,x}{3}\right)}{3} - \frac{\mathrm{e}^{-\frac{x}{3}} \sin\left(\frac{\sqrt{3}\,x}{3}\right) c_2}{3} + \frac{\mathrm{e}^{-\frac{x}{3}} \sqrt{3} \cos\left(\frac{\sqrt{3}\,x}{3}\right) c_2}{3}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 4$

    $4 = -\frac{c_1}{3} + \frac{\sqrt{3}\,c_2}{3}$

  - Solve for c_1 and c_2

    $\left\{c_1 = 3, c_2 = 5\sqrt{3}\right\}$

  - Substitute constant values into general solution and simplify

    $y = \left(5\sqrt{3} \sin\left(\frac{\sqrt{3}\,x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\,x}{3}\right)\right) \mathrm{e}^{-\frac{x}{3}}$

- Solution to the IVP

  $y = \left(5\sqrt{3} \sin\left(\frac{\sqrt{3}\,x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\,x}{3}\right)\right) \mathrm{e}^{-\frac{x}{3}}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 31

```
dsolve([9*diff(y(x),x$2)+6*diff(y(x),x)+4*y(x)=0,y(0) = 3, D(y)(0) = 4],y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{3}} \left(5\sqrt{3}\, \sin\left(\frac{\sqrt{3}\, x}{3}\right) + 3\cos\left(\frac{\sqrt{3}\, x}{3}\right)\right)$$

✓ Solution by Mathematica

Time used: 0.023 (sec). Leaf size: 39

```
DSolve[{9*y''[x]+6*y'[x]+4*y[x]==0,{y[0]==3,y'[0]==4}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to e^{-x/3} \left(5\sqrt{3} \sin\left(\frac{x}{\sqrt{3}}\right) + 3\cos\left(\frac{x}{\sqrt{3}}\right)\right)$$

9.12 problem 23

Internal problem ID [206]
Internal file name [`OUTPUT/206_Sunday_June_05_2022_01_36_43_AM_21669027/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 6y' + 25y = 0}$$

With initial conditions

$$[y(0) = 4, y'(0) = 1]$$

9.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -6 \\ q(x) &= 25 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$y'' - 6y' + 25y = 0$$

The domain of $p(x) = -6$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 25$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

9.12.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 25$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda\, e^{\lambda x} + 25\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-6^2 - (4)(1)(25)}\\ &= 3 \pm 4i\end{aligned}$$

Hence

$$\lambda_1 = 3 + 4i$$
$$\lambda_2 = 3 - 4i$$

Which simplifies to

$$\lambda_1 = 3 + 4i$$
$$\lambda_2 = 3 - 4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 3$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{3x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{3x}(c_1 \cos(4x) + c_2 \sin(4x)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3\,\mathrm{e}^{3x}(c_1 \cos(4x) + c_2 \sin(4x)) + \mathrm{e}^{3x}(-4c_1 \sin(4x) + 4c_2 \cos(4x))$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 3c_1 + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

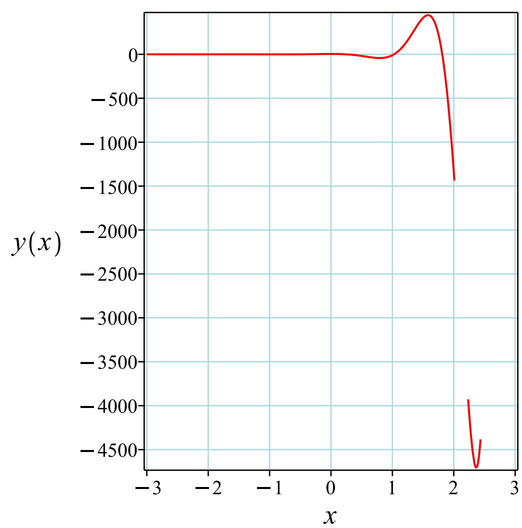
$$c_1 = 4$$
$$c_2 = -\frac{11}{4}$$

Substituting these values back in above solution results in

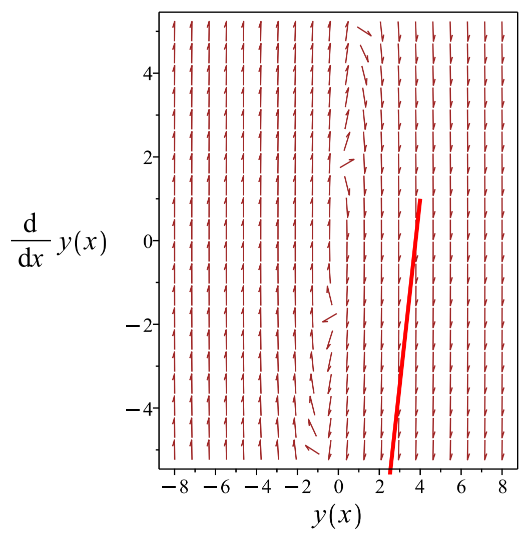
$$y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4}$$

Verified OK.

9.12.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 421: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx} \\ &= z_1 e^{3x} \\ &= z_1\left(\mathrm{e}^{3x}\right) \end{aligned}$$

Which simplifies to

$$y_1 = e^{3x} \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{6x}}{(y_1)^2}\,dx \\ &= y_1 \left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(e^{3x} \cos(4x)\right) + c_2 \left(e^{3x} \cos(4x) \left(\frac{\tan(4x)}{4}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{3x} \cos(4x) + \frac{c_2 e^{3x} \sin(4x)}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 4$ and $x = 0$ in the above gives

$$4 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = 3c_1 e^{3x} \cos(4x) - 4c_1 e^{3x} \sin(4x) + \frac{3c_2 e^{3x} \sin(4x)}{4} + c_2 e^{3x} \cos(4x)$$

substituting $y' = 1$ and $x = 0$ in the above gives

$$1 = 3c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = -11$$

Substituting these values back in above solution results in

$$y = 4\,e^{3x}\cos(4x) - \frac{11\,e^{3x}\sin(4x)}{4}$$

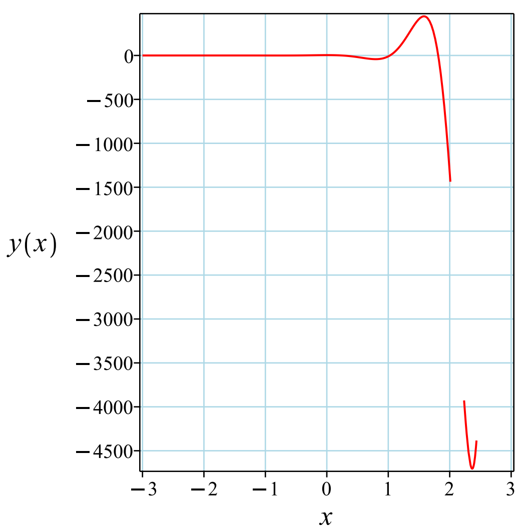
Which simplifies to

$$y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4}$$

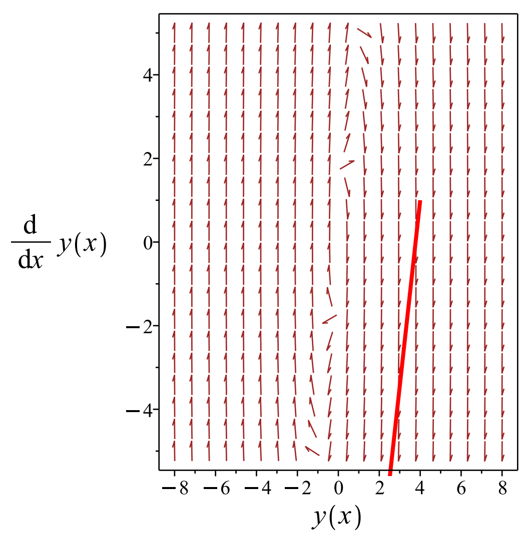
Summary

The solution(s) found are the following

$$y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4}$$

Verified OK.

9.12.4 Maple step by step solution

Let's solve

$$\left[y'' - 6y' + 25y = 0, y(0) = 4, y'\Big|_{\{x=0\}} = 1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 6r + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{6 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (3 - 4\,\mathrm{I}, 3 + 4\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^{3x} \cos(4x)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^{3x} \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^{3x} \cos(4x) + c_2 \mathrm{e}^{3x} \sin(4x)$

□ Check validity of solution $y = c_1 \mathrm{e}^{3x} \cos(4x) + c_2 \mathrm{e}^{3x} \sin(4x)$

- ○ Use initial condition $y(0) = 4$

  $4 = c_1$

- ○ Compute derivative of the solution

  $y' = 3c_1 \mathrm{e}^{3x} \cos(4x) - 4c_1 \mathrm{e}^{3x} \sin(4x) + 3c_2 \mathrm{e}^{3x} \sin(4x) + 4c_2 \mathrm{e}^{3x} \cos(4x)$

- ○ Use the initial condition $y'\Big|_{\{x=0\}} = 1$

  $1 = 3c_1 + 4c_2$

- ○ Solve for c_1 and c_2

  $\left\{c_1 = 4, c_2 = -\frac{11}{4}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4}$

- Solution to the IVP

  $y = \frac{e^{3x}(16\cos(4x) - 11\sin(4x))}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(y(x),x$2)-6*diff(y(x),x)+25*y(x)=0,y(0) = 4, D(y)(0) = 1],y(x), singsol=all)
```

$$y(x) = -\frac{e^{3x}(11\sin(4x) - 16\cos(4x))}{4}$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 27

```
DSolve[{y''[x]-6*y'[x]+25*y[x]==0,{y[0]==4,y'[0]==1}},y[x],x,IncludeSingularSolutions -> Tru
```

$$y(x) \to \frac{1}{4}e^{3x}(16\cos(4x) - 11\sin(4x))$$

9.13 problem 45

Internal problem ID [207]
Internal file name [`OUTPUT/207_Sunday_June_05_2022_01_36_44_AM_77074291/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - 2iy' + 3y = 0}$$

9.13.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2i, C = 3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2i\lambda\, e^{\lambda x} + 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2i\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2i, C = 3$ into the above gives

$$
\begin{aligned}
\lambda_{1,2} &= \frac{2i}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2i^2 - (4)(1)(3)} \\
&= i \pm 2i
\end{aligned}
$$

Hence

$$
\begin{aligned}
\lambda_1 &= i + 2i \\
\lambda_2 &= i - 2i
\end{aligned}
$$

Which simplifies to

$$
\begin{aligned}
\lambda_1 &= 3i \\
\lambda_2 &= -i
\end{aligned}
$$

The roots are complex but they are not conjugate of each others. Hence simplification using Euler relation is not possible here. Therefore the final solution is

$$
\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
&= c_1 e^{3ix} + c_2 e^{-ix}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{3ix} + c_2 \mathrm{e}^{-ix} \tag{1}
$$

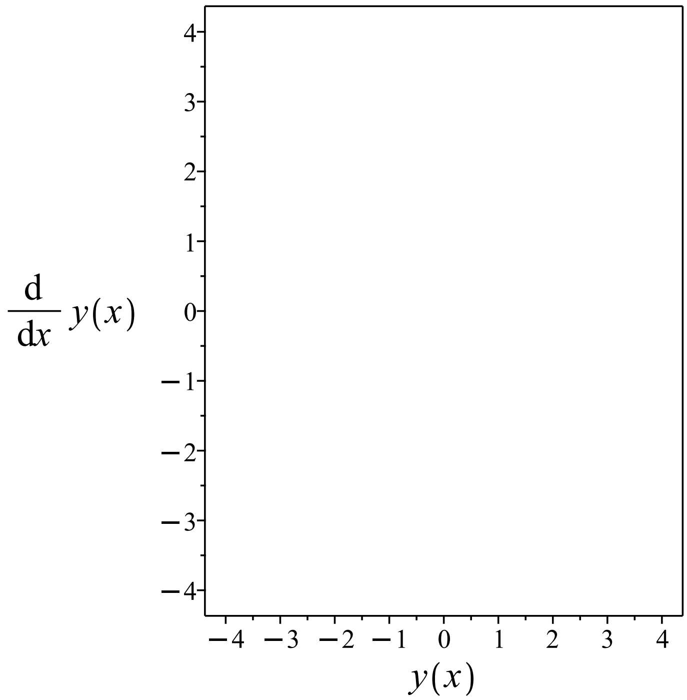


Figure 579: Slope field plot

Verification of solutions

$$y = c_1 e^{3ix} + c_2 e^{-ix}$$

Verified OK.

9.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2iy' + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2i \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 423: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2i}{1}\,dx} \\
&= z_1 e^{ix} \\
&= z_1\left(\mathrm{e}^{ix}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(2x)\,\mathrm{e}^{ix}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2i}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2ix}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos(2x)\, e^{ix}\right) + c_2 \left(\cos(2x)\, e^{ix} \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(2x)\, e^{ix} + \frac{c_2 \sin(2x)\, e^{ix}}{2} \tag{1}
$$

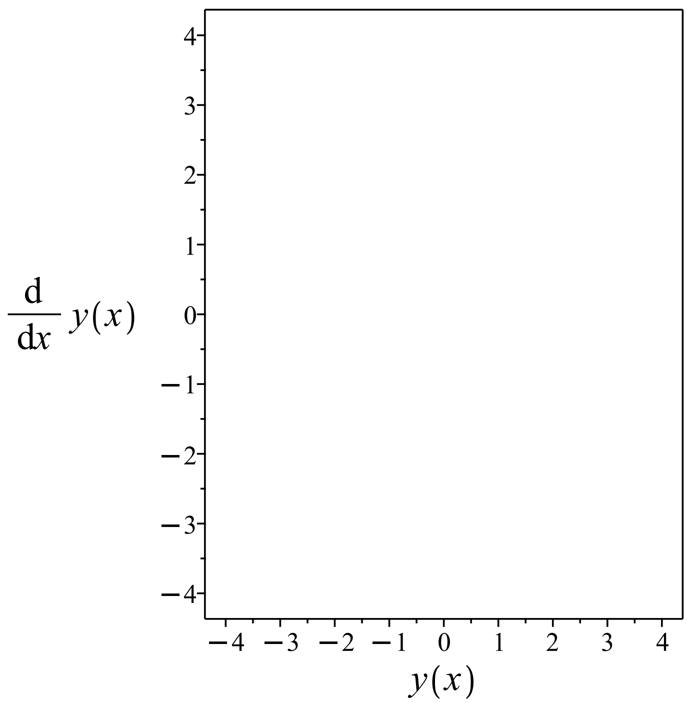


Figure 580: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x)\, e^{ix} + \frac{c_2 \sin(2x)\, e^{ix}}{2}
$$

Verified OK.

9.13.3 Maple step by step solution

Let's solve

$y'' - 2\,\mathrm{I}y' + 3y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - 2\,\mathrm{I}r + 3 = 0$

- Factor the characteristic polynomial

  $-(r + \mathrm{I})\,(-r + 3\,\mathrm{I}) = 0$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(x) + c_2 \sin(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-2*I*diff(y(x),x)+3*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{3ix} + c_2 \mathrm{e}^{-ix}$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 26

```
DSolve[y''[x]-2*I*y'[x]+3*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-ix}(c_1 e^{4ix} + c_2)$$

9.14 problem 46

Internal problem ID [208]
Internal file name [`OUTPUT/208_Sunday_June_05_2022_01_36_44_AM_46166529/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - iy' + 6y = 0}$$

9.14.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -i, C = 6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - i\lambda\, e^{\lambda x} + 6\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - i\lambda + 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -i, C = 6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{i}{(2)(1)} \pm \frac{1}{(2)(1)} \sqrt{-i^2 - (4)(1)(6)} \\
&= \frac{i}{2} \pm \frac{5i}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{i}{2} + \frac{5i}{2} \\
\lambda_2 &= \frac{i}{2} - \frac{5i}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3i \\
\lambda_2 &= -2i
\end{aligned}$$

The roots are complex but they are not conjugate of each others. Hence simplification using Euler relation is not possible here. Therefore the final solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
&= c_1 e^{3ix} + c_2 e^{-2ix}
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3ix} + c_2 e^{-2ix} \tag{1}$$

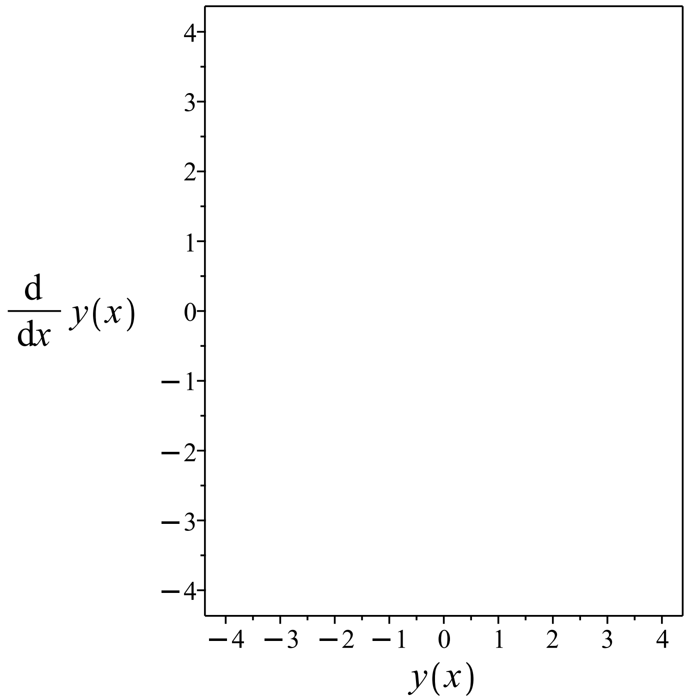


Figure 581: Slope field plot

Verification of solutions

$$y = c_1 e^{3ix} + c_2 e^{-2ix}$$

Verified OK.

9.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - iy' + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -i \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 425: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{5x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-i}{1}\,dx} \\ &= z_1 e^{\frac{ix}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{ix}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \cos\left(\frac{5x}{2}\right)\mathrm{e}^{\frac{ix}{2}}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-i}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{ix}}{(y_1)^2}\,dx \\ &= y_1\left(\frac{2\tan\left(\frac{5x}{2}\right)}{5}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\cos\left(\frac{5x}{2}\right) e^{\frac{ix}{2}}\right) + c_2 \left(\cos\left(\frac{5x}{2}\right) e^{\frac{ix}{2}} \left(\frac{2\tan\left(\frac{5x}{2}\right)}{5}\right)\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
y = c_1 \cos\left(\frac{5x}{2}\right) e^{\frac{ix}{2}} + \frac{2c_2 \sin\left(\frac{5x}{2}\right) e^{\frac{ix}{2}}}{5} \tag{1}
$$

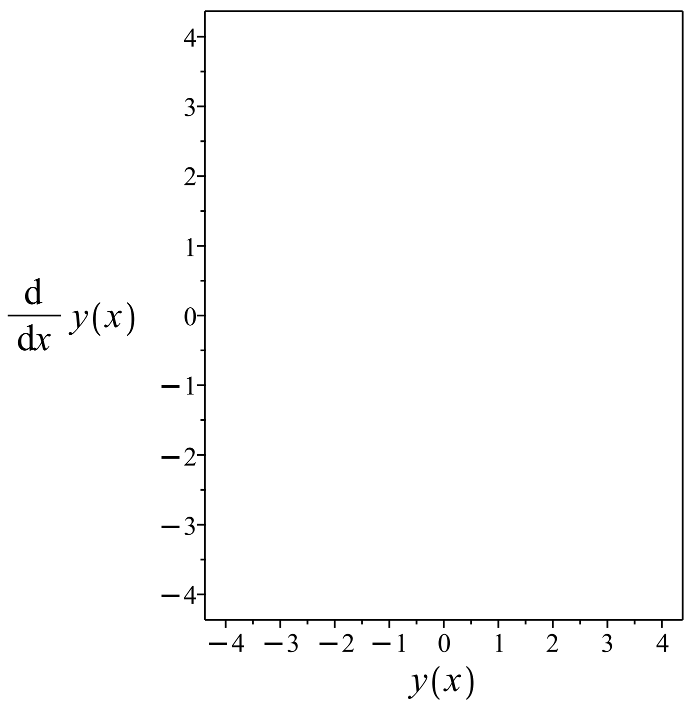


Figure 582: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \cos\left(\frac{5x}{2}\right) e^{\frac{ix}{2}} + \frac{2c_2 \sin\left(\frac{5x}{2}\right) e^{\frac{ix}{2}}}{5}
$$

Verified OK.

9.14.3 Maple step by step solution

Let's solve

$y'' - \mathrm{I}y' + 6y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of ODE

  $r^2 - \mathrm{I}r + 6 = 0$

- Factor the characteristic polynomial

  $-(-r + 3\,\mathrm{I})\,(r + 2\,\mathrm{I}) = 0$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \cos(2x) + c_2 \sin(2x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(diff(y(x),x$2)-I*diff(y(x),x)+6*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 e^{3ix} + c_2 e^{-2ix}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 26

```
DSolve[y''[x]-I*y'[x]+6*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{-2ix}(c_1 e^{5ix} + c_2)$$

9.15 problem 47

Internal problem ID [209]
Internal file name [`OUTPUT/209_Sunday_June_05_2022_01_36_45_AM_17138043/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "second_order_ode_can_be_made_integrable"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{y'' - \left(-2 + 2i\sqrt{3}\right) y = 0}$$

9.15.1 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = i, B = 0, C = 2i + 2\sqrt{3}$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$i\lambda^2 e^{\lambda x} + \left(2i + 2\sqrt{3}\right) e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$i\lambda^2 + 2i + 2\sqrt{3} = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = i, B = 0, C = 2i + 2\sqrt{3}$ into the above gives

$$\lambda_{1,2} = \frac{0}{(2)(i)} \pm \frac{1}{(2)(i)}\sqrt{0^2 - (4)(i)\left(2i + 2\sqrt{3}\right)}$$
$$= \pm - i\sqrt{-i\left(2i + 2\sqrt{3}\right)}$$

Hence

$$\lambda_1 = + - i\sqrt{-i\left(2i + 2\sqrt{3}\right)}$$
$$\lambda_2 = - - i\sqrt{-i\left(2i + 2\sqrt{3}\right)}$$

Which simplifies to

$$\lambda_1 = -i\sqrt{3} - 1$$
$$\lambda_2 = 1 + i\sqrt{3}$$

The roots are complex but they are not conjugate of each others. Hence simplification using Euler relation is not possible here. Therefore the final solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$$
$$= c_1 e^{\left(-i\sqrt{3}-1\right)x} + c_2 e^{\left(1+i\sqrt{3}\right)x}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{\left(-i\sqrt{3}-1\right)x} + c_2 \mathrm{e}^{\left(1+i\sqrt{3}\right)x} \tag{1}$$

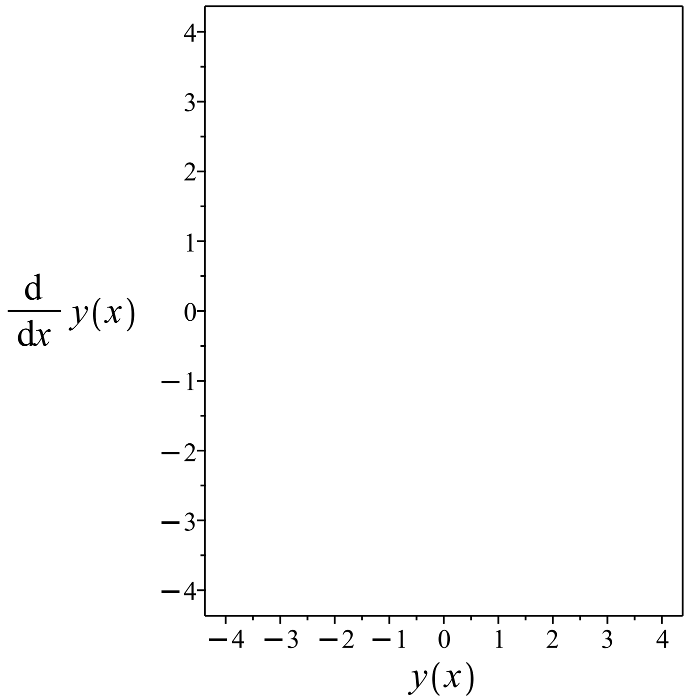


Figure 583: Slope field plot

Verification of solutions

$$y = c_1 e^{\left(-i\sqrt{3}-1\right)x} + c_2 e^{\left(1+i\sqrt{3}\right)x}$$

Verified OK.

9.15.2 Solving as second order ode can be made integrable ode

Multiplying the ode by y' gives

$$iy'y'' + \left(2i + 2\sqrt{3}\right) y'y = 0$$

Integrating the above w.r.t x gives

$$\int \left(iy'y'' + \left(2i + 2\sqrt{3}\right) y'y\right) dx = 0$$
$$\frac{iy'^2}{2} + \frac{\left(2i + 2\sqrt{3}\right) y^2}{2} = c_2$$

Which is now solved for y. Solving the given ode for y' results in 2 differential equations to solve. Each one of these will generate a solution. The equations generated are

$$y' = -i\sqrt{2}\,\sqrt{-i\left(iy^2 + y^2\sqrt{3} - c_1\right)} \tag{1}$$

$$y' = i\sqrt{2}\,\sqrt{-i\left(iy^2 + y^2\sqrt{3} - c_1\right)} \tag{2}$$

Now each one of the above ODE is solved.

Solving equation (1)

Integrating both sides gives

$$\int \frac{i\sqrt{2}}{2\sqrt{-i\left(iy^2 + y^2\sqrt{3} - c_1\right)}}dy = \int dx$$

$$\frac{i\sqrt{2}\,\arctan\left(\frac{\sqrt{i\left(\sqrt{3}+i\right)}\,y}{\sqrt{-i\left(\sqrt{3}+i\right)y^2+ic_1}}\right)}{2\sqrt{i\left(\sqrt{3}+i\right)}} = x + c_2$$

Solving equation (2)

Integrating both sides gives

$$\int -\frac{i\sqrt{2}}{2\sqrt{-i\left(iy^2 + y^2\sqrt{3} - c_1\right)}}dy = \int dx$$

$$-\frac{i\sqrt{2}\,\arctan\left(\frac{\sqrt{i\left(\sqrt{3}+i\right)}\,y}{\sqrt{-i\left(\sqrt{3}+i\right)y^2+ic_1}}\right)}{2\sqrt{i\left(\sqrt{3}+i\right)}} = c_3 + x$$

Summary
The solution(s) found are the following

$$\frac{i\sqrt{2}\,\arctan\left(\frac{\sqrt{i\left(\sqrt{3}+i\right)}\,y}{\sqrt{-i\left(\sqrt{3}+i\right)y^{2}+ic_{1}}}\right)}{2\sqrt{i\left(\sqrt{3}+i\right)}} = x + c_{2} \tag{1}$$

$$-\frac{i\sqrt{2}\,\arctan\left(\frac{\sqrt{i\left(\sqrt{3}+i\right)}\,y}{\sqrt{-i\left(\sqrt{3}+i\right)y^{2}+ic_{1}}}\right)}{2\sqrt{i\left(\sqrt{3}+i\right)}} = c_{3} + x \tag{2}$$

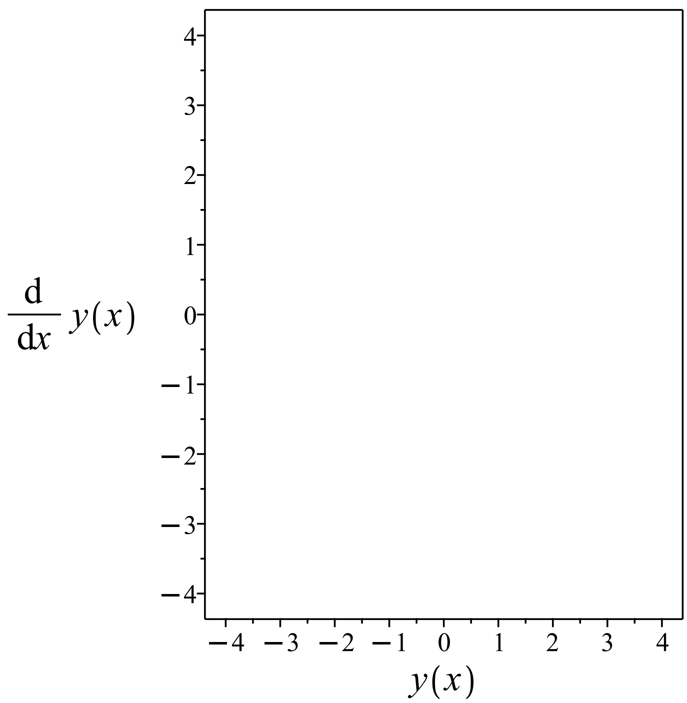


Figure 584: Slope field plot

Verification of solutions

$$\frac{i\sqrt{2}\, \arctan\left(\frac{\sqrt{i\left(\sqrt{3}+i\right)}\, y}{\sqrt{-i\left(\sqrt{3}+i\right) y^{2}+i c_{1}}}\right)}{2 \sqrt{i \left(\sqrt{3}+i\right)}} = x + c_{2}$$

Verified OK.

$$-\frac{i\sqrt{2}\, \arctan\left(\frac{\sqrt{i\left(\sqrt{3}+i\right)}\, y}{\sqrt{-i\left(\sqrt{3}+i\right) y^{2}+i c_{1}}}\right)}{2 \sqrt{i \left(\sqrt{3}+i\right)}} = c_{3} + x$$

Verified OK.

9.15.3 Solving using Kovacic algorithm

Writing the ode as

$$iy'' + \left(2i + 2\sqrt{3}\right) y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= i \\ B &= 0 \\ C &= 2i + 2\sqrt{3} \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^{2} - 4AC}{4A^{2}} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{2i(\sqrt{3}+i)}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = 2i\left(\sqrt{3}+i\right)$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = \left(2i\left(\sqrt{3}+i\right)\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 427: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 2i\left(\sqrt{3}+i\right)$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{\sqrt{2}\,\sqrt{i\left(\sqrt{3}+i\right)}\,x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{\sqrt{2}\,\sqrt{i\left(\sqrt{3}+i\right)}\,x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{\sqrt{2}\,\sqrt{i\sqrt{3}-1}\,x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{\sqrt{2}\,\sqrt{i\sqrt{3}-1}\,x} \int \frac{1}{\mathrm{e}^{2x\sqrt{-2+2i\sqrt{3}}}}\,dx \\ &= \mathrm{e}^{\sqrt{2}\,\sqrt{i\sqrt{3}-1}\,x} \left(\frac{\left(i\sqrt{3}-1\right)\mathrm{e}^{-2\left(1+i\sqrt{3}\right)x}}{8} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{\sqrt{2}\,\sqrt{i\sqrt{3}-1}\,x}\right) + c_2 \left(\mathrm{e}^{\sqrt{2}\,\sqrt{i\sqrt{3}-1}\,x} \left(\frac{\left(i\sqrt{3}-1\right) \mathrm{e}^{-2\left(1+i\sqrt{3}\right)x}}{8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{\sqrt{2}\,\sqrt{i\sqrt{3}-1}\,x} + \frac{c_2 \left(i\sqrt{3}-1\right) \mathrm{e}^{-\left(1+i\sqrt{3}\right)x}}{8} \tag{1}
$$

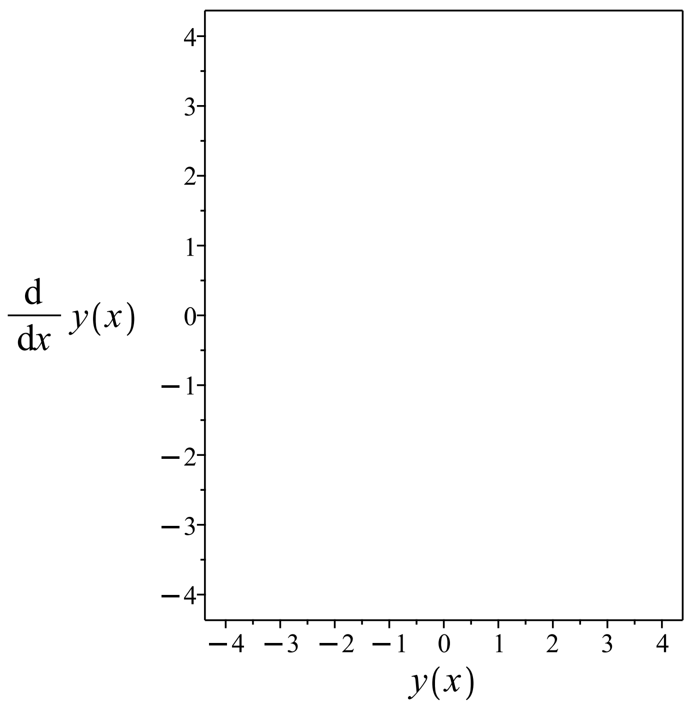


Figure 585: Slope field plot

Verification of solutions

$$
y = c_1 \mathrm{e}^{\sqrt{2}\,\sqrt{i\sqrt{3}-1}\,x} + \frac{c_2 \left(i\sqrt{3}-1\right) \mathrm{e}^{-\left(1+i\sqrt{3}\right)x}}{8}
$$

Verified OK.

9.15.4 Maple step by step solution

Let's solve

$\mathrm{I}y'' + \left(2\,\mathrm{I} + 2\sqrt{3}\right) y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = 2\,\mathrm{I}\left(\sqrt{3} + \mathrm{I}\right) y$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' - 2\,\mathrm{I}\left(\sqrt{3} + \mathrm{I}\right) y = 0$

- Characteristic polynomial of ODE

  $r^2 - 2\,\mathrm{I}\left(\sqrt{3} + \mathrm{I}\right) = 0$

- Factor the characteristic polynomial

  $-\left(r + 1 + \mathrm{I}\sqrt{3}\right)\left(\mathrm{I}\sqrt{3} - r + 1\right) = 0$

- Roots of the characteristic polynomial

  $r = \left(1 + \mathrm{I}\sqrt{3}, -\mathrm{I}\sqrt{3} - 1\right)$

- 1st solution of the ODE

  $y_1(x) = \mathrm{e}^x \cos\left(\sqrt{3}\,x\right)$

- 2nd solution of the ODE

  $y_2(x) = \mathrm{e}^x \sin\left(\sqrt{3}\,x\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x)$

- Substitute in solutions

  $y = c_1 \mathrm{e}^x \cos\left(\sqrt{3}\,x\right) + c_2 \mathrm{e}^x \sin\left(\sqrt{3}\,x\right)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)=(-2+2*I*sqrt(3))*y(x),y(x), singsol=all)
```

$$y(x) = c_1 \mathrm{e}^{-\left(1+i\sqrt{3}\right)x} + c_2 \mathrm{e}^{\left(1+i\sqrt{3}\right)x}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 41

```
DSolve[y''[x]==(-2+2*I*Sqrt[3])*y[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 e^{x+i\sqrt{3}x} + c_2 e^{\left(-1-i\sqrt{3}\right)x}$$

9.16 problem 52

Internal problem ID [210]
Internal file name [`OUTPUT/210_Sunday_June_05_2022_01_36_46_AM_41684352/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 52.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler], [_2nd_order, _linear, `_with_symmetry_[0,F(
   x)]`]]
```

$$\boxed{x^2y'' + y'x + 9y = 0}$$

9.16.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + 9x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + 9x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 9 = 0$$

Or

$$r^2 + 9 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3i \\ r_2 &= 3i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -3$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha + i\beta} + c_2 x^{\alpha - i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -3$, the above becomes

$$y = x^0 \left(c_1 e^{-3i \ln(x)} + c_2 e^{3i \ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(3 \ln(x)) + c_2 \sin(3 \ln(x))$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3 \ln(x)) + c_2 \sin(3 \ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(3 \ln(x)) + c_2 \sin(3 \ln(x))$$

Verified OK.

9.16.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{9}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'\left(x\right)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'\left(x\right)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned} \tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\ &= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)}dx \\ &= \int e^{-\ln(x)}\,dx \\ &= \int \frac{1}{x}dx \\ &= \ln\left(x\right) \end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{9}{x^2}}{\frac{1}{x^2}} \\
&= 9 \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + 9y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} + 9\,\mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\
&= \pm 3i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +3i \\
\lambda_2 &= -3i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau) = e^{\alpha\tau}(c_1 \cos(\beta\tau) + c_2 \sin(\beta\tau))$$

Which becomes

$$y(\tau) = e^0(c_1 \cos(3\tau) + c_2 \sin(3\tau))$$

Or

$$y(\tau) = c_1 \cos(3\tau) + c_2 \sin(3\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Verified OK.

9.16.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$
\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{9}{x^2}
\end{aligned}
$$

Applying change of variables $\tau = g(x)$ to (2) results

$$
\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}
$$

Where τ is the new independent variable, and

$$
p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}
$$

$$
q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}
$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$
\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{3\sqrt{\frac{1}{x^2}}}{c} \\
\tau'' &= -\frac{3}{c\sqrt{\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}
$$

Substituting the above into (4) results in

$$
\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{3}{c\sqrt{\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{3\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{3\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}
$$

Therefore ode (3) now becomes

$$
\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}
$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 3\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{3\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x)) \tag{1}$$

Verification of solutions

$$y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$$

Verified OK.

9.16.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + y'x + 9y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{1}{x} \\
q(x) &= \frac{9}{x^2}
\end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{9}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{6i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+6i)\,v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+6i)\,u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-6i)\,u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-6i}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{-1-6i}{x}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{-1-6i}{x}\, dx \\
\ln(u) &= (-1-6i)\ln(x) + c_1 \\
u &= \mathrm{e}^{(-1-6i)\ln(x)+c_1} \\
&= c_1 \mathrm{e}^{(-1-6i)\ln(x)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = \frac{c_1 x^{-6i}}{x}
$$

Now that $u(x)$ is known, then

$$
\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= \frac{i c_1 x^{-6i}}{6} + c_2
\end{aligned}
$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{3i} \\
&= x^{3i} c_2 + \frac{i x^{-3i} c_1}{6}
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{3i} \tag{1}
$$

Verification of solutions

$$
y = \left(\frac{i c_1 x^{-6i}}{6} + c_2\right) x^{3i}
$$

Verified OK.

9.16.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-37}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -37 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{37}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 429: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{37}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{37}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{37}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{37}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 3i \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 3i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{37}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2} + 3i$	$\frac{1}{2} - 3i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2} + 3i$	$\frac{1}{2} - 3i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2} - 3i$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - 3i - \left(\frac{1}{2} - 3i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - 3i}{x} + (-)(0) \\ &= \frac{\frac{1}{2} - 3i}{x} \\ &= \frac{\frac{1}{2} - 3i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + (\omega' + \omega^2 - r)\, p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{\frac{1}{2} - 3i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + 3i}{x^2}\right) + \left(\frac{\frac{1}{2} - 3i}{x}\right)^2 - \left(-\frac{37}{4x^2}\right) \right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\, dx} \\ &= e^{\int \frac{\frac{1}{2} - 3i}{x} dx} \\ &= x^{\frac{1}{2} - 3i} \end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-3i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{6i}}{6}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-3i}\right) + c_2 \left(x^{-3i}\left(-\frac{i x^{6i}}{6}\right)\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6} \tag{1}$$

Verification of solutions

$$y = x^{-3i} c_1 - \frac{i c_2 x^{3i}}{6}$$

Verified OK.

9.16.6 Maple step by step solution

Let's solve

$x^2y'' + y'x + 9y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{y'}{x} - \frac{9y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{y'}{x} + \frac{9y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2y'' + y'x + 9y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt}y(t)\right)t'(x)$

  - Compute derivative

    $y' = \frac{\frac{d}{dt}y(t)}{x}$

  - Calculate the 2nd derivative of y with respect to x , using the chain rule

    $y'' = \left(\frac{d^2}{dt^2}y(t)\right)t'(x)^2 + t''(x)\left(\frac{d}{dt}y(t)\right)$

  - Compute derivative

    $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2\left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + \frac{d}{dt}y(t) + 9y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 9y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$
- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$
- 1st solution of the ODE

  $y_1(t) = \cos(3t)$
- 2nd solution of the ODE

  $y_2(t) = \sin(3t)$
- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$
- Substitute in solutions

  $y(t) = c_1 \cos(3t) + c_2 \sin(3t)$
- Change variables back using $t = \ln(x)$

  $y = c_1 \cos(3\ln(x)) + c_2 \sin(3\ln(x))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+9*y(x)=0,y(x), singsol=all)
```

$$y(x) = c_1 \sin(3\ln(x)) + c_2 \cos(3\ln(x))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]+x*y'[x]+9*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1 \cos(3\log(x)) + c_2 \sin(3\log(x))$$

9.17 problem 53

Internal problem ID [211]
Internal file name [`OUTPUT/211_Sunday_June_05_2022_01_36_46_AM_36350048/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.3, second order linear equations. Page 323
Problem number: 53.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_Emden, _Fowler]]
```

$$\boxed{x^2y'' + 7y'x + 25y = 0}$$

9.17.1 Solving as second order euler ode ode

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + 7xrx^{r-1} + 25x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + 7r\,x^r + 25x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + 7r + 25 = 0$$

Or

$$r^2 + 6r + 25 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -3 - 4i \\ r_2 &= -3 + 4i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = -3$ and $\beta = -4$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^{\alpha}\left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^{\alpha}\left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^{\alpha}\left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = -3, \beta = -4$, the above becomes

$$y = x^{-3}\left(c_1 e^{-4i\ln(x)} + c_2 e^{4i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = \frac{1}{x^3}\left(c_1 \cos\left(4\ln\left(x\right)\right) + c_2 \sin\left(4\ln\left(x\right)\right)\right)$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos\left(4\ln\left(x\right)\right) + c_2 \sin\left(4\ln\left(x\right)\right)}{x^3} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 \cos\left(4\ln\left(x\right)\right) + c_2 \sin\left(4\ln\left(x\right)\right)}{x^3}$$

Verified OK.

9.17.2 Solving as second order change of variable on x method 2 ode

In normal form the ode

$$x^2y'' + 7y'x + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned}
p(x) &= \frac{7}{x}\\
q(x) &= \frac{25}{x^2}
\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx\\
&= \int \mathrm{e}^{-\left(\int \frac{7}{x}dx\right)}dx\\
&= \int e^{-7\ln(x)}\,dx\\
&= \int \frac{1}{x^7}dx\\
&= -\frac{1}{6x^6}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned} q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\ &= \frac{\frac{25}{x^2}}{\frac{1}{x^{14}}} \\ &= 25x^{12} \end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) = 0$$
$$\frac{d^2}{d\tau^2}y(\tau) + 25x^{12}y(\tau) = 0$$

But in terms of τ

$$25x^{12} = \frac{25}{36\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{25y(\tau)}{36\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$36\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 25y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$36\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 25\tau^r = 0$$

Simplifying gives

$$36r(r-1)\,\tau^r + 0\,\tau^r + 25\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$36r(r-1) + 0 + 25 = 0$$

Or

$$36r^2 - 36r + 25 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = \frac{1}{2} - \frac{2i}{3}$$
$$r_2 = \frac{1}{2} + \frac{2i}{3}$$

The roots are complex conjugate of each others. Let the roots be

$$r_1 = \alpha + i\beta$$
$$r_2 = \alpha - i\beta$$

Where in this case $\alpha = \frac{1}{2}$ and $\beta = -\frac{2}{3}$. Hence the solution becomes

$$\begin{aligned} y(\tau) &= c_1 \tau^{r_1} + c_2 \tau^{r_2} \\ &= c_1 \tau^{\alpha+i\beta} + c_2 \tau^{\alpha-i\beta} \\ &= \tau^\alpha \left(c_1 \tau^{i\beta} + c_2 \tau^{-i\beta}\right) \\ &= \tau^\alpha \left(c_1 e^{\ln(\tau^{i\beta})} + c_2 e^{\ln(\tau^{-i\beta})}\right) \\ &= \tau^\alpha \left(c_1 e^{i(\beta \ln \tau)} + c_2 e^{-i(\beta \ln \tau)}\right) \end{aligned}$$

Using the values for $\alpha = \frac{1}{2}, \beta = -\frac{2}{3}$, the above becomes

$$y(\tau) = \tau^{\frac{1}{2}} \left(c_1 e^{-\frac{2i \ln(\tau)}{3}} + c_2 e^{\frac{2i \ln(\tau)}{3}}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y(\tau) = \sqrt{\tau}\left(c_1 \cos\left(\frac{2\ln(\tau)}{3}\right) + c_2 \sin\left(\frac{2\ln(\tau)}{3}\right)\right)$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{\left(c_1 \cos\left(-\frac{2\ln(2)}{3} - \frac{2\ln(3)}{3} + \frac{2\ln\left(-\frac{1}{x^6}\right)}{3}\right) + c_2 \sin\left(-\frac{2\ln(2)}{3} - \frac{2\ln(3)}{3} + \frac{2\ln\left(-\frac{1}{x^6}\right)}{3}\right)\right)\sqrt{6}\sqrt{-\frac{1}{x^6}}}{6}$$

Summary

The solution(s) found are the following

$$y = \frac{\left(c_1 \cos\left(-\frac{2\ln(2)}{3} - \frac{2\ln(3)}{3} + \frac{2\ln\left(-\frac{1}{x^6}\right)}{3}\right) + c_2 \sin\left(-\frac{2\ln(2)}{3} - \frac{2\ln(3)}{3} + \frac{2\ln\left(-\frac{1}{x^6}\right)}{3}\right)\right)\sqrt{6}\sqrt{-\frac{1}{x^6}}}{6} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_1 \cos\left(-\frac{2\ln(2)}{3} - \frac{2\ln(3)}{3} + \frac{2\ln\left(-\frac{1}{x^6}\right)}{3}\right) + c_2 \sin\left(-\frac{2\ln(2)}{3} - \frac{2\ln(3)}{3} + \frac{2\ln\left(-\frac{1}{x^6}\right)}{3}\right)\right)\sqrt{6}\,\sqrt{-\frac{1}{x^6}}}{6}$$

Verified OK.

9.17.3 Solving as second order change of variable on x method 1 ode

In normal form the ode

$$x^2y'' + 7y'x + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{7}{x} \\ q(x) &= \frac{25}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{5\sqrt{\frac{1}{x^2}}}{c} \end{aligned} \tag{6}$$

$$\tau'' = -\frac{5}{c\sqrt{\frac{1}{x^2}}\,x^3}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{5}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{7}{x}\frac{5\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{5\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= \frac{6c}{5}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6c\left(\frac{d}{d\tau}y(\tau)\right)}{5} + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = e^{-\frac{3c\tau}{5}}\left(c_1 \cos\left(\frac{4c\tau}{5}\right) + c_2 \sin\left(\frac{4c\tau}{5}\right)\right)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 5\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{5\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))}{x^3}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))}{x^3} \qquad (1)$$

Verification of solutions

$$y = \frac{c_1 \cos(4\ln(x)) + c_2 \sin(4\ln(x))}{x^3}$$

Verified OK.

9.17.4 Solving as second order change of variable on y method 2 ode

In normal form the ode

$$x^2y'' + 7y'x + 25y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{7}{x} \\ q(x) &= \frac{25}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{7n}{x^2} + \frac{25}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = -3 + 4i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{-6+8i}{x} + \frac{7}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+8i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x)+\frac{(1+8i)\,u(x)}{x}=0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned}u' &= F(x,u)\\ &= f(x)g(u)\\ &= \frac{(-1-8i)\,u}{x}\end{aligned}$$

Where $f(x)=\frac{-1-8i}{x}$ and $g(u)=u$. Integrating both sides gives

$$\begin{aligned}\frac{1}{u}\,du &= \frac{-1-8i}{x}\,dx\\ \int\frac{1}{u}\,du &= \int\frac{-1-8i}{x}\,dx\\ \ln(u) &= (-1-8i)\ln(x)+c_1\\ u &= \mathrm{e}^{(-1-8i)\ln(x)+c_1}\\ &= c_1\mathrm{e}^{(-1-8i)\ln(x)}\end{aligned}$$

Which simplifies to

$$u(x)=\frac{c_1x^{-8i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned}v'(x) &= u(x)\\ v(x) &= \int u(x)\,dx+c_2\\ &= \frac{ic_1x^{-8i}}{8}+c_2\end{aligned}$$

Hence

$$\begin{aligned}y &= v(x)\,x^n\\ &= \left(\frac{ic_1x^{-8i}}{8}+c_2\right)x^{-3+4i}\\ &= c_2x^{-3+4i}+\frac{ic_1x^{-3-4i}}{8}\end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(\frac{i c_1 x^{-8i}}{8} + c_2\right) x^{-3+4i} \tag{1}$$

Verification of solutions

$$y = \left(\frac{i c_1 x^{-8i}}{8} + c_2\right) x^{-3+4i}$$

Verified OK.

9.17.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + 7y'x + 25y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= 7x \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-65}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$s = -65$$
$$t = 4x^2$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{65}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 431: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{65}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{65}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 4i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 4i
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = -\frac{65}{4x^2}$$

Since the $\gcd(s, t) = 1$. This gives $b = -\frac{65}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + 4i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - 4i
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = -\frac{65}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + 4i$	$\frac{1}{2} - 4i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}+4i$	$\frac{1}{2}-4i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = \frac{1}{2} - 4i$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - (\alpha_{c_1}^{-}) \\ &= \frac{1}{2} - 4i - \left(\frac{1}{2} - 4i\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{\frac{1}{2} - 4i}{x} + (-)\,(0) \\ &= \frac{\frac{1}{2} - 4i}{x} \\ &= \frac{\frac{1}{2} - 4i}{x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0)+2\left(\frac{\frac{1}{2}-4i}{x}\right)(0)+\left(\left(\frac{-\frac{1}{2}+4i}{x^2}\right)+\left(\frac{\frac{1}{2}-4i}{x}\right)^2-\left(-\frac{65}{4x^2}\right)\right)=0$$
$$0=0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2}-4i}{x} dx} \\
&= x^{\frac{1}{2}-4i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{7x}{x^2} \, dx} \\
&= z_1 e^{-\frac{7\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{x^{\frac{7}{2}}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = x^{-3-4i}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{7x}{x^2} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{-7\ln(x)}}{(y_1)^2} \, dx \\
&= y_1 \left(-\frac{i x^{8i}}{8}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-3-4i}\right) + c_2 \left(x^{-3-4i} \left(-\frac{i x^{8i}}{8}\right)\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 x^{-3-4i} - \frac{i c_2 x^{-3+4i}}{8} \tag{1}
$$

Verification of solutions

$$
y = c_1 x^{-3-4i} - \frac{i c_2 x^{-3+4i}}{8}
$$

Verified OK.

9.17.6 Maple step by step solution

Let's solve

$x^2 y'' + 7y' x + 25y = 0$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -\frac{7y'}{x} - \frac{25y}{x^2}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + \frac{7y'}{x} + \frac{25y}{x^2} = 0$

- Multiply by denominators of the ODE

  $x^2 y'' + 7y' x + 25y = 0$

- Make a change of variables

  $t = \ln(x)$

□ Substitute the change of variables back into the ODE

  - Calculate the 1st derivative of y with respect to x , using the chain rule

    $y' = \left(\frac{d}{dt} y(t)\right) t'(x)$

- Compute derivative

  $y' = \frac{\frac{d}{dt}y(t)}{x}$

- Calculate the 2nd derivative of y with respect to x , using the chain rule

  $y'' = \left(\frac{d^2}{dt^2}y(t)\right) t'(x)^2 + t''(x) \left(\frac{d}{dt}y(t)\right)$

- Compute derivative

  $y'' = \frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}$

  Substitute the change of variables back into the ODE

  $x^2 \left(\frac{\frac{d^2}{dt^2}y(t)}{x^2} - \frac{\frac{d}{dt}y(t)}{x^2}\right) + 7\frac{d}{dt}y(t) + 25y(t) = 0$

- Simplify

  $\frac{d^2}{dt^2}y(t) + 6\frac{d}{dt}y(t) + 25y(t) = 0$

- Characteristic polynomial of ODE

  $r^2 + 6r + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 4\,\mathrm{I}, -3 + 4\,\mathrm{I})$

- 1st solution of the ODE

  $y_1(t) = \mathrm{e}^{-3t} \cos(4t)$

- 2nd solution of the ODE

  $y_2(t) = \mathrm{e}^{-3t} \sin(4t)$

- General solution of the ODE

  $y(t) = c_1 y_1(t) + c_2 y_2(t)$

- Substitute in solutions

  $y(t) = c_1 \mathrm{e}^{-3t} \cos(4t) + c_2 \mathrm{e}^{-3t} \sin(4t)$

- Change variables back using $t = \ln(x)$

  $y = \frac{c_1 \cos(4\ln(x))}{x^3} + \frac{c_2 \sin(4\ln(x))}{x^3}$

- Simplify

  $y = \frac{c_1 \cos(4\ln(x))}{x^3} + \frac{c_2 \sin(4\ln(x))}{x^3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
checking if the LODE is of Euler type
<- LODE of Euler type successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(x^2*diff(y(x),x$2)+7*x*diff(y(x),x)+25*y(x)=0,y(x), singsol=all)
```

$$y(x) = \frac{c_1 \sin(4\ln(x)) + c_2 \cos(4\ln(x))}{x^3}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]+7*x*y'[x]+25*y[x]==0,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{c_2 \cos(4\log(x)) + c_1 \sin(4\log(x))}{x^3}$$

10 Section 5.4, Mechanical Vibrations. Page 337

10.1 problem 15

Internal problem ID [212]
Internal file name [`OUTPUT/212_Sunday_June_05_2022_01_36_47_AM_48138665/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.4, Mechanical Vibrations. Page 337
Problem number: 15.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{\frac{x''}{2} + 3x' + 4x = 0}$$

With initial conditions

$$[x(0) = 2, x'(0) = 0]$$

10.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 8 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 6x' + 8x = 0$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.1.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = \frac{1}{2}, B = 3, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\frac{\lambda^2 e^{\lambda t}}{2} + 3\lambda e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\frac{1}{2}\lambda^2 + 3\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = \frac{1}{2}, B = 3, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\left(\frac{1}{2}\right)} \pm \frac{1}{(2)\left(\frac{1}{2}\right)}\sqrt{3^2 - (4)\left(\frac{1}{2}\right)(4)} \\ &= -3 \pm 1\end{aligned}$$

Hence

$$\lambda_1 = -3 + 1$$
$$\lambda_2 = -3 - 1$$

Which simplifies to

$$\lambda_1 = -2$$
$$\lambda_2 = -4$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(-2)t} + c_2 e^{(-4)t}$$

Or

$$x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-2t} + c_2 \mathrm{e}^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2c_1 \mathrm{e}^{-2t} - 4c_2 \mathrm{e}^{-4t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 - 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = -2$$

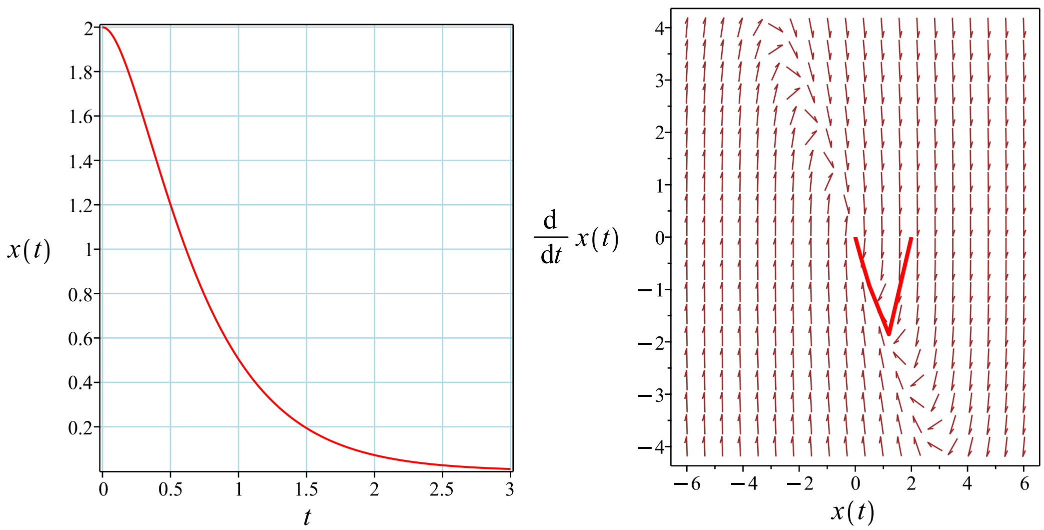
Substituting these values back in above solution results in

$$x = 4\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-4t}$$

Summary

The solution(s) found are the following

$$x = 4\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-4t} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 4\,\mathrm{e}^{-2t} - 2\,\mathrm{e}^{-4t}$$

Verified OK.

10.1.3 Solving using Kovacic algorithm

Writing the ode as

$$\frac{x''}{2} + 3x' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= \frac{1}{2} \\ B &= 3 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 433: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \mathrm{e}^{-t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{\frac{1}{2}}\,dt} \\ &= z_1 e^{-3t} \\ &= z_1\left(\mathrm{e}^{-3t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-4t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{3}{\frac{1}{2}}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-6t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\mathrm{e}^{2t}}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t} \left(\frac{\mathrm{e}^{2t}}{2}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-4t} + \frac{c_2 \mathrm{e}^{-2t}}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$2 = c_1 + \frac{c_2}{2} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4c_1 \mathrm{e}^{-4t} - c_2 \mathrm{e}^{-2t}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -4c_1 - c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 8$$

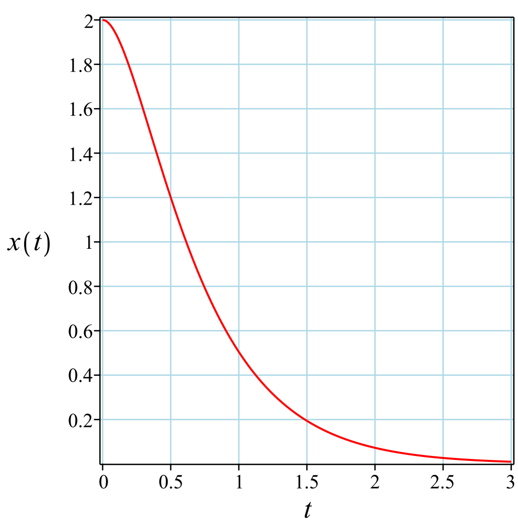
Substituting these values back in above solution results in

$$x = 4\,e^{-2t} - 2\,e^{-4t}$$

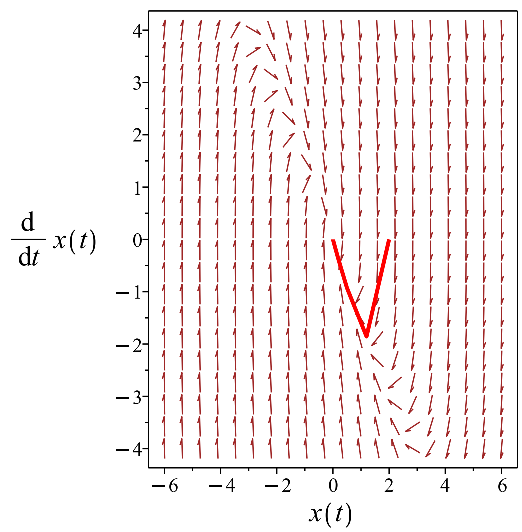
Summary

The solution(s) found are the following

$$x = 4\,e^{-2t} - 2\,e^{-4t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 4\,e^{-2t} - 2\,e^{-4t}$$

Verified OK.

10.1.4 Maple step by step solution

Let's solve

$$\left[\frac{x''}{2}+3x'+4x=0, x(0)=2, x'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x''=-6x'-8x$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x''+6x'+8x=0$

- Characteristic polynomial of ODE

  $r^2+6r+8=0$

- Factor the characteristic polynomial

  $(r+4)(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-4,-2)$

- 1st solution of the ODE

  $x_1(t)=\mathrm{e}^{-4t}$

- 2nd solution of the ODE

  $x_2(t)=\mathrm{e}^{-2t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)$

- Substitute in solutions

  $x=c_1\mathrm{e}^{-4t}+c_2\mathrm{e}^{-2t}$

□ Check validity of solution $x=c_1\mathrm{e}^{-4t}+c_2\mathrm{e}^{-2t}$

  ◦ Use initial condition $x(0)=2$

    $2=c_1+c_2$

  ◦ Compute derivative of the solution

    $x'=-4c_1\mathrm{e}^{-4t}-2c_2\mathrm{e}^{-2t}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -4c_1 - 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = -2, c_2 = 4\}$

- Substitute constant values into general solution and simplify

  $x = 4\,e^{-2t} - 2\,e^{-4t}$

- Solution to the IVP

  $x = 4\,e^{-2t} - 2\,e^{-4t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([1/2*diff(x(t),t$2)+3*diff(x(t),t)+4*x(t)=0,x(0) = 2, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = -2\,e^{-4t} + 4\,e^{-2t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[{1/2*x''[t]+3*x'[t]+4*x[t]==0,{x[0]==2,x'[0]==0}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to e^{-4t}\left(4e^{2t} - 2\right)$$

10.2 problem 16

Internal problem ID [213]
Internal file name [`OUTPUT/213_Sunday_June_05_2022_01_36_48_AM_60553072/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.4, Mechanical Vibrations. Page 337
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{3x'' + 30x' + 63x = 0}$$

With initial conditions

$$[x(0) = 2, x'(0) = 2]$$

10.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 10 \\ q(t) &= 21 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 10x' + 21x = 0$$

The domain of $p(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 21$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.2.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 3, B = 30, C = 63$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$3\lambda^2 e^{\lambda t} + 30\lambda\, e^{\lambda t} + 63\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$3\lambda^2 + 30\lambda + 63 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 3, B = 30, C = 63$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-30}{(2)(3)} \pm \frac{1}{(2)(3)}\sqrt{30^2 - (4)(3)(63)} \\ &= -5 \pm 2\end{aligned}$$

Hence

$$\lambda_1 = -5 + 2$$
$$\lambda_2 = -5 - 2$$

Which simplifies to

$$\lambda_1 = -3$$
$$\lambda_2 = -7$$

Since roots are real and distinct, then the solution is

$$x = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$
$$x = c_1 e^{(-3)t} + c_2 e^{(-7)t}$$

Or

$$x = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-7t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-3t} + c_2 \mathrm{e}^{-7t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$2 = c_1 + c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \mathrm{e}^{-3t} - 7c_2 \mathrm{e}^{-7t}$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$2 = -3c_1 - 7c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = -2$$

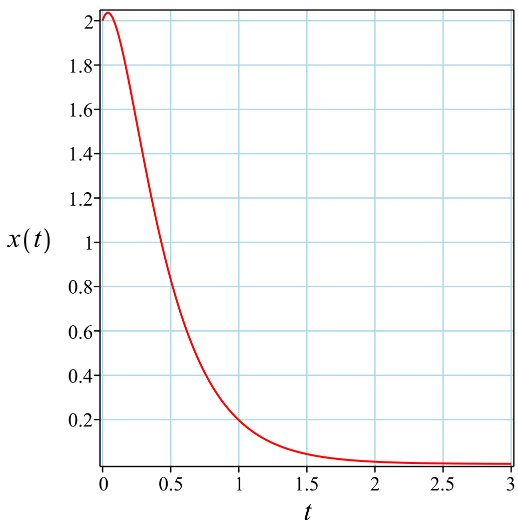
Substituting these values back in above solution results in

$$x = 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-7t}$$

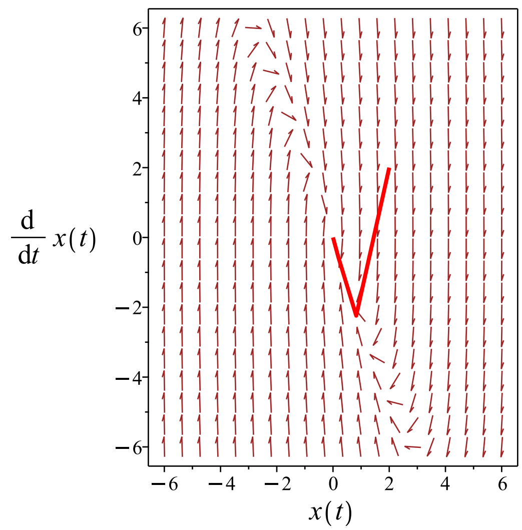
Summary

The solution(s) found are the following

$$x = 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-7t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-7t}$$

Verified OK.

10.2.3 Solving using Kovacic algorithm

Writing the ode as

$$3x'' + 30x' + 63x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 3 \\ B &= 30 \\ C &= 63 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 435: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \mathrm{e}^{-2t}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{30}{3}\,dt} \\
&= z_1 e^{-5t} \\
&= z_1 \left(\mathrm{e}^{-5t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-7t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{30}{3}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{\mathrm{e}^{-10t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{\mathrm{e}^{4t}}{4}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-7t}\right) + c_2 \left(\mathrm{e}^{-7t} \left(\frac{\mathrm{e}^{4t}}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{-7t} + \frac{c_2 \mathrm{e}^{-3t}}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 2$ and $t = 0$ in the above gives

$$
2 = c_1 + \frac{c_2}{4} \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -7c_1 \mathrm{e}^{-7t} - \frac{3c_2 \mathrm{e}^{-3t}}{4}
$$

substituting $x' = 2$ and $t = 0$ in the above gives

$$
2 = -7c_1 - \frac{3c_2}{4} \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= -2 \\
c_2 &= 16
\end{aligned}
$$

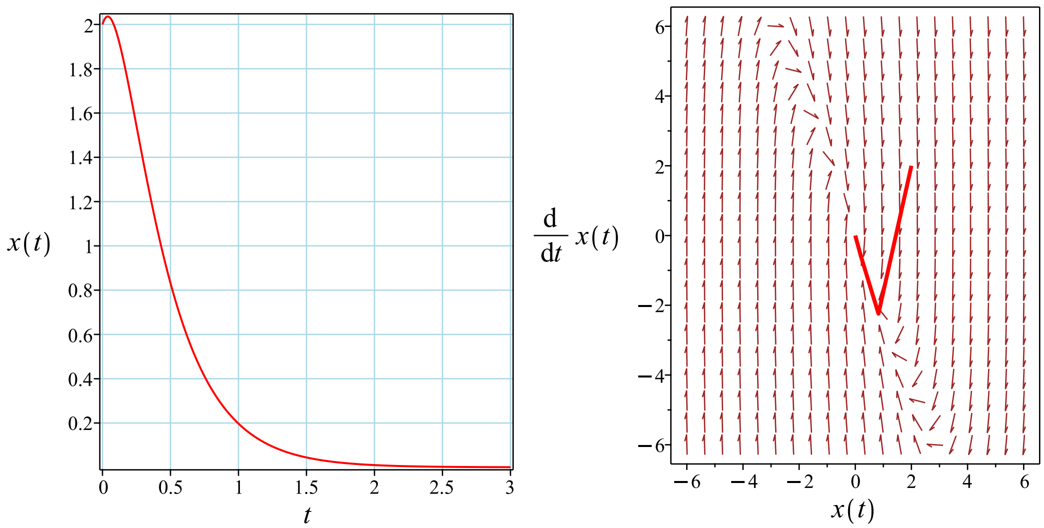
Substituting these values back in above solution results in

$$
x = 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-7t}
$$

Summary

The solution(s) found are the following

$$
x = 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-7t} \tag{1}
$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-7t}$$

Verified OK.

10.2.4 Maple step by step solution

Let's solve

$$\left[3x'' + 30x' + 63x = 0, x(0) = 2, x'\Big|_{\{t=0\}} = 2\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -10x' - 21x$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + 10x' + 21x = 0$

- Characteristic polynomial of ODE

  $r^2 + 10r + 21 = 0$

- Factor the characteristic polynomial

$(r+7)(r+3)=0$

- Roots of the characteristic polynomial

  $r=(-7,-3)$

- 1st solution of the ODE

  $x_1(t)=\mathrm{e}^{-7t}$

- 2nd solution of the ODE

  $x_2(t)=\mathrm{e}^{-3t}$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)$

- Substitute in solutions

  $x=c_1\mathrm{e}^{-7t}+c_2\mathrm{e}^{-3t}$

☐ Check validity of solution $x=c_1\mathrm{e}^{-7t}+c_2\mathrm{e}^{-3t}$

  ◦ Use initial condition $x(0)=2$

    $2=c_1+c_2$

  ◦ Compute derivative of the solution

    $x'=-7c_1\mathrm{e}^{-7t}-3c_2\mathrm{e}^{-3t}$

  ◦ Use the initial condition $x'\Big|_{\{t=0\}}=2$

    $2=-7c_1-3c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1=-2,c_2=4\}$

  ◦ Substitute constant values into general solution and simplify

    $x=4\,\mathrm{e}^{-3t}-2\,\mathrm{e}^{-7t}$

- Solution to the IVP

  $x=4\,\mathrm{e}^{-3t}-2\,\mathrm{e}^{-7t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 17

```
dsolve([3*diff(x(t),t$2)+30*diff(x(t),t)+63*x(t)=0,x(0) = 2, D(x)(0) = 2],x(t), singsol=all)
```

$$x(t) = 4\,\mathrm{e}^{-3t} - 2\,\mathrm{e}^{-7t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 20

```
DSolve[{3*x''[t]+30*x'[t]+63*x[t]==0,{x[0]==2,x'[0]==2}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to e^{-7t}\left(4e^{4t} - 2\right)$$

10.3 problem 17

Internal problem ID [214]
Internal file name [`OUTPUT/214_Sunday_June_05_2022_01_36_48_AM_90769389/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.4, Mechanical Vibrations. Page 337
Problem number: 17.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 8x' + 16x = 0}$$

With initial conditions

$$[x(0) = 5, x'(0) = -10]$$

10.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 8 \\ q(t) &= 16 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 8x' + 16x = 0$$

The domain of $p(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 16$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.3.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 8, C = 16$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 8\lambda\, e^{\lambda t} + 16\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 8\lambda + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(8)^2 - (4)\,(1)\,(16)}\\ &= -4\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 4$. Therefore the solution is

$$x = c_1 e^{-4t} + c_2 t\, e^{-4t} \tag{1}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-4t} + c_2 t\, e^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 5$ and $t = 0$ in the above gives

$$5 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4c_1 e^{-4t} + c_2 e^{-4t} - 4c_2 t\, e^{-4t}$$

substituting $x' = -10$ and $t = 0$ in the above gives

$$-10 = -4c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 5 \\ c_2 &= 10 \end{aligned}$$

Substituting these values back in above solution results in

$$x = 10t\, e^{-4t} + 5\, e^{-4t}$$

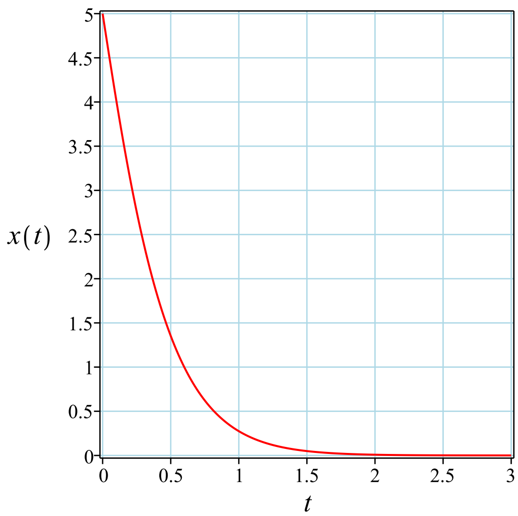
Which simplifies to

$$x = (5 + 10t)\, e^{-4t}$$

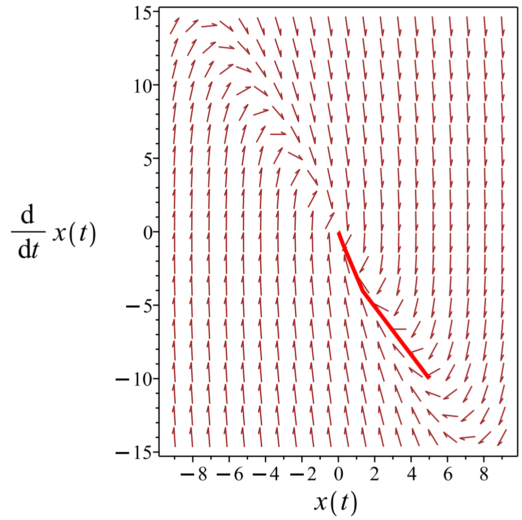
Summary

The solution(s) found are the following

$$x = (5 + 10t)\, e^{-4t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = (5 + 10t)\,\mathrm{e}^{-4t}$$

Verified OK.

10.3.3 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = 8$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 8\,dx} \\ &= \mathrm{e}^{4t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)x)'' &= 0 \\ \left(\mathrm{e}^{4t}x\right)'' &= 0 \end{aligned}$$

Integrating once gives

$$\left(e^{4t}x\right)' = c_1$$

Integrating again gives

$$\left(e^{4t}x\right) = c_1t + c_2$$

Hence the solution is

$$x = \frac{c_1t + c_2}{e^{4t}}$$

Or

$$x = c_1t\,e^{-4t} + c_2e^{-4t}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1t\,e^{-4t} + c_2e^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 5$ and $t = 0$ in the above gives

$$5 = c_2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = c_1e^{-4t} - 4c_1t\,e^{-4t} - 4c_2e^{-4t}$$

substituting $x' = -10$ and $t = 0$ in the above gives

$$-10 = c_1 - 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 10 \\ c_2 &= 5 \end{aligned}$$

Substituting these values back in above solution results in

$$x = 10t\,e^{-4t} + 5\,e^{-4t}$$

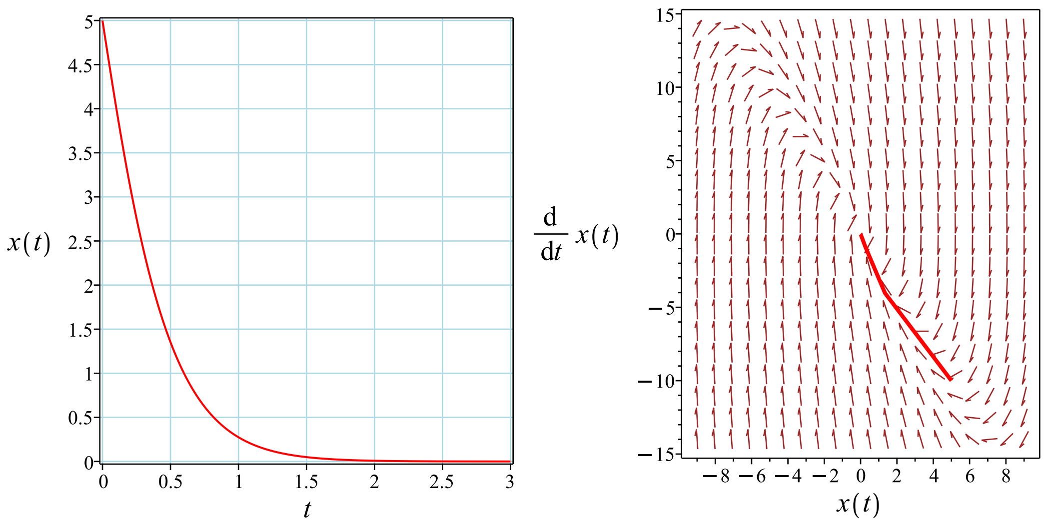
Which simplifies to

$$x = (5 + 10t)\,e^{-4t}$$

Summary

The solution(s) found are the following

$$x = (5 + 10t)\,\mathrm{e}^{-4t} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = (5 + 10t)\,\mathrm{e}^{-4t}$$

Verified OK.

10.3.4 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 8x' + 16x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = x e^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 437: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - -\infty\\ &= \infty\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\,dt}\\ &= z_1 e^{-4t}\\ &= z_1\left(\mathrm{e}^{-4t}\right)\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-4t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}x_2 &= x_1 \int \frac{e^{\int -\frac{8}{1}\,dt}}{(x_1)^2}\,dt\\ &= x_1 \int \frac{e^{-8t}}{(x_1)^2}\,dt\\ &= x_1(t)\end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-4t}\right) + c_2 \left(\mathrm{e}^{-4t}(t)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-4t} + c_2 t\,\mathrm{e}^{-4t} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 5$ and $t = 0$ in the above gives

$$5 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-4t} - 4c_2 t\,\mathrm{e}^{-4t}$$

substituting $x' = -10$ and $t = 0$ in the above gives

$$-10 = -4c_1 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 5 \\ c_2 &= 10 \end{aligned}$$

Substituting these values back in above solution results in

$$x = 10t\,\mathrm{e}^{-4t} + 5\,\mathrm{e}^{-4t}$$

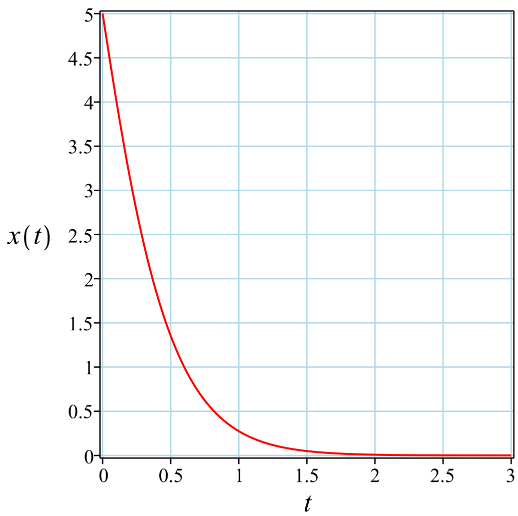
Which simplifies to

$$x = (5 + 10t)\,\mathrm{e}^{-4t}$$

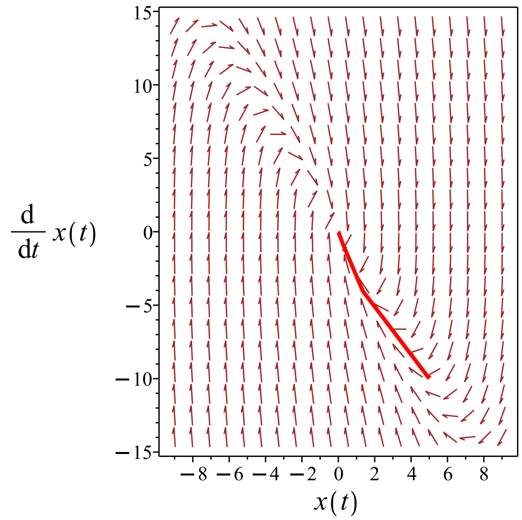
Summary

The solution(s) found are the following

$$x = (5 + 10t)\,\mathrm{e}^{-4t} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = (5 + 10t)\,e^{-4t}$$

Verified OK.

10.3.5 Maple step by step solution

Let's solve

$$\left[x'' + 8x' + 16x = 0, x(0) = 5, x'\Big|_{\{t=0\}} = -10\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 8r + 16 = 0$

- Factor the characteristic polynomial

  $(r + 4)^2 = 0$

- Root of the characteristic polynomial

  $r = -4$

- 1st solution of the ODE

  $x_1(t) = e^{-4t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t\,\mathrm{e}^{-4t}$
- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$
- Substitute in solutions

  $x = c_1 \mathrm{e}^{-4t} + c_2 t\,\mathrm{e}^{-4t}$

☐ Check validity of solution $x = c_1 \mathrm{e}^{-4t} + c_2 t \mathrm{e}^{-4t}$

- ◦ Use initial condition $x(0) = 5$

  $5 = c_1$
- ◦ Compute derivative of the solution

  $x' = -4c_1 \mathrm{e}^{-4t} + c_2 \mathrm{e}^{-4t} - 4c_2 t\,\mathrm{e}^{-4t}$
- ◦ Use the initial condition $x'\Big|_{\{t=0\}} = -10$

  $-10 = -4c_1 + c_2$
- ◦ Solve for c_1 and c_2

  $\{c_1 = 5, c_2 = 10\}$
- ◦ Substitute constant values into general solution and simplify

  $x = (5 + 10t)\,\mathrm{e}^{-4t}$

- Solution to the IVP

  $x = (5 + 10t)\,\mathrm{e}^{-4t}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(x(t),t$2)+8*diff(x(t),t)+16*x(t)=0,x(0) = 5, D(x)(0) = -10],x(t), singsol=all)
```

$$x(t) = (5 + 10t)\,e^{-4t}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 17

```
DSolve[{x''[t]+8*x'[t]+16*x[t]==0,{x[0]==5,x'[0]==-10}},x[t],t,IncludeSingularSolutions -> T
```

$$x(t) \to 5e^{-4t}(2t + 1)$$

10.4 problem 18

Internal problem ID [215]
Internal file name [`OUTPUT/215_Sunday_June_05_2022_01_36_49_AM_38051970/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.4, Mechanical Vibrations. Page 337
Problem number: 18.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2x'' + 12x' + 50x = 0}$$

With initial conditions

$$[x(0) = 0, x'(0) = -8]$$

10.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 25 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 6x' + 25x = 0$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 25$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.4.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 2, B = 12, C = 50$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda t} + 12\lambda\, e^{\lambda t} + 50\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2\lambda^2 + 12\lambda + 50 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 12, C = 50$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-12}{(2)(2)} \pm \frac{1}{(2)(2)}\sqrt{12^2 - (4)(2)(50)} \\ &= -3 \pm 4i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 4i \\ \lambda_2 &= -3 - 4i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -3 + 4i$$
$$\lambda_2 = -3 - 4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-3t}(c_1 \cos(4t) + c_2 \sin(4t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-3t}(c_1 \cos(4t) + c_2 \sin(4t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3\,\mathrm{e}^{-3t}(c_1 \cos(4t) + c_2 \sin(4t)) + \mathrm{e}^{-3t}(-4c_1 \sin(4t) + 4c_2 \cos(4t))$$

substituting $x' = -8$ and $t = 0$ in the above gives

$$-8 = -3c_1 + 4c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = -2$$

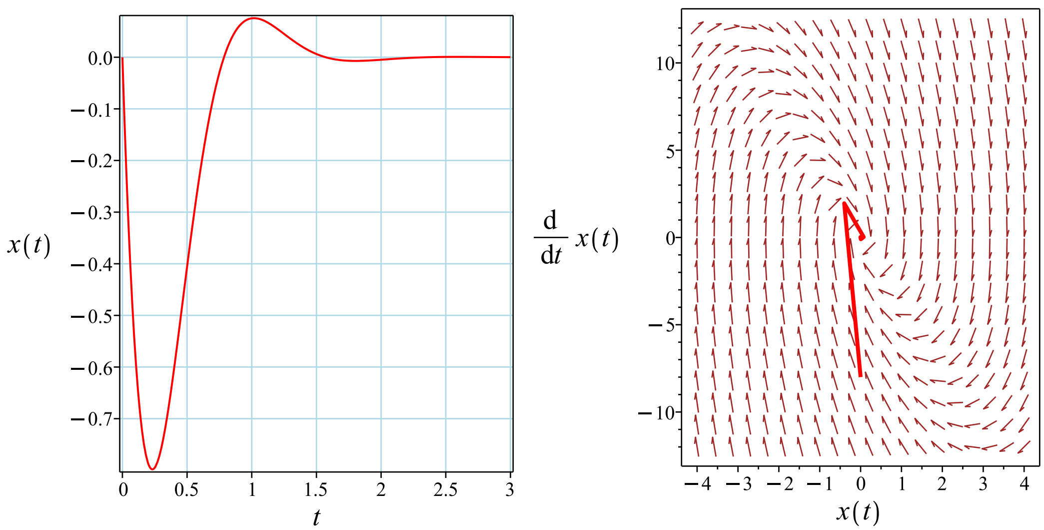
Substituting these values back in above solution results in

$$x = -2\,\mathrm{e}^{-3t} \sin(4t)$$

Summary

The solution(s) found are the following

$$x = -2\,\mathrm{e}^{-3t} \sin(4t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = -2\,\mathrm{e}^{-3t} \sin(4t)$$

Verified OK.

10.4.3 Solving using Kovacic algorithm

Writing the ode as

$$2x'' + 12x' + 50x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 12 \\ C &= 50 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -16z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 439: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(4t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{12}{2}\, dt} \\ &= z_1 e^{-3t} \\ &= z_1 \left(\mathrm{e}^{-3t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t} \cos(4t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{12}{2}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-6t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\tan(4t)}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\mathrm{e}^{-3t}\cos(4t)\right) + c_2\left(\mathrm{e}^{-3t}\cos(4t)\left(\frac{\tan(4t)}{4}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1\mathrm{e}^{-3t}\cos(4t) + \frac{\mathrm{e}^{-3t}c_2\sin(4t)}{4} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$
0 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -3c_1\mathrm{e}^{-3t}\cos(4t) - 4c_1\mathrm{e}^{-3t}\sin(4t) - \frac{3\,\mathrm{e}^{-3t}c_2\sin(4t)}{4} + \mathrm{e}^{-3t}c_2\cos(4t)
$$

substituting $x' = -8$ and $t = 0$ in the above gives

$$
-8 = -3c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 0 \\
c_2 &= -8
\end{aligned}
$$

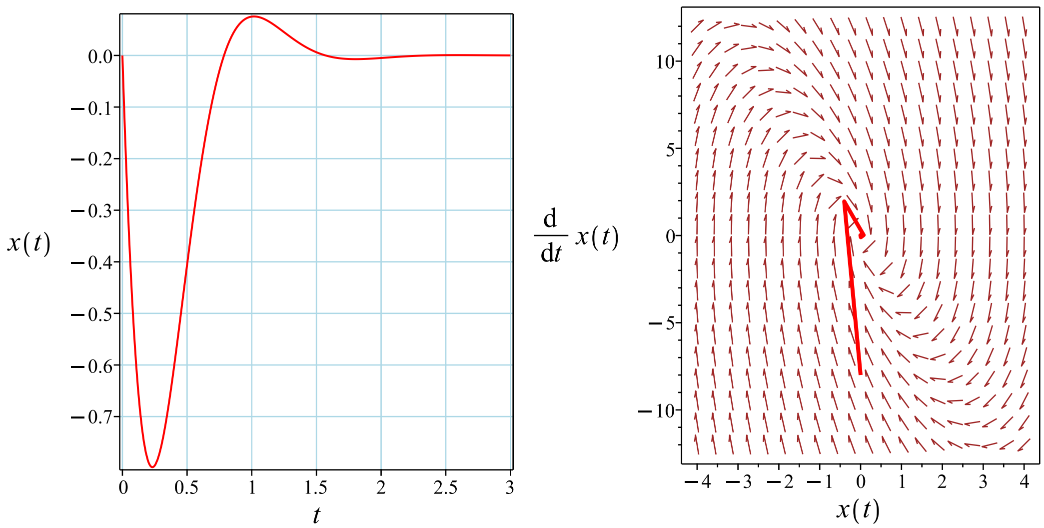
Substituting these values back in above solution results in

$$
x = -2\,\mathrm{e}^{-3t}\sin(4t)
$$

<u>Summary</u>

The solution(s) found are the following

$$
x = -2\,\mathrm{e}^{-3t}\sin(4t) \tag{1}
$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = -2\,\mathrm{e}^{-3t} \sin(4t)$$

Verified OK.

10.4.4 Maple step by step solution

Let's solve

$$\left[2x'' + 12x' + 50x = 0, x(0) = 0, x'\Big|_{\{t=0\}} = -8\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -6x' - 25x$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + 6x' + 25x = 0$

- Characteristic polynomial of ODE

  $r^2 + 6r + 25 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-6)\pm(\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

  $r = (-3-4\,\mathrm{I}, -3+4\,\mathrm{I})$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-3t}\cos(4t)$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-3t}\sin(4t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1\mathrm{e}^{-3t}\cos(4t) + \mathrm{e}^{-3t}c_2\sin(4t)$

☐ Check validity of solution $x = c_1\mathrm{e}^{-3t}\cos(4t) + \mathrm{e}^{-3t}c_2\sin(4t)$

  - Use initial condition $x(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $x' = -3c_1\mathrm{e}^{-3t}\cos(4t) - 4c_1\mathrm{e}^{-3t}\sin(4t) - 3\,\mathrm{e}^{-3t}c_2\sin(4t) + 4\,\mathrm{e}^{-3t}c_2\cos(4t)$

  - Use the initial condition $x'\Big|_{\{t=0\}} = -8$

    $-8 = -3c_1 + 4c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 0, c_2 = -2\}$

  - Substitute constant values into general solution and simplify

    $x = -2\,\mathrm{e}^{-3t}\sin(4t)$

- Solution to the IVP

  $x = -2\,\mathrm{e}^{-3t}\sin(4t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([2*diff(x(t),t$2)+12*diff(x(t),t)+50*x(t)=0,x(0) = 0, D(x)(0) = -8],x(t), singsol=all
```

$$x(t) = -2\,e^{-3t}\sin(4t)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 16

```
DSolve[{2*x''[t]+12*x'[t]+50*x[t]==0,{x[0]==0,x'[0]==-8}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to -2e^{-3t}\sin(4t)$$

10.5 problem 19

Internal problem ID [216]
Internal file name [`OUTPUT/216_Sunday_June_05_2022_01_36_50_AM_3008356/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.4, Mechanical Vibrations. Page 337
Problem number: 19.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _missing_x]]`

$$\boxed{4x'' + 20x' + 169x = 0}$$

With initial conditions

$$[x(0) = 4, x'(0) = 16]$$

10.5.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 5 \\ q(t) &= \frac{169}{4} \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 5x' + \frac{169x}{4} = 0$$

The domain of $p(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = \frac{169}{4}$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.5.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 4, B = 20, C = 169$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda t} + 20\lambda\, e^{\lambda t} + 169\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$4\lambda^2 + 20\lambda + 169 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 20, C = 169$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-20}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{20^2 - (4)(4)(169)} \\
&= -\frac{5}{2} \pm 6i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{5}{2} + 6i \\
\lambda_2 &= -\frac{5}{2} - 6i
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{5}{2} + 6i$$
$$\lambda_2 = -\frac{5}{2} - 6i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{5}{2}$ and $\beta = 6$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{5t}{2}}(c_1 \cos(6t) + c_2 \sin(6t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-\frac{5t}{2}}(c_1 \cos(6t) + c_2 \sin(6t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 4$ and $t = 0$ in the above gives

$$4 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{5\,\mathrm{e}^{-\frac{5t}{2}}(c_1 \cos(6t) + c_2 \sin(6t))}{2} + \mathrm{e}^{-\frac{5t}{2}}(-6c_1 \sin(6t) + 6c_2 \cos(6t))$$

substituting $x' = 16$ and $t = 0$ in the above gives

$$16 = -\frac{5c_1}{2} + 6c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = \frac{13}{3}$$

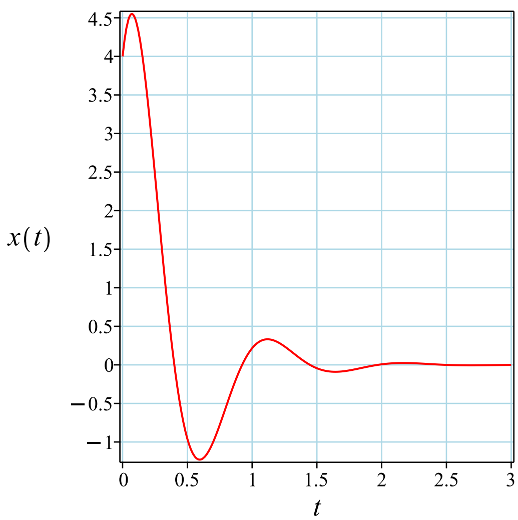
Substituting these values back in above solution results in

$$x = \frac{e^{-\frac{5t}{2}} (12\cos(6t) + 13\sin(6t))}{3}$$

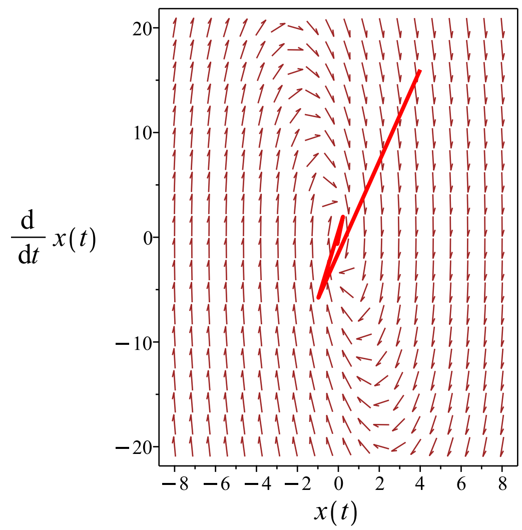
Summary

The solution(s) found are the following

$$x = \frac{e^{-\frac{5t}{2}} (12\cos(6t) + 13\sin(6t))}{3} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{e^{-\frac{5t}{2}} (12\cos(6t) + 13\sin(6t))}{3}$$

Verified OK.

10.5.3 Solving using Kovacic algorithm

Writing the ode as

$$4x'' + 20x' + 169x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 20 \\ C &= 169 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-36}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -36 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -36z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 441: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}O(\infty) &= \deg(t) - \deg(s)\\ &= 0 - 0\\ &= 0\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -36$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(6t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}\\ &= z_1 e^{-\int \frac{1}{2}\frac{20}{4}\, dt}\\ &= z_1 e^{-\frac{5t}{2}}\\ &= z_1\left(\mathrm{e}^{-\frac{5t}{2}}\right)\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{5t}{2}} \cos(6t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{20}{4}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-5t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\tan(6t)}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{5t}{2}} \cos(6t)\right) + c_2 \left(\mathrm{e}^{-\frac{5t}{2}} \cos(6t) \left(\frac{\tan(6t)}{6}\right)\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-\frac{5t}{2}} \cos(6t) + \frac{c_2 \mathrm{e}^{-\frac{5t}{2}} \sin(6t)}{6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 4$ and $t = 0$ in the above gives

$$4 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\frac{5c_1 \mathrm{e}^{-\frac{5t}{2}} \cos(6t)}{2} - 6c_1 \mathrm{e}^{-\frac{5t}{2}} \sin(6t) - \frac{5c_2 \mathrm{e}^{-\frac{5t}{2}} \sin(6t)}{12} + c_2 \mathrm{e}^{-\frac{5t}{2}} \cos(6t)$$

substituting $x' = 16$ and $t = 0$ in the above gives

$$16 = -\frac{5c_1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 4$$
$$c_2 = 26$$

Substituting these values back in above solution results in

$$x = 4\,\mathrm{e}^{-\frac{5t}{2}} \cos(6t) + \frac{13\,\mathrm{e}^{-\frac{5t}{2}} \sin(6t)}{3}$$

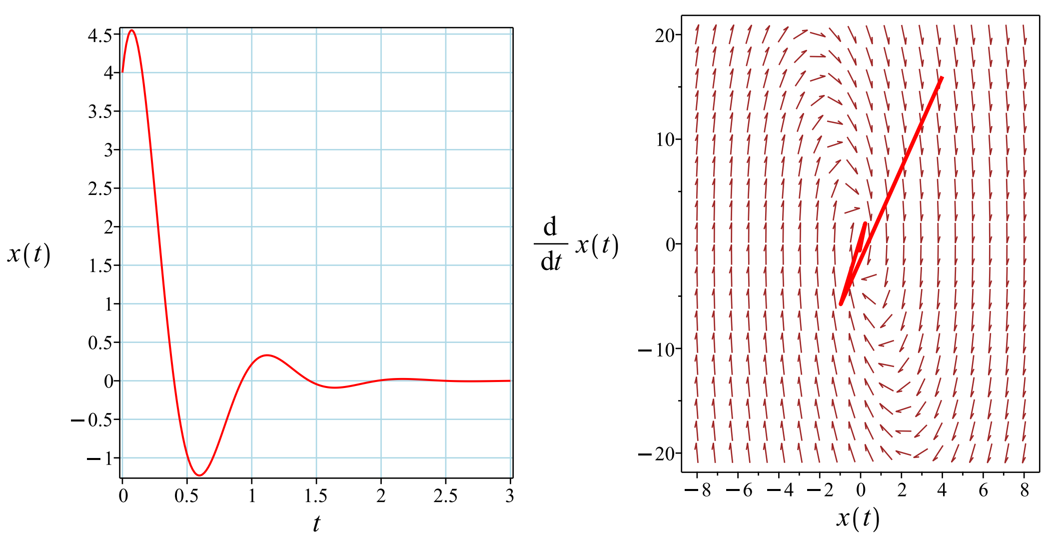
Which simplifies to

$$x = \frac{\mathrm{e}^{-\frac{5t}{2}} (12 \cos(6t) + 13 \sin(6t))}{3}$$

Summary

The solution(s) found are the following

$$x = \frac{\mathrm{e}^{-\frac{5t}{2}} (12 \cos(6t) + 13 \sin(6t))}{3} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{\mathrm{e}^{-\frac{5t}{2}} (12 \cos(6t) + 13 \sin(6t))}{3}$$

Verified OK.

10.5.4 Maple step by step solution

Let's solve

$\left[4x''+20x'+169x=0, x(0)=4, x'\Big|_{\{t=0\}}=16\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x''=-5x'-\frac{169x}{4}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x''+5x'+\frac{169x}{4}=0$

- Characteristic polynomial of ODE

  $r^2+5r+\frac{169}{4}=0$

- Use quadratic formula to solve for r

  $r=\frac{(-5)\pm(\sqrt{-144})}{2}$

- Roots of the characteristic polynomial

  $r=\left(-\frac{5}{2}-6\,\mathrm{I}, -\frac{5}{2}+6\,\mathrm{I}\right)$

- 1st solution of the ODE

  $x_1(t)=\mathrm{e}^{-\frac{5t}{2}}\cos(6t)$

- 2nd solution of the ODE

  $x_2(t)=\mathrm{e}^{-\frac{5t}{2}}\sin(6t)$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)$

- Substitute in solutions

  $x=c_1\mathrm{e}^{-\frac{5t}{2}}\cos(6t)+c_2\mathrm{e}^{-\frac{5t}{2}}\sin(6t)$

□ Check validity of solution $x=c_1\mathrm{e}^{-\frac{5t}{2}}\cos(6t)+c_2\mathrm{e}^{-\frac{5t}{2}}\sin(6t)$

  - Use initial condition $x(0)=4$

    $4=c_1$

  - Compute derivative of the solution

    $x'=-\frac{5c_1\mathrm{e}^{-\frac{5t}{2}}\cos(6t)}{2}-6c_1\mathrm{e}^{-\frac{5t}{2}}\sin(6t)-\frac{5c_2\mathrm{e}^{-\frac{5t}{2}}\sin(6t)}{2}+6c_2\mathrm{e}^{-\frac{5t}{2}}\cos(6t)$

- Use the initial condition $x'\Big|_{\{t=0\}} = 16$

  $16 = -\frac{5c_1}{2} + 6c_2$

- Solve for c_1 and c_2

  $\{c_1 = 4, c_2 = \frac{13}{3}\}$

- Substitute constant values into general solution and simplify

  $x = \frac{e^{-\frac{5t}{2}}(12\cos(6t)+13\sin(6t))}{3}$

- Solution to the IVP

  $x = \frac{e^{-\frac{5t}{2}}(12\cos(6t)+13\sin(6t))}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 23

```
dsolve([4*diff(x(t),t$2)+20*diff(x(t),t)+169*x(t)=0,x(0) = 4, D(x)(0) = 16],x(t), singsol=al
```

$$x(t) = \frac{e^{-\frac{5t}{2}}(13\sin(6t) + 12\cos(6t))}{3}$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 29

```
DSolve[{4*x''[t]+20*x'[t]+169*x[t]==0,{x[0]==4,x'[0]==16}},x[t],t,IncludeSingularSolutions -
```

$$x(t) \to \frac{1}{3}e^{-5t/2}(13\sin(6t) + 12\cos(6t))$$

10.6 problem 20

Internal problem ID [217]
Internal file name [`OUTPUT/217_Sunday_June_05_2022_01_36_51_AM_35899030/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.4, Mechanical Vibrations. Page 337
Problem number: 20.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{2x'' + 16x' + 40x = 0}$$

With initial conditions

$$[x(0) = 5, x'(0) = 4]$$

10.6.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 8 \\ q(t) &= 20 \\ F &= 0 \end{aligned}$$

Hence the ode is

$$x'' + 8x' + 20x = 0$$

The domain of $p(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 20$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.6.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 2, B = 16, C = 40$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda t} + 16\lambda\, e^{\lambda t} + 40\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2\lambda^2 + 16\lambda + 40 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 16, C = 40$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-16}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{16^2 - (4)\,(2)\,(40)}\\ &= -4 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -4 + 2i\\ \lambda_2 &= -4 - 2i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -4 + 2i$$
$$\lambda_2 = -4 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -4$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-4t}(c_1 \cos(2t) + c_2 \sin(2t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-4t}(c_1 \cos(2t) + c_2 \sin(2t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 5$ and $t = 0$ in the above gives

$$5 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4\,\mathrm{e}^{-4t}(c_1 \cos(2t) + c_2 \sin(2t)) + \mathrm{e}^{-4t}(-2c_1 \sin(2t) + 2c_2 \cos(2t))$$

substituting $x' = 4$ and $t = 0$ in the above gives

$$4 = -4c_1 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

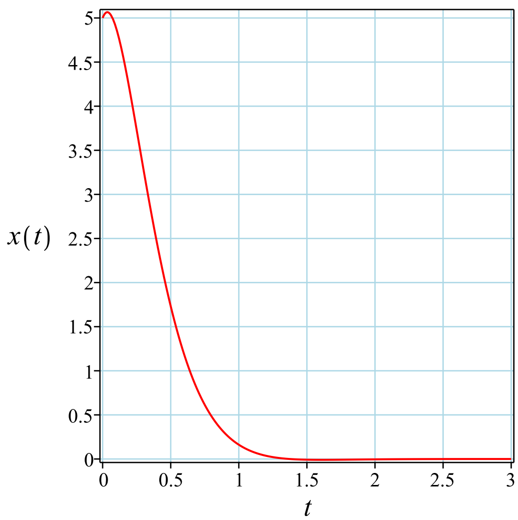
$$c_1 = 5$$
$$c_2 = 12$$

Substituting these values back in above solution results in

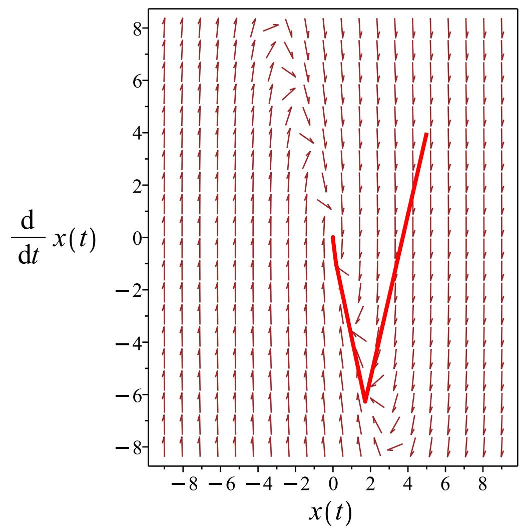
$$x = \mathrm{e}^{-4t}(5\cos(2t) + 12\sin(2t))$$

Summary
The solution(s) found are the following

$$x = e^{-4t}(5\cos(2t) + 12\sin(2t)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-4t}(5\cos(2t) + 12\sin(2t))$$

Verified OK.

10.6.3 Solving using Kovacic algorithm

Writing the ode as

$$2x'' + 16x' + 40x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 16 \\ C &= 40 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 443: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{16}{2}\, dt} \\ &= z_1 e^{-4t} \\ &= z_1 \left(\mathrm{e}^{-4t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-4t} \cos(2t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{16}{2}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-8t}}{(x_1)^2}\, dt \\ &= x_1 \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-4t} \cos(2t)\right) + c_2 \left(\mathrm{e}^{-4t} \cos(2t) \left(\frac{\tan(2t)}{2}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{-4t} \cos(2t) + \frac{c_2 \mathrm{e}^{-4t} \sin(2t)}{2} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 5$ and $t = 0$ in the above gives

$$
5 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -4c_1 \mathrm{e}^{-4t} \cos(2t) - 2c_1 \mathrm{e}^{-4t} \sin(2t) - 2c_2 \mathrm{e}^{-4t} \sin(2t) + c_2 \mathrm{e}^{-4t} \cos(2t)
$$

substituting $x' = 4$ and $t = 0$ in the above gives

$$
4 = -4c_1 + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 5 \\
c_2 &= 24
\end{aligned}
$$

Substituting these values back in above solution results in

$$
x = 5\,\mathrm{e}^{-4t} \cos(2t) + 12\,\mathrm{e}^{-4t} \sin(2t)
$$

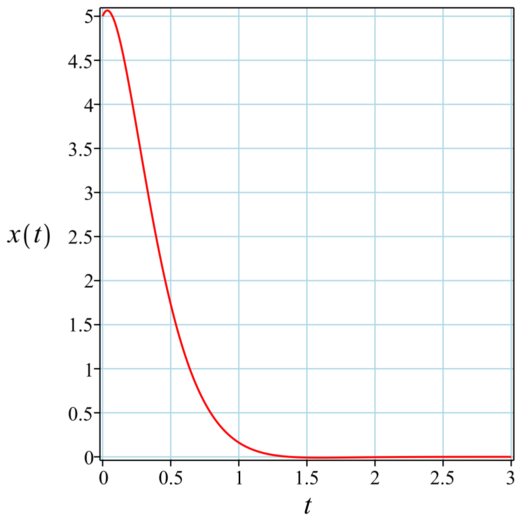
Which simplifies to

$$
x = \mathrm{e}^{-4t}(5\cos(2t) + 12\sin(2t))
$$

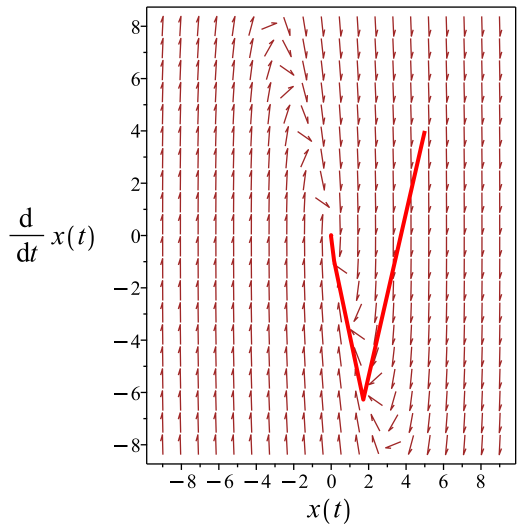
Summary

The solution(s) found are the following

$$
x = \mathrm{e}^{-4t}(5\cos(2t) + 12\sin(2t)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = e^{-4t}(5\cos(2t) + 12\sin(2t))$$

Verified OK.

10.6.4 Maple step by step solution

Let's solve

$$\left[2x'' + 16x' + 40x = 0, x(0) = 5, x'\Big|_{\{t=0\}} = 4\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -8x' - 20x$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + 8x' + 20x = 0$

- Characteristic polynomial of ODE

  $r^2 + 8r + 20 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-8)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-4-2\,\mathrm{I}, -4+2\,\mathrm{I})$

- 1st solution of the ODE

  $x_1(t) = \mathrm{e}^{-4t}\cos(2t)$

- 2nd solution of the ODE

  $x_2(t) = \mathrm{e}^{-4t}\sin(2t)$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t)$

- Substitute in solutions

  $x = c_1\mathrm{e}^{-4t}\cos(2t) + c_2\mathrm{e}^{-4t}\sin(2t)$

☐ Check validity of solution $x = c_1\mathrm{e}^{-4t}\cos(2t) + c_2\mathrm{e}^{-4t}\sin(2t)$

  - Use initial condition $x(0) = 5$

    $5 = c_1$

  - Compute derivative of the solution

    $x' = -4c_1\mathrm{e}^{-4t}\cos(2t) - 2c_1\mathrm{e}^{-4t}\sin(2t) - 4c_2\mathrm{e}^{-4t}\sin(2t) + 2c_2\mathrm{e}^{-4t}\cos(2t)$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 4$

    $4 = -4c_1 + 2c_2$

  - Solve for c_1 and c_2

    $\{c_1 = 5, c_2 = 12\}$

  - Substitute constant values into general solution and simplify

    $x = \mathrm{e}^{-4t}(5\cos(2t) + 12\sin(2t))$

- Solution to the IVP

  $x = \mathrm{e}^{-4t}(5\cos(2t) + 12\sin(2t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 22

```
dsolve([2*diff(x(t),t$2)+16*diff(x(t),t)+40*x(t)=0,x(0) = 5, D(x)(0) = 4],x(t), singsol=all)
```

$$x(t) = e^{-4t}(12\sin(2t) + 5\cos(2t))$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 24

```
DSolve[{2*x''[t]+16*x'[t]+40*x[t]==0,{x[0]==5,x'[0]==4}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to e^{-4t}(12\sin(2t) + 5\cos(2t))$$

10.7 problem 21

Internal problem ID [218]
Internal file name [`OUTPUT/218_Sunday_June_05_2022_01_36_52_AM_8777435/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.4, Mechanical Vibrations. Page 337
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _missing_x]]
```

$$\boxed{x'' + 10x' + 125x = 0}$$

With initial conditions

$$[x(0) = 6, x'(0) = 50]$$

10.7.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned}p(t) &= 10\\ q(t) &= 125\\ F &= 0\end{aligned}$$

Hence the ode is

$$x'' + 10x' + 125x = 0$$

The domain of $p(t) = 10$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 125$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

10.7.2 Solving as second order linear constant coeff ode

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 10, C = 125$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 10\lambda\, \mathrm{e}^{\lambda t} + 125\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 10\lambda + 125 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 10, C = 125$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-10}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{10^2 - (4)\,(1)\,(125)}\\ &= -5 \pm 10i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -5 + 10i\\ \lambda_2 &= -5 - 10i\end{aligned}$$

Which simplifies to

$$\lambda_1 = -5 + 10i$$
$$\lambda_2 = -5 - 10i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -5$ and $\beta = 10$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-5t}(c_1 \cos(10t) + c_2 \sin(10t))$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = \mathrm{e}^{-5t}(c_1 \cos(10t) + c_2 \sin(10t)) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 6$ and $t = 0$ in the above gives

$$6 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -5\,\mathrm{e}^{-5t}(c_1 \cos(10t) + c_2 \sin(10t)) + \mathrm{e}^{-5t}(-10c_1 \sin(10t) + 10c_2 \cos(10t))$$

substituting $x' = 50$ and $t = 0$ in the above gives

$$50 = -5c_1 + 10c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 6$$
$$c_2 = 8$$

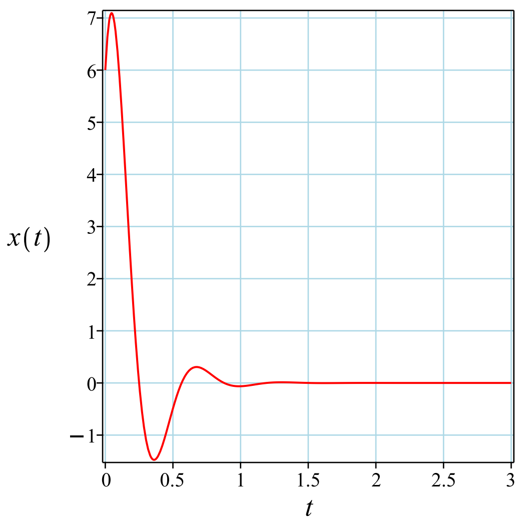
Substituting these values back in above solution results in

$$x = 2\,\mathrm{e}^{-5t}(3\cos(10t) + 4\sin(10t))$$

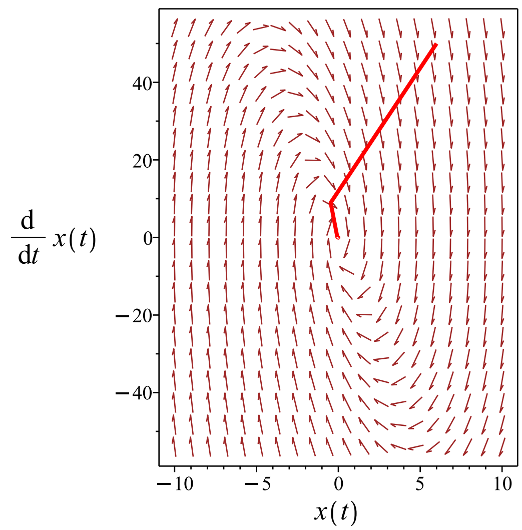
Summary

The solution(s) found are the following

$$x = 2\,e^{-5t}(3\cos(10t) + 4\sin(10t)) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 2\,e^{-5t}(3\cos(10t) + 4\sin(10t))$$

Verified OK.

10.7.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 10x' + 125x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 10 \\ C &= 125 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-100}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -100 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -100z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 445: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -100$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(10t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{10}{1}\,dt} \\ &= z_1 e^{-5t} \\ &= z_1\left(\mathrm{e}^{-5t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-5t}\cos(10t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{10}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-10t}}{(x_1)^2}\,dt \\ &= x_1\left(\frac{\tan(10t)}{10}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-5t} \cos (10t)\right) + c_2 \left(\mathrm{e}^{-5t} \cos (10t) \left(\frac{\tan (10t)}{10}\right)\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
x = c_1 \mathrm{e}^{-5t} \cos (10t) + \frac{c_2 \mathrm{e}^{-5t} \sin (10t)}{10} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 6$ and $t = 0$ in the above gives

$$
6 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
x' = -5c_1 \mathrm{e}^{-5t} \cos (10t) - 10c_1 \mathrm{e}^{-5t} \sin (10t) - \frac{c_2 \mathrm{e}^{-5t} \sin (10t)}{2} + c_2 \mathrm{e}^{-5t} \cos (10t)
$$

substituting $x' = 50$ and $t = 0$ in the above gives

$$
50 = -5c_1 + c_2 \tag{2A}
$$

Equations $\{1\text{A}, 2\text{A}\}$ are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 6 \\
c_2 &= 80
\end{aligned}
$$

Substituting these values back in above solution results in

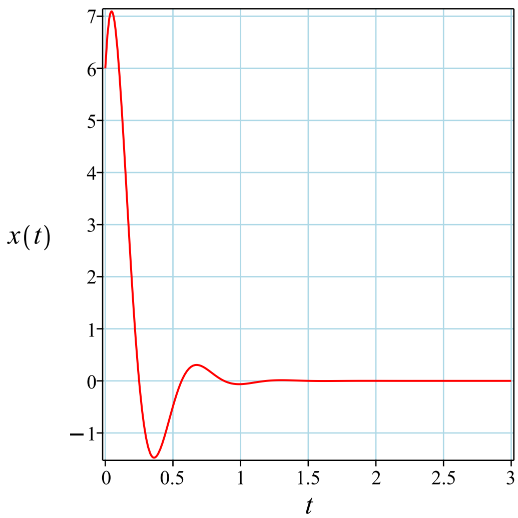
$$
x = 6\,\mathrm{e}^{-5t} \cos (10t) + 8\,\mathrm{e}^{-5t} \sin (10t)
$$

Which simplifies to

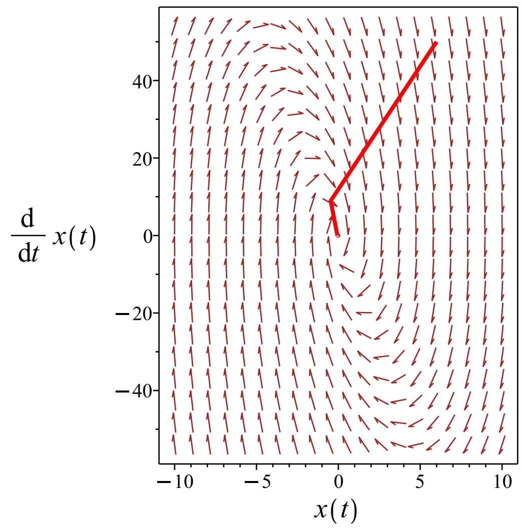
$$
x = 2\,\mathrm{e}^{-5t} (3 \cos (10t) + 4 \sin (10t))
$$

<u>Summary</u>
The solution(s) found are the following

$$
x = 2\,\mathrm{e}^{-5t} (3 \cos (10t) + 4 \sin (10t)) \tag{1}
$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = 2\,\mathrm{e}^{-5t}(3\cos(10t) + 4\sin(10t))$$

Verified OK.

10.7.4 Maple step by step solution

Let's solve

$$\left[x'' + 10x' + 125x = 0, x(0) = 6, x'\Big|_{\{t=0\}} = 50\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of ODE

  $r^2 + 10r + 125 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-10)\pm(\sqrt{-400})}{2}$

- Roots of the characteristic polynomial

  $r = (-5 - 10\,\mathrm{I}, -5 + 10\,\mathrm{I})$

- 1st solution of the ODE

$x_1(t) = e^{-5t} \cos(10t)$

- 2nd solution of the ODE

  $x_2(t) = e^{-5t} \sin(10t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t)$

- Substitute in solutions

  $x = c_1 e^{-5t} \cos(10t) + c_2 e^{-5t} \sin(10t)$

□ Check validity of solution $x = c_1 e^{-5t} \cos(10t) + c_2 e^{-5t} \sin(10t)$

  ◦ Use initial condition $x(0) = 6$

    $6 = c_1$

  ◦ Compute derivative of the solution

    $x' = -5c_1 e^{-5t} \cos(10t) - 10c_1 e^{-5t} \sin(10t) - 5c_2 e^{-5t} \sin(10t) + 10c_2 e^{-5t} \cos(10t)$

  ◦ Use the initial condition $x'\Big|_{\{t=0\}} = 50$

    $50 = -5c_1 + 10c_2$

  ◦ Solve for c_1 and c_2

    $\{c_1 = 6, c_2 = 8\}$

  ◦ Substitute constant values into general solution and simplify

    $x = 2\,e^{-5t}(3\cos(10t) + 4\sin(10t))$

- Solution to the IVP

  $x = 2\,e^{-5t}(3\cos(10t) + 4\sin(10t))$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
checking if the LODE has constant coefficients
<- constant coefficients successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 23

```
dsolve([diff(x(t),t$2)+10*diff(x(t),t)+125*x(t)=0,x(0) = 6, D(x)(0) = 50],x(t), singsol=all)
```

$$x(t) = 2\,\mathrm{e}^{-5t}(4\sin(10t) + 3\cos(10t))$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 24

```
DSolve[{x''[t]+10*x'[t]+125*x[t]==0,{x[0]==6,x'[0]==50}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to e^{-5t}(8\sin(10t) + 6\cos(10t))$$

11 Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351

11.1 problem 1

Internal problem ID [219]
Internal file name [`OUTPUT/219_Sunday_June_05_2022_01_36_53_AM_52974580/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 16y = e^{3x}}$$

11.1.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 16, f(x) = e^{3x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 16$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 16\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 16 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 16$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(16)} \\ &= \pm 4i\end{aligned}$$

Hence

$$\lambda_1 = +4i$$
$$\lambda_2 = -4i$$

Which simplifies to

$$\lambda_1 = 4i$$
$$\lambda_2 = -4i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 4$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(4x) + c_2 \sin(4x))$$

Or

$$y = c_1 \cos(4x) + c_2 \sin(4x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(4x) + c_2 \sin(4x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(4x), \sin(4x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$25A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x}}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(4x) + c_2 \sin(4x)) + \left(\frac{e^{3x}}{25}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{e^{3x}}{25} \tag{1}$$

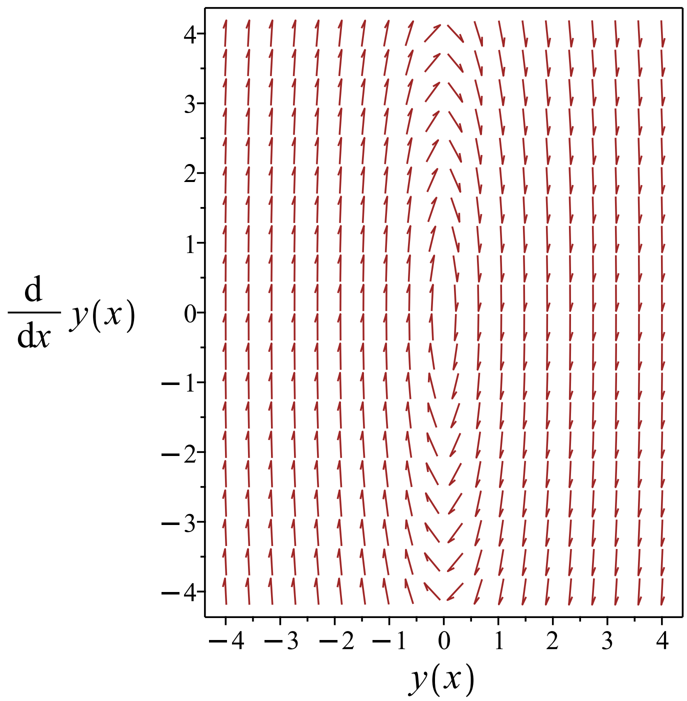


Figure 601: Slope field plot

Verification of solutions

$$y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{e^{3x}}{25}$$

Verified OK.

11.1.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 16y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 16 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-16}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -16 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -16 z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 447: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -16$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(4x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(4x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(4x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(4x) \int \frac{1}{\cos(4x)^2}\, dx \\ &= \cos(4x) \left(\frac{\tan(4x)}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(4x)) + c_2\left(\cos(4x)\left(\frac{\tan(4x)}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 16y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$e^{3x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(4x)}{4}, \cos(4x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^{3x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$25A_1 e^{3x} = e^{3x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{25}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^{3x}}{25}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4}\right) + \left(\frac{e^{3x}}{25}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + \frac{e^{3x}}{25} \tag{1}$$

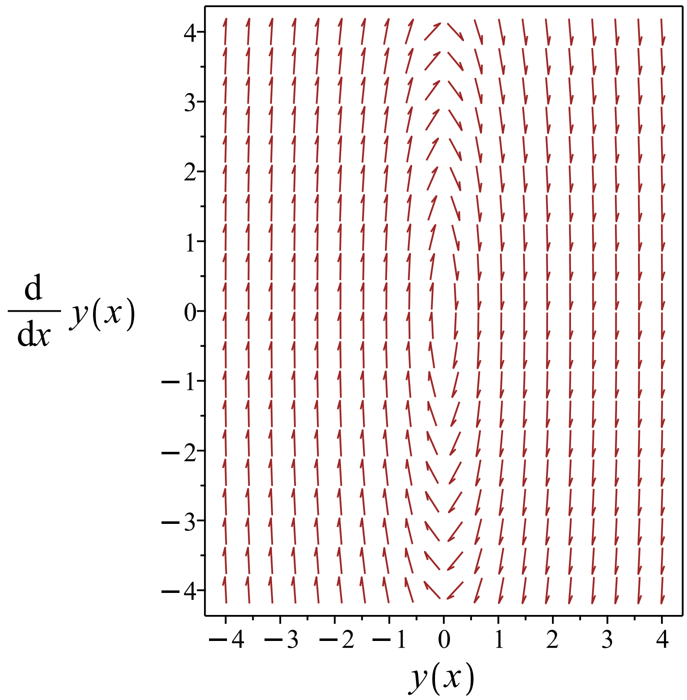


Figure 602: Slope field plot

Verification of solutions

$$y = c_1 \cos(4x) + \frac{c_2 \sin(4x)}{4} + \frac{e^{3x}}{25}$$

Verified OK.

11.1.3 Maple step by step solution

Let's solve

$y'' + 16y = e^{3x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 16 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-64})}{2}$

- Roots of the characteristic polynomial

$r = (-4\,\mathrm{I}, 4\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(4x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(4x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \mathrm{e}^{3x}\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(4x) & \sin(4x) \\ -4\sin(4x) & 4\cos(4x) \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{\cos(4x)\left(\int \mathrm{e}^{3x} \sin(4x) dx\right)}{4} + \frac{\sin(4x)\left(\int \mathrm{e}^{3x} \cos(4x) dx\right)}{4}$

  ○ Compute integrals

  $y_p(x) = \frac{\mathrm{e}^{3x}}{25}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(4x) + c_2 \sin(4x) + \frac{\mathrm{e}^{3x}}{25}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+16*y(x)=exp(3*x),y(x), singsol=all)
```

$$y(x) = \sin(4x) c_2 + \cos(4x) c_1 + \frac{e^{3x}}{25}$$

✓ Solution by Mathematica

Time used: 0.144 (sec). Leaf size: 29

```
DSolve[y''[x]+16*y[x]==Exp[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{e^{3x}}{25} + c_1 \cos(4x) + c_2 \sin(4x)$$

11.2 problem 2

Internal problem ID [220]
Internal file name [`OUTPUT/220_Sunday_June_05_2022_01_36_54_AM_59806796/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - y' - 2y = 3x + 4}$$

11.2.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -2, f(x) = 3x + 4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - \lambda e^{\lambda x} - 2 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-1^2 - (4)(1)(-2)} \\ &= \frac{1}{2} \pm \frac{3}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= \frac{1}{2} + \frac{3}{2} \\ \lambda_2 &= \frac{1}{2} - \frac{3}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-1)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_2x - 2A_1 - A_2 = 3x + 4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{4}, A_2 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3x}{2} - \frac{5}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-x}\right) + \left(-\frac{3x}{2} - \frac{5}{4}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-x} - \frac{3x}{2} - \frac{5}{4} \tag{1}$$

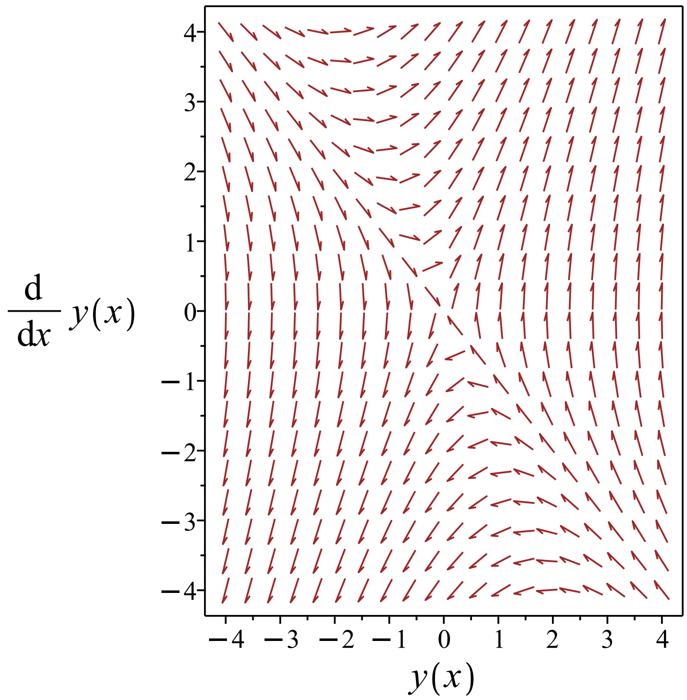


Figure 603: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-x} - \frac{3x}{2} - \frac{5}{4}$$

Verified OK.

11.2.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 9$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{9z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 449: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{9}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{3x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\ &= z_1 e^{-\int \frac{1}{2} \frac{-1}{1} \, dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1 \left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1} \, dx}}{(y_1)^2} \, dx \\ &= y_1 \int \frac{e^{x}}{(y_1)^2} \, dx \\ &= y_1 \left(\frac{\mathrm{e}^{3x}}{3}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-x}\right) + c_2 \left(\mathrm{e}^{-x} \left(\frac{\mathrm{e}^{3x}}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - y' - 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x + 1
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{1, x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{3}, \mathrm{e}^{-x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_2 x + A_1
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_2x - 2A_1 - A_2 = 3x + 4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{5}{4}, A_2 = -\frac{3}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3x}{2} - \frac{5}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3}\right) + \left(-\frac{3x}{2} - \frac{5}{4}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3} - \frac{3x}{2} - \frac{5}{4} \tag{1}$$

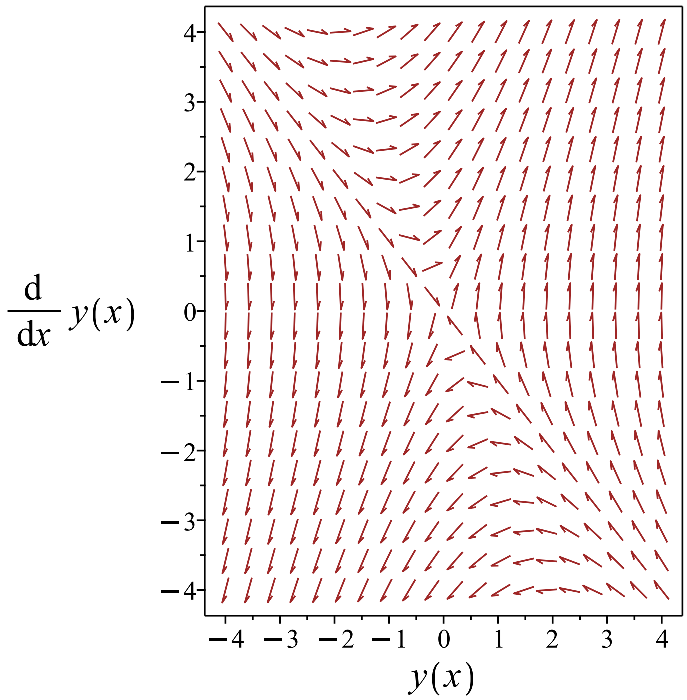


Figure 604: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{-x} + \frac{c_2 \mathrm{e}^{2x}}{3} - \frac{3x}{2} - \frac{5}{4}$$

Verified OK.

11.2.3 Maple step by step solution

Let's solve

$y'' - y' - 2y = 3x + 4$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 2 = 0$

- Factor the characteristic polynomial

  $(r + 1)(r - 2) = 0$

- Roots of the characteristic polynomial

$r = (-1, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 3x + 4\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2\,e^{2x} \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3\,e^{x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{e^{-x}\left(\int (3x+4)e^{x}dx\right)}{3} + \frac{e^{2x}\left(\int (3x+4)e^{-2x}dx\right)}{3}$

  ○ Compute integrals

  $y_p(x) = -\frac{3x}{2} - \frac{5}{4}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-x} + c_2 e^{2x} - \frac{3x}{2} - \frac{5}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-2*y(x)=3*x+4,y(x), singsol=all)
```

$$y(x) = c_2 e^{-x} + e^{2x} c_1 - \frac{3x}{2} - \frac{5}{4}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 30

```
DSolve[y''[x]-y'[x]-2*y[x]==3*x+4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{3x}{2} + c_1 e^{-x} + c_2 e^{2x} - \frac{5}{4}$$

11.3 problem 3

Internal problem ID [221]
Internal file name [`OUTPUT/221_Sunday_June_05_2022_01_36_54_AM_9534624/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - y' - 6y = 2\sin(3x)}$$

11.3.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -1, C = -6, f(x) = 2\sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -1, C = -6$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - \lambda\, \mathrm{e}^{\lambda x} - 6\, \mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - \lambda - 6 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -1, C = -6$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-1^2 - (4)\,(1)\,(-6)} \\
&= \frac{1}{2} \pm \frac{5}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= \frac{1}{2} + \frac{5}{2} \\
\lambda_2 &= \frac{1}{2} - \frac{5}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 3 \\
\lambda_2 &= -2
\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}
y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\
y &= c_1 e^{(3)x} + c_2 e^{(-2)x}
\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{3x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{3x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1\cos(3x) + A_2\sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-15A_1\cos(3x) - 15A_2\sin(3x) + 3A_1\sin(3x) - 3A_2\cos(3x) = 2\sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{39}, A_2 = -\frac{5}{39}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{3x} + c_2 e^{-2x}\right) + \left(\frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{3x} + c_2 e^{-2x} + \frac{\cos(3x)}{39} - \frac{5 \sin(3x)}{39} \tag{1}$$

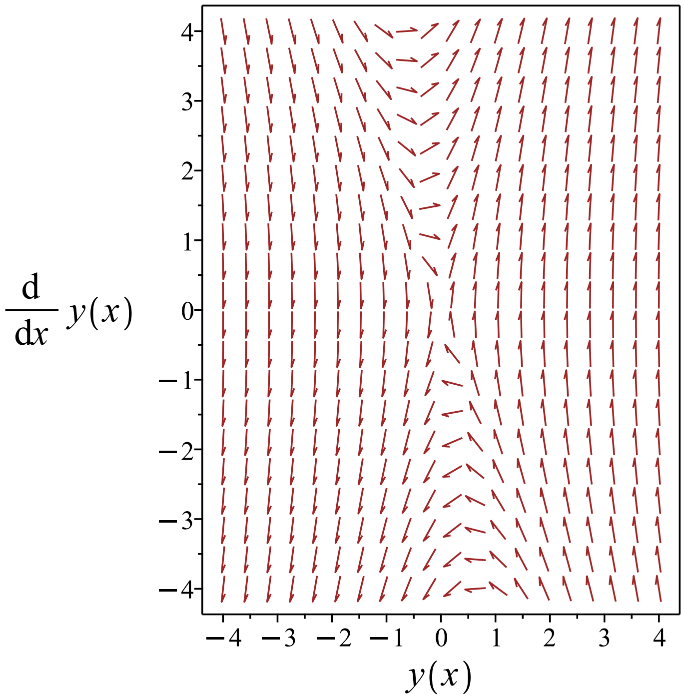


Figure 605: Slope field plot

Verification of solutions

$$y = c_1 e^{3x} + c_2 e^{-2x} + \frac{\cos(3x)}{39} - \frac{5 \sin(3x)}{39}$$

Verified OK.

11.3.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - y' - 6y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -1 \\ C &= -6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{25}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 25 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{25z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 451: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{25}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{5x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-1}{1}\,dx} \\ &= z_1 e^{\frac{x}{2}} \\ &= z_1\left(\mathrm{e}^{\frac{x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-1}{1}\, dx}}{(y_1)^2}\, dx \\
&= y_1 \int \frac{e^{x}}{(y_1)^2}\, dx \\
&= y_1 \left(\frac{\mathrm{e}^{5x}}{5}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{5x}}{5}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - y' - 6y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{3x}}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2 \sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{e^{3x}}{5}, e^{-2x}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3x) + A_2 \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-15A_1 \cos(3x) - 15A_2 \sin(3x) + 3A_1 \sin(3x) - 3A_2 \cos(3x) = 2\sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{39}, A_2 = -\frac{5}{39}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{3x}}{5}\right) + \left(\frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{3x}}{5} + \frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39} \tag{1}$$

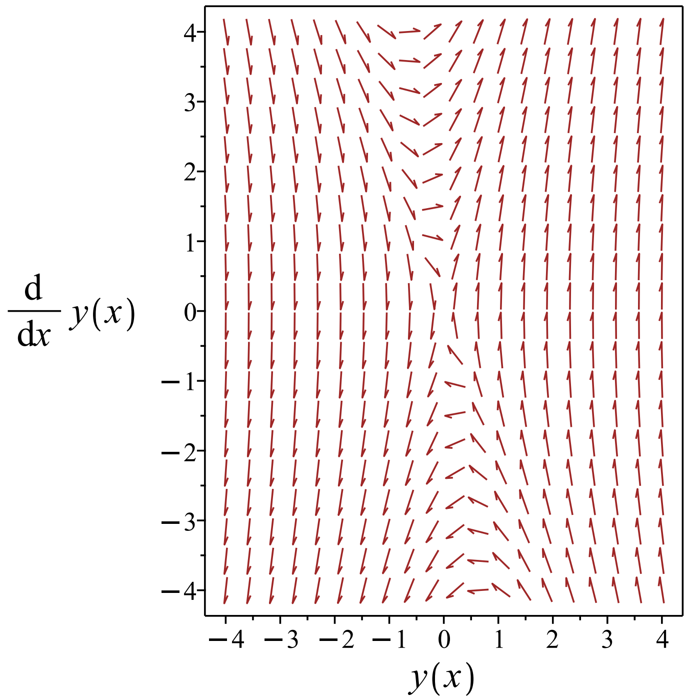


Figure 606: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{3x}}{5} + \frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39}$$

Verified OK.

11.3.3 Maple step by step solution

Let's solve

$y'' - y' - 6y = 2\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - r - 6 = 0$

- Factor the characteristic polynomial

  $(r + 2)(r - 3) = 0$

- Roots of the characteristic polynomial

$r = (-2, 3)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{3x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{3x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\sin(3x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{3x} \\ -2\,e^{-2x} & 3\,e^{3x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 5\,e^{x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = \frac{2\left(e^{5x}\left(\int \sin(3x) e^{-3x} dx\right) - \left(\int \sin(3x) e^{2x} dx\right)\right) e^{-2x}}{5}$

  - Compute integrals

    $y_p(x) = \frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{3x} + \frac{\cos(3x)}{39} - \frac{5\sin(3x)}{39}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)-diff(y(x),x)-6*y(x)=2*sin(3*x),y(x), singsol=all)
```

$$y(x) = \mathrm{e}^{-2x}\left(\frac{(\cos(3x) - 5\sin(3x))\,\mathrm{e}^{2x}}{39} + c_2\mathrm{e}^{5x} + c_1\right)$$

✓ Solution by Mathematica

Time used: 0.021 (sec). Leaf size: 37

```
DSolve[y''[x]-y'[x]-6*y[x]==2*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to c_1e^{-2x} + c_2e^{3x} + \frac{1}{39}(\cos(3x) - 5\sin(3x))$$

11.4 problem 4

Internal problem ID [222]
Internal file name [`OUTPUT/222_Sunday_June_05_2022_01_36_55_AM_76986988/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{4y'' + 4y' + y = 3x\,e^{x}}$$

11.4.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4, B = 4, C = 1, f(x) = 3x\,e^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4y'' + 4y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 4, B = 4, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$4\lambda^2 e^{\lambda x} + 4\lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$4\lambda^2 + 4\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 4, B = 4, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(4)} \pm \frac{1}{(2)(4)}\sqrt{(4)^2 - (4)(4)(1)}\\ &= -\frac{1}{2}\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = \frac{1}{2}$. Therefore the solution is

$$y = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-\frac{x}{2}} + c_2 x e^{-\frac{x}{2}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x e^x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x e^x, e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{x e^{-\frac{x}{2}}, e^{-\frac{x}{2}}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x \, e^x + A_2 e^x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$12A_1 e^x + 9A_1 x \, e^x + 9A_2 e^x = 3x \, e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -\frac{4}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x \, e^x}{3} - \frac{4 \, e^x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{x}{2}} + c_2 x \, e^{-\frac{x}{2}}\right) + \left(\frac{x \, e^x}{3} - \frac{4 \, e^x}{9}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-\frac{x}{2}} (c_2 x + c_1) + \frac{x \, e^x}{3} - \frac{4 \, e^x}{9}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}} (c_2 x + c_1) + \frac{x \, e^x}{3} - \frac{4 \, e^x}{9} \tag{1}$$

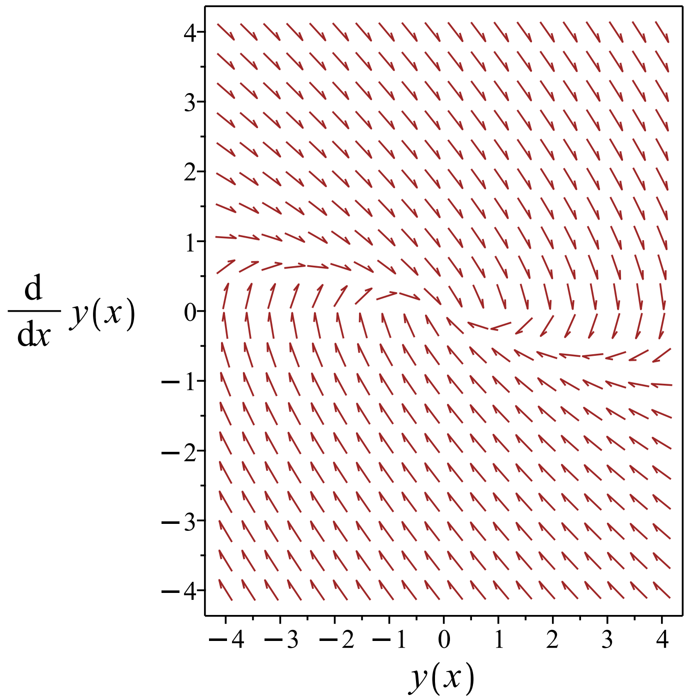


Figure 607: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}}(c_2 x + c_1) + \frac{x\,e^{x}}{3} - \frac{4\,e^{x}}{9}$$

Verified OK.

11.4.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\,y' + \frac{\left(p(x)^2 + p'(x)\right)y}{2} = f(x)$$

Where $p(x) = 1$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 1\,dx} \\ &= e^{\frac{x}{2}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = \frac{3\,e^{\frac{x}{2}} x\,e^{x}}{4}$$

$$\left(e^{\frac{x}{2}} y\right)'' = \frac{3\,e^{\frac{x}{2}} x\,e^{x}}{4}$$

Integrating once gives

$$\left(e^{\frac{x}{2}} y\right)' = \frac{(3x-2)\,e^{\frac{3x}{2}}}{6} + c_1$$

Integrating again gives

$$\left(e^{\frac{x}{2}} y\right) = \frac{(3x-4)\,e^{\frac{3x}{2}}}{9} + c_1 x + c_2$$

Hence the solution is

$$y = \frac{\frac{(3x-4)e^{\frac{3x}{2}}}{9} + c_1 x + c_2}{e^{\frac{x}{2}}}$$

Or

$$y = c_1 x\,e^{-\frac{x}{2}} + \frac{x\,e^{-\frac{x}{2}} e^{\frac{3x}{2}}}{3} + c_2 e^{-\frac{x}{2}} - \frac{4\,e^{-\frac{x}{2}} e^{\frac{3x}{2}}}{9}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,e^{-\frac{x}{2}} + \frac{x\,e^{-\frac{x}{2}} e^{\frac{3x}{2}}}{3} + c_2 e^{-\frac{x}{2}} - \frac{4\,e^{-\frac{x}{2}} e^{\frac{3x}{2}}}{9} \tag{1}$$

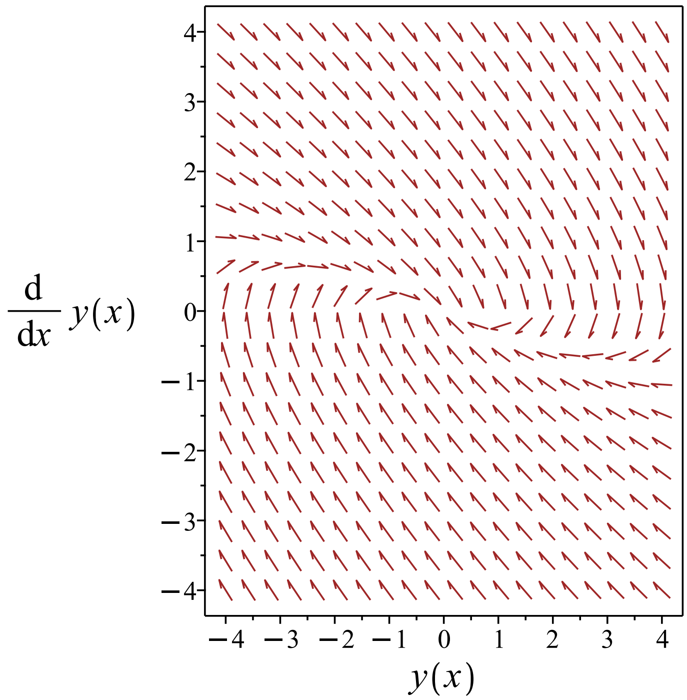


Figure 608: Slope field plot

Verification of solutions

$$y = c_1 x \,\mathrm{e}^{-\frac{x}{2}} + \frac{x\,\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{\frac{3x}{2}}}{3} + c_2 \mathrm{e}^{-\frac{x}{2}} - \frac{4\,\mathrm{e}^{-\frac{x}{2}}\mathrm{e}^{\frac{3x}{2}}}{9}$$

Verified OK.

11.4.3 Solving using Kovacic algorithm

Writing the ode as

$$4y'' + 4y' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4 \\ B &= 4 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 453: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{4}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{4}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}}\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
4y'' + 4y' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-\frac{x}{2}} + c_2 x\,\mathrm{e}^{-\frac{x}{2}}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
3x\,\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{x\,\mathrm{e}^{-\frac{x}{2}}, \mathrm{e}^{-\frac{x}{2}}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\,\mathrm{e}^{x} + A_2 \mathrm{e}^{x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
12A_1\mathrm{e}^{x} + 9A_1 x\,\mathrm{e}^{x} + 9A_2\mathrm{e}^{x} = 3x\,\mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{3}, A_2 = -\frac{4}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,e^x}{3} - \frac{4\,e^x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-\frac{x}{2}} + c_2 x\,e^{-\frac{x}{2}}\right) + \left(\frac{x\,e^x}{3} - \frac{4\,e^x}{9}\right) \end{aligned}$$

Which simplifies to

$$y = e^{-\frac{x}{2}}(c_2 x + c_1) + \frac{x\,e^x}{3} - \frac{4\,e^x}{9}$$

Summary

The solution(s) found are the following

$$y = e^{-\frac{x}{2}}(c_2 x + c_1) + \frac{x\,e^x}{3} - \frac{4\,e^x}{9} \tag{1}$$

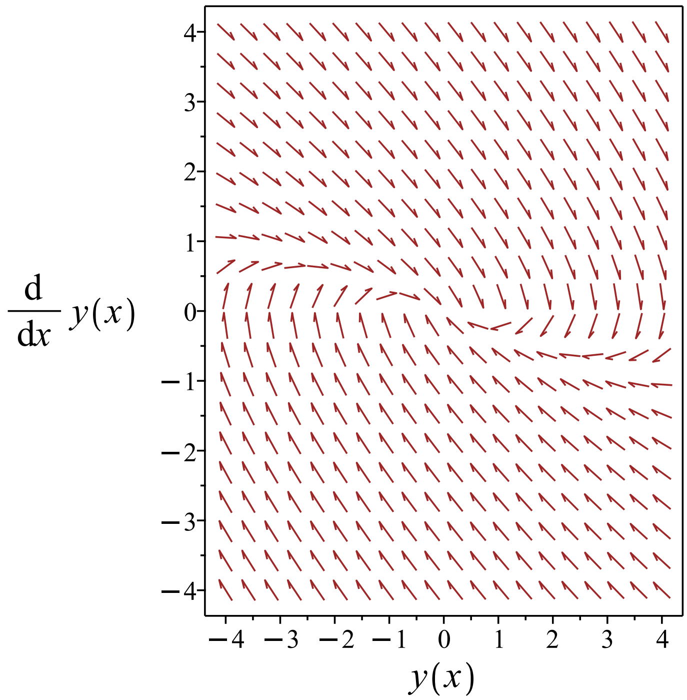


Figure 609: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}}(c_2x + c_1) + \frac{x\,e^x}{3} - \frac{4\,e^x}{9}$$

Verified OK.

11.4.4 Maple step by step solution

Let's solve

$4y'' + 4y' + y = 3x\,e^x$

- Highest derivative means the order of the ODE is 2

  y''

- Isolate 2nd derivative

  $y'' = -y' - \frac{y}{4} + \frac{3x\,e^x}{4}$

- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + y' + \frac{y}{4} = \frac{3x\,e^x}{4}$

- Characteristic polynomial of homogeneous ODE

$r^2 + r + \frac{1}{4} = 0$

- Factor the characteristic polynomial

  $\frac{(2r+1)^2}{4} = 0$

- Root of the characteristic polynomial

  $r = -\frac{1}{2}$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{-\frac{x}{2}}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-\frac{x}{2}} + c_2 x\,e^{-\frac{x}{2}} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = \frac{3x\,e^x}{4}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} & x\,e^{-\frac{x}{2}} \\ -\frac{e^{-\frac{x}{2}}}{2} & e^{-\frac{x}{2}} - \frac{x\,e^{-\frac{x}{2}}}{2} \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{3\,e^{-\frac{x}{2}}\left(\int x^2 e^{\frac{3x}{2}} dx - \left(\int x\,e^{\frac{3x}{2}} dx\right) x\right)}{4}$

  ◦ Compute integrals

  $y_p(x) = \frac{e^x(3x-4)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-\frac{x}{2}} + c_2 x\,e^{-\frac{x}{2}} + \frac{e^x(3x-4)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 24

```
dsolve(4*diff(y(x),x$2)+4*diff(y(x),x)+y(x)=3*x*exp(x),y(x), singsol=all)
```

$$y(x) = (c_1 x + c_2)\,\mathrm{e}^{-\frac{x}{2}} + \frac{(3x-4)\,\mathrm{e}^{x}}{9}$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 33

```
DSolve[4*y''[x]+4*y'[x]+y[x]==3*x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{9}e^x(3x-4) + e^{-x/2}(c_2 x + c_1)$$

11.5 problem 5

Internal problem ID [223]
Internal file name [`OUTPUT/223_Sunday_June_05_2022_01_36_56_AM_87598048/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = \sin(x)^2}$$

11.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-1}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{1^2 - (4)\,(1)\,(1)} \\
&= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$
$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right), e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2 \cos(2x) - 3A_3 \sin(2x) - 2A_2 \sin(2x) + 2A_3 \cos(2x) + A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{2}, A_2 = \frac{3}{26}, A_3 = -\frac{1}{13} \right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right)\right) + \left(\frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + \frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13} \tag{1}
$$

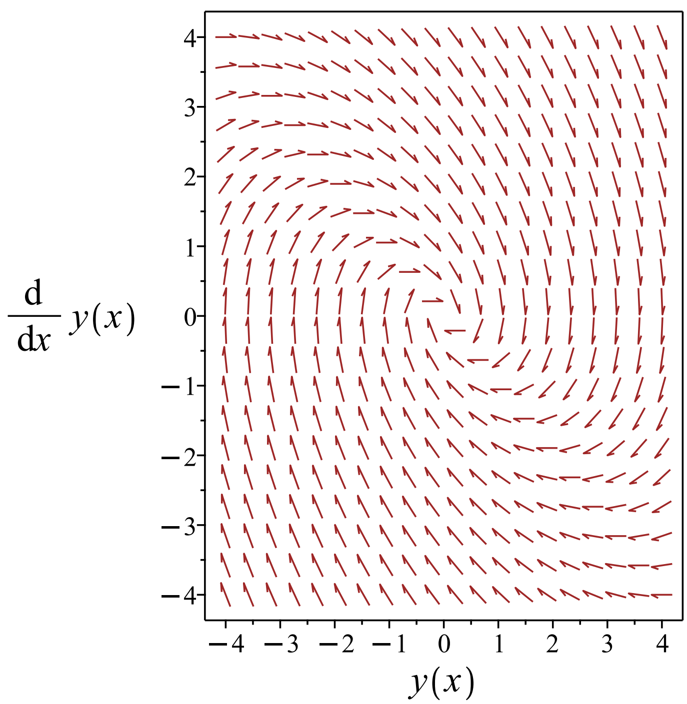


Figure 610: Slope field plot

Verification of solutions

$$
y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right)\right) + \frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13}
$$

Verified OK.

11.5.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 455: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''+y'+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h=\cos\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}c_1+\frac{2c_2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin\left(x\right)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\},\{\cos\left(2x\right),\sin\left(2x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right),\frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right)\mathrm{e}^{-\frac{x}{2}}\sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p=A_1+A_2\cos\left(2x\right)+A_3\sin\left(2x\right)$$

The unknowns $\{A_1,A_2,A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_2\cos\left(2x\right)-3A_3\sin\left(2x\right)-2A_2\sin\left(2x\right)+2A_3\cos\left(2x\right)+A_1=\sin\left(x\right)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{1}{2},A_2=\frac{3}{26},A_3=-\frac{1}{13}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) + \left(\frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} + \frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13} \tag{1}$$

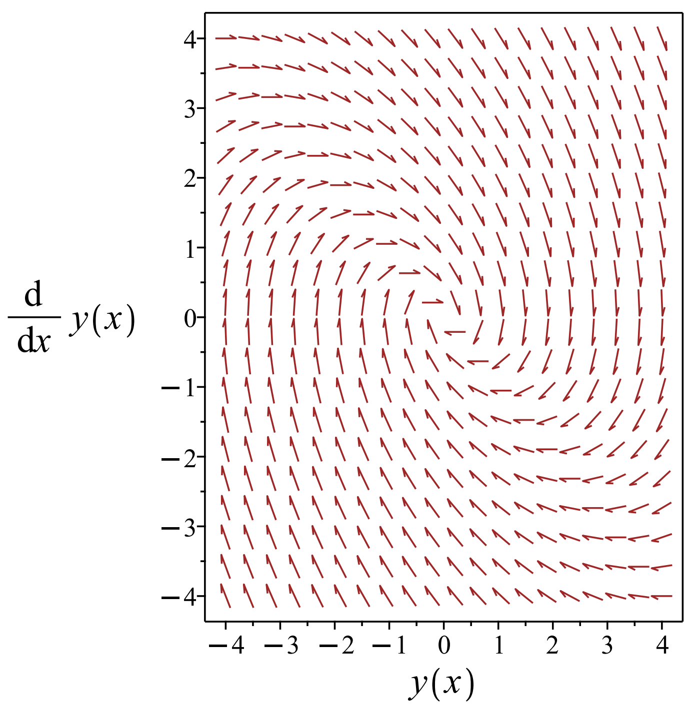


Figure 611: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} + \frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13}$$

Verified OK.

11.5.3 Maple step by step solution

Let's solve

$y'' + y' + y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{1}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) & e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) \\ -\frac{e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}} \sqrt{3}\cos\left(\frac{\sqrt{3}\, x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = \frac{\sqrt{3}\, e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\frac{2\, e^{-\frac{x}{2}} \sqrt{3} \left(\cos\left(\frac{\sqrt{3}\, x}{2}\right)\left(\int e^{\frac{x}{2}} \sin(x)^2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) dx\right) - \sin\left(\frac{\sqrt{3}\, x}{2}\right)\left(\int e^{\frac{x}{2}} \sin(x)^2 \cos\left(\frac{\sqrt{3}\, x}{2}\right) dx\right)\right)}{3}$

- Compute integrals

  $y_p(x) = \frac{1}{2} + \frac{3\cos(2x)}{26} - \frac{\sin(2x)}{13}$

- Substitute particular solution into general solution to ODE

  $y = \cos\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} c_1 + \sin\left(\frac{\sqrt{3}\, x}{2}\right) e^{-\frac{x}{2}} c_2 - \frac{\sin(2x)}{13} + \frac{3\cos(2x)}{26} + \frac{1}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 44

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 - \frac{\sin(2x)}{13} + \frac{3\cos(2x)}{26} + \frac{1}{2}$$

✓ Solution by Mathematica

Time used: 1.827 (sec). Leaf size: 67

```
DSolve[y''[x]+y'[x]+y[x]==Sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{13}\sin(2x) + \frac{3}{26}\cos(2x) + c_2 e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right) + \frac{1}{2}$$

11.6 problem 6

Internal problem ID [224]
Internal file name [`OUTPUT/224_Sunday_June_05_2022_01_36_58_AM_69153529/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 6.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{2y'' + 4y' + 7y = x^2}$$

11.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 2, B = 4, C = 7, f(x) = x^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$2y'' + 4y' + 7y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 2, B = 4, C = 7$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda x} + 4\lambda\, e^{\lambda x} + 7\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$2\lambda^2 + 4\lambda + 7 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 4, C = 7$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{4^2 - (4)\,(2)\,(7)}\\ &= -1 \pm \frac{i\sqrt{10}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + \frac{i\sqrt{10}}{2}\\ \lambda_2 &= -1 - \frac{i\sqrt{10}}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + \frac{i\sqrt{10}}{2}\\ \lambda_2 &= -1 - \frac{i\sqrt{10}}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = \frac{\sqrt{10}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-x}\left(c_1 \cos\left(\frac{\sqrt{10}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{10}\, x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = e^{-x} \left(c_1 \cos\left(\frac{\sqrt{10}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{10}\, x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-x} \cos\left(\frac{\sqrt{10}\, x}{2}\right), e^{-x} \sin\left(\frac{\sqrt{10}\, x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_3 x^2 + 7A_2 x + 8x A_3 + 7A_1 + 4A_2 + 4A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{343}, A_2 = -\frac{8}{49}, A_3 = \frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{7} x^2 - \frac{8}{49} x + \frac{4}{343}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(e^{-x}\left(c_1 \cos\left(\frac{\sqrt{10}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{10}\, x}{2}\right)\right)\right) + \left(\frac{1}{7}x^2 - \frac{8}{49}x + \frac{4}{343}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = e^{-x}\left(c_1 \cos\left(\frac{\sqrt{10}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{10}\, x}{2}\right)\right) + \frac{x^2}{7} - \frac{8x}{49} + \frac{4}{343} \tag{1}
$$

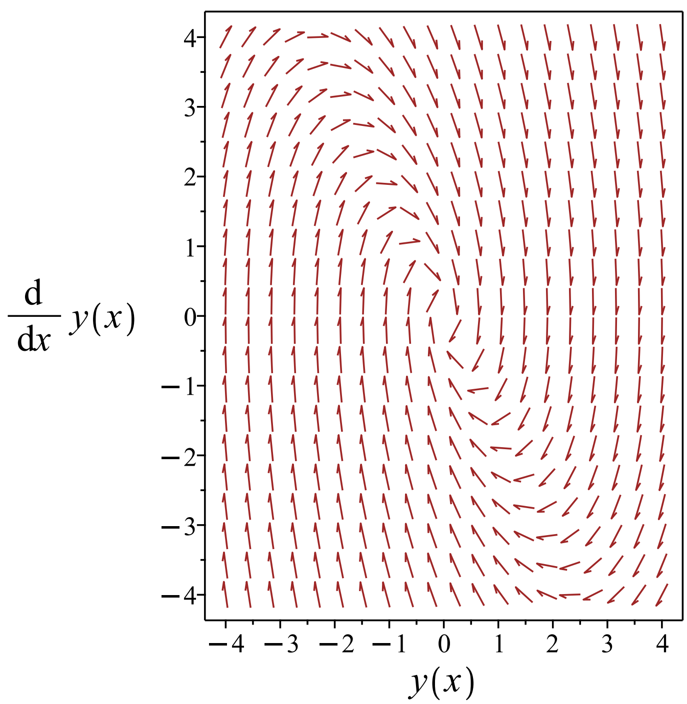


Figure 612: Slope field plot

Verification of solutions

$$
y = e^{-x}\left(c_1 \cos\left(\frac{\sqrt{10}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{10}\, x}{2}\right)\right) + \frac{x^2}{7} - \frac{8x}{49} + \frac{4}{343}
$$

Verified OK.

11.6.2 Solving using Kovacic algorithm

Writing the ode as

$$2y'' + 4y' + 7y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 4 \\ C &= 7 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{5z(x)}{2} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 457: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{5}{2}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{10}\, x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{2}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-x}\cos\left(\frac{\sqrt{10}\,x}{2}\right)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{4}{2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-2x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{\sqrt{10}\,\tan\left(\frac{\sqrt{10}\,x}{2}\right)}{5}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-x}\cos\left(\frac{\sqrt{10}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-x}\cos\left(\frac{\sqrt{10}\,x}{2}\right)\left(\frac{\sqrt{10}\,\tan\left(\frac{\sqrt{10}\,x}{2}\right)}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$2y'' + 4y' + 7y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\frac{\sqrt{10}\,x}{2}\right) \mathrm{e}^{-x} c_1 + \frac{c_2 \sin\left(\frac{\sqrt{10}\,x}{2}\right) \mathrm{e}^{-x} \sqrt{10}}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x, x^2\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-x} \cos\left(\frac{\sqrt{10}\,x}{2}\right), \frac{\sin\left(\frac{\sqrt{10}\,x}{2}\right) \mathrm{e}^{-x} \sqrt{10}}{5}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_3 x^2 + A_2 x + A_1$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$7A_3x^2 + 7A_2x + 8xA_3 + 7A_1 + 4A_2 + 4A_3 = x^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{4}{343}, A_2 = -\frac{8}{49}, A_3 = \frac{1}{7}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{7}x^2 - \frac{8}{49}x + \frac{4}{343}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{\sqrt{10}\,x}{2}\right) e^{-x} c_1 + \frac{c_2 \sin\left(\frac{\sqrt{10}\,x}{2}\right) e^{-x} \sqrt{10}}{5}\right) + \left(\frac{1}{7}x^2 - \frac{8}{49}x + \frac{4}{343}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \cos\left(\frac{\sqrt{10}\,x}{2}\right) e^{-x} c_1 + \frac{c_2 \sin\left(\frac{\sqrt{10}\,x}{2}\right) e^{-x} \sqrt{10}}{5} + \frac{x^2}{7} - \frac{8x}{49} + \frac{4}{343} \tag{1}$$

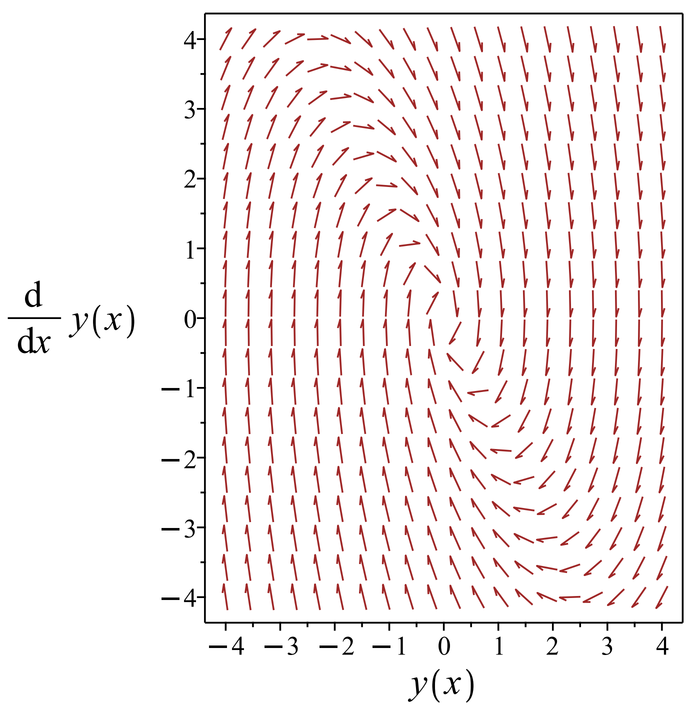


Figure 613: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{\sqrt{10}\, x}{2}\right) \mathrm{e}^{-x} c_1 + \frac{c_2 \sin\left(\frac{\sqrt{10}\, x}{2}\right) \mathrm{e}^{-x} \sqrt{10}}{5} + \frac{x^2}{7} - \frac{8x}{49} + \frac{4}{343}$$

Verified OK.

11.6.3 Maple step by step solution

Let's solve

$2y'' + 4y' + 7y = x^2$

- Highest derivative means the order of the ODE is 2

  y''
- Isolate 2nd derivative

  $y'' = -2y' - \frac{7y}{2} + \frac{x^2}{2}$
- Group terms with y on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $y'' + 2y' + \frac{7y}{2} = \frac{x^2}{2}$
- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r + \frac{7}{2} = 0$
- Use quadratic formula to solve for r

  $r = \frac{(-2) \pm (\sqrt{-10})}{2}$
- Roots of the characteristic polynomial

  $r = \left(-1 - \frac{\mathrm{I}\sqrt{10}}{2}, -1 + \frac{\mathrm{I}\sqrt{10}}{2}\right)$
- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-x} \cos\left(\frac{\sqrt{10}\, x}{2}\right)$
- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x} \sin\left(\frac{\sqrt{10}\, x}{2}\right)$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = \cos\left(\frac{\sqrt{10}\, x}{2}\right) \mathrm{e}^{-x} c_1 + \sin\left(\frac{\sqrt{10}\, x}{2}\right) \mathrm{e}^{-x} c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))}dx\right), f(x) = \frac{x^2}{2}\right]$$

- Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-x}\cos\left(\frac{\sqrt{10}\,x}{2}\right) & e^{-x}\sin\left(\frac{\sqrt{10}\,x}{2}\right) \\ -e^{-x}\cos\left(\frac{\sqrt{10}\,x}{2}\right) - \frac{\sin\left(\frac{\sqrt{10}\,x}{2}\right)e^{-x}\sqrt{10}}{2} & -e^{-x}\sin\left(\frac{\sqrt{10}\,x}{2}\right) + \frac{e^{-x}\sqrt{10}\,\cos\left(\frac{\sqrt{10}\,x}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(y_1(x), y_2(x)) = \frac{\sqrt{10}\,e^{-2x}}{2}$$

- Substitute functions into equation for $y_p(x)$

  $$y_p(x) = -\frac{e^{-x}\sqrt{10}\left(\cos\left(\frac{\sqrt{10}\,x}{2}\right)\left(\int x^2 e^x \sin\left(\frac{\sqrt{10}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{10}\,x}{2}\right)\left(\int x^2 e^x \cos\left(\frac{\sqrt{10}\,x}{2}\right)dx\right)\right)}{10}$$

- Compute integrals

  $$y_p(x) = \frac{1}{7}x^2 - \frac{8}{49}x + \frac{4}{343}$$

- Substitute particular solution into general solution to ODE

  $$y = \cos\left(\frac{\sqrt{10}\,x}{2}\right)e^{-x}c_1 + \sin\left(\frac{\sqrt{10}\,x}{2}\right)e^{-x}c_2 + \frac{x^2}{7} - \frac{8x}{49} + \frac{4}{343}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ <u>Solution by Maple</u>

Time used: 0.016 (sec). Leaf size: 40

```
dsolve(2*diff(y(x),x$2)+4*diff(y(x),x)+7*y(x)=x^2,y(x), singsol=all)
```

$$y(x) = e^{-x} \sin\left(\frac{\sqrt{10}\, x}{2}\right) c_2 + e^{-x} \cos\left(\frac{\sqrt{10}\, x}{2}\right) c_1 + \frac{x^2}{7} - \frac{8x}{49} + \frac{4}{343}$$

✓ <u>Solution by Mathematica</u>

Time used: 0.029 (sec). Leaf size: 56

```
DSolve[2*y''[x]+4*y'[x]+7*y[x]==x^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{343}\left(49x^2 - 56x + 4\right) + c_2 e^{-x} \cos\left(\sqrt{\frac{5}{2}} x\right) + c_1 e^{-x} \sin\left(\sqrt{\frac{5}{2}} x\right)$$

11.7 problem 7

Internal problem ID [225]
Internal file name [`OUTPUT/225_Sunday_June_05_2022_01_36_59_AM_18542131/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = \sinh(x)}$$

11.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = \sinh(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = e^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(e^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix}$$

Therefore

$$W = (e^{2x})(-2e^{-2x}) - (e^{-2x})(2e^{2x})$$

Which simplifies to

$$W = -4e^{2x}e^{-2x}$$

Which simplifies to

$$W = -4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-2x} \sinh(x)}{-4} dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x} \sinh(x)}{4} dx$$

Hence

$$u_1 = \frac{\sinh(x)}{8} - \frac{\sinh(3x)}{24} - \frac{\cosh(x)}{8} + \frac{\cosh(3x)}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \sinh(x)}{-4} dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{2x} \sinh(x)}{4} dx$$

Hence

$$u_2 = \frac{\sinh(x)}{8} - \frac{\sinh(3x)}{24} + \frac{\cosh(x)}{8} - \frac{\cosh(3x)}{24}$$

Which simplifies to

$$u_1 = \frac{(\cosh(x) - \sinh(x)) \sinh(x)^2}{6} - \frac{\cosh(x)}{12}$$

$$u_2 = -\frac{(\cosh(x) + \sinh(x)) \sinh(x)^2}{6} + \frac{\cosh(x)}{12}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{(\cosh(x) - \sinh(x)) \sinh(x)^2}{6} - \frac{\cosh(x)}{12}\right) e^{2x} + \left(-\frac{(\cosh(x) + \sinh(x)) \sinh(x)^2}{6} + \frac{\cosh(x)}{12}\right) e^{-2x}$$

Which simplifies to

$$y_p(x) = \frac{\left(-2\sinh\left(x\right)^{3}-2\sinh\left(x\right)^{2}\cosh\left(x\right)+\cosh\left(x\right)\right){\mathrm e}^{-2x}}{12} + \frac{\left(-\sinh\left(x\right)^{3}+\sinh\left(x\right)^{2}\cosh\left(x\right)-\frac{\cosh\left(x\right)}{2}\right){\mathrm e}^{2x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 {\mathrm e}^{2x} + c_2 {\mathrm e}^{-2x}\right) + \left(\frac{\left(-2\sinh\left(x\right)^{3}-2\sinh\left(x\right)^{2}\cosh\left(x\right)+\cosh\left(x\right)\right){\mathrm e}^{-2x}}{12} + \frac{\left(-\sinh\left(x\right)^{3}+\sinh\left(x\right)^{2}\cosh\left(x\right)-\frac{\cosh\left(x\right)}{2}\right){\mathrm e}^{2x}}{6}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 {\mathrm e}^{2x} + c_2 {\mathrm e}^{-2x} + \frac{\left(-2\sinh\left(x\right)^{3}-2\sinh\left(x\right)^{2}\cosh\left(x\right)+\cosh\left(x\right)\right){\mathrm e}^{-2x}}{12} + \frac{\left(-\sinh\left(x\right)^{3}+\sinh\left(x\right)^{2}\cosh\left(x\right)-\frac{\cosh\left(x\right)}{2}\right){\mathrm e}^{2x}}{6} \tag{1}$$

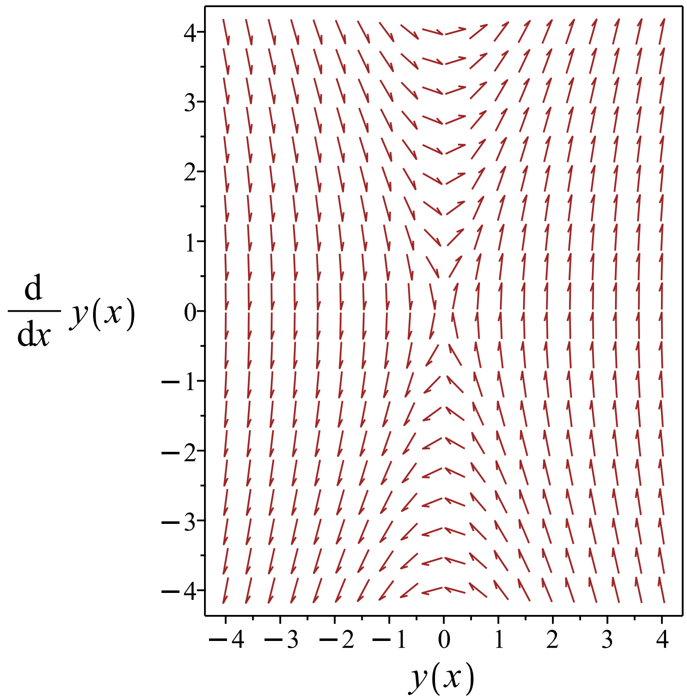


Figure 614: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{(-2\sinh(x)^3 - 2\sinh(x)^2\cosh(x) + \cosh(x))\,e^{-2x}}{12} + \frac{\left(-\sinh(x)^3 + \sinh(x)^2\cosh(x) - \frac{\cosh(x)}{2}\right)e^{2x}}{6}$$

Verified OK.

11.7.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 459: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \mathrm{e}^{-2x}
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{\mathrm{e}^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\, dx \\ &= \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \frac{\mathrm{e}^{2x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{2x}}{4} \\ \frac{d}{dx}(\mathrm{e}^{-2x}) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{2x}}{4} \\ -2\,\mathrm{e}^{-2x} & \frac{\mathrm{e}^{2x}}{2} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{-2x})\left(\frac{\mathrm{e}^{2x}}{2}\right) - \left(\frac{\mathrm{e}^{2x}}{4}\right)(-2\,\mathrm{e}^{-2x})$$

Which simplifies to

$$W = \mathrm{e}^{2x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}\sinh(x)}{4}}{1} dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}\sinh(x)}{4} dx$$

Hence

$$u_1 = \frac{\sinh(x)}{8} - \frac{\sinh(3x)}{24} + \frac{\cosh(x)}{8} - \frac{\cosh(3x)}{24}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}\sinh(x)}{1} dx$$

Which simplifies to

$$u_2 = \int e^{-2x}\sinh(x)\, dx$$

Hence

$$u_2 = \frac{\sinh(x)}{2} - \frac{\sinh(3x)}{6} - \frac{\cosh(x)}{2} + \frac{\cosh(3x)}{6}$$

Which simplifies to

$$u_1 = -\frac{(\cosh(x)+\sinh(x))\sinh(x)^2}{6} + \frac{\cosh(x)}{12}$$

$$u_2 = \frac{2(\cosh(x)-\sinh(x))\sinh(x)^2}{3} - \frac{\cosh(x)}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{(\cosh(x)+\sinh(x))\sinh(x)^2}{6} + \frac{\cosh(x)}{12}\right)e^{-2x} + \frac{\left(\frac{2(\cosh(x)-\sinh(x))\sinh(x)^2}{3} - \frac{\cosh(x)}{3}\right)e^{2x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{\left(-2\sinh\left(x\right)^{3}-2\sinh\left(x\right)^{2}\cosh\left(x\right)+\cosh\left(x\right)\right){\mathrm e}^{-2x}}{12} + \frac{\left(-\sinh\left(x\right)^{3}+\sinh\left(x\right)^{2}\cosh\left(x\right)-\frac{\cosh\left(x\right)}{2}\right){\mathrm e}^{2x}}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 {\mathrm e}^{-2x}+\frac{c_2 {\mathrm e}^{2x}}{4}\right) + \left(\frac{\left(-2\sinh\left(x\right)^{3}-2\sinh\left(x\right)^{2}\cosh\left(x\right)+\cosh\left(x\right)\right){\mathrm e}^{-2x}}{12} + \frac{\left(-\sinh\left(x\right)^{3}+\sinh\left(x\right)^{2}\cosh\left(x\right)-\frac{\cosh\left(x\right)}{2}\right){\mathrm e}^{2x}}{6}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 {\mathrm e}^{-2x}+\frac{c_2 {\mathrm e}^{2x}}{4}+\frac{\left(-2\sinh\left(x\right)^{3}-2\sinh\left(x\right)^{2}\cosh\left(x\right)+\cosh\left(x\right)\right){\mathrm e}^{-2x}}{12} + \frac{\left(-\sinh\left(x\right)^{3}+\sinh\left(x\right)^{2}\cosh\left(x\right)-\frac{\cosh\left(x\right)}{2}\right){\mathrm e}^{2x}}{6} \tag{1}$$

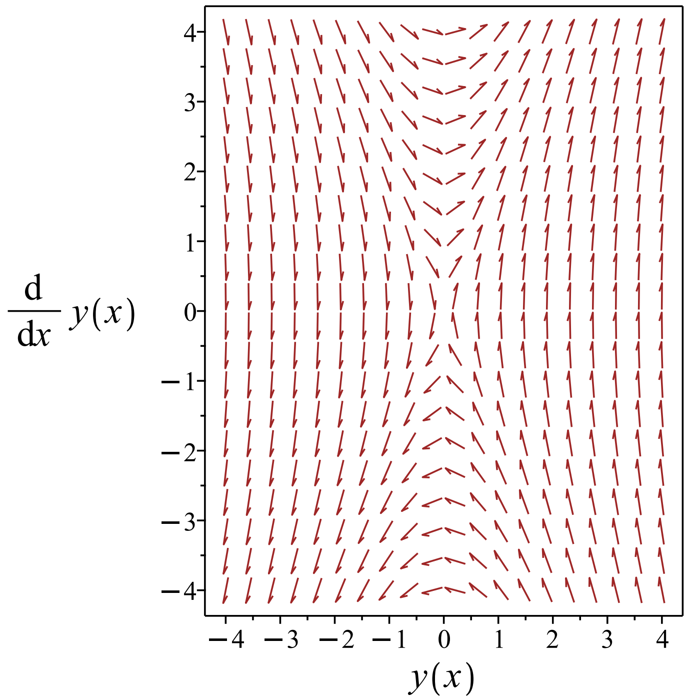


Figure 615: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + \frac{\left(-2\sinh(x)^3 - 2\sinh(x)^2\cosh(x) + \cosh(x)\right)e^{-2x}}{12} + \frac{\left(-\sinh(x)^3 + \sinh(x)^2\cosh(x) - \frac{\cosh(x)}{2}\right)e^{2x}}{6}$$

Verified OK.

11.7.3 Maple step by step solution

Let's solve

$y'' - 4y = \sinh(x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

$$(r-2)(r+2)=0$$

- Roots of the characteristic polynomial

$$r=(-2,2)$$

- 1st solution of the homogeneous ODE

$$y_1(x)=\mathrm{e}^{-2x}$$

- 2nd solution of the homogeneous ODE

$$y_2(x)=\mathrm{e}^{2x}$$

- General solution of the ODE

$$y=c_1y_1(x)+c_2y_2(x)+y_p(x)$$

- Substitute in solutions of the homogeneous ODE

$$y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}+y_p(x)$$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

$$\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right),f(x)=\sinh(x)\right]$$

  - Wronskian of solutions of the homogeneous equation

$$W(y_1(x),y_2(x))=\begin{bmatrix}\mathrm{e}^{-2x} & \mathrm{e}^{2x}\\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x}\end{bmatrix}$$

  - Compute Wronskian

$$W(y_1(x),y_2(x))=4$$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x)=-\frac{\mathrm{e}^{-2x}\left(\int\mathrm{e}^{2x}\sinh(x)dx\right)}{4}+\frac{\mathrm{e}^{2x}\left(\int\mathrm{e}^{-2x}\sinh(x)dx\right)}{4}$$

  - Compute integrals

$$y_p(x)=\frac{\left(-2\sinh(x)^3-2\sinh(x)^2\cosh(x)+\cosh(x)\right)\mathrm{e}^{-2x}}{12}+\frac{\left(-\sinh(x)^3+\sinh(x)^2\cosh(x)-\frac{\cosh(x)}{2}\right)\mathrm{e}^{2x}}{6}$$

- Substitute particular solution into general solution to ODE

$$y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}+\frac{\left(-2\sinh(x)^3-2\sinh(x)^2\cosh(x)+\cosh(x)\right)\mathrm{e}^{-2x}}{12}+\frac{\left(-\sinh(x)^3+\sinh(x)^2\cosh(x)-\frac{\cosh(x)}{2}\right)\mathrm{e}^{2x}}{6}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 56

```
dsolve(diff(y(x),x$2)-4*y(x)=sinh(x),y(x), singsol=all)
```

$$y(x) = \frac{\left(-2\sinh(x)^2\cosh(x) - 2\sinh(x)^3 + 12c_1 + \cosh(x)\right)e^{-2x}}{12} + e^{2x}\left(\frac{\sinh(x)^2\cosh(x)}{6} - \frac{\sinh(x)^3}{6} + c_2 - \frac{\cosh(x)}{12}\right)$$

✓ Solution by Mathematica

Time used: 0.049 (sec). Leaf size: 38

```
DSolve[y''[x]-4*y[x]==Sinh[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{6}e^{-2x}\left(e^x - e^{3x} + 6c_1e^{4x} + 6c_2\right)$$

11.8 problem 8

Internal problem ID [226]
Internal file name [`OUTPUT/226_Sunday_June_05_2022_01_37_00_AM_49499764/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = \cosh(2x)}$$

11.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = \cosh(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{2x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 e^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{2x}$$
$$y_2 = e^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ \frac{d}{dx}(e^{2x}) & \frac{d}{dx}(e^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2\,e^{2x} & -2\,e^{-2x} \end{vmatrix}$$

Therefore

$$W = (e^{2x})\left(-2\,e^{-2x}\right) - (e^{-2x})\left(2\,e^{2x}\right)$$

Which simplifies to

$$W = -4\,e^{2x}e^{-2x}$$

Which simplifies to

$$W = -4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-2x} \cosh(2x)}{-4} dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x} \cosh(2x)}{4} dx$$

Hence

$$u_1 = \frac{x}{8} + \frac{\sinh(4x)}{32} - \frac{\cosh(4x)}{32}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \cosh(2x)}{-4} dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{2x} \cosh(2x)}{4} dx$$

Hence

$$u_2 = -\frac{\cosh(2x)^2}{16} - \frac{\cosh(2x)\sinh(2x)}{16} - \frac{x}{8}$$

Which simplifies to

$$u_1 = \frac{x}{8} + \frac{\sinh(4x)}{32} - \frac{\cosh(4x)}{32}$$
$$u_2 = -\frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{8} + \frac{\sinh(4x)}{32} - \frac{\cosh(4x)}{32}\right) e^{2x} + \left(-\frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}\right) e^{-2x}$$

Which simplifies to

$$y_p(x) = \frac{(-4x - 1 - \cosh(4x) - \sinh(4x)) e^{-2x}}{32} + \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right) e^{2x}}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}\right) \\
&\quad + \left(\frac{(-4x - 1 - \cosh(4x) - \sinh(4x))\,\mathrm{e}^{-2x}}{32} + \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right)\mathrm{e}^{2x}}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
\begin{aligned}
y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x} + \frac{(-4x - 1 - \cosh(4x) - \sinh(4x))\,\mathrm{e}^{-2x}}{32} \\
+ \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right)\mathrm{e}^{2x}}{8}
\end{aligned}
\tag{1}
$$

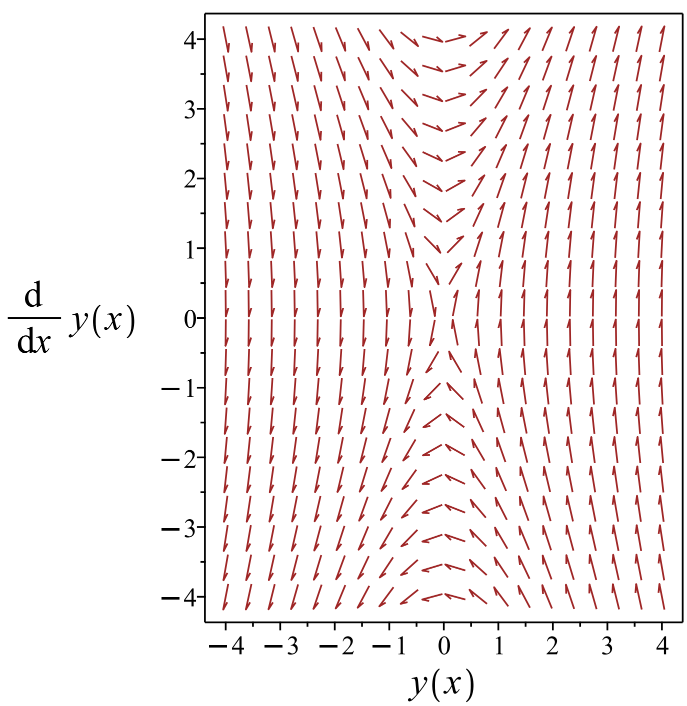


Figure 616: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{(-4x - 1 - \cosh(4x) - \sinh(4x)) e^{-2x}}{32} + \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right) e^{2x}}{8}$$

Verified OK.

11.8.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 461: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\,dx \\ &= \mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-2x}$$
$$y_2 = \frac{e^{2x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{2x}}{4} \\ \frac{d}{dx}\left(e^{-2x}\right) & \frac{d}{dx}\left(\frac{e^{2x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{2x}}{4} \\ -2\,e^{-2x} & \frac{e^{2x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(\frac{e^{2x}}{2}\right) - \left(\frac{e^{2x}}{4}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{2x} e^{-2x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}\cosh(2x)}{4}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}\cosh(2x)}{4}dx$$

Hence

$$u_1 = -\frac{\cosh(2x)^2}{16} - \frac{\cosh(2x)\sinh(2x)}{16} - \frac{x}{8}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}\cosh(2x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int e^{-2x}\cosh(2x)\,dx$$

Hence

$$u_2 = \frac{x}{2} + \frac{\sinh(4x)}{8} - \frac{\cosh(4x)}{8}$$

Which simplifies to

$$u_1 = -\frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}$$
$$u_2 = \frac{x}{2} + \frac{\sinh(4x)}{8} - \frac{\cosh(4x)}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(-\frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}\right) e^{-2x} + \frac{\left(\frac{x}{2} + \frac{\sinh(4x)}{8} - \frac{\cosh(4x)}{8}\right) e^{2x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{(-4x - 1 - \cosh(4x) - \sinh(4x))\, e^{-2x}}{32} + \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right) e^{2x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) \\ &\quad + \left(\frac{(-4x - 1 - \cosh(4x) - \sinh(4x))\, e^{-2x}}{32} + \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right) e^{2x}}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + \frac{(-4x - 1 - \cosh(4x) - \sinh(4x))\, e^{-2x}}{32} \\ + \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right) e^{2x}}{8} \end{aligned} \tag{1}$$

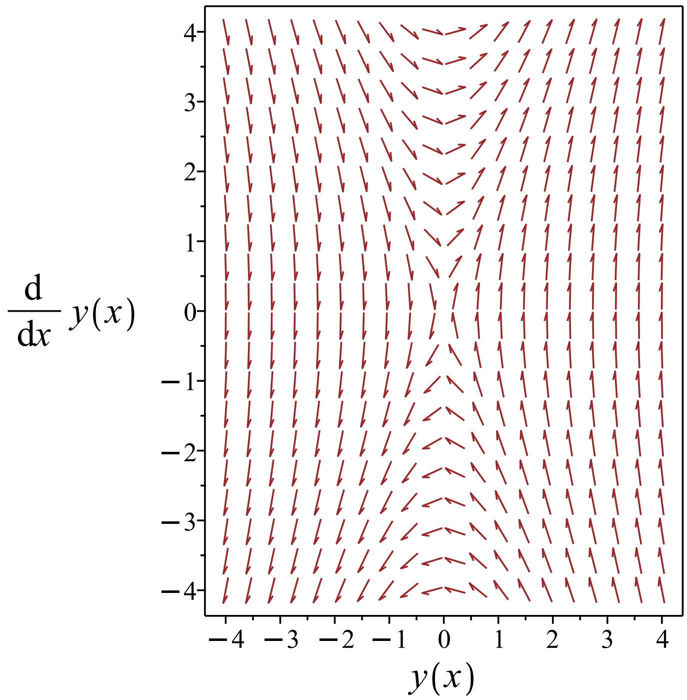


Figure 617: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + \frac{(-4x - 1 - \cosh(4x) - \sinh(4x)) e^{-2x}}{32} + \frac{\left(x + \frac{\sinh(4x)}{4} - \frac{\cosh(4x)}{4}\right) e^{2x}}{8}$$

Verified OK.

11.8.3 Maple step by step solution

Let's solve

$y'' - 4y = \cosh(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

$(r-2)(r+2)=0$

- Roots of the characteristic polynomial

  $r=(-2,2)$

- 1st solution of the homogeneous ODE

  $y_1(x)=\mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\mathrm{e}^{2x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\cosh(2x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x),y_2(x))=\begin{bmatrix}\mathrm{e}^{-2x} & \mathrm{e}^{2x}\\ -2\,\mathrm{e}^{-2x} & 2\,\mathrm{e}^{2x}\end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x),y_2(x))=4$

- ○ Substitute functions into equation for $y_p(x)$

  $y_p(x)=-\frac{\mathrm{e}^{-2x}\left(\int\mathrm{e}^{2x}\cosh(2x)dx\right)}{4}+\frac{\mathrm{e}^{2x}\left(\int\mathrm{e}^{-2x}\cosh(2x)dx\right)}{4}$

- ○ Compute integrals

  $y_p(x)=\frac{(-4x-1-\cosh(4x)-\sinh(4x))\mathrm{e}^{-2x}}{32}+\frac{\left(x+\frac{\sinh(4x)}{4}-\frac{\cosh(4x)}{4}\right)\mathrm{e}^{2x}}{8}$

- Substitute particular solution into general solution to ODE

  $y=c_1\mathrm{e}^{-2x}+c_2\mathrm{e}^{2x}+\frac{(-4x-1-\cosh(4x)-\sinh(4x))\mathrm{e}^{-2x}}{32}+\frac{\left(x+\frac{\sinh(4x)}{4}-\frac{\cosh(4x)}{4}\right)\mathrm{e}^{2x}}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 31

```
dsolve(diff(y(x),x$2)-4*y(x)=cosh(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-4x + 32c_1 - 2)\,\mathrm{e}^{-2x}}{32} + \frac{\mathrm{e}^{2x}\left(x + 8c_2 - \frac{1}{4}\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.048 (sec). Leaf size: 38

```
DSolve[y''[x]-4*y[x]==Cosh[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{32}e^{-2x}\left(-4x + e^{4x}(4x - 1 + 32c_1) - 1 + 32c_2\right)$$

11.9 problem 9

Internal problem ID [227]
Internal file name [`OUTPUT/227_Sunday_June_05_2022_01_37_01_AM_77252817/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 2y' - 3y = 1 + x\,\mathrm{e}^{x}}$$

11.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 2, C = -3, f(x) = 1 + x\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' - 3y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 2, C = -3$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 2\lambda\, e^{\lambda x} - 3\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 2\lambda - 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = -3$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{2^2 - (4)\,(1)\,(-3)} \\ &= -1 \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 2 \\ \lambda_2 &= -1 - 2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 \\ \lambda_2 &= -3\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(1)x} + c_2 e^{(-3)x}\end{aligned}$$

Or

$$y = c_1 e^{x} + e^{-3x} c_2$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{x} + e^{-3x} c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$1 + x\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{x}, \mathrm{e}^{-3x}\}$$

Since e^{x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\,\mathrm{e}^{x}, x^{2}\mathrm{e}^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\,\mathrm{e}^{x} + A_3 x^{2}\mathrm{e}^{x}$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_2\mathrm{e}^{x} + 2A_3\mathrm{e}^{x} + 8A_3 x\,\mathrm{e}^{x} - 3A_1 = 1 + x\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = -\frac{1}{16}, A_3 = \frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{1}{3} - \frac{x\,\mathrm{e}^{x}}{16} + \frac{x^{2}\mathrm{e}^{x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{x} + \mathrm{e}^{-3x}c_2\right) + \left(-\frac{1}{3} - \frac{x\,\mathrm{e}^{x}}{16} + \frac{x^{2}\mathrm{e}^{x}}{8}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^x + e^{-3x} c_2 - \frac{1}{3} - \frac{x e^x}{16} + \frac{x^2 e^x}{8} \tag{1}$$

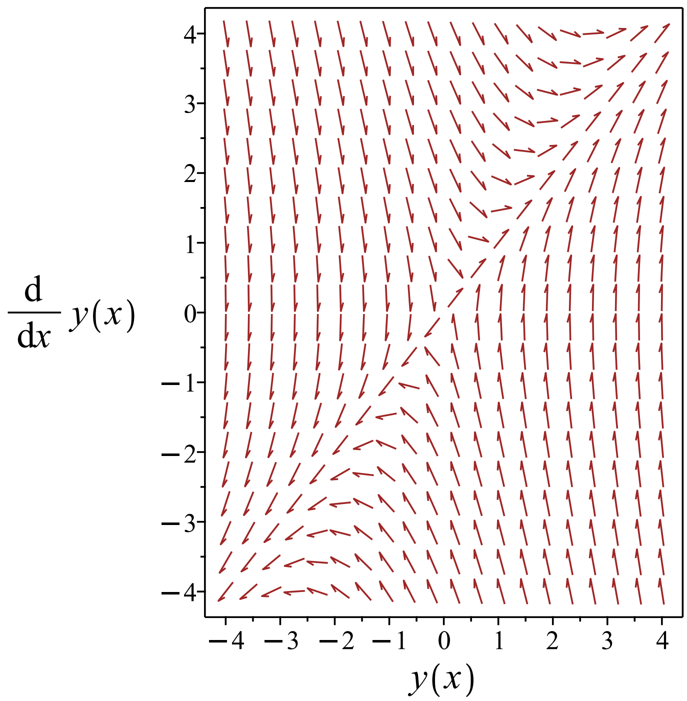


Figure 618: Slope field plot

Verification of solutions

$$y = c_1 e^x + e^{-3x} c_2 - \frac{1}{3} - \frac{x e^x}{16} + \frac{x^2 e^x}{8}$$

Verified OK.

11.9.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 2y' - 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 2 \\
C &= -3
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A} \, dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{4}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= 4 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = 4z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x) \, e^{-\int \frac{B}{2A} \, dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 463: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \mathrm{e}^{-2x}
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dx} \\
&= z_1 e^{-x} \\
&= z_1\left(\mathrm{e}^{-x}\right)
\end{aligned}
$$

Which simplifies to

$$y_1 = \mathrm{e}^{-3x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-2x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-3x}\right) + c_2 \left(\mathrm{e}^{-3x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 2y' - 3y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-3x} + \frac{c_2 \mathrm{e}^{x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = e^{-3x}$$
$$y_2 = \frac{e^{x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} e^{-3x} & \frac{e^{x}}{4} \\ \frac{d}{dx}\left(e^{-3x}\right) & \frac{d}{dx}\left(\frac{e^{x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-3x} & \frac{e^{x}}{4} \\ -3\,e^{-3x} & \frac{e^{x}}{4} \end{vmatrix}$$

Therefore

$$W = \left(e^{-3x}\right)\left(\frac{e^{x}}{4}\right) - \left(\frac{e^{x}}{4}\right)\left(-3\,e^{-3x}\right)$$

Which simplifies to

$$W = e^{-3x} e^{x}$$

Which simplifies to

$$W = \mathrm{e}^{-2x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{\mathrm{e}^x(1+x\,\mathrm{e}^x)}{4}}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{(1+x\,\mathrm{e}^x)\,\mathrm{e}^{3x}}{4}dx$$

Hence

$$u_1 = -\frac{\mathrm{e}^{3x}}{12} - \frac{x\,\mathrm{e}^{4x}}{16} + \frac{\mathrm{e}^{4x}}{64}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\mathrm{e}^{-3x}(1+x\,\mathrm{e}^x)}{\mathrm{e}^{-2x}}\,dx$$

Which simplifies to

$$u_2 = \int \left(\mathrm{e}^{-x} + x\right) dx$$

Hence

$$u_2 = \frac{x^2}{2} - \mathrm{e}^{-x}$$

Which simplifies to

$$u_1 = \frac{(1-4x)\,\mathrm{e}^{4x}}{64} - \frac{\mathrm{e}^{3x}}{12}$$

$$u_2 = \frac{x^2}{2} - \mathrm{e}^{-x}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{(1-4x)\,\mathrm{e}^{4x}}{64} - \frac{\mathrm{e}^{3x}}{12}\right)\mathrm{e}^{-3x} + \frac{\left(\frac{x^2}{2} - \mathrm{e}^{-x}\right)\mathrm{e}^{x}}{4}$$

Which simplifies to

$$y_p(x) = -\frac{1}{3} + \frac{(8x^2 - 4x + 1)\,e^x}{64}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-3x} + \frac{c_2 e^x}{4}\right) + \left(-\frac{1}{3} + \frac{(8x^2 - 4x + 1)\,e^x}{64}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{-3x} + \frac{c_2 e^x}{4} - \frac{1}{3} + \frac{(8x^2 - 4x + 1)\,e^x}{64} \tag{1}$$

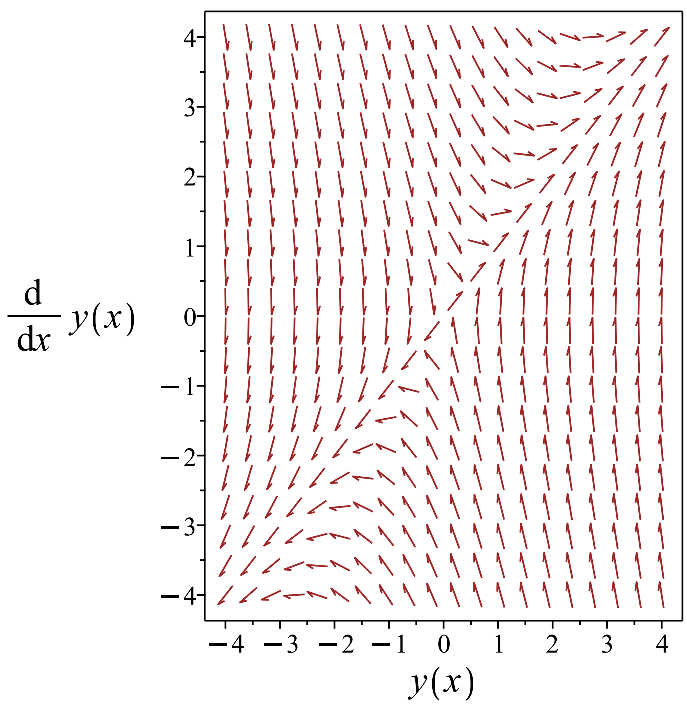


Figure 619: Slope field plot

Verification of solutions

$$y = c_1 e^{-3x} + \frac{c_2 e^x}{4} - \frac{1}{3} + \frac{(8x^2 - 4x + 1)\,e^x}{64}$$

Verified OK.

11.9.3 Maple step by step solution

Let's solve

$y'' + 2y' - 3y = 1 + x\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 2r - 3 = 0$

- Factor the characteristic polynomial

  $(r+3)(r-1) = 0$

- Roots of the characteristic polynomial

  $r = (-3, 1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-3x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-3x} + c_2 \mathrm{e}^{x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 1 + x\,\mathrm{e}^{x}\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-3x} & \mathrm{e}^{x} \\ -3\,\mathrm{e}^{-3x} & \mathrm{e}^{x} \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 4\,\mathrm{e}^{-2x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{\left(\mathrm{e}^{4x} \left(\int \left(\mathrm{e}^{-x}+x\right)dx\right)-\left(\int \left(1+x\,\mathrm{e}^{x}\right)\mathrm{e}^{3x}dx\right)\right)\mathrm{e}^{-3x}}{4}$$

- ◦ Compute integrals

$$y_p(x) = -\frac{1}{3} + \frac{(8x^2-4x+1)\,\mathrm{e}^{x}}{64}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\mathrm{e}^{-3x} + c_2\mathrm{e}^{x} - \frac{1}{3} + \frac{(8x^2-4x+1)\,\mathrm{e}^{x}}{64}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+2*diff(y(x),x)-3*y(x)=1+x*exp(x),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{-3x}\left(\left(x^2-\frac{1}{2}x+8c_2+\frac{1}{8}\right)\mathrm{e}^{4x}+8c_1-\frac{8\,\mathrm{e}^{3x}}{3}\right)}{8}$$

✓ Solution by Mathematica

Time used: 0.092 (sec). Leaf size: 38

```
DSolve[y''[x]+2*y'[x]-3*y[x]==1+x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{64}e^x\left(8x^2-4x+1+64c_2\right)+c_1e^{-3x}-\frac{1}{3}$$

11.10 problem 10

Internal problem ID [228]
Internal file name [`OUTPUT/228_Sunday_June_05_2022_01_37_02_AM_22619764/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 2\cos(3x) + 3\sin(3x)}$$

11.10.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 2\cos(3x) + 3\sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(3x) + 3\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(3x), x\sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(3x) + A_2 x\sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1\sin(3x) + 6A_2\cos(3x) = 2\cos(3x) + 3\sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(3x)}{2} + \frac{x\sin(3x)}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(-\frac{x\cos(3x)}{2} + \frac{x\sin(3x)}{3}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{x\cos(3x)}{2} + \frac{x\sin(3x)}{3} \tag{1}
$$

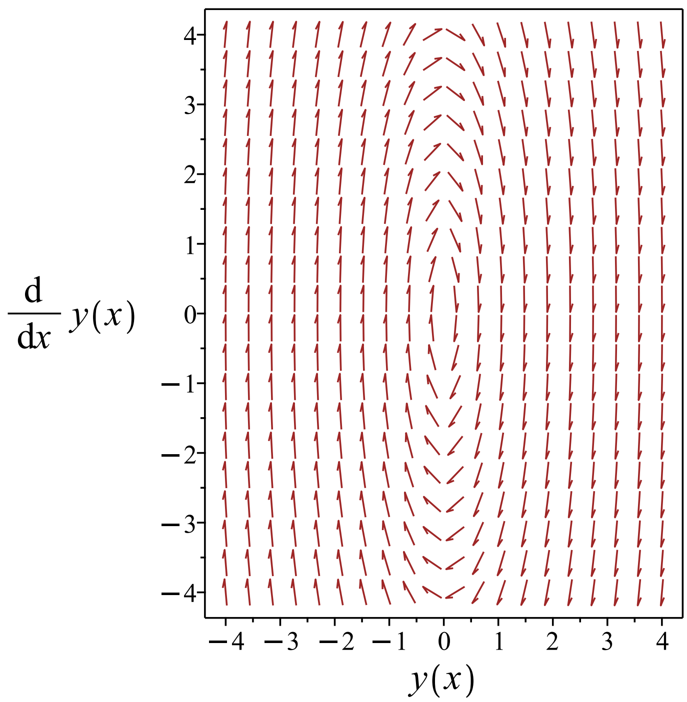


Figure 620: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{x\cos(3x)}{2} + \frac{x\sin(3x)}{3}
$$

Verified OK.

11.10.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 465: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\
&= \cos(3x) \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\cos(3x) + 3\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(3x), x\sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(3x) + A_2 x \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3x) + 6A_2 \cos(3x) = 2\cos(3x) + 3\sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = \frac{1}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(3x)}{2} + \frac{x\sin(3x)}{3}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(-\frac{x \cos(3x)}{2} + \frac{x \sin(3x)}{3}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - \frac{x \cos(3x)}{2} + \frac{x \sin(3x)}{3} \tag{1}
$$

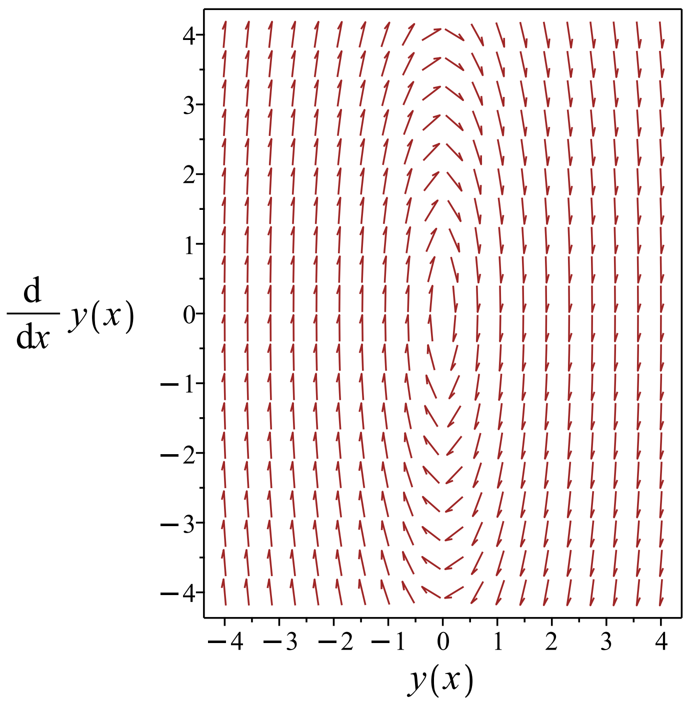


Figure 621: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - \frac{x \cos(3x)}{2} + \frac{x \sin(3x)}{3}
$$

Verified OK.

11.10.3 Maple step by step solution

Let's solve

$y'' + 9y = 2\cos(3x) + 3\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\cos(3x) + 3\sin(3x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(3x)\left(\int (3-3\cos(6x)+2\sin(6x))dx\right)}{6} + \frac{\sin(3x)\left(\int (3\sin(6x)+2+2\cos(6x))dx\right)}{6}$$

- Compute integrals

$$y_p(x) = \frac{(-9x+1)\cos(3x)}{18} + \frac{(4x+1)\sin(3x)}{12}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{(-9x+1)\cos(3x)}{18} + \frac{(4x+1)\sin(3x)}{12}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+9*y(x)=2*cos(3*x)+3*sin(3*x),y(x), singsol=all)
```

$$y(x) = \frac{(18c_1 - 9x + 2)\cos(3x)}{18} + \frac{\sin(3x)(x + 3c_2)}{3}$$

✓ Solution by Mathematica

Time used: 0.231 (sec). Leaf size: 39

```
DSolve[y''[x]+9*y[x]==2*Cos[3*x]+3*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x}{2} + \frac{1}{9} + c_1\right)\cos(3x) + \frac{1}{12}(4x + 1 + 12c_2)\sin(3x)$$

11.11 problem 16

Internal problem ID [229]
Internal file name [`OUTPUT/229_Sunday_June_05_2022_01_37_03_AM_65712223/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 16.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 2x^2 e^{3x} + 5}$$

11.11.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 2x^2 e^{3x} + 5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x^2 e^{3x} + 5$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{x\, e^{3x}, x^2 e^{3x}, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 x\, e^{3x} + A_3 x^2 e^{3x} + A_4 e^{3x}$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_2 e^{3x} + 18A_2 x\, e^{3x} + 2A_3 e^{3x} + 12A_3 x\, e^{3x} + 18A_3 x^2 e^{3x} + 18A_4 e^{3x} + 9A_1 = 2x^2 e^{3x} + 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{9}, A_2 = -\frac{2}{27}, A_3 = \frac{1}{9}, A_4 = \frac{1}{81}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5}{9} - \frac{2x\, e^{3x}}{27} + \frac{x^2 e^{3x}}{9} + \frac{e^{3x}}{81}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{5}{9} - \frac{2x\, e^{3x}}{27} + \frac{x^2 e^{3x}}{9} + \frac{e^{3x}}{81}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{5}{9} - \frac{2x\,e^{3x}}{27} + \frac{x^2 e^{3x}}{9} + \frac{e^{3x}}{81} \tag{1}$$

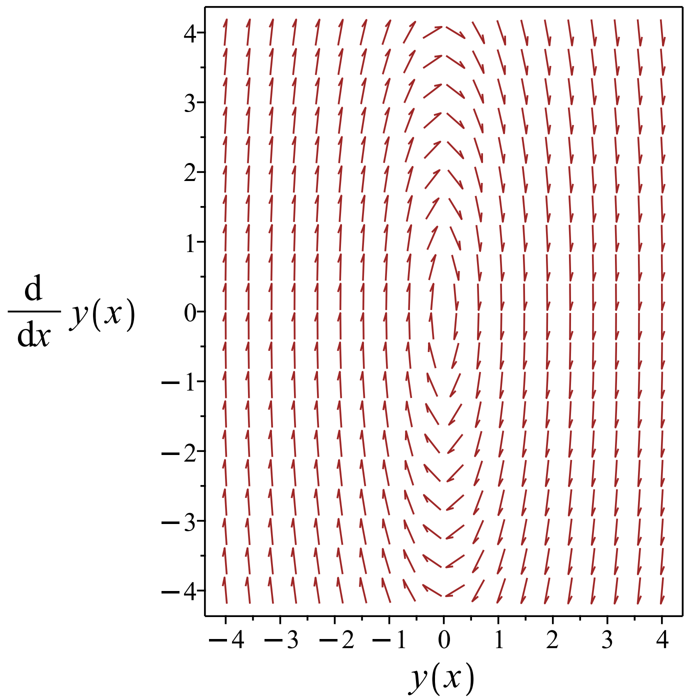


Figure 622: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{5}{9} - \frac{2x\,e^{3x}}{27} + \frac{x^2 e^{3x}}{9} + \frac{e^{3x}}{81}$$

Verified OK.

11.11.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 467: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\
&= \cos(3x) \left(\frac{\tan(3x)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2x^2 e^{3x} + 5$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{x\, e^{3x}, x^2 e^{3x}, e^{3x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 x\, e^{3x} + A_3 x^2 e^{3x} + A_4 e^{3x}$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_2 e^{3x} + 18A_2 x\, e^{3x} + 2A_3 e^{3x} + 12A_3 x\, e^{3x} + 18A_3 x^2 e^{3x} + 18A_4 e^{3x} + 9A_1 = 2x^2 e^{3x} + 5$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{5}{9}, A_2 = -\frac{2}{27}, A_3 = \frac{1}{9}, A_4 = \frac{1}{81}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{5}{9} - \frac{2x\, e^{3x}}{27} + \frac{x^2 e^{3x}}{9} + \frac{e^{3x}}{81}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{5}{9} - \frac{2x\, e^{3x}}{27} + \frac{x^2 e^{3x}}{9} + \frac{e^{3x}}{81}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{5}{9} - \frac{2x\,\mathrm{e}^{3x}}{27} + \frac{x^2\mathrm{e}^{3x}}{9} + \frac{\mathrm{e}^{3x}}{81} \tag{1}$$

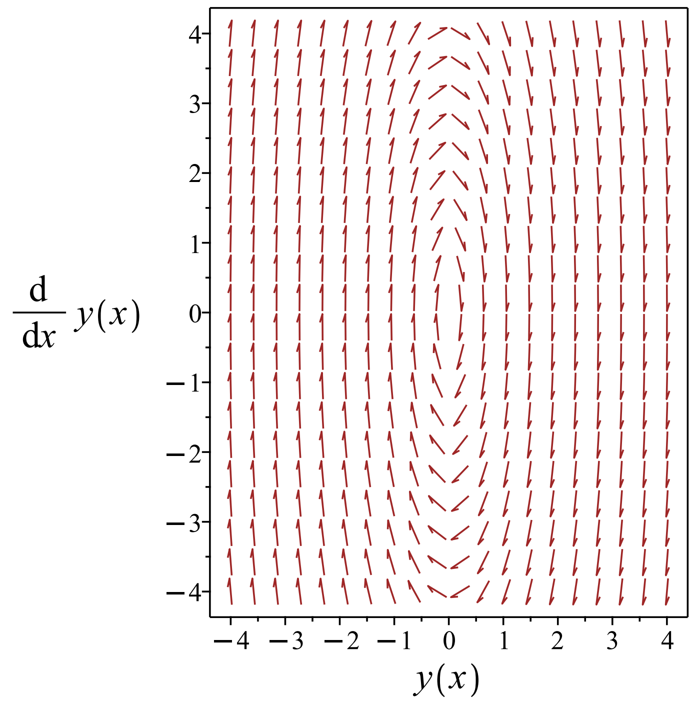


Figure 623: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{5}{9} - \frac{2x\,\mathrm{e}^{3x}}{27} + \frac{x^2\mathrm{e}^{3x}}{9} + \frac{\mathrm{e}^{3x}}{81}$$

Verified OK.

11.11.3 Maple step by step solution

Let's solve

$y'' + 9y = 2x^2\mathrm{e}^{3x} + 5$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2x^2 \mathrm{e}^{3x} + 5\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(3x)\left(\int (2x^2 \mathrm{e}^{3x} \sin(3x) + 5\sin(3x)) dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x)(2x^2 \mathrm{e}^{3x} + 5) dx\right)}{3}$

  - Compute integrals

    $y_p(x) = \frac{5}{9} + \frac{\left(x - \frac{1}{3}\right)^2 \mathrm{e}^{3x}}{9}$

- Substitute particular solution into general solution to ODE

  $y = \frac{\left(x - \frac{1}{3}\right)^2 \mathrm{e}^{3x}}{9} + c_1 \cos(3x) + c_2 \sin(3x) + \frac{5}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 29

```
dsolve(diff(y(x),x$2)+9*y(x)=2*x^2*exp(3*x)+5,y(x), singsol=all)
```

$$y(x) = \sin(3x)\, c_2 + \cos(3x)\, c_1 + \frac{5}{9} + \frac{\left(x - \frac{1}{3}\right)^2 e^{3x}}{9}$$

✓ Solution by Mathematica

Time used: 0.251 (sec). Leaf size: 50

```
DSolve[y''[x]+9*y[x]==2*x^2*Exp[3*x]+5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{81}\left(9e^{3x}x^2 - 6e^{3x}x + e^{3x} + 81c_1\cos(3x) + 81c_2\sin(3x) + 45\right)$$

11.12 problem 21

Internal problem ID [230]
Internal file name [`OUTPUT/230_Sunday_June_05_2022_01_37_04_AM_46500331/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 21.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 2y' + 2y = \sin(x)\,\mathrm{e}^{x}}$$

11.12.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = \sin(x)\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 2\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-2^2 - (4)(1)(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,e^{x}, \sin(x)\,e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,e^{x}, \sin(x)\,e^{x}\}$$

Since $\cos(x)\,e^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x)\,e^{x}, x\sin(x)\,e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(x)\,e^{x} + A_2 x\sin(x)\,e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x)\,e^{x} + 2A_2\cos(x)\,e^{x} = \sin(x)\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(x)\,e^{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{x}(c_1\cos(x) + c_2\sin(x))) + \left(-\frac{x\cos(x)\,e^{x}}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x \cos(x) e^{x}}{2} \tag{1}$$

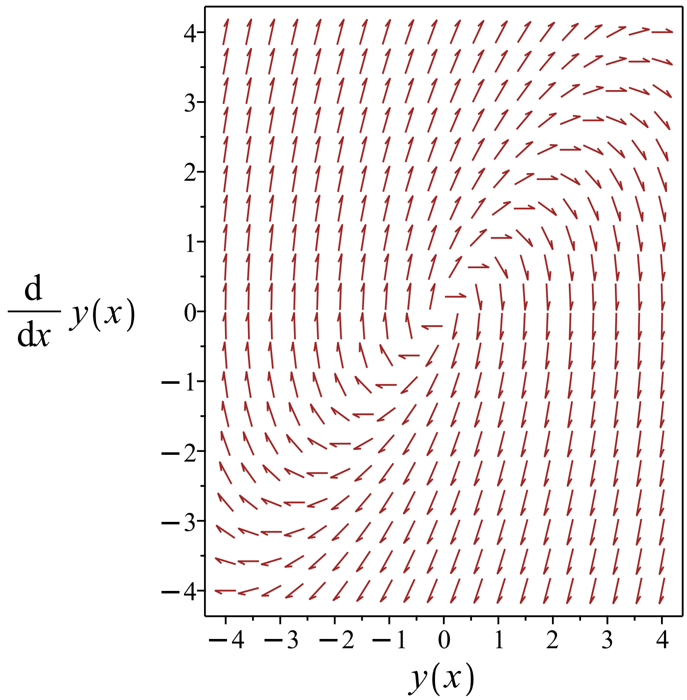


Figure 624: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x \cos(x) e^{x}}{2}$$

Verified OK.

11.12.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 469: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\
&= z_1 e^{x} \\
&= z_1(\mathrm{e}^{x})
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)\,\mathrm{e}^{x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\ &= y_1(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)\,\mathrm{e}^{x}) + c_2(\cos(x)\,\mathrm{e}^{x}(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \mathrm{e}^{x}\cos(x)\,c_1 + \sin(x)\,\mathrm{e}^{x}c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x)\,e^{x}, \sin(x)\,e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,e^{x}, \sin(x)\,e^{x}\}$$

Since $\cos(x)\,e^{x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x)\,e^{x}, x\sin(x)\,e^{x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(x)\,e^{x} + A_2 x\sin(x)\,e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x)\,e^{x} + 2A_2\cos(x)\,e^{x} = \sin(x)\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(x)\,e^{x}}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{x}\cos(x)\,c_1 + \sin(x)\,e^{x}c_2) + \left(-\frac{x\cos(x)\,e^{x}}{2}\right) \end{aligned}$$

Which simplifies to

$$y = e^{x}(c_1\cos(x) + c_2\sin(x)) - \frac{x\cos(x)\,e^{x}}{2}$$

Summary

The solution(s) found are the following

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x \cos(x) e^{x}}{2} \tag{1}$$

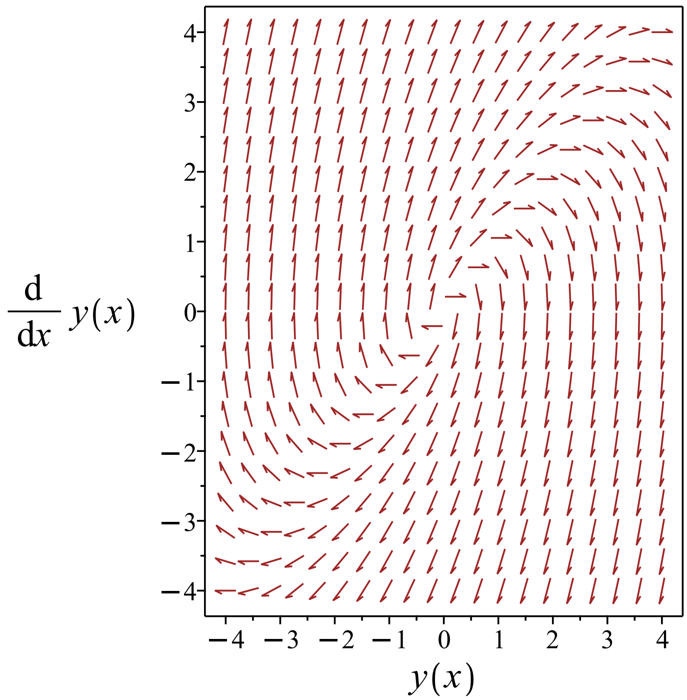


Figure 625: Slope field plot

Verification of solutions

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) - \frac{x \cos(x) e^{x}}{2}$$

Verified OK.

11.12.3 Maple step by step solution

Let's solve

$y'' - 2y' + 2y = \sin(x) e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)\, \mathrm{e}^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\, \mathrm{e}^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = \mathrm{e}^x \cos(x)\, c_1 + \sin(x)\, \mathrm{e}^x c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)\, \mathrm{e}^x\right]$

  ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\, \mathrm{e}^x & \sin(x)\, \mathrm{e}^x \\ -\sin(x)\, \mathrm{e}^x + \cos(x)\, \mathrm{e}^x & \cos(x)\, \mathrm{e}^x + \sin(x)\, \mathrm{e}^x \end{bmatrix}$

  ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = \mathrm{e}^{2x}$

  ○ Substitute functions into equation for $y_p(x)$

  $y_p(x) = \frac{\mathrm{e}^x \left(\sin(x) \left(\int \sin(2x) dx\right) - 2\cos(x) \left(\int \sin(x)^2 dx\right)\right)}{2}$

  ○ Compute integrals

  $y_p(x) = \frac{(\sin(x) - 2x\cos(x))\mathrm{e}^x}{4}$

- Substitute particular solution into general solution to ODE

  $y = \mathrm{e}^x \cos(x)\, c_1 + \sin(x)\, \mathrm{e}^x c_2 + \frac{(\sin(x) - 2x\cos(x))\mathrm{e}^x}{4}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 25

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=exp(x)*sin(x),y(x), singsol=all)
```

$$y(x) = -\frac{((x - 2c_1)\cos(x) + (-2c_2 - 1)\sin(x))\,e^x}{2}$$

✓ Solution by Mathematica

Time used: 0.046 (sec). Leaf size: 28

```
DSolve[y''[x]-2*y'[x]+2*y[x]==Exp[x]*Sin[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{2}e^x((x - 2c_2)\cos(x) - 2c_1\sin(x))$$

11.13 problem 23

Internal problem ID [231]
Internal file name [`OUTPUT/231_Sunday_June_05_2022_01_37_05_AM_26728427/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 23.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = 3x\cos(2x)}$$

11.13.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 3x\cos(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\
&= \pm 2i
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= +2i \\
\lambda_2 &= -2i
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= 2i \\
\lambda_2 &= -2i
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(2x), x \sin(2x), x^2 \cos(2x), x^2 \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x) + A_3 x^2 \cos(2x) + A_4 x^2 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-4A_1 \sin(2x) + 4A_2 \cos(2x) + 2A_3 \cos(2x) - 8A_3 x \sin(2x)\\&+ 2A_4 \sin(2x) + 8A_4 x \cos(2x) = 3x \cos(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{16}, A_2 = 0, A_3 = 0, A_4 = \frac{3}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x \cos(2x)}{16} + \frac{3x^2 \sin(2x)}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{3x \cos(2x)}{16} + \frac{3x^2 \sin(2x)}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{3x \cos(2x)}{16} + \frac{3x^2 \sin(2x)}{8} \tag{1}
$$

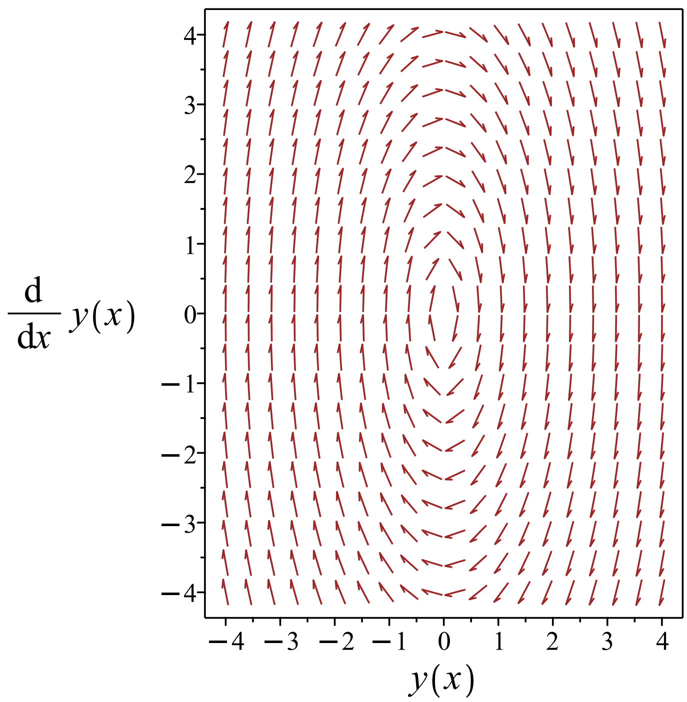


Figure 626: Slope field plot

Verification of solutions

$$
y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{3x \cos(2x)}{16} + \frac{3x^2 \sin(2x)}{8}
$$

Verified OK.

11.13.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 471: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned}y_1 &= z_1\\&= \cos(2x)\end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned}y_2 &= y_1 \int \frac{1}{y_1^2}\,dx\\&= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx\\&= \cos(2x)\left(\frac{\tan(2x)}{2}\right)\end{aligned}$$

Therefore the solution is

$$\begin{aligned}y &= c_1 y_1 + c_2 y_2\\&= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right)\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3x \cos(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(2x), x \sin(2x), \cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(2x), x \sin(2x), x^2 \cos(2x), x^2 \sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(2x) + A_2 x \sin(2x) + A_3 x^2 \cos(2x) + A_4 x^2 \sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-4A_1 \sin(2x) + 4A_2 \cos(2x) + 2A_3 \cos(2x) - 8A_3 x \sin(2x)\\ &+ 2A_4 \sin(2x) + 8A_4 x \cos(2x) = 3x \cos(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{16}, A_2 = 0, A_3 = 0, A_4 = \frac{3}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x\cos(2x)}{16} + \frac{3x^2\sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(2x) + \frac{c_2\sin(2x)}{2}\right) + \left(\frac{3x\cos(2x)}{16} + \frac{3x^2\sin(2x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{3x\cos(2x)}{16} + \frac{3x^2\sin(2x)}{8} \tag{1}$$

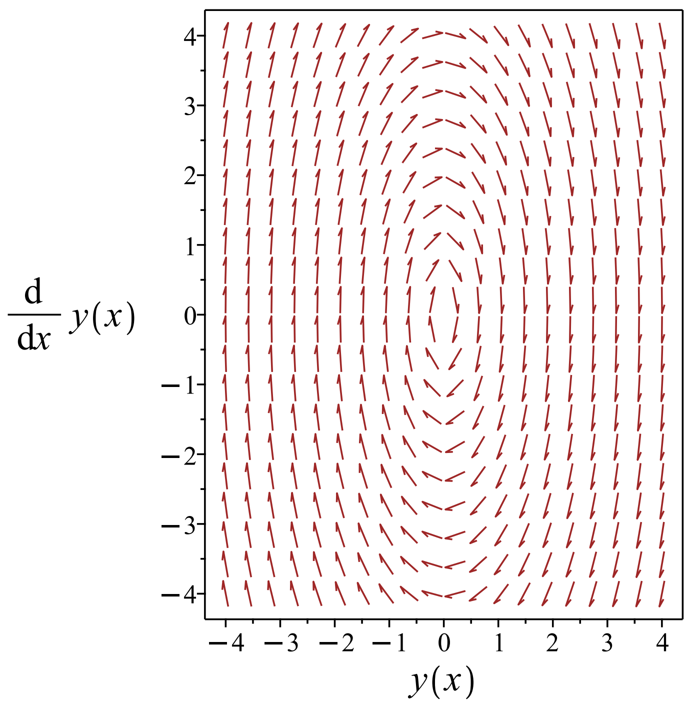


Figure 627: Slope field plot

Verification of solutions

$$y = c_1\cos(2x) + \frac{c_2\sin(2x)}{2} + \frac{3x\cos(2x)}{16} + \frac{3x^2\sin(2x)}{8}$$

Verified OK.

11.13.3 Maple step by step solution

Let's solve

$y'' + 4y = 3x\cos(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 3x\cos(2x)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

- ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{3\cos(2x)\left(\int \sin(4x)x dx\right)}{4} + \frac{3\sin(2x)\left(\int \cos(2x)^2 x dx\right)}{2}$$

- Compute integrals

$$y_p(x) = \frac{3x^2 \sin(2x)}{8} - \frac{3\sin(2x)}{32} + \frac{3x\cos(2x)}{16}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{3x^2 \sin(2x)}{8} - \frac{3\sin(2x)}{32} + \frac{3x\cos(2x)}{16}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+4*y(x)=3*x*cos(2*x),y(x), singsol=all)
```

$$y(x) = \frac{(24x^2 + 64c_2 - 3)\sin(2x)}{64} + \frac{3\left(x + \frac{16c_1}{3}\right)\cos(2x)}{16}$$

✓ Solution by Mathematica

Time used: 0.124 (sec). Leaf size: 38

```
DSolve[y''[x]+4*y[x]==3*x*Cos[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{64}(24x^2 - 3 + 64c_2)\sin(2x) + \left(\frac{3x}{16} + c_1\right)\cos(2x)$$

11.14 problem 25

Internal problem ID [232]
Internal file name [`OUTPUT/232_Sunday_June_05_2022_01_37_06_AM_91972790/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 25.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 3y' + 2y = x\left(\mathrm{e}^{-x} - \mathrm{e}^{-2x}\right)}$$

11.14.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = -x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 3\lambda\,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-x\,\mathrm{e}^{-x}\left(\mathrm{e}^{-x}-1\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x},\mathrm{e}^{-x}\},\{\mathrm{e}^{-2x}x,\mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x},\mathrm{e}^{-x}\}$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x},x^2\mathrm{e}^{-x}\},\{\mathrm{e}^{-2x}x,\mathrm{e}^{-2x}\}]$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x},x^2\mathrm{e}^{-x}\},\{x^2\mathrm{e}^{-2x},\mathrm{e}^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1x\,\mathrm{e}^{-x}+A_2x^2\mathrm{e}^{-x}+A_3x^2\mathrm{e}^{-2x}+A_4\mathrm{e}^{-2x}x$$

The unknowns $\{A_1,A_2,A_3,A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\mathrm{e}^{-x}+2A_2\mathrm{e}^{-x}+2A_2x\,\mathrm{e}^{-x}+2A_3\mathrm{e}^{-2x}-2A_3x\,\mathrm{e}^{-2x}-A_4\mathrm{e}^{-2x}=-x\,\mathrm{e}^{-x}\left(\mathrm{e}^{-x}-1\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-1,A_2=\frac{1}{2},A_3=\frac{1}{2},A_4=1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p=-x\,\mathrm{e}^{-x}+\frac{x^2\mathrm{e}^{-x}}{2}+\frac{x^2\mathrm{e}^{-2x}}{2}+\mathrm{e}^{-2x}x$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}\right) + \left(-x\,\mathrm{e}^{-x} + \frac{x^2 \mathrm{e}^{-x}}{2} + \frac{x^2 \mathrm{e}^{-2x}}{2} + \mathrm{e}^{-2x} x\right)$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} - x\,\mathrm{e}^{-x} + \frac{x^2 \mathrm{e}^{-x}}{2} + \frac{x^2 \mathrm{e}^{-2x}}{2} + \mathrm{e}^{-2x} x \tag{1}$$

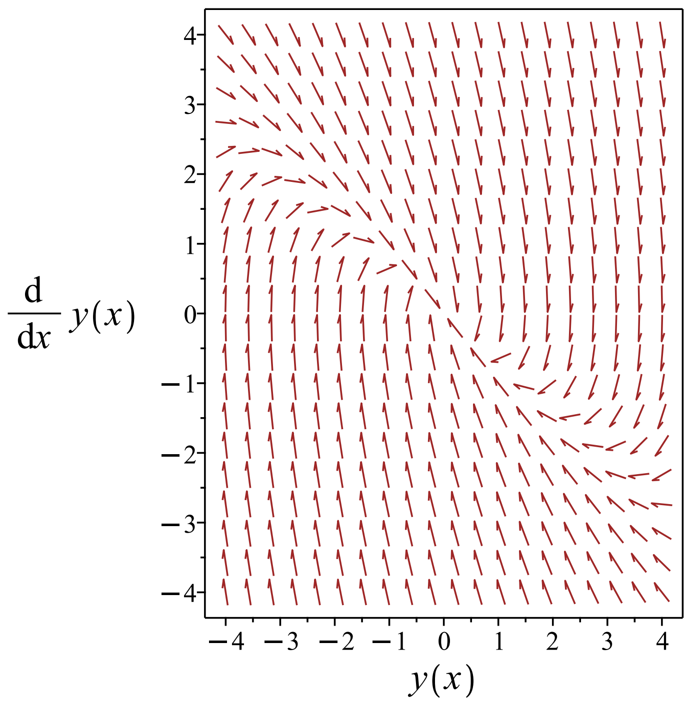


Figure 628: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} - x\,\mathrm{e}^{-x} + \frac{x^2 \mathrm{e}^{-x}}{2} + \frac{x^2 \mathrm{e}^{-2x}}{2} + \mathrm{e}^{-2x} x$$

Verified OK.

11.14.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 473: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^{-\frac{3x}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{-2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\
&= y_1(\mathrm{e}^{x})
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-x\,\mathrm{e}^{-x}\left(\mathrm{e}^{-x}-1\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}, \{\mathrm{e}^{-2x}x, \mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, \mathrm{e}^{-x}\}, \{x^2\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x\}]$$

Since e^{-x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,\mathrm{e}^{-x}, x^2\mathrm{e}^{-x}\}, \{x^2\mathrm{e}^{-2x}, \mathrm{e}^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\,\mathrm{e}^{-x} + A_2 x^2\mathrm{e}^{-x} + A_3 x^2\mathrm{e}^{-2x} + A_4\mathrm{e}^{-2x}x$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$A_1\mathrm{e}^{-x} + 2A_2\mathrm{e}^{-x} + 2A_2 x\,\mathrm{e}^{-x} + 2A_3\mathrm{e}^{-2x} - 2A_3 x\,\mathrm{e}^{-2x} - A_4\mathrm{e}^{-2x} = -x\,\mathrm{e}^{-x}\left(\mathrm{e}^{-x}-1\right)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -1, A_2 = \frac{1}{2}, A_3 = \frac{1}{2}, A_4 = 1\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -x\,\mathrm{e}^{-x} + \frac{x^2\mathrm{e}^{-x}}{2} + \frac{x^2\mathrm{e}^{-2x}}{2} + \mathrm{e}^{-2x}x$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}\right) + \left(-x\,\mathrm{e}^{-x} + \frac{x^2 \mathrm{e}^{-x}}{2} + \frac{x^2 \mathrm{e}^{-2x}}{2} + \mathrm{e}^{-2x} x\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} + \frac{x^2 \mathrm{e}^{-x}}{2} + \frac{x^2 \mathrm{e}^{-2x}}{2} + \mathrm{e}^{-2x} x \tag{1}
$$

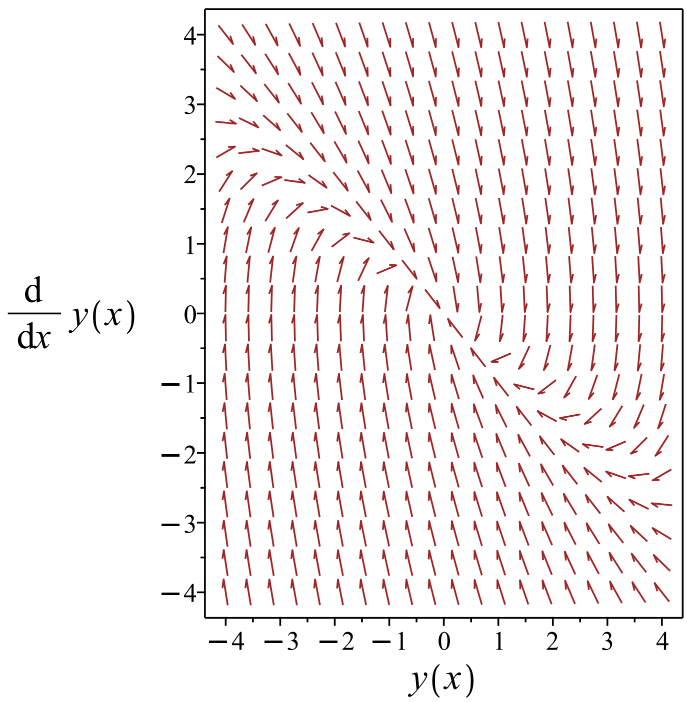


Figure 629: Slope field plot

<u>Verification of solutions</u>

$$
y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} - x\,\mathrm{e}^{-x} + \frac{x^2 \mathrm{e}^{-x}}{2} + \frac{x^2 \mathrm{e}^{-2x}}{2} + \mathrm{e}^{-2x} x
$$

Verified OK.

11.14.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = -x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))} dx\right), f(x) = -x\,\mathrm{e}^{-x}(\mathrm{e}^{-x} - 1)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = \mathrm{e}^{-3x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\mathrm{e}^{-2x}\left(\int x(\mathrm{e}^{x} - 1)\, dx\right) - \mathrm{e}^{-x}\left(\int (\mathrm{e}^{-x} - 1)\, x dx\right)$

- Compute integrals

  $y_p(x) = \frac{(x^2+2x+2)\,\mathrm{e}^{-2x}}{2} + \frac{\mathrm{e}^{-x}(x^2-2x+2)}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + \frac{(x^2+2x+2)\,\mathrm{e}^{-2x}}{2} + \frac{\mathrm{e}^{-x}(x^2-2x+2)}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 36

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=x*(exp(-x)-exp(-2*x)),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{-x}\left((x^2-2c_1+2x+2)\,\mathrm{e}^{-x}+x^2-2x+2c_2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.086 (sec). Leaf size: 42

```
DSolve[y''[x]+3*y'[x]+2*y[x]==x*(Exp[-x]-Exp[-2*x]),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}e^{-2x}\left(x^2+e^x\left(x^2-2x+2+2c_2\right)+2x+2+2c_1\right)$$

11.15 problem 26

Internal problem ID [233]
Internal file name [`OUTPUT/233_Sunday_June_05_2022_01_37_07_AM_82096344/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 26.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 6y' + 13y = x\,e^{3x} \sin(2x)}$$

11.15.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -6, C = 13, f(x) = x\,e^{3x} \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 6y' + 13y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -6, C = 13$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 6\lambda e^{\lambda x} + 13 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{-6^2 - (4)(1)(13)}\\ &= 3 \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = 3 + 2i$$
$$\lambda_2 = 3 - 2i$$

Which simplifies to

$$\lambda_1 = 3 + 2i$$
$$\lambda_2 = 3 - 2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

Therefore the homogeneous solution y_h is

$$y_h = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{3x}\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x}\cos(2x), \mathrm{e}^{3x}\sin(2x), x\cos(2x)\,\mathrm{e}^{3x}, x\,\mathrm{e}^{3x}\sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{3x}\cos(2x), \mathrm{e}^{3x}\sin(2x)\}$$

Since $\mathrm{e}^{3x}\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(2x)\,\mathrm{e}^{3x}, x\,\mathrm{e}^{3x}\sin(2x), x^2\cos(2x)\,\mathrm{e}^{3x}, x^2\mathrm{e}^{3x}\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(2x)\,\mathrm{e}^{3x} + A_2 x\,\mathrm{e}^{3x}\sin(2x) + A_3 x^2\cos(2x)\,\mathrm{e}^{3x} + A_4 x^2\mathrm{e}^{3x}\sin(2x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&8A_4 x\,\mathrm{e}^{3x}\cos(2x) + 2A_3\cos(2x)\,\mathrm{e}^{3x} - 8A_3 x\sin(2x)\,\mathrm{e}^{3x}\\&+ 2A_4\mathrm{e}^{3x}\sin(2x) + 4A_2\mathrm{e}^{3x}\cos(2x) - 4A_1\sin(2x)\,\mathrm{e}^{3x} = x\,\mathrm{e}^{3x}\sin(2x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}, A_3 = -\frac{1}{8}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{3x}\sin(2x)}{16} - \frac{x^2\cos(2x)\,\mathrm{e}^{3x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(e^{3x}(c_1 \cos(2x) + c_2 \sin(2x))\right) + \left(\frac{x\,e^{3x}\sin(2x)}{16} - \frac{x^2\cos(2x)\,e^{3x}}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) + \frac{x\,e^{3x}\sin(2x)}{16} - \frac{x^2\cos(2x)\,e^{3x}}{8} \tag{1}$$

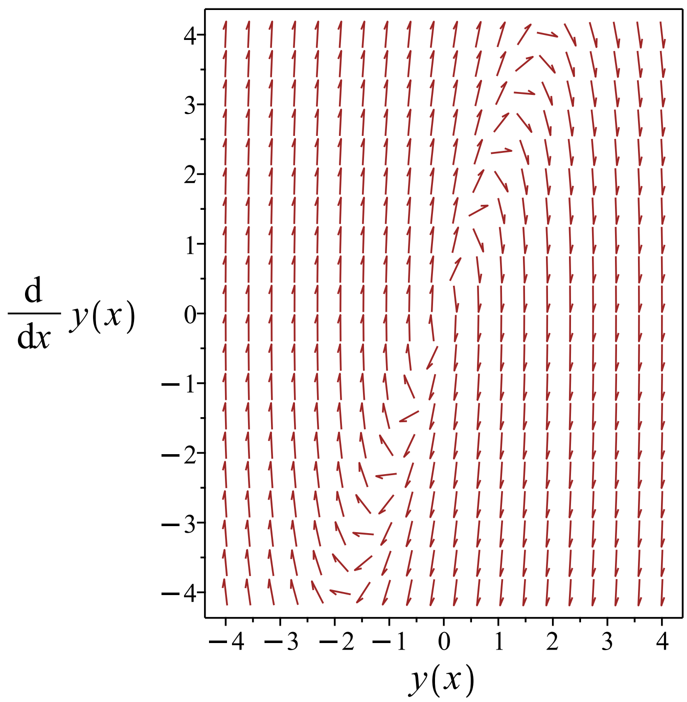


Figure 630: Slope field plot

Verification of solutions

$$y = e^{3x}(c_1 \cos(2x) + c_2 \sin(2x)) + \frac{x\,e^{3x}\sin(2x)}{16} - \frac{x^2\cos(2x)\,e^{3x}}{8}$$

Verified OK.

11.15.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 6y' + 13y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 475: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-6}{1}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{3x}\\
&= z_1 \left(\mathrm{e}^{3x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{3x} \cos(2x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-6}{1}\, dx}}{(y_1)^2}\, dx\\
&= y_1 \int \frac{e^{6x}}{(y_1)^2}\, dx\\
&= y_1 \left(\frac{\tan(2x)}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2\\
&= c_1 \left(\mathrm{e}^{3x} \cos(2x)\right) + c_2 \left(\mathrm{e}^{3x} \cos(2x) \left(\frac{\tan(2x)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 6y' + 13y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{3x} \cos(2x) + \frac{c_2 \mathrm{e}^{3x} \sin(2x)}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,\mathrm{e}^{3x} \sin\left(2x\right)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{3x} \cos\left(2x\right), \mathrm{e}^{3x} \sin\left(2x\right), x \cos\left(2x\right) \mathrm{e}^{3x}, x\,\mathrm{e}^{3x} \sin\left(2x\right)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{3x} \cos\left(2x\right), \frac{\mathrm{e}^{3x} \sin\left(2x\right)}{2}\right\}$$

Since $\mathrm{e}^{3x} \cos\left(2x\right)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos\left(2x\right) \mathrm{e}^{3x}, x\,\mathrm{e}^{3x} \sin\left(2x\right), x^{2} \cos\left(2x\right) \mathrm{e}^{3x}, x^{2} \mathrm{e}^{3x} \sin\left(2x\right)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos\left(2x\right) \mathrm{e}^{3x} + A_2 x\,\mathrm{e}^{3x} \sin\left(2x\right) + A_3 x^{2} \cos\left(2x\right) \mathrm{e}^{3x} + A_4 x^{2} \mathrm{e}^{3x} \sin\left(2x\right)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&2A_3 \cos\left(2x\right) \mathrm{e}^{3x} - 8A_3 x \sin\left(2x\right) \mathrm{e}^{3x} + 2A_4 \mathrm{e}^{3x} \sin\left(2x\right)\\ &\quad + 8A_4 x\,\mathrm{e}^{3x} \cos\left(2x\right) + 4A_2 \mathrm{e}^{3x} \cos\left(2x\right) - 4A_1 \sin\left(2x\right) \mathrm{e}^{3x} = x\,\mathrm{e}^{3x} \sin\left(2x\right)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{16}, A_3 = -\frac{1}{8}, A_4 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x\,\mathrm{e}^{3x} \sin\left(2x\right)}{16} - \frac{x^{2} \cos\left(2x\right) \mathrm{e}^{3x}}{8}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{3x} \cos(2x) + \frac{c_2 e^{3x} \sin(2x)}{2}\right) + \left(\frac{x\, e^{3x} \sin(2x)}{16} - \frac{x^2 \cos(2x)\, e^{3x}}{8}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 e^{3x} \cos(2x) + \frac{c_2 e^{3x} \sin(2x)}{2} + \frac{x\, e^{3x} \sin(2x)}{16} - \frac{x^2 \cos(2x)\, e^{3x}}{8} \tag{1}
$$

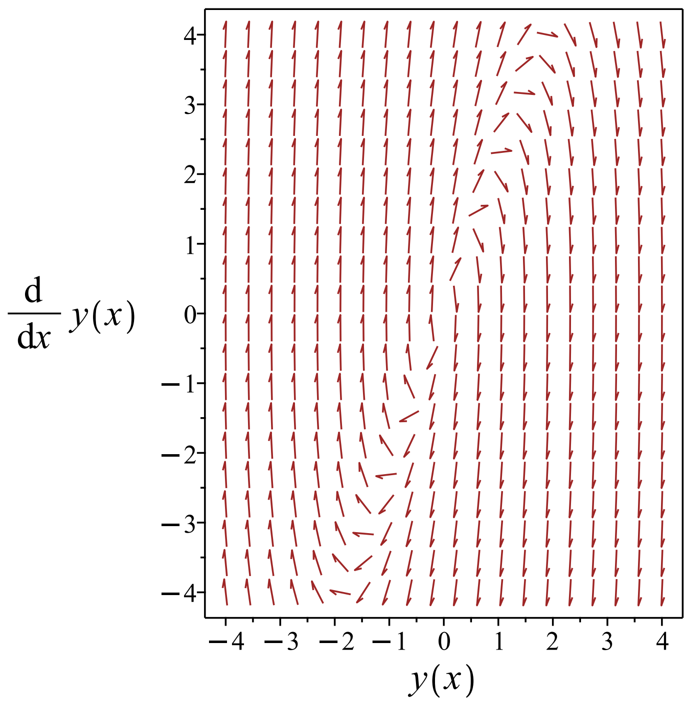


Figure 631: Slope field plot

Verification of solutions

$$
y = c_1 e^{3x} \cos(2x) + \frac{c_2 e^{3x} \sin(2x)}{2} + \frac{x\, e^{3x} \sin(2x)}{16} - \frac{x^2 \cos(2x)\, e^{3x}}{8}
$$

Verified OK.

11.15.3 Maple step by step solution

Let's solve

$y'' - 6y' + 13y = x\,\mathrm{e}^{3x} \sin(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{6 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (3 - 2\,\mathrm{I}, 3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{3x} \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{3x} \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{3x} \cos(2x) + c_2 \mathrm{e}^{3x} \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\,\mathrm{e}^{3x} \sin(2x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{3x} \cos(2x) & \mathrm{e}^{3x} \sin(2x) \\ 3\,\mathrm{e}^{3x} \cos(2x) - 2\,\mathrm{e}^{3x} \sin(2x) & 3\,\mathrm{e}^{3x} \sin(2x) + 2\,\mathrm{e}^{3x} \cos(2x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2\,\mathrm{e}^{6x}$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = \frac{e^{3x}\left(\sin(2x)\left(\int \sin(4x)x dx\right)-2\cos(2x)\left(\int x \sin(2x)^2 dx\right)\right)}{4}$$

- Compute integrals

$$y_p(x) = -\frac{\left(x\cos(2x)-\frac{\sin(2x)}{2}\right)x\,e^{3x}}{8}$$

- Substitute particular solution into general solution to ODE

$$y = c_1e^{3x}\cos(2x) + c_2e^{3x}\sin(2x) - \frac{\left(x\cos(2x)-\frac{\sin(2x)}{2}\right)x\,e^{3x}}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)-6*diff(y(x),x)+13*y(x)=x*exp(3*x)*sin(2*x),y(x), singsol=all)
```

$$y(x) = -\frac{\left((x^2-8c_1)\cos(2x) - \frac{\sin(2x)(x+16c_2)}{2}\right)e^{3x}}{8}$$

✓ Solution by Mathematica

Time used: 0.105 (sec). Leaf size: 43

```
DSolve[y''[x]-6*y'[x]+13*y[x]==x*Exp[3*x]*Sin[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{64}e^{3x}\left((-8x^2+1+64c_2)\cos(2x)+4(x+16c_1)\sin(2x)\right)$$

11.16 problem 31

Internal problem ID [234]
Internal file name [`OUTPUT/234_Sunday_June_05_2022_01_37_08_AM_19462801/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 31.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 4y = 2x}$$

With initial conditions

$$[y(0) = 1, y'(0) = 2]$$

11.16.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 4 \\ F &= 2x \end{aligned}$$

Hence the ode is

$$y'' + 4y = 2x$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 4$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = 2x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.16.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = 2x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_2 x + 4A_1 = 2x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{x}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \frac{1}{2}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = \frac{1}{2} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = \frac{3}{4}$$

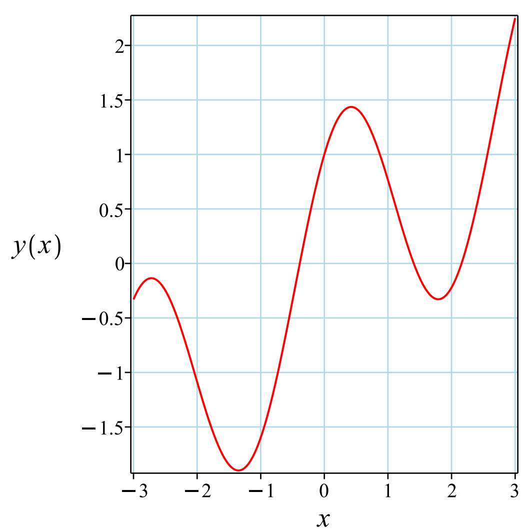
Substituting these values back in above solution results in

$$y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2}$$

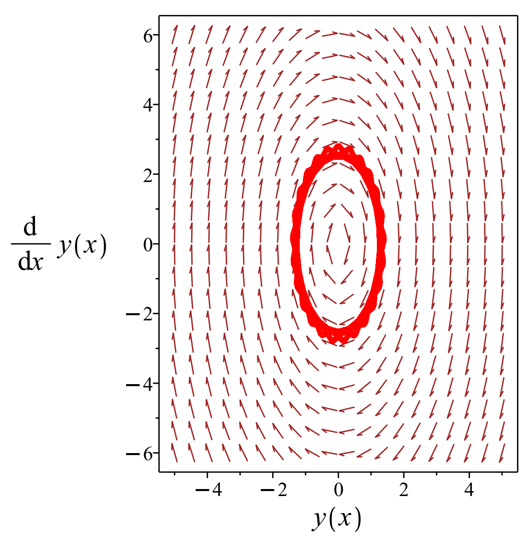
Summary

The solution(s) found are the following

$$y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2}$$

Verified OK.

11.16.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 477: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(2x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$4A_2 x + 4A_1 = 2x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{x}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{x}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 \sin(2x) + c_2 \cos(2x) + \frac{1}{2}$$

substituting $y' = 2$ and $x = 0$ in the above gives

$$2 = \frac{1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

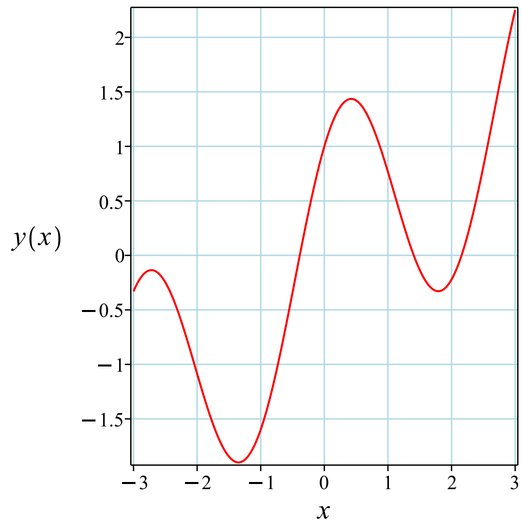
$$c_1 = 1$$
$$c_2 = \frac{3}{2}$$

Substituting these values back in above solution results in

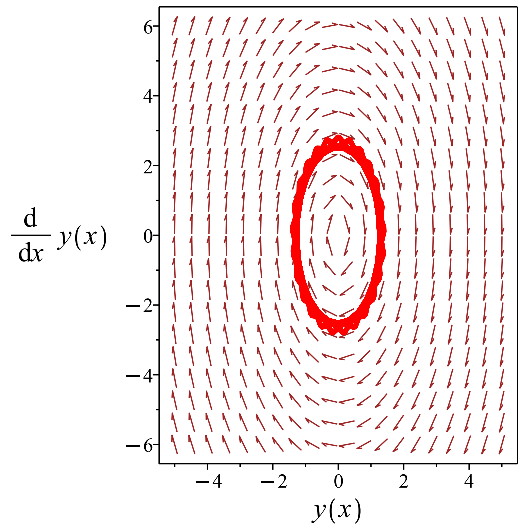
$$y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2}$$

Summary
The solution(s) found are the following

$$y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2}$$

Verified OK.

11.16.4 Maple step by step solution

Let's solve

$\left[y'' + 4y = 2x, y(0) = 1, y'\Big|_{\{x=0\}} = 2\right]$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

$y_2(x) = \sin(2x)$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2x\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 2$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -\cos(2x)\left(\int x\sin(2x)\,dx\right) + \sin(2x)\left(\int x\cos(2x)\,dx\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{x}{2}$

- Substitute particular solution into general solution to ODE

$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{2}$

□ Check validity of solution $y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{x}{2}$

  ◦ Use initial condition $y(0) = 1$

  $1 = c_1$

  ◦ Compute derivative of the solution

  $y' = -2c_1 \sin(2x) + 2c_2 \cos(2x) + \frac{1}{2}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 2$

  $2 = \frac{1}{2} + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = \frac{3}{4}\}$
- Substitute constant values into general solution and simplify

  $y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2}$

- Solution to the IVP

  $y = \cos(2x) + \frac{3\sin(2x)}{4} + \frac{x}{2}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 18

```
dsolve([diff(y(x),x$2)+4*y(x)=2*x,y(0) = 1, D(y)(0) = 2],y(x), singsol=all)
```

$$y(x) = \frac{3\sin(2x)}{4} + \cos(2x) + \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[{y''[x]+4*y[x]==2*x,{y[0]==1,y'[0]==2}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(2x) + \frac{1}{2}(x + 3\sin(x)\cos(x))$$

11.17 problem 32

Internal problem ID [235]
Internal file name [`OUTPUT/235_Sunday_June_05_2022_01_37_09_AM_2830899/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 32.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = e^{x}}$$

With initial conditions

$$[y(0) = 0, y'(0) = 3]$$

11.17.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 3 \\ q(x) &= 2 \\ F &= e^{x} \end{aligned}$$

Hence the ode is

$$y'' + 3y' + 2y = e^x$$

The domain of $p(x) = 3$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = e^x$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.17.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = e^x$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(2)}\\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2}\\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 e^x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 e^x = e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-x} + c_2 e^{-2x}\right) + \left(\frac{e^x}{6}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{e^x}{6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{1}{6} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 e^{-x} - 2c_2 e^{-2x} + \frac{e^x}{6}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = -c_1 - 2c_2 + \frac{1}{6} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= \frac{5}{2} \\ c_2 &= -\frac{8}{3} \end{aligned}$$

Substituting these values back in above solution results in

$$y = \frac{5\,e^{-x}}{2} - \frac{8\,e^{-2x}}{3} + \frac{e^x}{6}$$

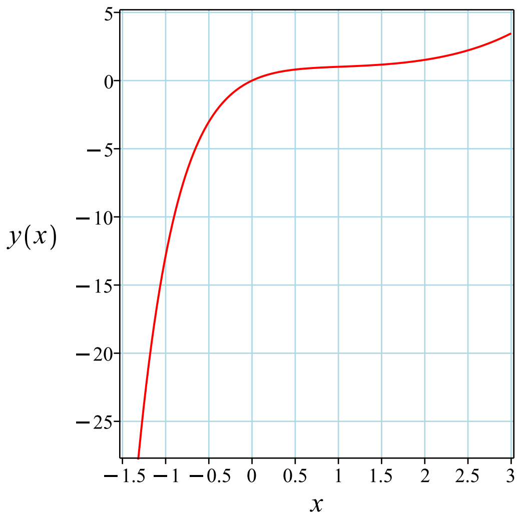
Which simplifies to

$$y = \frac{(e^{3x} + 15\,e^x - 16)\,e^{-2x}}{6}$$

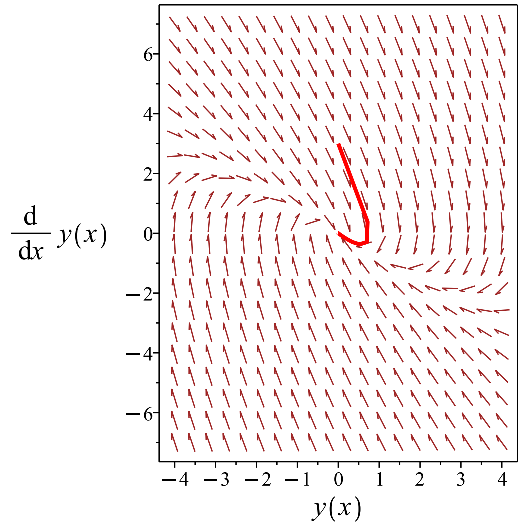
Summary

The solution(s) found are the following

$$y = \frac{(e^{3x} + 15\,e^x - 16)\,e^{-2x}}{6} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(e^{3x} + 15\,e^{x} - 16)\,e^{-2x}}{6}$$

Verified OK.

11.17.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 479: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\,dx}}{(y_1)^2}\,dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\,dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(e^{-2x}\right) + c_2 \left(e^{-2x}(e^x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 e^{-2x} + c_2 e^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
e^x
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{e^x\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{e^{-2x}, e^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 e^x
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
6A_1 e^x = e^x
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{6}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{e^x}{6}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + c_2 e^{-x}\right) + \left(\frac{e^x}{6}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 e^{-2x} + c_2 e^{-x} + \frac{e^x}{6} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 0$ and $x = 0$ in the above gives

$$0 = c_1 + c_2 + \frac{1}{6} \tag{1A}$$

Taking derivative of the solution gives

$$y' = -2c_1 e^{-2x} - c_2 e^{-x} + \frac{e^x}{6}$$

substituting $y' = 3$ and $x = 0$ in the above gives

$$3 = -2c_1 - c_2 + \frac{1}{6} \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -\frac{8}{3} \\ c_2 &= \frac{5}{2} \end{aligned}$$

Substituting these values back in above solution results in

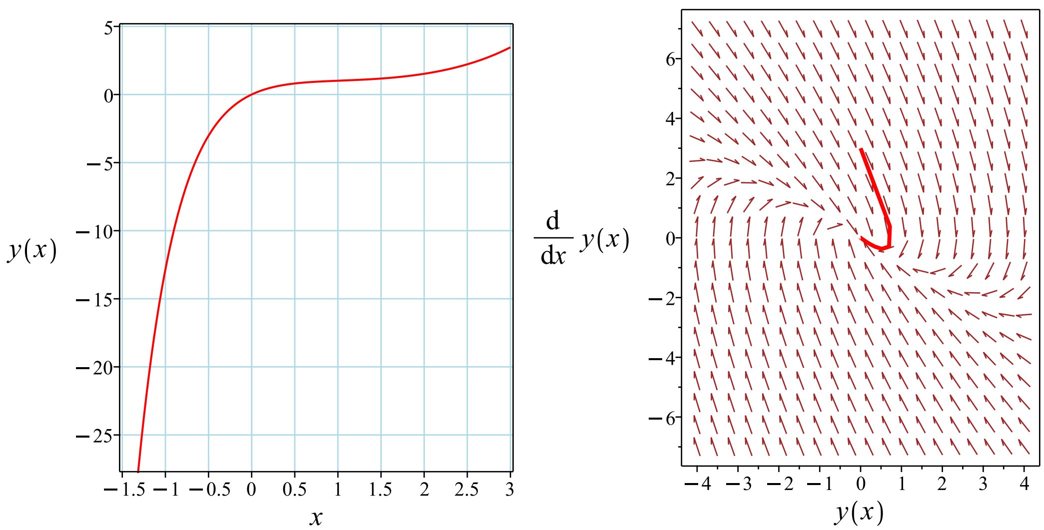
$$y = \frac{5\,e^{-x}}{2} - \frac{8\,e^{-2x}}{3} + \frac{e^{x}}{6}$$

Which simplifies to

$$y = \frac{(e^{3x} + 15\,e^{x} - 16)\,e^{-2x}}{6}$$

Summary
The solution(s) found are the following

$$y = \frac{(e^{3x} + 15\,e^{x} - 16)\,e^{-2x}}{6} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = \frac{(e^{3x} + 15\,e^{x} - 16)\,e^{-2x}}{6}$$

Verified OK.

11.17.4 Maple step by step solution

Let's solve

$$\left[y''+3y'+2y=e^{x}, y(0)=0, y'\Big|_{\{x=0\}}=3\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+3r+2=0$

- Factor the characteristic polynomial

  $(r+2)(r+1)=0$

- Roots of the characteristic polynomial

  $r=(-2,-1)$

- 1st solution of the homogeneous ODE

  $y_1(x)=e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x)=e^{-x}$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1e^{-2x}+c_2e^{-x}+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $$\left[y_p(x)=-y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=e^{x}\right]$$

  - Wronskian of solutions of the homogeneous equation

    $$W(y_1(x), y_2(x))=\begin{bmatrix} e^{-2x} & e^{-x} \\ -2\,e^{-2x} & -e^{-x} \end{bmatrix}$$

  - Compute Wronskian

    $W(y_1(x), y_2(x))=e^{-3x}$

  - Substitute functions into equation for $y_p(x)$

$y_p(x) = -\mathrm{e}^{-2x}\left(\int \mathrm{e}^{3x}dx\right) + \mathrm{e}^{-x}\left(\int \mathrm{e}^{2x}dx\right)$

- Compute integrals

  $y_p(x) = \frac{\mathrm{e}^x}{6}$

- Substitute particular solution into general solution to ODE

  $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + \frac{\mathrm{e}^x}{6}$

☐ Check validity of solution $y = c_1\mathrm{e}^{-2x} + c_2\mathrm{e}^{-x} + \frac{\mathrm{e}^x}{6}$

- Use initial condition $y(0) = 0$

  $0 = c_1 + c_2 + \frac{1}{6}$

- Compute derivative of the solution

  $y' = -2c_1\mathrm{e}^{-2x} - c_2\mathrm{e}^{-x} + \frac{\mathrm{e}^x}{6}$

- Use the initial condition $y'\Big|_{\{x=0\}} = 3$

  $3 = -2c_1 - c_2 + \frac{1}{6}$

- Solve for c_1 and c_2

  $\left\{c_1 = -\frac{8}{3}, c_2 = \frac{5}{2}\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(\mathrm{e}^{3x}+15\,\mathrm{e}^x-16)\,\mathrm{e}^{-2x}}{6}$

- Solution to the IVP

  $y = \frac{(\mathrm{e}^{3x}+15\,\mathrm{e}^x-16)\,\mathrm{e}^{-2x}}{6}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 20

```
dsolve([diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=exp(x),y(0) = 0, D(y)(0) = 3],y(x), singsol=all
```

$$y(x) = \frac{(e^{3x} + 15\,e^{x} - 16)\,e^{-2x}}{6}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 26

```
DSolve[{y''[x]+3*y'[x]+2*y[x]==Exp[x],{y[0]==0,y'[0]==3}},y[x],x,IncludeSingularSolutions ->
```

$$y(x) \to \frac{1}{6}e^{-2x}\left(15e^{x} + e^{3x} - 16\right)$$

11.18 problem 33

Internal problem ID [236]
Internal file name [`OUTPUT/236_Sunday_June_05_2022_01_37_11_AM_67843875/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 33.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \sin(2x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = 0]$$

11.18.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 9 \\ F &= \sin(2x) \end{aligned}$$

Hence the ode is

$$y'' + 9y = \sin(2x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 9$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \sin(2x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.18.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \sin(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2x) + 5A_2 \sin(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(2x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{\sin(2x)}{5}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\sin(2x)}{5} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + \frac{2\cos(2x)}{5}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = \frac{2}{5} + 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -\frac{2}{15}$$

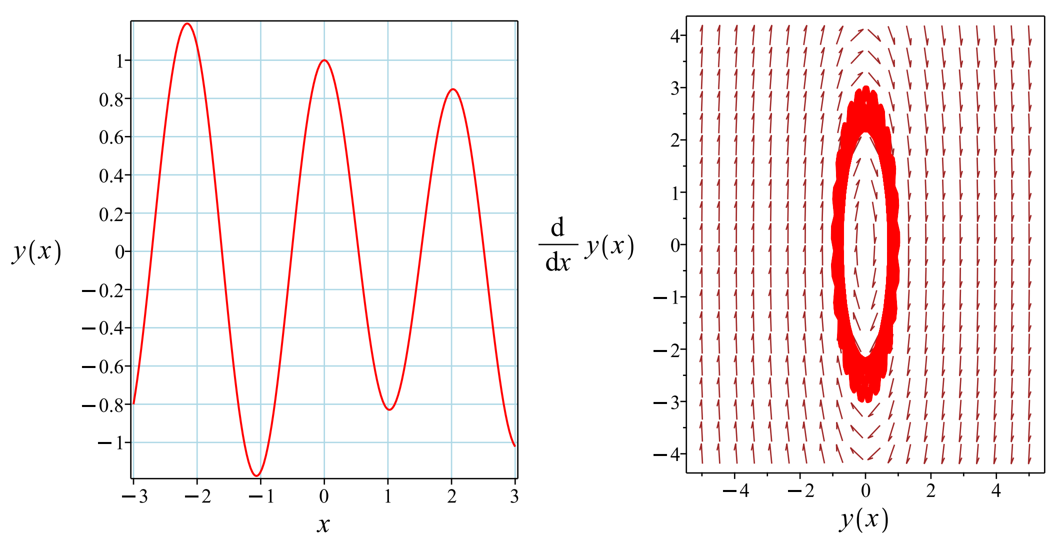
Substituting these values back in above solution results in

$$y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5}$$

Summary

The solution(s) found are the following

$$y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5}$$

Verified OK.

11.18.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 481: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x)\left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(2x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2x) + 5A_2 \sin(2x) = \sin(2x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{5}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(2x)}{5}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{\sin(2x)}{5}\right)
\end{aligned}
$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$
y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{\sin(2x)}{5} \tag{1}
$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$
1 = c_1 \tag{1A}
$$

Taking derivative of the solution gives

$$
y' = -3c_1 \sin(3x) + c_2 \cos(3x) + \frac{2\cos(2x)}{5}
$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$
0 = \frac{2}{5} + c_2 \tag{2A}
$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$
\begin{aligned}
c_1 &= 1 \\
c_2 &= -\frac{2}{5}
\end{aligned}
$$

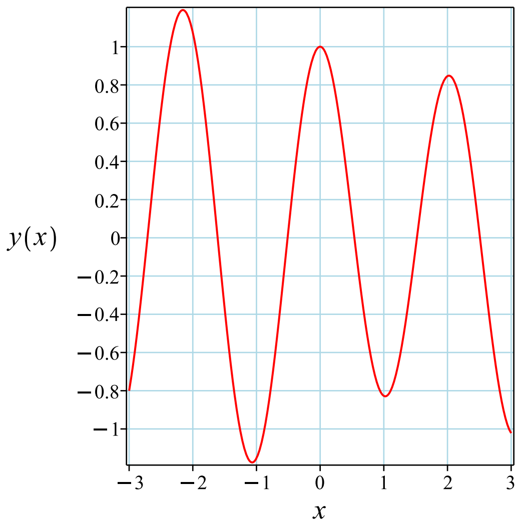
Substituting these values back in above solution results in

$$
y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5}
$$

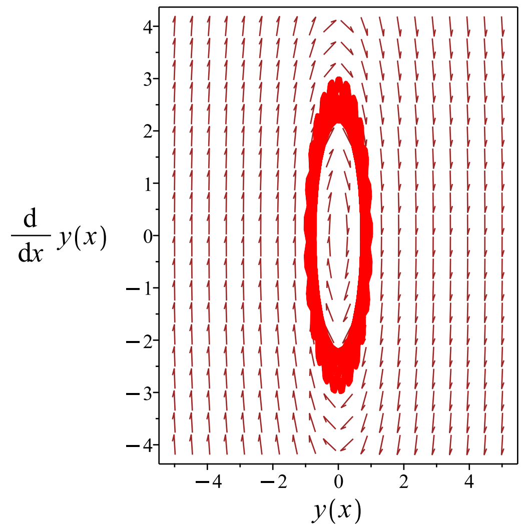
Summary

The solution(s) found are the following

$$
y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5} \tag{1}
$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5}$$

Verified OK.

11.18.4 Maple step by step solution

Let's solve

$$\left[y'' + 9y = \sin(2x), y(0) = 1, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(2x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(3x)\left(\int (\cos(x) - \cos(5x)) dx\right)}{6} + \frac{\sin(3x)\left(\int (\sin(5x) - \sin(x)) dx\right)}{6}$

  - Compute integrals

    $y_p(x) = \frac{\sin(2x)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\sin(2x)}{5}$

☐ Check validity of solution $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\sin(2x)}{5}$

  - Use initial condition $y(0) = 1$

    $1 = c_1$

  - Compute derivative of the solution

    $y' = -3c_1 \sin(3x) + 3c_2 \cos(3x) + \frac{2\cos(2x)}{5}$

  - Use the initial condition $y'\Big|_{\{x=0\}} = 0$

    $0 = \frac{2}{5} + 3c_2$

- Solve for c_1 and c_2

  $\{c_1 = 1, c_2 = -\frac{2}{15}\}$
- Substitute constant values into general solution and simplify

  $y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5}$

- Solution to the IVP

  $y = \cos(3x) - \frac{2\sin(3x)}{15} + \frac{\sin(2x)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)+9*y(x)=sin(2*x),y(0) = 1, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = -\frac{2\sin(3x)}{15} + \cos(3x) + \frac{\sin(2x)}{5}$$

✓ Solution by Mathematica

Time used: 0.094 (sec). Leaf size: 26

```
DSolve[{y''[x]+9*y[x]==Sin[2*x],{y[0]==1,y'[0]==0}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}\sin(2x) - \frac{2}{15}\sin(3x) + \cos(3x)$$

11.19 problem 34

Internal problem ID [237]
Internal file name [`OUTPUT/237_Sunday_June_05_2022_01_37_12_AM_59525582/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 34.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \cos(x)}$$

With initial conditions

$$[y(0) = 1, y'(0) = -1]$$

11.19.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= 0 \\ q(x) &= 1 \\ F &= \cos(x) \end{aligned}$$

Hence the ode is

$$y'' + y = \cos(x)$$

The domain of $p(x) = 0$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = \cos(x)$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.19.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \cos(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(x), \sin(x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), \sin(x)\, x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x\cos(x) + A_2 \sin(x)\, x$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1\sin(x) + 2A_2\cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{\sin(x)\, x}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x) x}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) + \frac{x \cos(x)}{2} + \frac{\sin(x)}{2}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

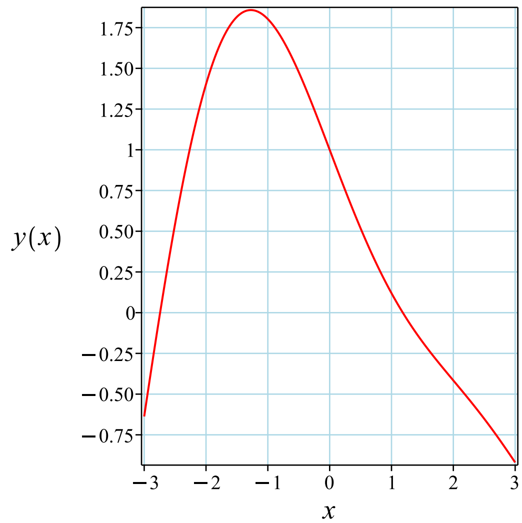
$$y = \cos(x) - \sin(x) + \frac{\sin(x) x}{2}$$

Which simplifies to

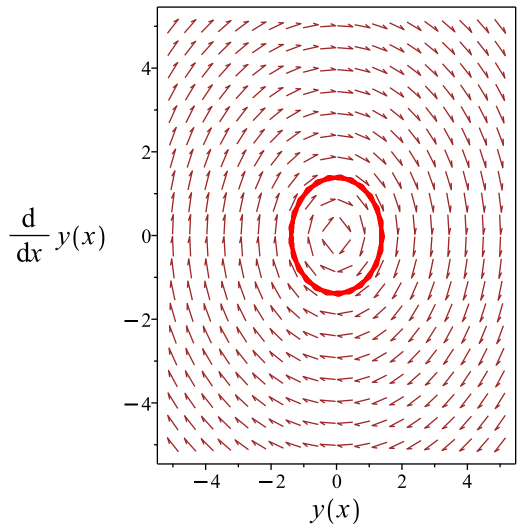
$$y = \frac{(-2 + x) \sin(x)}{2} + \cos(x)$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{(-2 + x) \sin(x)}{2} + \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(-2+x)\sin(x)}{2} + \cos(x)$$

Verified OK.

11.19.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -1$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 483: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\,dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$
y'' + y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \cos(x) + c_2 \sin(x)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
\cos(x)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(x), \sin(x)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\cos(x), \sin(x)\}
$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$
[\{x\cos(x), \sin(x)\, x\}]
$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$
y_p = A_1 x \cos(x) + A_2 \sin(x)\, x
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-2A_1 \sin(x) + 2A_2 \cos(x) = \cos(x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 0, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{\sin(x)\, x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{\sin(x)\, x}{2}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{\sin(x)\, x}{2} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 1$ and $x = 0$ in the above gives

$$1 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = -c_1 \sin(x) + c_2 \cos(x) + \frac{x \cos(x)}{2} + \frac{\sin(x)}{2}$$

substituting $y' = -1$ and $x = 0$ in the above gives

$$-1 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 1$$
$$c_2 = -1$$

Substituting these values back in above solution results in

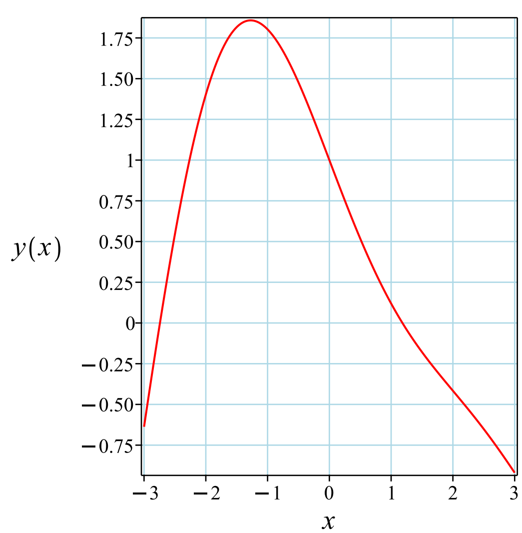
$$y = \cos(x) - \sin(x) + \frac{\sin(x)\, x}{2}$$

Which simplifies to

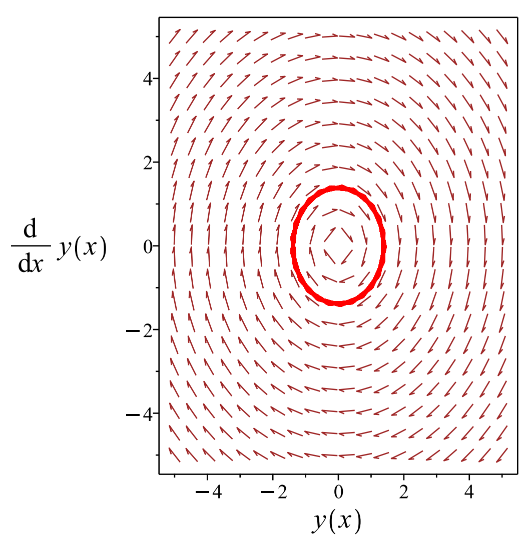
$$y = \frac{(-2 + x)\sin(x)}{2} + \cos(x)$$

Summary
The solution(s) found are the following

$$y = \frac{(-2 + x)\sin(x)}{2} + \cos(x) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = \frac{(-2 + x)\sin(x)}{2} + \cos(x)$$

Verified OK.

11.19.4 Maple step by step solution

Let's solve

$$\left[y''+y=\cos(x), y(0)=1, y'\Big|_{\{x=0\}}=-1\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2+1=0$

- Use quadratic formula to solve for r

  $r=\frac{0\pm(\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r=(-\mathrm{I},\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x)=\cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x)=\sin(x)$

- General solution of the ODE

  $y=c_1y_1(x)+c_2y_2(x)+y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y=c_1\cos(x)+c_2\sin(x)+y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x)=-y_1(x)\left(\int\frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right)+y_2(x)\left(\int\frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x)=\cos(x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x))=\begin{bmatrix}\cos(x) & \sin(x)\\ -\sin(x) & \cos(x)\end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x))=1$

  ◦ Substitute functions into equation for $y_p(x)$

$y_p(x) = -\frac{\cos(x)\left(\int \sin(2x)dx\right)}{2} + \sin(x)\left(\int \cos(x)^2 dx\right)$

- Compute integrals

  $y_p(x) = \frac{\cos(x)}{4} + \frac{\sin(x)x}{2}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)}{4} + \frac{\sin(x)x}{2}$

□ Check validity of solution $y = c_1 \cos(x) + c_2 \sin(x) + \frac{\cos(x)}{4} + \frac{\sin(x)x}{2}$

- Use initial condition $y(0) = 1$

  $1 = c_1 + \frac{1}{4}$

- Compute derivative of the solution

  $y' = -c_1 \sin(x) + c_2 \cos(x) + \frac{\sin(x)}{4} + \frac{x\cos(x)}{2}$

- Use the initial condition $y'\Big|_{\{x=0\}} = -1$

  $-1 = c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{3}{4}, c_2 = -1\right\}$

- Substitute constant values into general solution and simplify

  $y = \frac{(-2+x)\sin(x)}{2} + \cos(x)$

- Solution to the IVP

  $y = \frac{(-2+x)\sin(x)}{2} + \cos(x)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 14

```
dsolve([diff(y(x),x$2)+y(x)=cos(x),y(0) = 1, D(y)(0) = -1],y(x), singsol=all)
```

$$y(x) = \frac{(-2 + x) \sin (x)}{2} + \cos (x)$$

✓ Solution by Mathematica

Time used: 0.03 (sec). Leaf size: 17

```
DSolve[{y''[x]+y[x]==Cos[x],{y[0]==1,y'[0]==-1}},y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{2}(x - 2) \sin(x) + \cos(x)$$

11.20 problem 35

Internal problem ID [238]
Internal file name [`OUTPUT/238_Sunday_June_05_2022_01_37_13_AM_39482632/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 35.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _with_linear_symmetries]]`

$$\boxed{y'' - 2y' + 2y = x + 1}$$

With initial conditions

$$[y(0) = 3, y'(0) = 0]$$

11.20.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$y'' + p(x)y' + q(x)y = F$$

Where here

$$\begin{aligned} p(x) &= -2 \\ q(x) &= 2 \\ F &= x + 1 \end{aligned}$$

Hence the ode is

$$y'' - 2y' + 2y = x + 1$$

The domain of $p(x) = -2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is inside this domain. The domain of $q(x) = 2$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. The domain of $F = x + 1$ is

$$\{-\infty < x < \infty\}$$

And the point $x_0 = 0$ is also inside this domain. Hence solution exists and is unique.

11.20.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = 2, f(x) = x + 1$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda \,\mathrm{e}^{\lambda x} + 2\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(2)} \\ &= 1 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1 + i \\ \lambda_2 &= 1 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 1$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x))$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{x}(c_1 \cos(x) + c_2 \sin(x))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,\mathrm{e}^{x}, \sin(x)\,\mathrm{e}^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 - 2A_2 = x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (\mathrm{e}^{x}(c_1 \cos(x) + c_2 \sin(x))) + \left(\frac{x}{2} + 1\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = \mathrm{e}^{x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{x}{2} + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^x(c_1 \cos(x) + c_2 \sin(x)) + e^x(-c_1 \sin(x) + c_2 \cos(x)) + \frac{1}{2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 2$$
$$c_2 = -\frac{5}{2}$$

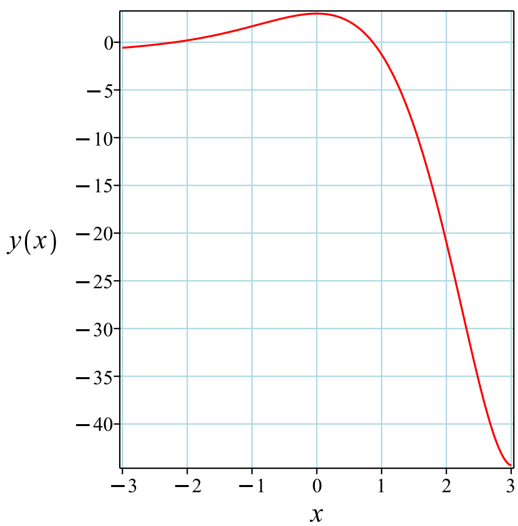
Substituting these values back in above solution results in

$$y = 1 - \frac{5 \sin(x) e^x}{2} + 2 \cos(x) e^x + \frac{x}{2}$$

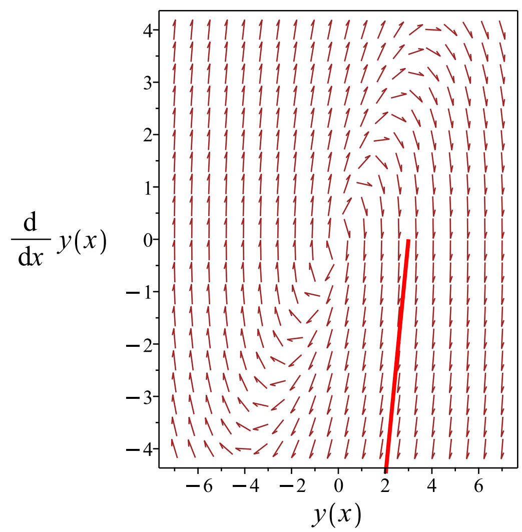
Summary

The solution(s) found are the following

$$y = 1 - \frac{5 \sin(x) e^x}{2} + 2 \cos(x) e^x + \frac{x}{2} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$y = 1 - \frac{5 \sin(x) e^x}{2} + 2 \cos(x) e^x + \frac{x}{2}$$

Verified OK.

11.20.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 485: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2} \frac{-2}{1} \, dx}
\end{aligned}$$

$$
\begin{aligned}
&= z_1 e^x \\
&= z_1(\mathrm{e}^x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(x)\,\mathrm{e}^x
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{2x}}{(y_1)^2}\,dx \\
&= y_1(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)\,\mathrm{e}^x) + c_2(\cos(x)\,\mathrm{e}^x(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 2y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = \mathrm{e}^x \cos(x)\,c_1 + \sin(x)\,\mathrm{e}^x c_2
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x + 1$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1, x\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x)\,e^{x}, \sin(x)\,e^{x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_2 x + A_1$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_2 x + 2A_1 - 2A_2 = x + 1$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = 1, A_2 = \frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{x}{2} + 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (e^{x} \cos(x)\, c_1 + \sin(x)\, e^{x} c_2) + \left(\frac{x}{2} + 1\right) \end{aligned}$$

Which simplifies to

$$y = e^{x}(c_1 \cos(x) + c_2 \sin(x)) + \frac{x}{2} + 1$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$y = e^x(c_1 \cos(x) + c_2 \sin(x)) + \frac{x}{2} + 1 \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $y = 3$ and $x = 0$ in the above gives

$$3 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$y' = e^x(c_1 \cos(x) + c_2 \sin(x)) + e^x(-c_1 \sin(x) + c_2 \cos(x)) + \frac{1}{2}$$

substituting $y' = 0$ and $x = 0$ in the above gives

$$0 = c_1 + \frac{1}{2} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

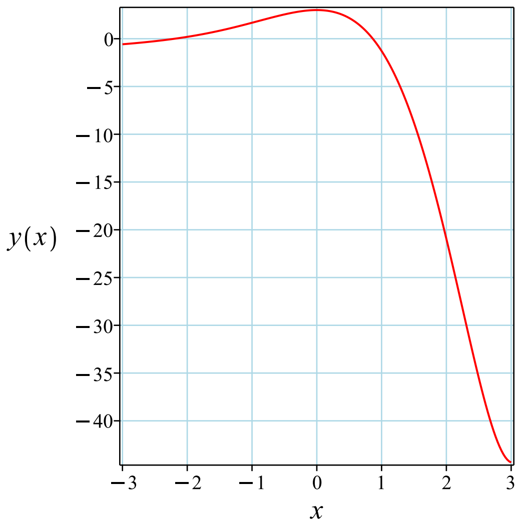
$$\begin{aligned} c_1 &= 2 \\ c_2 &= -\frac{5}{2} \end{aligned}$$

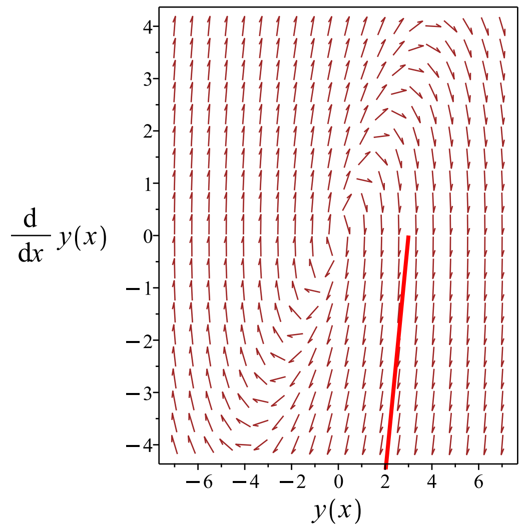
Substituting these values back in above solution results in

$$y = 1 - \frac{5 \sin(x) e^x}{2} + 2 \cos(x) e^x + \frac{x}{2}$$

Summary
The solution(s) found are the following

$$y = 1 - \frac{5 \sin(x) e^x}{2} + 2 \cos(x) e^x + \frac{x}{2} \tag{1}$$





(a) Solution plot (b) Slope field plot

Verification of solutions

$$y = 1 - \frac{5\sin(x)\,\mathrm{e}^{x}}{2} + 2\cos(x)\,\mathrm{e}^{x} + \frac{x}{2}$$

Verified OK.

11.20.4 Maple step by step solution

Let's solve

$$\left[y'' - 2y' + 2y = x + 1, y(0) = 3, y'\Big|_{\{x=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 2r + 2 = 0$

- Use quadratic formula to solve for r

  $r = \frac{2 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (1 - \mathrm{I}, 1 + \mathrm{I})$

- 1st solution of the homogeneous ODE

$y_1(x) = \cos(x)\, e^x$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)\, e^x$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = e^x \cos(x)\, c_1 + \sin(x)\, e^x c_2 + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x+1\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x)\, e^x & \sin(x)\, e^x \\ -\sin(x)\, e^x + \cos(x)\, e^x & \cos(x)\, e^x + \sin(x)\, e^x \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{2x}$

  ◦ Substitute functions into equation for $y_p(x)$

  $y_p(x) = -e^x\left(\cos(x)\left(\int (x+1)\sin(x)\, e^{-x} dx\right) - \sin(x)\left(\int (x+1)\cos(x)\, e^{-x} dx\right)\right)$

  ◦ Compute integrals

  $y_p(x) = \frac{x}{2} + 1$

- Substitute particular solution into general solution to ODE

  $y = e^x \cos(x)\, c_1 + \sin(x)\, e^x c_2 + \frac{x}{2} + 1$

☐ Check validity of solution $y = e^x \cos(x)\, c_1 + \sin(x)\, e^x c_2 + \frac{x}{2} + 1$

  ◦ Use initial condition $y(0) = 3$

  $3 = 1 + c_1$

  ◦ Compute derivative of the solution

  $y' = e^x \cos(x)\, c_1 - e^x \sin(x)\, c_1 + \cos(x)\, e^x c_2 + \sin(x)\, e^x c_2 + \frac{1}{2}$

  ◦ Use the initial condition $y'\Big|_{\{x=0\}} = 0$

  $0 = c_1 + \frac{1}{2} + c_2$

- Solve for c_1 and c_2

  $\{c_1 = 2, c_2 = -\frac{5}{2}\}$
- Substitute constant values into general solution and simplify

  $y = \frac{(4\cos(x)-5\sin(x))\mathrm{e}^x}{2} + \frac{x}{2} + 1$

- Solution to the IVP

  $y = \frac{(4\cos(x)-5\sin(x))\mathrm{e}^x}{2} + \frac{x}{2} + 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 21

```
dsolve([diff(y(x),x$2)-2*diff(y(x),x)+2*y(x)=x+1,y(0) = 3, D(y)(0) = 0],y(x), singsol=all)
```

$$y(x) = \frac{(-5\sin(x) + 4\cos(x))\,\mathrm{e}^x}{2} + 1 + \frac{x}{2}$$

✓ Solution by Mathematica

Time used: 0.014 (sec). Leaf size: 26

```
DSolve[{y''[x]-2*y'[x]+2*y[x]==x+1,{y[0]==3,y'[0]==0}},y[x],x,IncludeSingularSolutions -> Tr
```

$$y(x) \to \frac{1}{2}(x - 5e^x\sin(x) + 4e^x\cos(x) + 2)$$

11.21 problem 44

Internal problem ID [239]
Internal file name [`OUTPUT/239_Sunday_June_05_2022_01_37_14_AM_39087720/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 44.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y' + y = \sin(x)\sin(3x)}$$

11.21.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 1, C = 1, f(x) = \sin(x)\sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 1, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + \lambda e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + \lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 1, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-1}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{1^2 - (4)(1)(1)} \\ &= -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \lambda_2 &= -\frac{1}{2} - \frac{i\sqrt{3}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

$$\lambda_2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

Therefore the homogeneous solution y_h is

$$y_h = \mathrm{e}^{-\frac{x}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right), \mathrm{e}^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x) + A_3 \cos(4x) + A_4 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned}&-3A_1\cos(2x) - 3A_2\sin(2x) - 15A_3\cos(4x) - 15A_4\sin(4x) - 2A_1\sin(2x)\\ &+ 2A_2\cos(2x) - 4A_3\sin(4x) + 4A_4\cos(4x) = \sin(x)\sin(3x)\end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{26}, A_2 = \frac{1}{13}, A_3 = \frac{15}{482}, A_4 = -\frac{2}{241}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\mathrm{e}^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right)\right) \\
&\quad + \left(-\frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}\right)
\end{aligned}
$$

<u>Summary</u>
The solution(s) found are the following

$$
\begin{aligned}
y = \mathrm{e}^{-\frac{x}{2}} &\left(c_1 \cos\left(\frac{\sqrt{3}\,x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right)\right) \\
&- \frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}
\end{aligned}
\tag{1}
$$

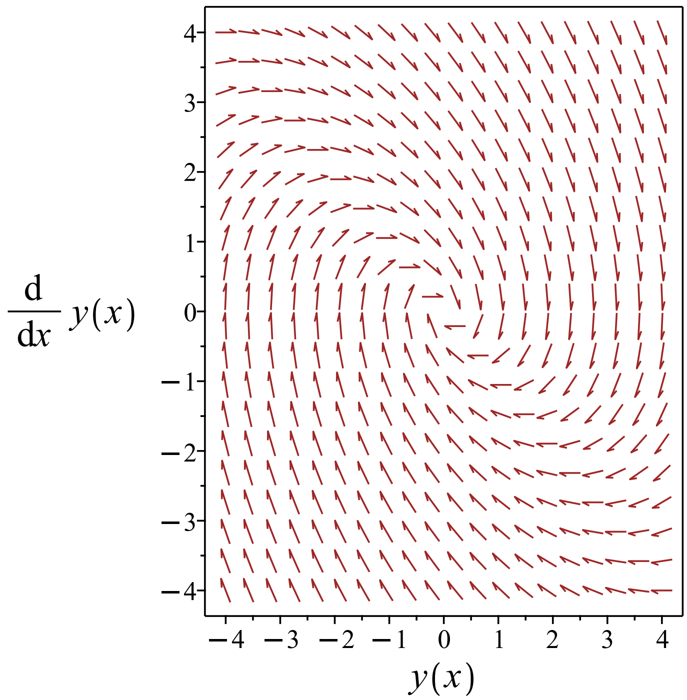


Figure 642: Slope field plot

Verification of solutions

$$y = e^{-\frac{x}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, x}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, x}{2}\right) \right) - \frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}$$

Verified OK.

11.21.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y' + y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -\frac{3z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 487: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{3}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{1}{1}\,dx} \\
&= z_1 e^{-\frac{x}{2}} \\
&= z_1\left(\mathrm{e}^{-\frac{x}{2}}\right)
\end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Substituting gives

$$\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{1}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-x}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\right) + c_2\left(\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,x}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \cos\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2x), \sin(2x)\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\,x}{2}\right), \frac{2\sin\left(\frac{\sqrt{3}\,x}{2}\right) \mathrm{e}^{-\frac{x}{2}} \sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(2x) + A_2 \sin(2x) + A_3 \cos(4x) + A_4 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3, A_4\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-3A_1 \cos(2x) - 3A_2 \sin(2x) - 15A_3 \cos(4x) - 15A_4 \sin(4x) - 2A_1 \sin(2x) + 2A_2 \cos(2x) - 4A_3 \sin(4x) + 4A_4 \cos(4x) = \sin(x)\sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{3}{26}, A_2 = \frac{1}{13}, A_3 = \frac{15}{482}, A_4 = -\frac{2}{241}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3}\right) \\ &\quad + \left(-\frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned} y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} \\ - \frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241} \end{aligned} \tag{1}$$

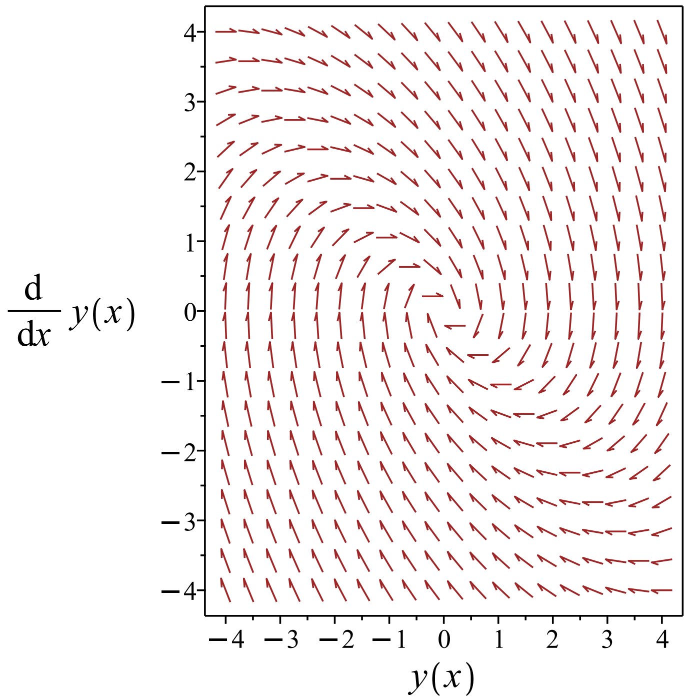


Figure 643: Slope field plot

Verification of solutions

$$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,x}{2}\right) e^{-\frac{x}{2}} \sqrt{3}}{3} - \frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}$$

Verified OK.

11.21.3 Maple step by step solution

Let's solve

$y'' + y' + y = \sin(x)\sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + 1 = 0$

- Use quadratic formula to solve for r

$r = \frac{(-1)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

$r = \left(-\frac{1}{2}-\frac{I\sqrt{3}}{2}, -\frac{1}{2}+\frac{I\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

$y_1(x) = e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)$

- 2nd solution of the homogeneous ODE

$y_2(x) = e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)$

- General solution of the ODE

$y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}c_1 + \sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}c_2 + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

$\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sin(x)\sin(3x)\right]$

- Wronskian of solutions of the homogeneous equation

$W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right) & e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right) \\ -\frac{e^{-\frac{x}{2}}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}\sqrt{3}}{2} & -\frac{e^{-\frac{x}{2}}\sin\left(\frac{\sqrt{3}\,x}{2}\right)}{2} + \frac{e^{-\frac{x}{2}}\sqrt{3}\cos\left(\frac{\sqrt{3}\,x}{2}\right)}{2} \end{bmatrix}$

- Compute Wronskian

$W(y_1(x), y_2(x)) = \frac{\sqrt{3}\,e^{-x}}{2}$

- Substitute functions into equation for $y_p(x)$

$y_p(x) = -\frac{2\,e^{-\frac{x}{2}}\sqrt{3}\left(\cos\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\sin(3x)\sin\left(\frac{\sqrt{3}\,x}{2}\right)dx\right) - \sin\left(\frac{\sqrt{3}\,x}{2}\right)\left(\int e^{\frac{x}{2}}\sin(x)\sin(3x)\cos\left(\frac{\sqrt{3}\,x}{2}\right)dx\right)\right)}{3}$

- Compute integrals

$y_p(x) = -\frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} + \frac{15\cos(4x)}{482} - \frac{2\sin(4x)}{241}$

- Substitute particular solution into general solution to ODE

$y = \cos\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}c_1 + \sin\left(\frac{\sqrt{3}\,x}{2}\right)e^{-\frac{x}{2}}c_2 + \frac{\sin(2x)}{13} - \frac{2\sin(4x)}{241} - \frac{3\cos(2x)}{26} + \frac{15\cos(4x)}{482}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 55

```
dsolve(diff(y(x),x$2)+diff(y(x),x)+y(x)=sin(x)*sin(3*x),y(x), singsol=all)
```

$$y(x) = e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{3}\, x}{2}\right) c_2 + e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{3}\, x}{2}\right) c_1 - \frac{3\cos(2x)}{26} + \frac{\sin(2x)}{13} - \frac{2\sin(4x)}{241} + \frac{15\cos(4x)}{482}$$

✓ Solution by Mathematica

Time used: 5.225 (sec). Leaf size: 80

```
DSolve[y''[x]+y'[x]+y[x]==Sin[x]*Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{13}\sin(2x) - \frac{2}{241}\sin(4x) - \frac{3}{26}\cos(2x) + \frac{15}{482}\cos(4x) + c_2 e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right) + c_1 e^{-x/2}\sin\left(\frac{\sqrt{3}x}{2}\right)$$

11.22 problem 45

Internal problem ID [240]
Internal file name [`OUTPUT/240_Sunday_June_05_2022_01_37_16_AM_36621995/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 45.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \sin(x)^4}$$

11.22.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \sin(x)^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos (3x) + c_2 \sin (3x))$$

Or

$$y = c_1 \cos (3x) + c_2 \sin (3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^4$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x) + A_4 \cos(4x) + A_5 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_2 \cos(2x) + 5A_3 \sin(2x) - 7A_4 \cos(4x) - 7A_5 \sin(4x) + 9A_1 = \sin(x)^4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}, A_2 = -\frac{1}{10}, A_3 = 0, A_4 = -\frac{1}{56}, A_5 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(\frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56} \tag{1}$$

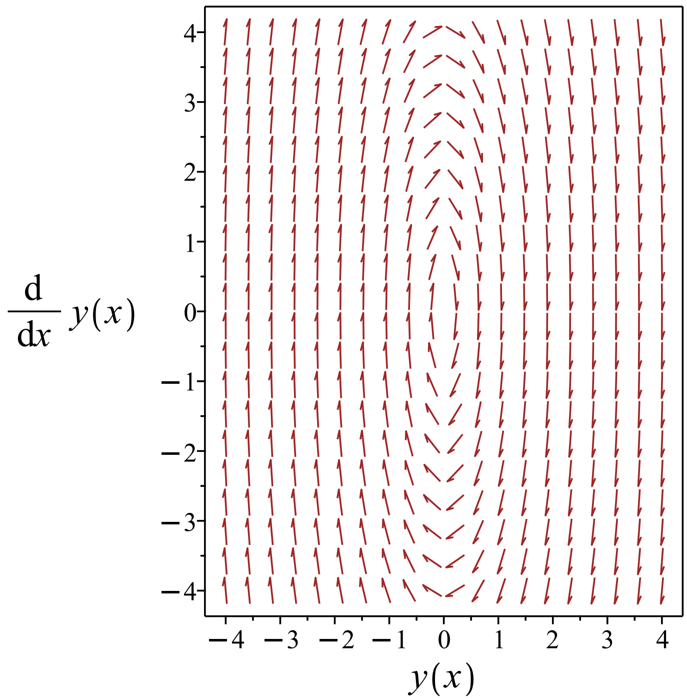


Figure 644: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56}$$

Verified OK.

11.22.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$
\begin{aligned}
A &= 1 \\
B &= 0 \\
C &= 9
\end{aligned} \tag{3}
$$

Applying the Liouville transformation on the dependent variable gives

$$
z(x) = ye^{\int \frac{B}{2A}\,dx}
$$

Then (2) becomes

$$
z''(x) = rz(x) \tag{4}
$$

Where r is given by

$$
\begin{aligned}
r &= \frac{s}{t} \\
&= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2}
\end{aligned} \tag{5}
$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$
r = \frac{-9}{1} \tag{6}
$$

Comparing the above to (5) shows that

$$
\begin{aligned}
s &= -9 \\
t &= 1
\end{aligned}
$$

Therefore eq. (4) becomes

$$
z''(x) = -9z(x) \tag{7}
$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$
y = z(x)\, e^{-\int \frac{B}{2A}\,dx}
$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 489: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(3x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(3x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(3x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(3x) \int \frac{1}{\cos(3x)^2}\,dx \\ &= \cos(3x)\left(\frac{\tan(3x)}{3}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^4$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}, \{\cos(4x), \sin(4x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 + A_2 \cos(2x) + A_3 \sin(2x) + A_4 \cos(4x) + A_5 \sin(4x)$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_2 \cos(2x) + 5A_3 \sin(2x) - 7A_4 \cos(4x) - 7A_5 \sin(4x) + 9A_1 = \sin(x)^4$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{24}, A_2 = -\frac{1}{10}, A_3 = 0, A_4 = -\frac{1}{56}, A_5 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(\frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56} \tag{1}$$

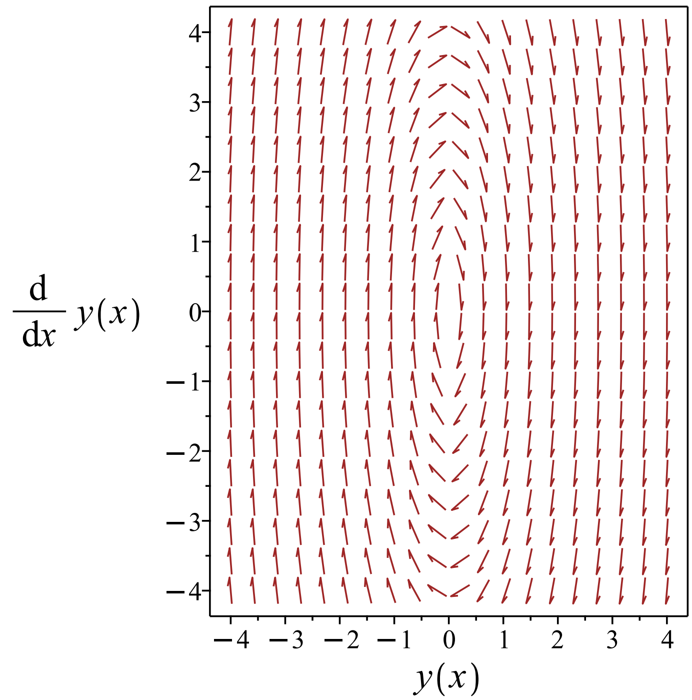


Figure 645: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} + \frac{1}{24} - \frac{\cos(2x)}{10} - \frac{\cos(4x)}{56}$$

Verified OK.

11.22.3 Maple step by step solution

Let's solve

$y'' + 9y = \sin(x)^4$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^4\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 3$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x) \sin(x)^4 dx\right)}{3} + \frac{\sin(3x)\left(\int \cos(3x) \sin(x)^4 dx\right)}{3}$

  - Compute integrals

    $y_p(x) = -\frac{\cos(x)^4}{7} - \frac{2\cos(x)^2}{35} + \frac{13}{105}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{\cos(x)^4}{7} - \frac{2\cos(x)^2}{35} + \frac{13}{105}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 32

```
dsolve(diff(y(x),x$2)+9*y(x)=sin(x)^4,y(x), singsol=all)
```

$$y(x) = \sin(3x) c_2 + \cos(3x) c_1 - \frac{\cos(2x)}{10} - \frac{\cos(2x)^2}{28} + \frac{5}{84}$$

✓ Solution by Mathematica

Time used: 0.215 (sec). Leaf size: 39

```
DSolve[y''[x]+9*y[x]==Sin[x]^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{10}\cos(2x) - \frac{1}{56}\cos(4x) + c_1\cos(3x) + c_2\sin(3x) + \frac{1}{24}$$

11.23 problem 46

Internal problem ID [241]
Internal file name [`OUTPUT/241_Sunday_June_05_2022_01_37_17_AM_79584620/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 46.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = x\cos(x)^3}$$

11.23.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = x\cos(x)^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)} \\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i \\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i \\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \cos(x)^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x \cos(x), \sin(x) x, \cos(x), \sin(x)\}, \{x \cos(3x), x \sin(3x), \cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x \cos(x), x^2 \cos(x), \sin(x) x, \sin(x) x^2\}, \{x \cos(3x), x \sin(3x), \cos(3x), \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$\begin{aligned} y_p = & A_1 x \cos(x) + A_2 x^2 \cos(x) + A_3 \sin(x) x + A_4 \sin(x) x^2 \\ & + A_5 x \cos(3x) + A_6 x \sin(3x) + A_7 \cos(3x) + A_8 \sin(3x) \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & -2A_1 \sin(x) + 2A_2 \cos(x) - 4A_2 x \sin(x) + 2A_3 \cos(x) + 4A_4 \cos(x) x \\ & + 2A_4 \sin(x) - 6A_5 \sin(3x) - 8A_5 x \cos(3x) + 6A_6 \cos(3x) \\ & - 8A_6 x \sin(3x) - 8A_7 \cos(3x) - 8A_8 \sin(3x) = x \cos(x)^3 \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{16}, A_2 = 0, A_3 = 0, A_4 = \frac{3}{16}, A_5 = -\frac{1}{32}, A_6 = 0, A_7 = 0, A_8 = \frac{3}{128}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x\cos(x)}{16} + \frac{3\sin(x)\,x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + \left(\frac{3x\cos(x)}{16} + \frac{3\sin(x)\,x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(x) + c_2\sin(x) + \frac{3x\cos(x)}{16} + \frac{3\sin(x)\,x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128} \quad (1)$$

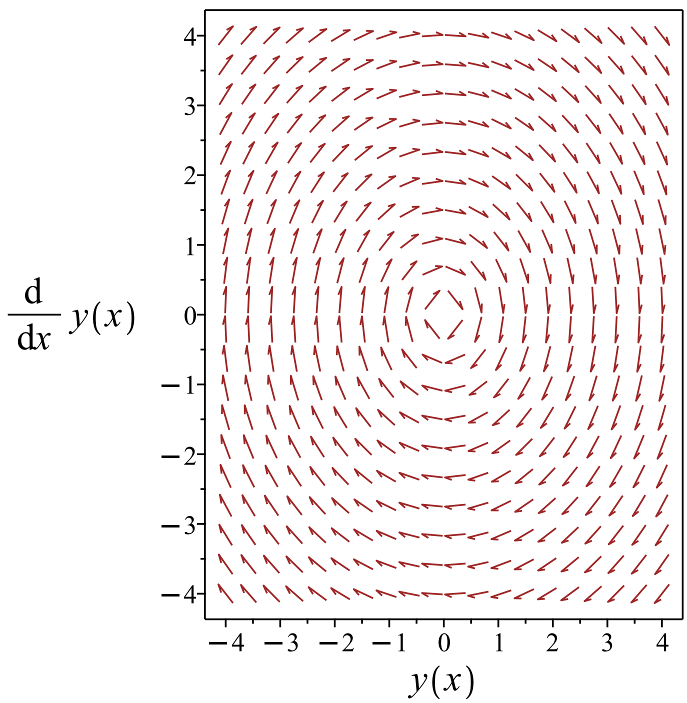


Figure 646: Slope field plot

Verification of solutions

$$y = c_1\cos(x) + c_2\sin(x) + \frac{3x\cos(x)}{16} + \frac{3\sin(x)\,x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128}$$

Verified OK.

11.23.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 491: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(x)
\end{aligned}
$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\
&= \cos(x)(\tan(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(x)) + c_2(\cos(x)(\tan(x)))
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x \cos(x)^3$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\cos(x), \sin(x)x, \cos(x), \sin(x)\}, \{x\cos(3x), x\sin(3x), \cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(x), \sin(x)\}$$

Since $\cos(x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(x), x^2\cos(x), \sin(x)x, \sin(x)x^2\}, \{x\cos(3x), x\sin(3x), \cos(3x), \sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$\begin{aligned} y_p = {} & A_1x\cos(x) + A_2x^2\cos(x) + A_3\sin(x)x + A_4\sin(x)x^2 \\ & + A_5x\cos(3x) + A_6x\sin(3x) + A_7\cos(3x) + A_8\sin(3x) \end{aligned}$$

The unknowns $\{A_1, A_2, A_3, A_4, A_5, A_6, A_7, A_8\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$\begin{aligned} & -2A_1\sin(x) + 2A_2\cos(x) - 4A_2x\sin(x) + 2A_3\cos(x) + 4A_4\cos(x)x \\ & + 2A_4\sin(x) - 6A_5\sin(3x) - 8A_5x\cos(3x) + 6A_6\cos(3x) \\ & - 8A_6x\sin(3x) - 8A_7\cos(3x) - 8A_8\sin(3x) = x\cos(x)^3 \end{aligned}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{3}{16}, A_2 = 0, A_3 = 0, A_4 = \frac{3}{16}, A_5 = -\frac{1}{32}, A_6 = 0, A_7 = 0, A_8 = \frac{3}{128}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{3x\cos(x)}{16} + \frac{3\sin(x)x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128}$$

Therefore the general solution is

$$y = y_h + y_p$$
$$= (c_1 \cos(x) + c_2 \sin(x)) + \left(\frac{3x\cos(x)}{16} + \frac{3\sin(x)x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128}\right)$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3x\cos(x)}{16} + \frac{3\sin(x)x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128} \quad (1)$$

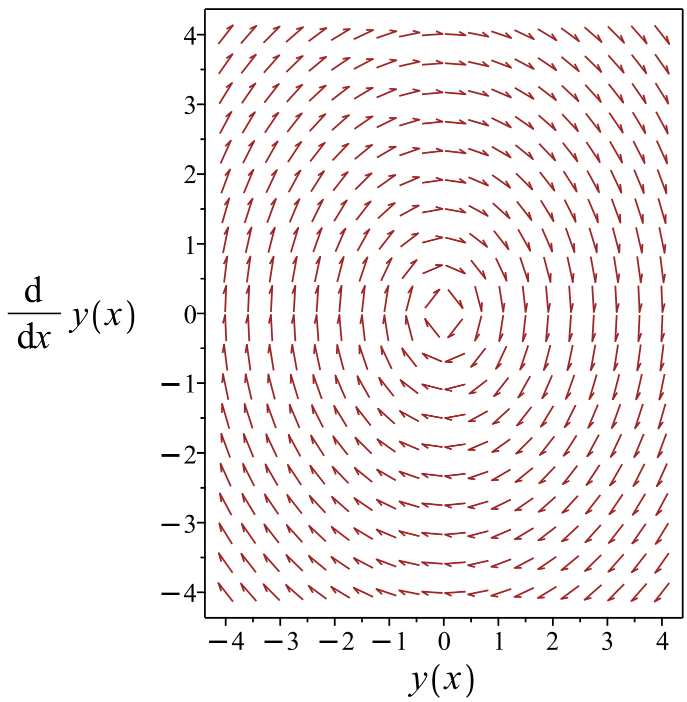


Figure 647: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \frac{3x\cos(x)}{16} + \frac{3\sin(x)x^2}{16} - \frac{x\cos(3x)}{32} + \frac{3\sin(3x)}{128}$$

Verified OK.

11.23.3 Maple step by step solution

Let's solve

$y'' + y = x\cos(x)^3$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1\cos(x) + c_2\sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = x\cos(x)^3\right]$

- Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = 1$

- Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\cos(x)\left(\int \sin(x)\, x\cos(x)^3\, dx\right) + \sin(x)\left(\int \cos(x)^4\, x dx\right)$$

- Compute integrals

$$y_p(x) = \frac{\left(6\cos(x)^2+12x^2+9\right)\sin(x)}{64} - \frac{x\cos(x)^3}{8} + \frac{9x\cos(x)}{32}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(x) + c_2\sin(x) + \frac{\left(6\cos(x)^2+12x^2+9\right)\sin(x)}{64} - \frac{x\cos(x)^3}{8} + \frac{9x\cos(x)}{32}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(y(x),x$2)+y(x)=x*cos(x)^3,y(x), singsol=all)
```

$$y(x) = -\frac{x\cos(x)^3}{8} + \frac{3\sin(x)\cos(x)^2}{32} + \frac{(9x+32c_1)\cos(x)}{32} + \frac{3\left(x^2+\frac{16c_2}{3}+\frac{3}{4}\right)\sin(x)}{16}$$

✓ Solution by Mathematica

Time used: 0.116 (sec). Leaf size: 49

```
DSolve[y''[x]+y[x]==x*Cos[x]^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{128}\left(\sin(x)\left(24x^2+6\cos(2x)-9+128c_2\right)-4x\cos(3x)+8(3x+16c_1)\cos(x)\right)$$

11.24 problem 47

Internal problem ID [242]
Internal file name [`OUTPUT/242_Sunday_June_05_2022_01_37_18_AM_30887006/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 47.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' + 3y' + 2y = 4\,\mathrm{e}^{x}}$$

11.24.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 3, C = 2, f(x) = 4\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 3y' + 2y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 3, C = 2$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 3\lambda\, e^{\lambda x} + 2\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 3\lambda + 2 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 2$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{3^2 - (4)\,(1)\,(2)} \\ &= -\frac{3}{2} \pm \frac{1}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{1}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{1}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(-1)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 e^{-x} + c_2 e^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$4\,\mathrm{e}^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \mathrm{e}^{x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$6A_1 \mathrm{e}^{x} = 4\,\mathrm{e}^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2\,\mathrm{e}^{x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x}\right) + \left(\frac{2\,\mathrm{e}^{x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-x} + c_2 \mathrm{e}^{-2x} + \frac{2\,\mathrm{e}^{x}}{3} \tag{1}$$

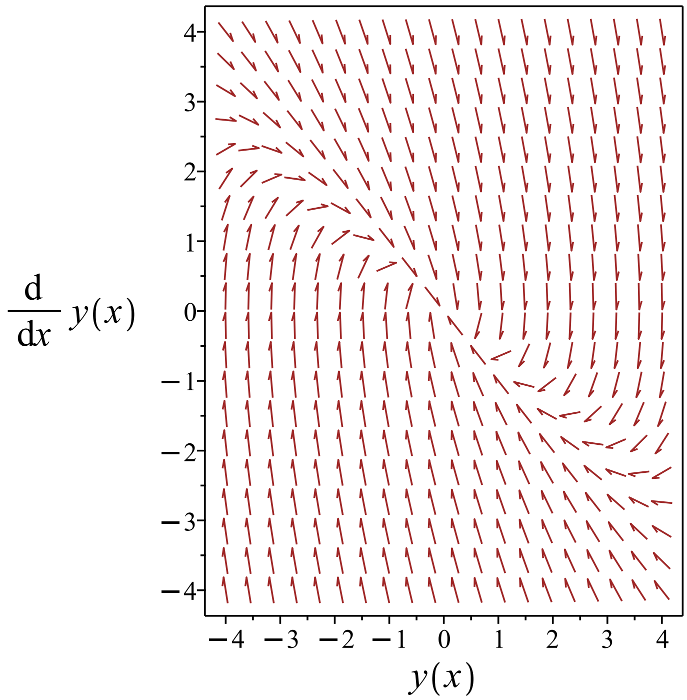


Figure 648: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{2 e^{x}}{3}$$

Verified OK.

11.24.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 3y' + 2y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$s = 1$$
$$t = 4$$

Therefore eq. (4) becomes

$$z''(x) = \frac{z(x)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 493: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = \frac{1}{4}$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-\frac{x}{2}}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\, dx} \\ &= z_1 e^{-\frac{3x}{2}} \\ &= z_1 \left(\mathrm{e}^{-\frac{3x}{2}}\right) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{3}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{-3x}}{(y_1)^2}\, dx \\ &= y_1(\mathrm{e}^{x}) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}(\mathrm{e}^{x})\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 3y' + 2y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
4\,\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2x}, \mathrm{e}^{-x}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 \mathrm{e}^{x}
$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
6A_1 \mathrm{e}^{x} = 4\,\mathrm{e}^{x}
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{2}{3}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{2\,\mathrm{e}^{x}}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x}\right) + \left(\frac{2\,\mathrm{e}^{x}}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \frac{2\,\mathrm{e}^{x}}{3} \tag{1}$$

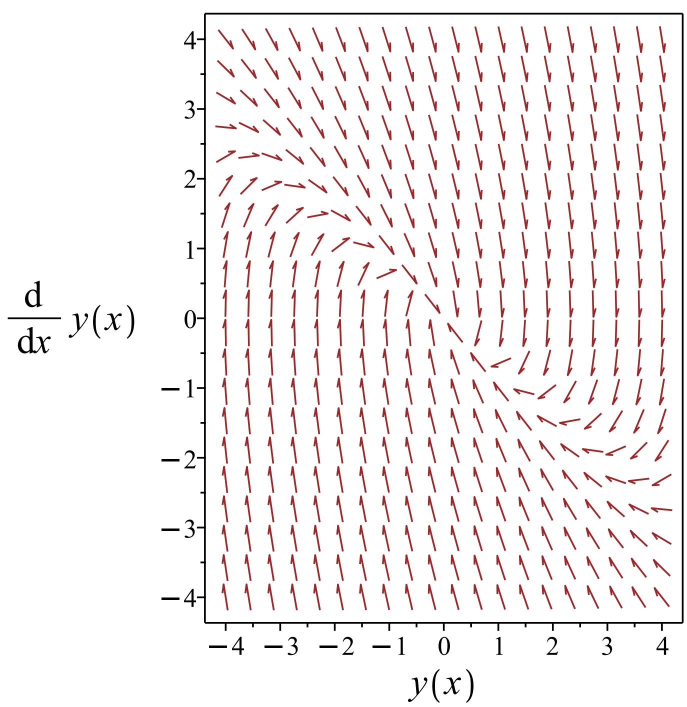


Figure 649: Slope field plot

Verification of solutions

$$y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + \frac{2\,\mathrm{e}^{x}}{3}$$

Verified OK.

11.24.3 Maple step by step solution

Let's solve

$y'' + 3y' + 2y = 4\,\mathrm{e}^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 2 = 0$

- Factor the characteristic polynomial

  $(r+2)(r+1) = 0$

- Roots of the characteristic polynomial

  $r = (-2, -1)$

- 1st solution of the homogeneous ODE

  $y_1(x) = \mathrm{e}^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \mathrm{e}^{-x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \mathrm{e}^{-2x} + c_2 \mathrm{e}^{-x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 4\,\mathrm{e}^{x}\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \mathrm{e}^{-2x} & \mathrm{e}^{-x} \\ -2\,\mathrm{e}^{-2x} & -\mathrm{e}^{-x} \end{bmatrix}$

- Compute Wronskian

  $W(y_1(x), y_2(x)) = e^{-3x}$

- Substitute functions into equation for $y_p(x)$

  $y_p(x) = -4\,e^{-2x}\left(\int e^{3x}dx\right) + 4\,e^{-x}\left(\int e^{2x}dx\right)$

- Compute integrals

  $y_p(x) = \frac{2\,e^{x}}{3}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{-x} + \frac{2\,e^{x}}{3}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
<- double symmetry of the form [xi=0, eta=F(x)] successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+3*diff(y(x),x)+2*y(x)=4*exp(x),y(x), singsol=all)
```

$$y(x) = -\left(-e^{x} c_2 + c_1 - \frac{2\,e^{3x}}{3}\right) e^{-2x}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 29

```
DSolve[y''[x]+3*y'[x]+2*y[x]==4*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{2e^{x}}{3} + c_1 e^{-2x} + c_2 e^{-x}$$

11.25 problem 48

Internal problem ID [243]
Internal file name [`OUTPUT/243_Sunday_June_05_2022_01_37_19_AM_86049182/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 48.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 2y' - 8y = 3\,e^{-2x}}$$

11.25.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -2, C = -8, f(x) = 3\,e^{-2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 8y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -2, C = -8$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 2\lambda\,\mathrm{e}^{\lambda x} - 8\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 2\lambda - 8 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -2, C = -8$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{2}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{-2^2 - (4)\,(1)\,(-8)}\\ &= 1 \pm 3\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= 1 + 3\\ \lambda_2 &= 1 - 3\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 4\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(4)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{4x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,e^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{4x}\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{e^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 e^{-2x} x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 e^{-2x} = 3\,e^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{e^{-2x}x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{4x} + c_2 e^{-2x}\right) + \left(-\frac{e^{-2x}x}{2}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{4x} + c_2 e^{-2x} - \frac{e^{-2x} x}{2} \tag{1}$$

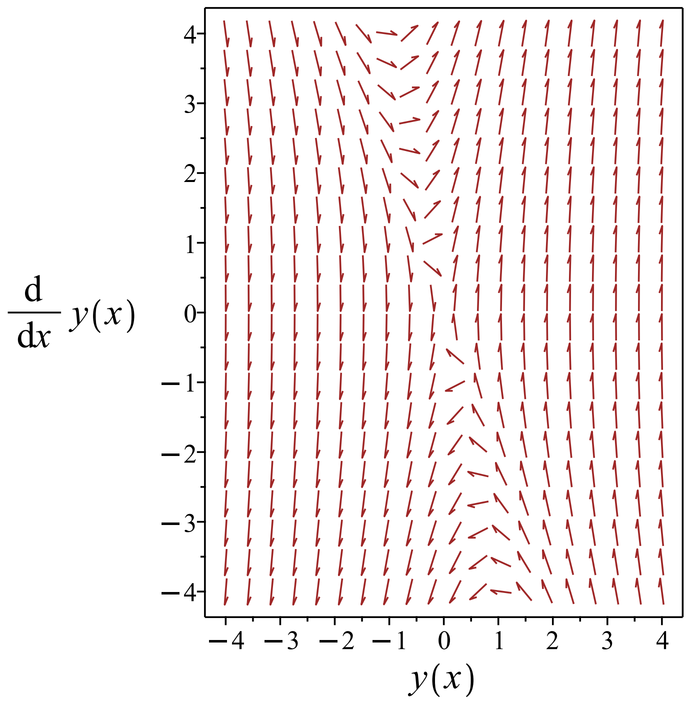


Figure 650: Slope field plot

Verification of solutions

$$y = c_1 e^{4x} + c_2 e^{-2x} - \frac{e^{-2x} x}{2}$$

Verified OK.

11.25.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 2y' - 8y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -2 \\ C &= -8 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 495: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = e^{-3x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$\begin{aligned} y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\ &= z_1 e^{-\int \frac{1}{2}\frac{-2}{1}\,dx} \\ &= z_1 e^{x} \\ &= z_1 (e^{x}) \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Substituting gives

$$\begin{aligned} y_2 &= y_1 \int \frac{e^{\int -\frac{-2}{1}\, dx}}{(y_1)^2}\, dx \\ &= y_1 \int \frac{e^{2x}}{(y_1)^2}\, dx \\ &= y_1 \left(\frac{\mathrm{e}^{6x}}{6}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{6x}}{6}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' - 2y' - 8y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{4x}}{6}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\,\mathrm{e}^{-2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\mathrm{e}^{-2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\mathrm{e}^{4x}}{6}, \mathrm{e}^{-2x}\right\}$$

Since e^{-2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{\mathrm{e}^{-2x}x\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1\mathrm{e}^{-2x}x$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1\mathrm{e}^{-2x} = 3\,\mathrm{e}^{-2x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{2}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\mathrm{e}^{-2x}x}{2}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\mathrm{e}^{-2x} + \frac{c_2\mathrm{e}^{4x}}{6}\right) + \left(-\frac{\mathrm{e}^{-2x}x}{2}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{4x}}{6} - \frac{e^{-2x} x}{2} \tag{1}$$

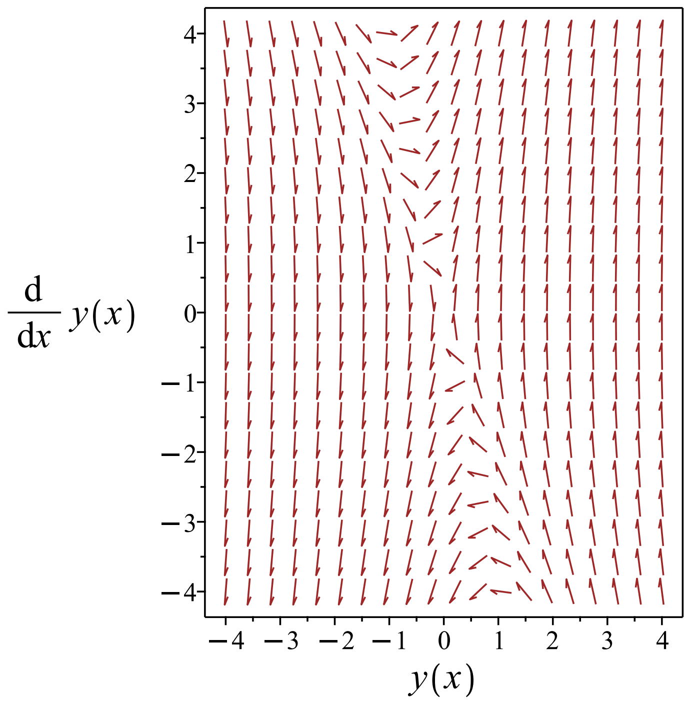


Figure 651: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{4x}}{6} - \frac{e^{-2x} x}{2}$$

Verified OK.

11.25.3 Maple step by step solution

Let's solve

$y'' - 2y' - 8y = 3\,e^{-2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 - 2r - 8 = 0$

- Factor the characteristic polynomial

  $(r+2)(r-4) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 4)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{4x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{4x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 3\, e^{-2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{4x} \\ -2\, e^{-2x} & 4\, e^{4x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 6\, e^{2x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-2x} \left(\int 1 dx\right)}{2} + \frac{e^{4x} \left(\int e^{-6x} dx\right)}{2}$

  - Compute integrals

    $y_p(x) = -\frac{e^{-2x}(1+6x)}{12}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{4x} - \frac{e^{-2x}(1+6x)}{12}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)-2*diff(y(x),x)-8*y(x)=3*exp(-2*x),y(x), singsol=all)
```

$$y(x) = \frac{(-x + 2c_1)\,\mathrm{e}^{-2x}}{2} + \mathrm{e}^{4x}c_2$$

✓ Solution by Mathematica

Time used: 0.036 (sec). Leaf size: 32

```
DSolve[y''[x]-2*y'[x]-8*y[x]==3*Exp[-2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{12}e^{-2x}\left(-6x + 12c_2e^{6x} - 1 + 12c_1\right)$$

11.26 problem 49

Internal problem ID [244]
Internal file name [`OUTPUT/244_Sunday_June_05_2022_01_37_20_AM_71664465/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 49.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{y'' - 4y' + 4y = 2\,e^{2x}}$$

11.26.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = -4, C = 4, f(x) = 2\,e^{2x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = -4, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} - 4\lambda e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = -4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{(-4)^2 - (4)(1)(4)}\\ &= 2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = -2$. Therefore the solution is

$$y = c_1 e^{2x} + c_2 x\, e^{2x} \tag{1}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 e^{2x} + c_2 x\, e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\,e^{2x}\}]$$

Since $x\,e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{2x} = 2\,e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^{2x}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 x\,e^{2x}\right) + \left(x^2 e^{2x}\right) \end{aligned}$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x} \tag{1}$$

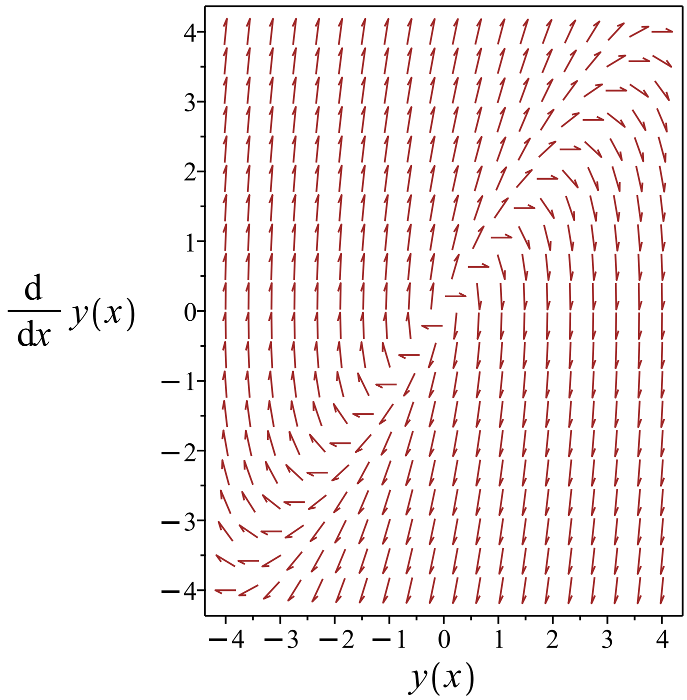


Figure 652: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2x + c_1) + x^2e^{2x}$$

Verified OK.

11.26.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{(p(x)^2 + p'(x))\, y}{2} = f(x)$$

Where $p(x) = -4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int -4\,dx} \\ &= e^{-2x} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$(M(x)y)'' = 2\,e^{2x}e^{-2x}$$
$$\left(e^{-2x}y\right)'' = 2\,e^{2x}e^{-2x}$$

Integrating once gives

$$\left(e^{-2x}y\right)' = 2x + c_1$$

Integrating again gives

$$\left(e^{-2x}y\right) = x(x + c_1) + c_2$$

Hence the solution is

$$y = \frac{x(x + c_1) + c_2}{e^{-2x}}$$

Or

$$y = c_1 x\,e^{2x} + x^2 e^{2x} + c_2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = c_1 x\,e^{2x} + x^2 e^{2x} + c_2 e^{2x} \tag{1}$$

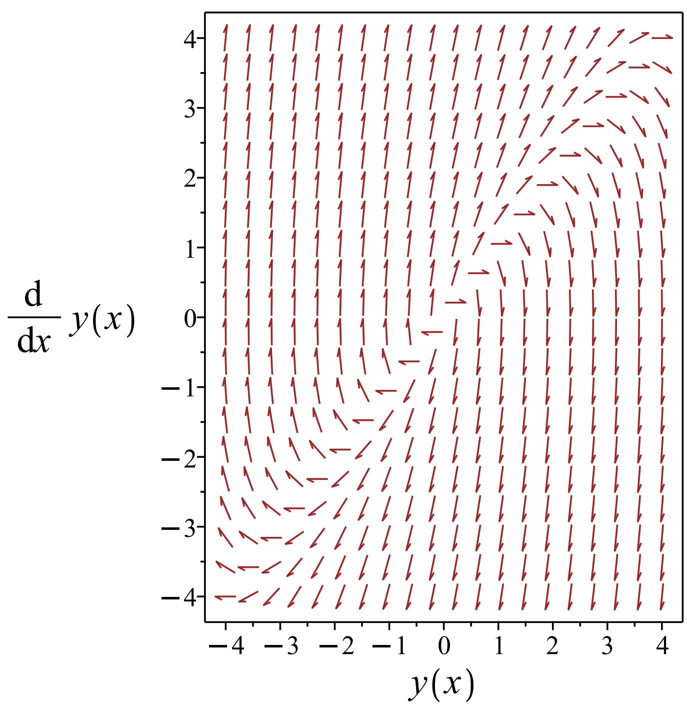


Figure 653: Slope field plot

Verification of solutions

$$y = c_1 x\,\mathrm{e}^{2x} + x^2 \mathrm{e}^{2x} + c_2 \mathrm{e}^{2x}$$

Verified OK.

11.26.3 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= -4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 497: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4}{1}\,dx} \\
&= z_1 e^{2x} \\
&= z_1 \left(\mathrm{e}^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \mathrm{e}^{2x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4}{1}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{4x}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{2x}\right) + c_2 \left(\mathrm{e}^{2x}(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y' + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 e^{2x} + c_2 x\, e^{2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$2\, e^{2x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{e^{2x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{x\, e^{2x}, e^{2x}\}$$

Since e^{2x} is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\, e^{2x}\}]$$

Since $x\, e^{2x}$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x^2 e^{2x}\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x^2 e^{2x}$$

The unknowns $\{A_1\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{2x} = 2\, e^{2x}$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = x^2 e^{2x}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} + c_2 x\, e^{2x}\right) + \left(x^2 e^{2x}\right)
\end{aligned}
$$

Which simplifies to

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x}$$

Summary

The solution(s) found are the following

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x} \tag{1}$$

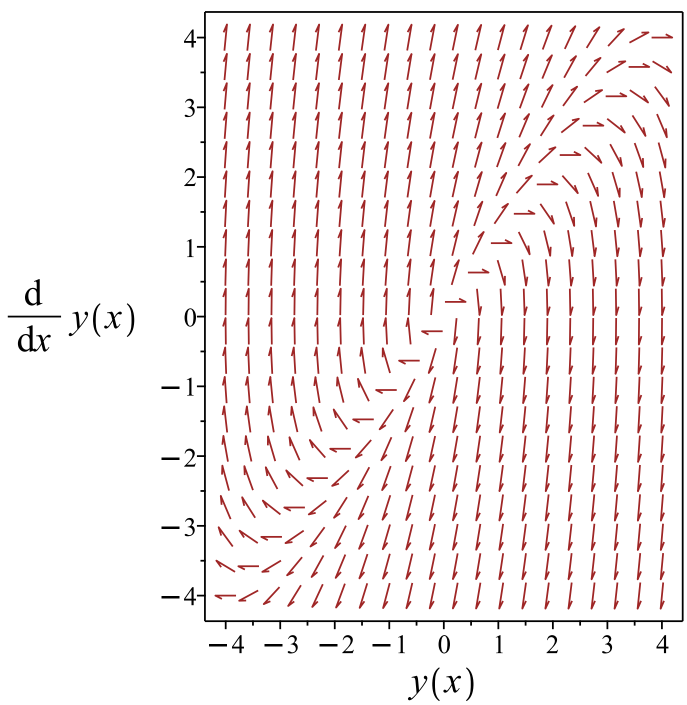


Figure 654: Slope field plot

Verification of solutions

$$y = e^{2x}(c_2 x + c_1) + x^2 e^{2x}$$

Verified OK.

11.26.4 Maple step by step solution

Let's solve

$y'' - 4y' + 4y = 2\,e^{2x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r-2)^2 = 0$

- Root of the characteristic polynomial

  $r = 2$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{2x}$

- Repeated root, multiply $y_1(x)$ by x to ensure linear independence

  $y_2(x) = x\,e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{2x} + c_2 x\,e^{2x} + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = 2\,e^{2x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{2x} & x\,e^{2x} \\ 2\,e^{2x} & e^{2x} + 2x\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = e^{4x}$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -2\,e^{2x}\left(\int x dx - \left(\int 1 dx\right) x\right)$

- Compute integrals

  $y_p(x) = x^2 e^{2x}$

- Substitute particular solution into general solution to ODE

  $y = c_2 x\, e^{2x} + x^2 e^{2x} + c_1 e^{2x}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(diff(y(x),x$2)-4*diff(y(x),x)+4*y(x)=2*exp(2*x),y(x), singsol=all)
```

$$y(x) = e^{2x}(c_1 x + x^2 + c_2)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 21

```
DSolve[y''[x]-4*y'[x]+4*y[x]==2*Exp[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to e^{2x}(x^2 + c_2 x + c_1)$$

11.27 problem 50

Internal problem ID [245]
Internal file name [`OUTPUT/245_Sunday_June_05_2022_01_37_21_AM_41560567/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 50.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = \sinh(2x)}$$

11.27.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = \sinh(2x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(-4)}\\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2\\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2\\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}\\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{2x}$$
$$y_2 = \mathrm{e}^{-2x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{2x} & \mathrm{e}^{-2x} \\ \frac{d}{dx}(\mathrm{e}^{2x}) & \frac{d}{dx}(\mathrm{e}^{-2x}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \mathrm{e}^{2x} & \mathrm{e}^{-2x} \\ 2\,\mathrm{e}^{2x} & -2\,\mathrm{e}^{-2x} \end{vmatrix}$$

Therefore

$$W = (\mathrm{e}^{2x})\left(-2\,\mathrm{e}^{-2x}\right) - \left(\mathrm{e}^{-2x}\right)\left(2\,\mathrm{e}^{2x}\right)$$

Which simplifies to

$$W = -4\,\mathrm{e}^{2x}\mathrm{e}^{-2x}$$

Which simplifies to

$$W = -4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{e^{-2x} \sinh(2x)}{-4} dx$$

Which simplifies to

$$u_1 = -\int -\frac{e^{-2x} \sinh(2x)}{4} dx$$

Hence

$$u_1 = \frac{x}{8} - \frac{\sinh(4x)}{32} + \frac{\cosh(4x)}{32}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{2x} \sinh(2x)}{-4} dx$$

Which simplifies to

$$u_2 = \int -\frac{e^{2x} \sinh(2x)}{4} dx$$

Hence

$$u_2 = -\frac{\cosh(2x)\sinh(2x)}{16} + \frac{x}{8} - \frac{\cosh(2x)^2}{16}$$

Which simplifies to

$$u_1 = \frac{x}{8} - \frac{\sinh(4x)}{32} + \frac{\cosh(4x)}{32}$$
$$u_2 = \frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{8} - \frac{\sinh(4x)}{32} + \frac{\cosh(4x)}{32}\right) e^{2x} + \left(\frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}\right) e^{-2x}$$

Which simplifies to

$$y_p(x) = \frac{(4x - 1 - \sinh(4x) - \cosh(4x)) e^{-2x}}{32} + \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right) e^{2x}}{8}$$

Therefore the general solution is

$$\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 e^{2x} + c_2 e^{-2x}\right) \\
&\quad + \left(\frac{(4x - 1 - \sinh(4x) - \cosh(4x))\, e^{-2x}}{32} + \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right) e^{2x}}{8}\right)
\end{aligned}$$

Summary

The solution(s) found are the following

$$\begin{aligned}
y = c_1 e^{2x} + c_2 e^{-2x} + \frac{(4x - 1 - \sinh(4x) - \cosh(4x))\, e^{-2x}}{32} \\
+ \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right) e^{2x}}{8}
\end{aligned} \tag{1}$$

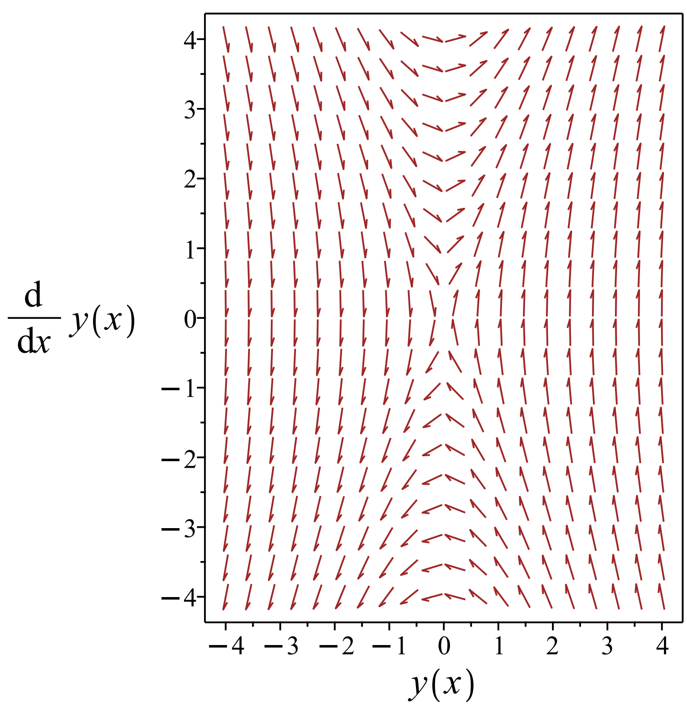


Figure 655: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{(4x - 1 - \sinh(4x) - \cosh(4x))\, e^{-2x}}{32} + \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right) e^{2x}}{8}$$

Verified OK.

11.27.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 499: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\,dx \\ &= \mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1\left(\mathrm{e}^{-2x}\right) + c_2\left(\mathrm{e}^{-2x}\left(\frac{\mathrm{e}^{4x}}{4}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$y''-4y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \mathrm{e}^{-2x}$$
$$y_2 = \frac{\mathrm{e}^{2x}}{4}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \mathrm{e}^{-2x} & \frac{\mathrm{e}^{2x}}{4} \\ \frac{d}{dx}\left(\mathrm{e}^{-2x}\right) & \frac{d}{dx}\left(\frac{\mathrm{e}^{2x}}{4}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} e^{-2x} & \frac{e^{2x}}{4} \\ -2\,e^{-2x} & \frac{e^{2x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(e^{-2x}\right)\left(\frac{e^{2x}}{2}\right) - \left(\frac{e^{2x}}{4}\right)\left(-2\,e^{-2x}\right)$$

Which simplifies to

$$W = e^{2x} e^{-2x}$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{e^{2x}\sinh(2x)}{4}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{e^{2x}\sinh(2x)}{4}dx$$

Hence

$$u_1 = -\frac{\cosh(2x)\sinh(2x)}{16} + \frac{x}{8} - \frac{\cosh(2x)^2}{16}$$

And Eq. (3) becomes

$$u_2 = \int \frac{e^{-2x}\sinh(2x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int e^{-2x}\sinh(2x)\,dx$$

Hence

$$u_2 = \frac{x}{2} - \frac{\sinh(4x)}{8} + \frac{\cosh(4x)}{8}$$

Which simplifies to

$$u_1 = \frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}$$
$$u_2 = \frac{x}{2} - \frac{\sinh(4x)}{8} + \frac{\cosh(4x)}{8}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{x}{8} - \frac{\cosh(4x)}{32} - \frac{1}{32} - \frac{\sinh(4x)}{32}\right)e^{-2x} + \frac{\left(\frac{x}{2} - \frac{\sinh(4x)}{8} + \frac{\cosh(4x)}{8}\right)e^{2x}}{4}$$

Which simplifies to

$$y_p(x) = \frac{(4x - 1 - \sinh(4x) - \cosh(4x))\,e^{-2x}}{32} + \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right)e^{2x}}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) \\ &\quad + \left(\frac{(4x - 1 - \sinh(4x) - \cosh(4x))\,e^{-2x}}{32} + \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right)e^{2x}}{8}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$\begin{aligned} y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + \frac{(4x - 1 - \sinh(4x) - \cosh(4x))\,e^{-2x}}{32} \\ + \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right)e^{2x}}{8} \end{aligned} \tag{1}$$

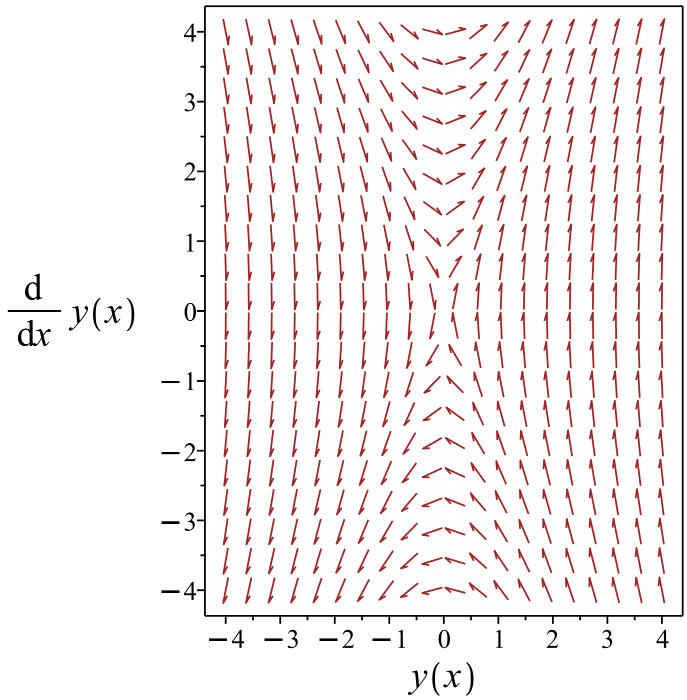


Figure 656: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} + \frac{(4x - 1 - \sinh(4x) - \cosh(4x))\, e^{-2x}}{32} + \frac{\left(x - \frac{\sinh(4x)}{4} + \frac{\cosh(4x)}{4}\right) e^{2x}}{8}$$

Verified OK.

11.27.3 Maple step by step solution

Let's solve

$y'' - 4y = \sinh(2x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r + 2) = 0$

- Roots of the characteristic polynomial

  $r = (-2, 2)$
- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$
- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$
- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$
- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x)f(x)}{W(y_1(x),y_2(x))}dx\right) + y_2(x)\left(\int \frac{y_1(x)f(x)}{W(y_1(x),y_2(x))}dx\right), f(x) = \sinh(2x)\right]$
  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{2x} \\ -2\,e^{-2x} & 2\,e^{2x} \end{bmatrix}$
  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$
  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-2x}\left(\int e^{2x}\sinh(2x)dx\right)}{4} + \frac{e^{2x}\left(\int e^{-2x}\sinh(2x)dx\right)}{4}$
  - Compute integrals

    $y_p(x) = \frac{(4x-1-\sinh(4x)-\cosh(4x))e^{-2x}}{32} + \frac{\left(x-\frac{\sinh(4x)}{4}+\frac{\cosh(4x)}{4}\right)e^{2x}}{8}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{2x} + \frac{(4x-1-\sinh(4x)-\cosh(4x))e^{-2x}}{32} + \frac{\left(x-\frac{\sinh(4x)}{4}+\frac{\cosh(4x)}{4}\right)e^{2x}}{8}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple
Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)-4*y(x)=sinh(2*x),y(x), singsol=all)
```

$$y(x) = \frac{\mathrm{e}^{2x}(4x + 32c_2 - 1)}{32} + \frac{\mathrm{e}^{-2x}(x + 8c_1)}{8}$$

✓ Solution by Mathematica
Time used: 0.049 (sec). Leaf size: 38

```
DSolve[y''[x]-4*y[x]==Sinh[2*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{32}e^{-2x}\left(4x + e^{4x}(4x - 1 + 32c_1) + 1 + 32c_2\right)$$

11.28 problem 51

Internal problem ID [246]
Internal file name [`OUTPUT/246_Sunday_June_05_2022_01_37_22_AM_52057044/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 51.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \cos(3x)}$$

11.28.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \cos(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i\\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i\\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3x) + A_2 \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(3x) - 5A_2 \sin(3x) = \cos(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(3x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(-\frac{\cos(3x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\cos(3x)}{5} \tag{1}$$

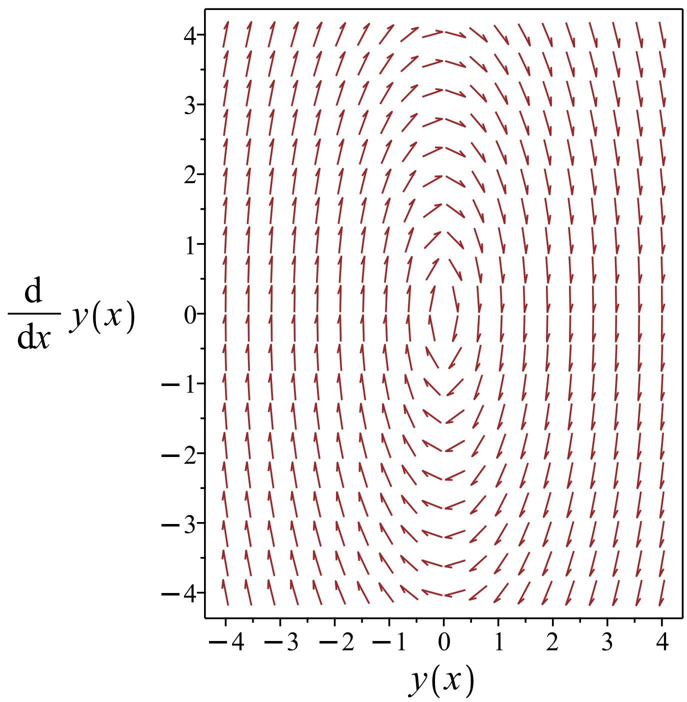


Figure 657: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\cos(3x)}{5}$$

Verified OK.

11.28.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 501: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\,dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\,dx \\ &= \cos(2x)\left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\cos(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 \cos(3x) + A_2 \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(3x) - 5A_2 \sin(3x) = \cos(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{5}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{\cos(3x)}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(-\frac{\cos(3x)}{5}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - \frac{\cos(3x)}{5} \tag{1}$$

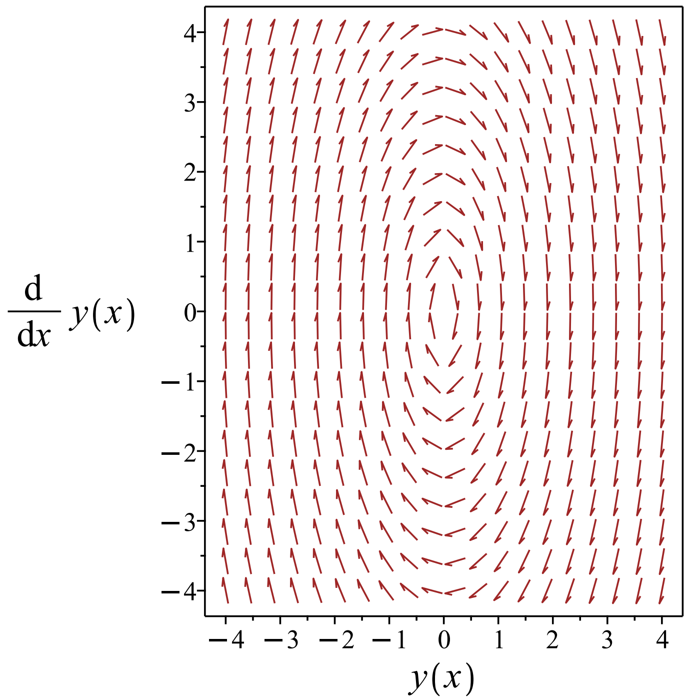


Figure 658: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} - \frac{\cos(3x)}{5}$$

Verified OK.

11.28.3 Maple step by step solution

Let's solve

$y'' + 4y = \cos(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

$r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \cos(3x)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{\cos(2x)\left(\int (\sin(5x) - \sin(x))dx\right)}{4} + \frac{\sin(2x)\left(\int (\cos(x) + \cos(5x))dx\right)}{4}$

  - Compute integrals

    $y_p(x) = -\frac{\cos(3x)}{5}$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{\cos(3x)}{5}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 23

```
dsolve(diff(y(x),x$2)+4*y(x)=cos(3*x),y(x), singsol=all)
```

$$y(x) = \sin(2x) c_2 + \cos(2x) c_1 - \frac{\cos(3x)}{5}$$

✓ Solution by Mathematica

Time used: 0.078 (sec). Leaf size: 28

```
DSolve[y''[x]+4*y[x]==Cos[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{5}\cos(3x) + c_1\cos(2x) + c_2\sin(2x)$$

11.29 problem 52

Internal problem ID [247]
Internal file name [`OUTPUT/247_Sunday_June_05_2022_01_37_23_AM_89292267/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 52.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = \sin(3x)}$$

11.29.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = \sin(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} + 9\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3x), \sin(3x)\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(3x), x\sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(3x) + A_2 x \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3x) + 6A_2 \cos(3x) = \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{6}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(3x)}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(3x) + c_2 \sin(3x)) + \left(-\frac{x \cos(3x)}{6}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{x \cos(3x)}{6} \tag{1}
$$

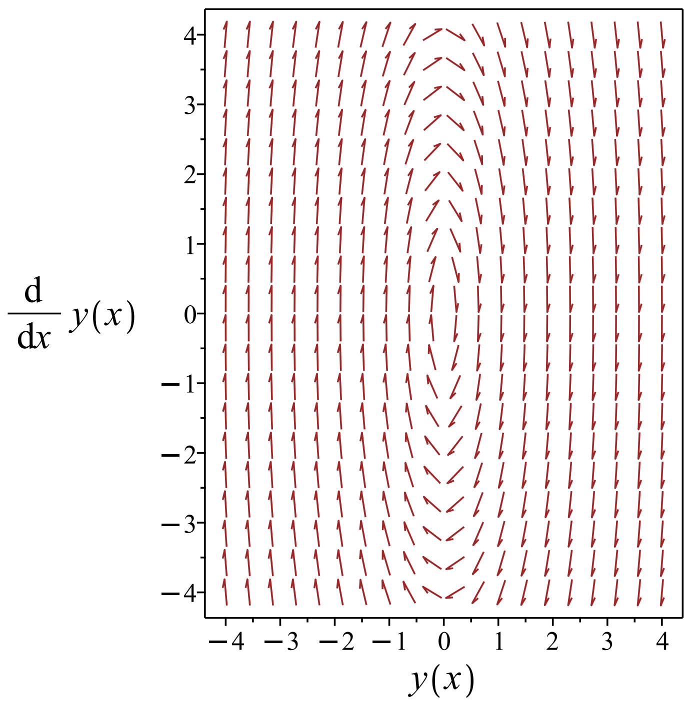


Figure 659: Slope field plot

Verification of solutions

$$
y = c_1 \cos(3x) + c_2 \sin(3x) - \frac{x \cos(3x)}{6}
$$

Verified OK.

11.29.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 503: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\
&= \cos(3x) \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2\left(\cos(3x) \left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(3x)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3x), \sin(3x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3x)}{3}, \cos(3x)\right\}$$

Since $\cos(3x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{x\cos(3x), x\sin(3x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 x \cos(3x) + A_2 x \sin(3x)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-6A_1 \sin(3x) + 6A_2 \cos(3x) = \sin(3x)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{6}, A_2 = 0\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\cos(3x)}{6}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}\right) + \left(-\frac{x\cos(3x)}{6}\right)
\end{aligned}
$$

Summary
The solution(s) found are the following

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - \frac{x\cos(3x)}{6} \tag{1}$$

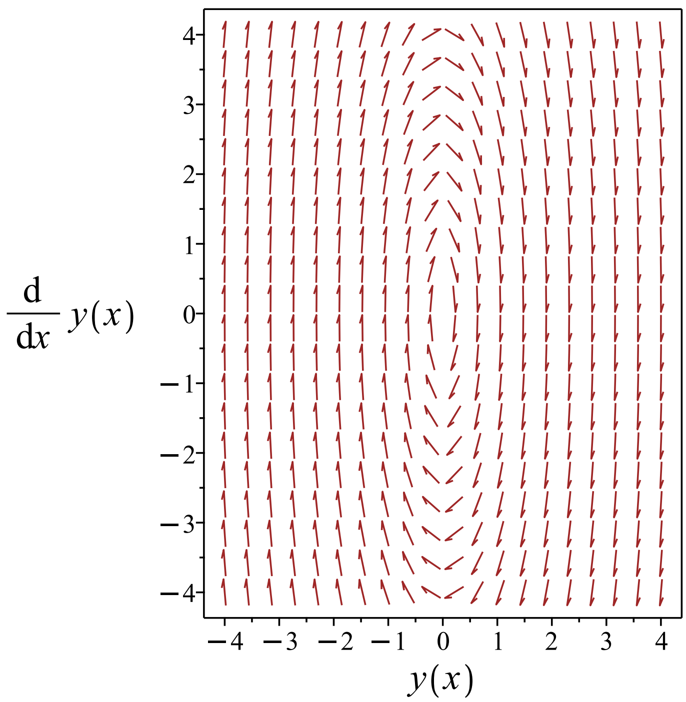


Figure 660: Slope field plot

Verification of solutions

$$y = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3} - \frac{x\cos(3x)}{6}$$

Verified OK.

11.29.3 Maple step by step solution

Let's solve

$y'' + 9y = \sin(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

- ○ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(3x)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

- ○ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

- ○ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(3x)\left(\int \sin(3x)^2 dx\right)}{3} + \frac{\sin(3x)\left(\int \sin(6x)dx\right)}{6}$$

- Compute integrals

$$y_p(x) = \frac{\sin(3x)}{36} - \frac{x\cos(3x)}{6}$$

- Substitute particular solution into general solution to ODE

$$y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{\sin(3x)}{36} - \frac{x\cos(3x)}{6}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 24

```
dsolve(diff(y(x),x$2)+9*y(x)=sin(3*x),y(x), singsol=all)
```

$$y(x) = \frac{(-x + 6c_1)\cos(3x)}{6} + \sin(3x)\, c_2$$

✓ Solution by Mathematica

Time used: 0.053 (sec). Leaf size: 33

```
DSolve[y''[x]+9*y[x]==Sin[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \left(-\frac{x}{6} + c_1\right)\cos(3x) + \frac{1}{36}(1 + 36c_2)\sin(3x)$$

11.30 problem 53

Internal problem ID [248]
Internal file name [`OUTPUT/248_Sunday_June_05_2022_01_37_24_AM_81343399/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 53.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 9y = 2\sec(3x)}$$

11.30.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 9, f(x) = 2\sec(3x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 9y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 9\, e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{0^2 - (4)\,(1)\,(9)}\\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i\\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i\\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(3x) + c_2 \sin(3x))$$

Or

$$y = c_1 \cos(3x) + c_2 \sin(3x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(3x) + c_2 \sin(3x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \sin(3x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}(\sin(3x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(3\cos(3x)) - (\sin(3x))(-3\sin(3x))$$

Which simplifies to

$$W = 3\cos(3x)^2 + 3\sin(3x)^2$$

Which simplifies to

$$W = 3$$

Therefore Eq. (2) becomes

$$u_1 = -\left(\int \frac{2\sin(3x)\sec(3x)}{3}dx\right)$$

Which simplifies to

$$u_1 = -\left(\int \frac{2\tan(3x)}{3}dx\right)$$

Hence

$$u_1 = -\frac{\ln(1+\tan(3x)^2)}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(3x)\sec(3x)}{3}dx$$

Which simplifies to

$$u_2 = \int \frac{2}{3}dx$$

Hence

$$u_2 = \frac{2x}{3}$$

Which simplifies to

$$u_1 = -\frac{\ln(\sec(3x)^2)}{9}$$
$$u_2 = \frac{2x}{3}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln\left(\sec\left(3x\right)^2\right)\cos\left(3x\right)}{9} + \frac{2x\sin\left(3x\right)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos\left(3x\right) + c_2\sin\left(3x\right)\right) + \left(-\frac{\ln\left(\sec\left(3x\right)^2\right)\cos\left(3x\right)}{9} + \frac{2x\sin\left(3x\right)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos\left(3x\right) + c_2\sin\left(3x\right) - \frac{\ln\left(\sec\left(3x\right)^2\right)\cos\left(3x\right)}{9} + \frac{2x\sin\left(3x\right)}{3} \tag{1}$$

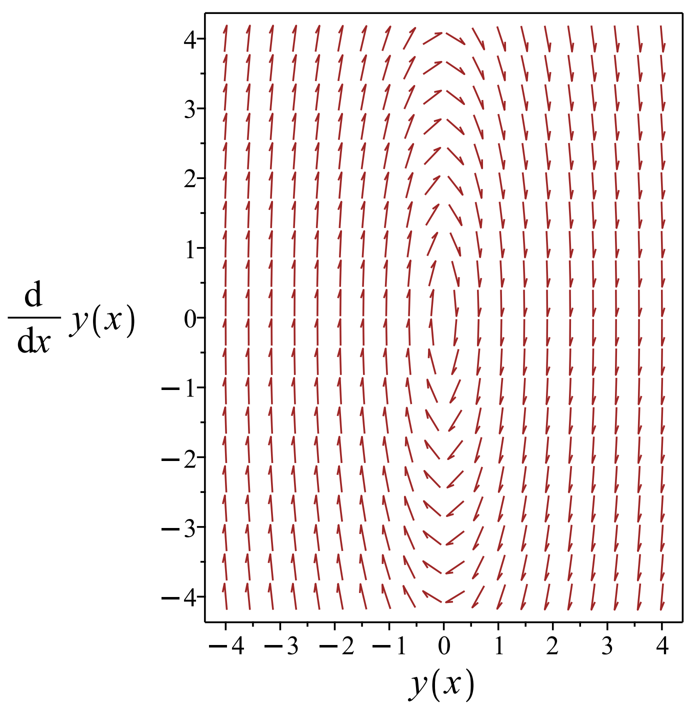


Figure 661: Slope field plot

Verification of solutions

$$y = c_1\cos\left(3x\right) + c_2\sin\left(3x\right) - \frac{\ln\left(\sec\left(3x\right)^2\right)\cos\left(3x\right)}{9} + \frac{2x\sin\left(3x\right)}{3}$$

Verified OK.

11.30.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 9y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -9z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\,e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 505: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(3x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
y_1 &= z_1 \\
&= \cos(3x)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \cos(3x)
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\
&= \cos(3x) \int \frac{1}{\cos(3x)^2}\, dx \\
&= \cos(3x) \left(\frac{\tan(3x)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1(\cos(3x)) + c_2\left(\cos(3x)\left(\frac{\tan(3x)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' + 9y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(3x) + \frac{c_2 \sin(3x)}{3}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(3x)$$
$$y_2 = \frac{\sin(3x)}{3}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ \frac{d}{dx}(\cos(3x)) & \frac{d}{dx}\left(\frac{\sin(3x)}{3}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(3x) & \frac{\sin(3x)}{3} \\ -3\sin(3x) & \cos(3x) \end{vmatrix}$$

Therefore

$$W = (\cos(3x))(\cos(3x)) - \left(\frac{\sin(3x)}{3}\right)(-3\sin(3x))$$

Which simplifies to

$$W = \cos(3x)^2 + \sin(3x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\frac{2\sin(3x)\sec(3x)}{3}}{1}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2\tan(3x)}{3}dx$$

Hence

$$u_1 = -\frac{\ln\left(1+\tan(3x)^2\right)}{9}$$

And Eq. (3) becomes

$$u_2 = \int \frac{2\cos(3x)\sec(3x)}{1}\,dx$$

Which simplifies to

$$u_2 = \int 2dx$$

Hence

$$u_2 = 2x$$

Which simplifies to

$$u_1 = -\frac{\ln\left(\sec(3x)^2\right)}{9}$$
$$u_2 = 2x$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{\ln\left(\sec(3x)^2\right)\cos(3x)}{9} + \frac{2x\sin(3x)}{3}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1\cos(3x) + \frac{c_2\sin(3x)}{3}\right) + \left(-\frac{\ln\left(\sec(3x)^2\right)\cos(3x)}{9} + \frac{2x\sin(3x)}{3}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1\cos(3x) + \frac{c_2\sin(3x)}{3} - \frac{\ln\left(\sec(3x)^2\right)\cos(3x)}{9} + \frac{2x\sin(3x)}{3} \tag{1}$$

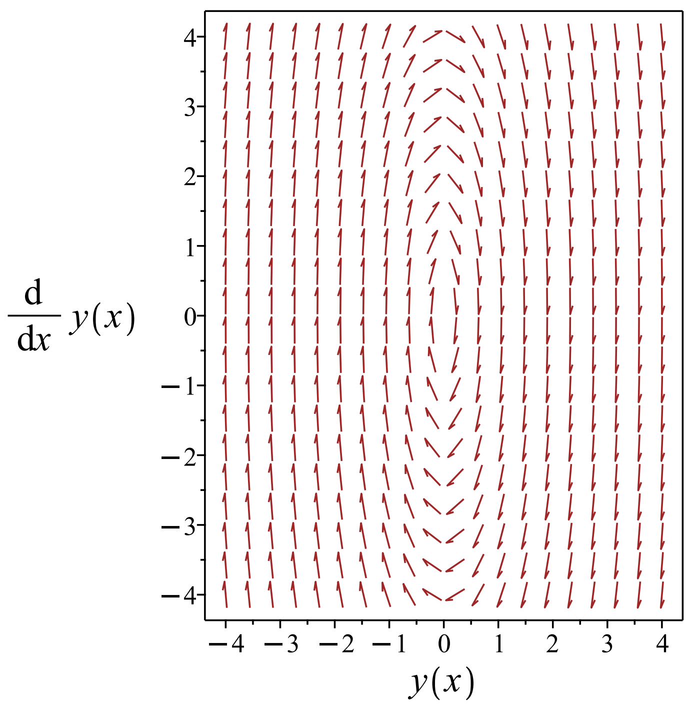


Figure 662: Slope field plot

Verification of solutions

$$y = c_1\cos(3x) + \frac{c_2\sin(3x)}{3} - \frac{\ln\left(\sec(3x)^2\right)\cos(3x)}{9} + \frac{2x\sin(3x)}{3}$$

Verified OK.

11.30.3 Maple step by step solution

Let's solve

$y'' + 9y = 2\sec(3x)$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(3x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(3x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(3x) + c_2 \sin(3x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

  $\left[y_p(x) = -y_1(x) \left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x) \left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = 2\sec(3x)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(y_1(x), y_2(x)) = 3$

  ◦ Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{2\cos(3x)\left(\int \tan(3x)dx\right)}{3} + \frac{2\sin(3x)\left(\int 1dx\right)}{3}$$

- Compute integrals

$$y_p(x) = -\frac{\ln\left(\sec(3x)^2\right)\cos(3x)}{9} + \frac{2x\sin(3x)}{3}$$

- Substitute particular solution into general solution to ODE

$$y = c_1\cos(3x) + c_2\sin(3x) - \frac{\ln\left(\sec(3x)^2\right)\cos(3x)}{9} + \frac{2x\sin(3x)}{3}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 33

```
dsolve(diff(y(x),x$2)+9*y(x)=2*sec(3*x),y(x), singsol=all)
```

$$y(x) = -\frac{2\ln(\sec(3x))\cos(3x)}{9} + \cos(3x)c_1 + \frac{2\sin(3x)\left(x + \frac{3c_2}{2}\right)}{3}$$

✓ Solution by Mathematica

Time used: 0.038 (sec). Leaf size: 39

```
DSolve[y''[x]+9*y[x]==2*Sec[3*x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{3}(2x + 3c_2)\sin(3x) + \cos(3x)\left(\frac{2}{9}\log(\cos(3x)) + c_1\right)$$

11.31 problem 54

Internal problem ID [249]
Internal file name [`OUTPUT/249_Sunday_June_05_2022_01_37_25_AM_14519892/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 54.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + y = \csc(x)^2}$$

11.31.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 1, f(x) = \csc(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 1$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(1)}\\ &= \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +i\\ \lambda_2 &= -i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= i\\ \lambda_2 &= -i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(x) + c_2 \sin(x))$$

Or

$$y = c_1 \cos(x) + c_2 \sin(x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \sin(x)^2 + \cos(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)^2}{1} dx$$

Which simplifies to

$$u_1 = -\int \csc(x)\, dx$$

Hence

$$u_1 = \ln(\csc(x) + \cot(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)^2}{1} dx$$

Which simplifies to

$$u_2 = \int \csc(x)\cot(x)\, dx$$

Hence

$$u_2 = -\csc(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\csc(x) + \cot(x))\cos(x) - \csc(x)\sin(x)$$

Which simplifies to

$$y_p(x) = \ln(\csc(x) + \cot(x))\cos(x) - 1$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 \cos(x) + c_2 \sin(x)) + (\ln(\csc(x) + \cot(x)) \cos(x) - 1)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
y = c_1 \cos(x) + c_2 \sin(x) + \ln(\csc(x) + \cot(x)) \cos(x) - 1 \tag{1}
$$

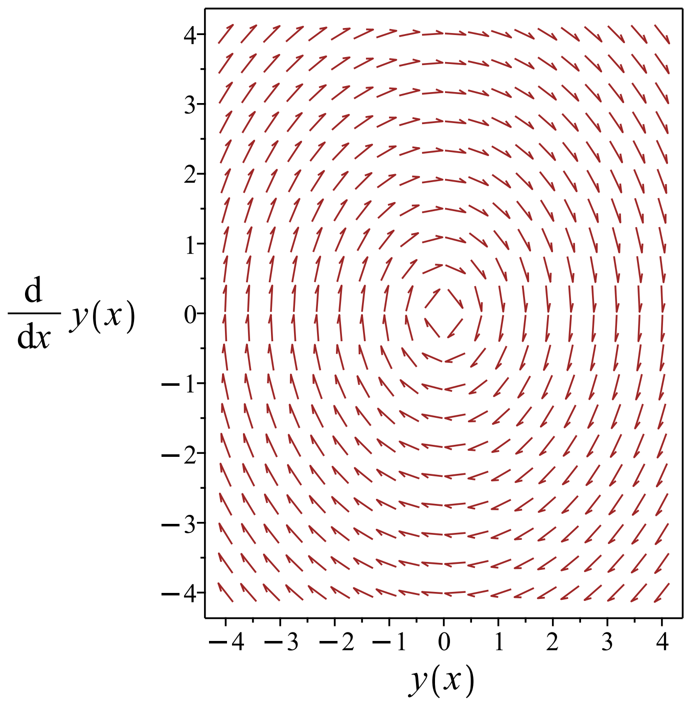


Figure 663: Slope field plot

Verification of solutions

$$
y = c_1 \cos(x) + c_2 \sin(x) + \ln(\csc(x) + \cot(x)) \cos(x) - 1
$$

Verified OK.

11.31.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} \, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x) \, e^{-\int \frac{B}{2A} \, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 507: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -1$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = \cos(x)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A}\, dx}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(x) \int \frac{1}{\cos(x)^2}\, dx \\ &= \cos(x)\,(\tan(x)) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(x)) + c_2(\cos(x)\,(\tan(x))) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(x)$$
$$y_2 = \sin(x)$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ \frac{d}{dx}(\cos(x)) & \frac{d}{dx}(\sin(x)) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix}$$

Therefore

$$W = (\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))$$

Which simplifies to

$$W = \sin(x)^2 + \cos(x)^2$$

Which simplifies to

$$W = 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(x)\csc(x)^2}{1} dx$$

Which simplifies to

$$u_1 = -\int \csc(x) dx$$

Hence

$$u_1 = \ln(\csc(x) + \cot(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(x)\csc(x)^2}{1} dx$$

Which simplifies to

$$u_2 = \int \csc(x)\cot(x) dx$$

Hence

$$u_2 = -\csc(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \ln(\csc(x) + \cot(x))\cos(x) - \csc(x)\sin(x)$$

Which simplifies to

$$y_p(x) = \ln(\csc(x) + \cot(x))\cos(x) - 1$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(x) + c_2\sin(x)) + (\ln(\csc(x) + \cot(x))\cos(x) - 1) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\csc(x) + \cot(x)) \cos(x) - 1 \tag{1}$$

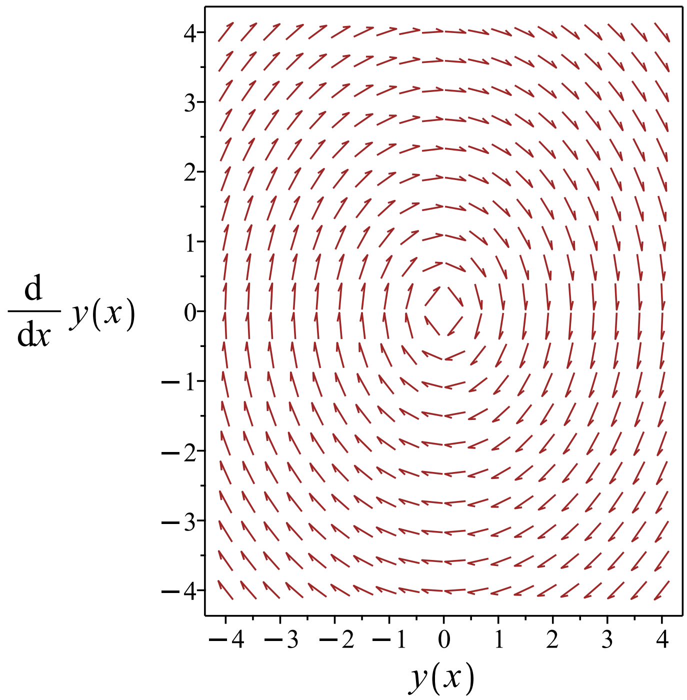


Figure 664: Slope field plot

Verification of solutions

$$y = c_1 \cos(x) + c_2 \sin(x) + \ln(\csc(x) + \cot(x)) \cos(x) - 1$$

Verified OK.

11.31.3 Maple step by step solution

Let's solve

$y'' + y = \csc(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 1 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-4})}{2}$

- Roots of the characteristic polynomial

  $r = (-\mathrm{I}, \mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  ◦ Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \csc(x)^2\right]$

  ◦ Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{bmatrix}$

  ◦ Compute Wronskian

    $W(y_1(x), y_2(x)) = 1$

  ◦ Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\cos(x)\left(\int \csc(x)\, dx\right) + \sin(x)\left(\int \csc(x)\cot(x)\, dx\right)$

  ◦ Compute integrals

    $y_p(x) = \ln(\csc(x) + \cot(x))\cos(x) - 1$

- Substitute particular solution into general solution to ODE

  $y = c_1 \cos(x) + c_2 \sin(x) + \ln(\csc(x) + \cot(x))\cos(x) - 1$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 26

```
dsolve(diff(y(x),x$2)+y(x)=csc(x)^2,y(x), singsol=all)
```

$$y(x) = \sin(x) c_2 + \cos(x) c_1 - 1 - \ln(\csc(x) - \cot(x)) \cos(x)$$

✓ Solution by Mathematica

Time used: 0.034 (sec). Leaf size: 23

```
DSolve[y''[x]+y[x]==Csc[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(x)\text{arctanh}(\cos(x)) + c_1 \cos(x) + c_2 \sin(x) - 1$$

11.32 problem 55

Internal problem ID [250]
Internal file name [`OUTPUT/250_Sunday_June_05_2022_01_37_26_AM_59411339/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 55.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' + 4y = \sin(x)^2}$$

11.32.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = 4, f(x) = \sin(x)^2$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda x} + 4 e^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)} \\ &= \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2i \\ \lambda_2 &= -2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2i \\ \lambda_2 &= -2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$y = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

Which becomes

$$y = e^0(c_1 \cos(2x) + c_2 \sin(2x))$$

Or

$$y = c_1 \cos(2x) + c_2 \sin(2x)$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos(2x) + c_2 \sin(2x)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2x), \sin(2x)\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x \cos(2x) + A_3 x \sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2 \sin(2x) + 4A_3 \cos(2x) + 4A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = -\frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{8} - \frac{x\sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(2x) + c_2 \sin(2x)) + \left(\frac{1}{8} - \frac{x \sin(2x)}{8}\right) \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{8} - \frac{x \sin(2x)}{8} \tag{1}$$

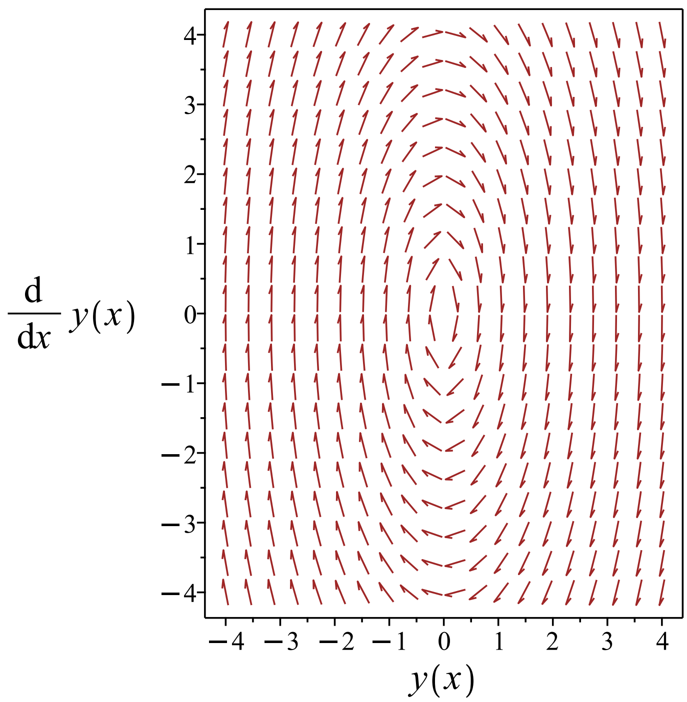


Figure 665: Slope field plot

<u>Verification of solutions</u>

$$y = c_1 \cos(2x) + c_2 \sin(2x) + \frac{1}{8} - \frac{x \sin(2x)}{8}$$

Verified OK.

11.32.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' + 4y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = -4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 509: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \cos(2x)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \cos(2x) \end{aligned}$$

Which simplifies to

$$y_1 = \cos(2x)$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \cos(2x) \int \frac{1}{\cos(2x)^2}\, dx \\ &= \cos(2x) \left(\frac{\tan(2x)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} y &= c_1 y_1 + c_2 y_2 \\ &= c_1(\cos(2x)) + c_2\left(\cos(2x)\left(\frac{\tan(2x)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$y'' + 4y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$\sin(x)^2$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{1\}, \{\cos(2x), \sin(2x)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2x)}{2}, \cos(2x)\right\}$$

Since $\cos(2x)$ is duplicated in the UC_set, then this basis is multiplied by extra x. The UC_set becomes

$$[\{1\}, \{x\cos(2x), x\sin(2x)\}]$$

Since there was duplication between the basis functions in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis function in the above updated UC_set.

$$y_p = A_1 + A_2 x\cos(2x) + A_3 x\sin(2x)$$

The unknowns $\{A_1, A_2, A_3\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_2\sin(2x) + 4A_3\cos(2x) + 4A_1 = \sin(x)^2$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{1}{8}, A_2 = 0, A_3 = -\frac{1}{8}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = \frac{1}{8} - \frac{x\sin(2x)}{8}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2}\right) + \left(\frac{1}{8} - \frac{x \sin(2x)}{8}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{1}{8} - \frac{x \sin(2x)}{8} \tag{1}$$

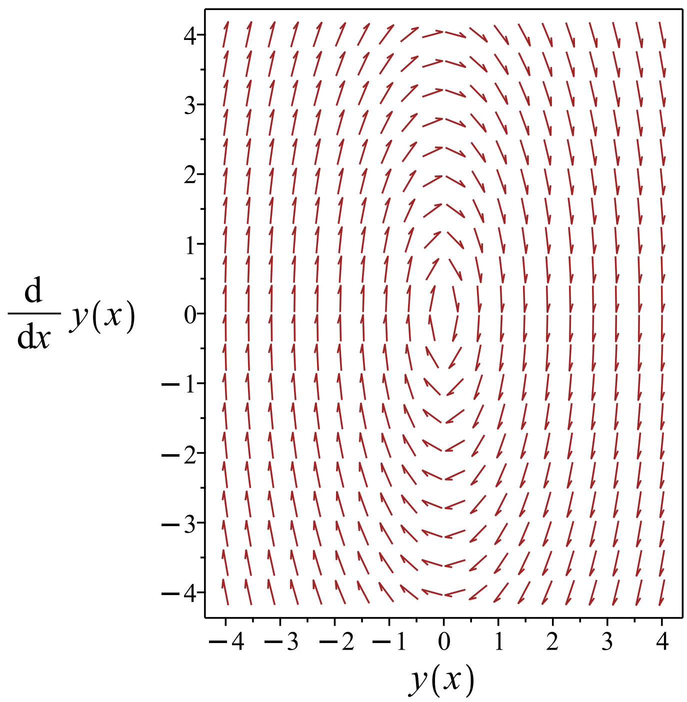


Figure 666: Slope field plot

Verification of solutions

$$y = c_1 \cos(2x) + \frac{c_2 \sin(2x)}{2} + \frac{1}{8} - \frac{x \sin(2x)}{8}$$

Verified OK.

11.32.3 Maple step by step solution

Let's solve

$y'' + 4y = \sin(x)^2$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $y_1(x) = \cos(2x)$

- 2nd solution of the homogeneous ODE

  $y_2(x) = \sin(2x)$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 \cos(2x) + c_2 \sin(2x) + y_p(x)$

□ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = \sin(x)^2\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 2$

  - Substitute functions into equation for $y_p(x)$

$$y_p(x) = -\frac{\cos(2x)\left(\int \sin(2x)\sin(x)^2 dx\right)}{2} + \frac{\sin(2x)\left(\int \cos(2x)\sin(x)^2 dx\right)}{2}$$

- Compute integrals

  $$y_p(x) = -\frac{x\sin(2x)}{8} + \frac{1}{8} - \frac{\cos(2x)}{8}$$

- Substitute particular solution into general solution to ODE

  $$y = c_1 \cos(2x) + c_2 \sin(2x) - \frac{x\sin(2x)}{8} + \frac{1}{8} - \frac{\cos(2x)}{8}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 30

```
dsolve(diff(y(x),x$2)+4*y(x)=sin(x)^2,y(x), singsol=all)
```

$$y(x) = \frac{(8c_1 - 1)\cos(2x)}{8} + \frac{1}{8} + \frac{(-x + 8c_2)\sin(2x)}{8}$$

✓ Solution by Mathematica

Time used: 0.111 (sec). Leaf size: 71

```
DSolve[y''[x]+4*y[x]==sin[x]^2,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \cos(2x)\int_1^x -\cos(K[1])\sin(K[1])^2\sin(K[1])dK[1] + \sin(2x)\int_1^x \frac{1}{2}\cos(2K[2])\sin(K[2])^2 dK[2] + c_1\cos(2x) + c_2\sin(2x)$$

11.33 problem 56

Internal problem ID [251]
Internal file name [`OUTPUT/251_Sunday_June_05_2022_01_37_28_AM_63226036/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 56.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{y'' - 4y = x\,\mathrm{e}^{x}}$$

11.33.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 1, B = 0, C = -4, f(x) = x\,\mathrm{e}^{x}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$y'' - 4y = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = 0$$

Where in the above $A = 1, B = 0, C = -4$. Let the solution be $y = e^{\lambda x}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda x} - 4\,\mathrm{e}^{\lambda x} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda x}$ gives

$$\lambda^2 - 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-4)} \\ &= \pm 2\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +2 \\ \lambda_2 &= -2\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 2 \\ \lambda_2 &= -2\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y &= c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \\ y &= c_1 e^{(2)x} + c_2 e^{(-2)x}\end{aligned}$$

Or

$$y = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \mathrm{e}^{2x} + c_2 \mathrm{e}^{-2x}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$x\,e^{x}$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{x\,e^{x}, e^{x}\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2x}, e^{2x}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$y_p = A_1 x\,e^{x} + A_2 e^{x}$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^{x} - 3A_1 x\,e^{x} - 3A_2 e^{x} = x\,e^{x}$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = -\frac{2}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\,e^{x}}{3} - \frac{2\,e^{x}}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{2x} + c_2 e^{-2x}\right) + \left(-\frac{x\,e^{x}}{3} - \frac{2\,e^{x}}{9}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x\,e^{x}}{3} - \frac{2\,e^{x}}{9} \tag{1}$$

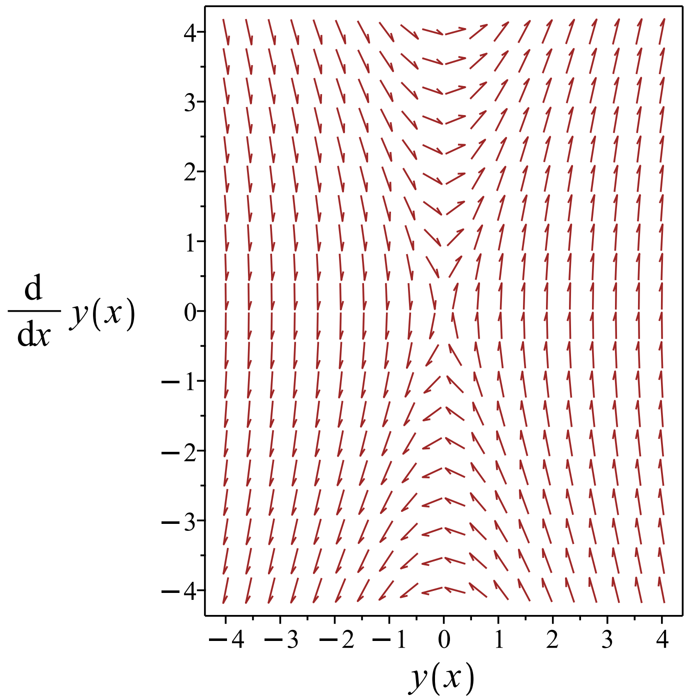


Figure 667: Slope field plot

Verification of solutions

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x\,e^{x}}{3} - \frac{2\,e^{x}}{9}$$

Verified OK.

11.33.2 Solving using Kovacic algorithm

Writing the ode as

$$y'' - 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= -4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 4z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 511: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 4$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = \mathrm{e}^{-2x}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$y_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dx}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} y_1 &= z_1 \\ &= \mathrm{e}^{-2x} \end{aligned}$$

Which simplifies to

$$y_1 = \mathrm{e}^{-2x}$$

The second solution y_2 to the original ode is found using reduction of order

$$y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx$$

Since $B = 0$ then the above becomes

$$\begin{aligned} y_2 &= y_1 \int \frac{1}{y_1^2}\, dx \\ &= \mathrm{e}^{-2x} \int \frac{1}{\mathrm{e}^{-4x}}\, dx \\ &= \mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\mathrm{e}^{-2x}\right) + c_2 \left(\mathrm{e}^{-2x} \left(\frac{\mathrm{e}^{4x}}{4}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
y'' - 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_1 \mathrm{e}^{-2x} + \frac{c_2 \mathrm{e}^{2x}}{4}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
x\,\mathrm{e}^{x}
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{x\,\mathrm{e}^{x}, \mathrm{e}^{x}\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{\mathrm{e}^{2x}}{4}, \mathrm{e}^{-2x}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
y_p = A_1 x\,\mathrm{e}^{x} + A_2 \mathrm{e}^{x}
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution y_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$2A_1 e^x - 3A_1 x\, e^x - 3A_2 e^x = x\, e^x$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{3}, A_2 = -\frac{2}{9}\right]$$

Substituting the above back in the above trial solution y_p, gives the particular solution

$$y_p = -\frac{x\, e^x}{3} - \frac{2\, e^x}{9}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 e^{-2x} + \frac{c_2 e^{2x}}{4}\right) + \left(-\frac{x\, e^x}{3} - \frac{2\, e^x}{9}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - \frac{x\, e^x}{3} - \frac{2\, e^x}{9} \tag{1}$$

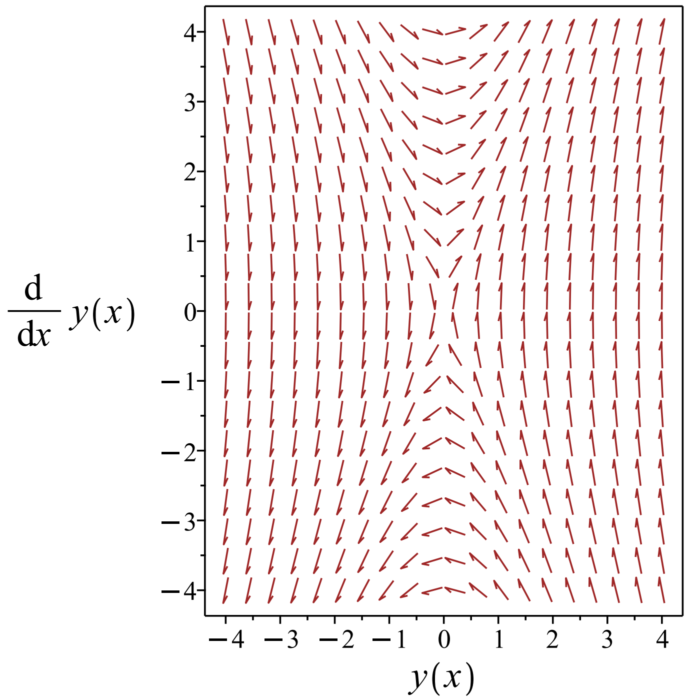


Figure 668: Slope field plot

Verification of solutions

$$y = c_1 e^{-2x} + \frac{c_2 e^{2x}}{4} - \frac{x\,e^{x}}{3} - \frac{2\,e^{x}}{9}$$

Verified OK.

11.33.3 Maple step by step solution

Let's solve

$y'' - 4y = x\,e^{x}$

- Highest derivative means the order of the ODE is 2

  y''

- Characteristic polynomial of homogeneous ODE

  $r^2 - 4 = 0$

- Factor the characteristic polynomial

  $(r - 2)(r + 2) = 0$

- Roots of the characteristic polynomial

$r = (-2, 2)$

- 1st solution of the homogeneous ODE

  $y_1(x) = e^{-2x}$

- 2nd solution of the homogeneous ODE

  $y_2(x) = e^{2x}$

- General solution of the ODE

  $y = c_1 y_1(x) + c_2 y_2(x) + y_p(x)$

- Substitute in solutions of the homogeneous ODE

  $y = c_1 e^{-2x} + c_2 e^{2x} + y_p(x)$

☐ Find a particular solution $y_p(x)$ of the ODE

  - Use variation of parameters to find y_p here $f(x)$ is the forcing function

    $\left[y_p(x) = -y_1(x)\left(\int \frac{y_2(x) f(x)}{W(y_1(x), y_2(x))} dx\right) + y_2(x)\left(\int \frac{y_1(x) f(x)}{W(y_1(x), y_2(x))} dx\right), f(x) = x\,e^{x}\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(y_1(x), y_2(x)) = \begin{bmatrix} e^{-2x} & e^{2x} \\ -2\,e^{-2x} & 2\,e^{2x} \end{bmatrix}$

  - Compute Wronskian

    $W(y_1(x), y_2(x)) = 4$

  - Substitute functions into equation for $y_p(x)$

    $y_p(x) = -\frac{e^{-2x}\left(\int x\,e^{3x} dx\right)}{4} + \frac{e^{2x}\left(\int x\,e^{-x} dx\right)}{4}$

  - Compute integrals

    $y_p(x) = -\frac{e^{x}(3x+2)}{9}$

- Substitute particular solution into general solution to ODE

  $y = c_1 e^{-2x} + c_2 e^{2x} - \frac{e^{x}(3x+2)}{9}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 34

```
dsolve(diff(y(x),x$2)-4*y(x)=x*exp(x),y(x), singsol=all)
```

$$y(x) = -\frac{\left(-9\,\mathrm{e}^{4x} c_2 + 3x\,\mathrm{e}^{3x} + 2\,\mathrm{e}^{3x} - 9c_1\right)\mathrm{e}^{-2x}}{9}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 34

```
DSolve[y''[x]-4*y[x]==x*Exp[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to -\frac{1}{9}e^x(3x+2) + c_1 e^{2x} + c_2 e^{-2x}$$

11.34 problem 57

Internal problem ID [252]
Internal file name [`OUTPUT/252_Sunday_June_05_2022_01_37_29_AM_43258265/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 57.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "exact linear second order ode", "second_order_integrable_as_is", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _exact, _linear, _nonhomogeneous]]
```

$$\boxed{x^2y'' + y'x - y = 72x^5}$$

11.34.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = 72x^5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x - y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} - x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r - x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r - 1 = 0$$

Or

$$r^2 - 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = -1$$
$$r_2 = 1$$

Since the roots are real and distinct, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. Hence

$$y = \frac{c_1}{x} + c_2x$$

Next, we find the particular solution to the ODE

$$x^2y'' + y'x - y = 72x^5$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = x$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & x \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}(x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & x \\ -\frac{1}{x^2} & 1 \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)(1) - (x)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{2}{x}$$

Which simplifies to

$$W = \frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{72x^6}{2x}\,dx$$

Which simplifies to

$$u_1 = -\int 36x^5 dx$$

Hence

$$u_1 = -6x^6$$

And Eq. (3) becomes

$$u_2 = \int \frac{72x^4}{2x}\,dx$$

Which simplifies to

$$u_2 = \int 36x^3 dx$$

Hence

$$u_2 = 9x^4$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^5$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= 3x^5 + \frac{c_1}{x} + c_2 x \end{aligned}$$

Summary

The solution(s) found are the following

$$y = 3x^5 + \frac{c_1}{x} + c_2 x \tag{1}$$

Verification of solutions

$$y = 3x^5 + \frac{c_1}{x} + c_2 x$$

Verified OK.

11.34.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2 y'' + y'x - y = 0$$

In normal form the ode

$$x^2 y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\, \tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{-\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= -1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A = 1, B = 0, C = -1$. Let the solution be $y(\tau) = e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau} - \mathrm{e}^{\lambda\tau} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2 - 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = -1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(-1)}\\ &= \pm 1\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +1\\ \lambda_2 &= -1\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 1\\ \lambda_2 &= -1\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}y(\tau) &= c_1 e^{\lambda_1 \tau} + c_2 e^{\lambda_2 \tau}\\ y(\tau) &= c_1 e^{(1)\tau} + c_2 e^{(-1)\tau}\end{aligned}$$

Or

$$y(\tau) = c_1 \mathrm{e}^{\tau} + c_2 \mathrm{e}^{-\tau}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 x^2 + c_2}{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 x^2 + c_2}{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{72x^4}{-2x} dx$$

Which simplifies to

$$u_1 = -\int -36x^3 dx$$

Hence

$$u_1 = 9x^4$$

And Eq. (3) becomes

$$u_2 = \int \frac{72x^6}{-2x} dx$$

Which simplifies to

$$u_2 = \int -36x^5 dx$$

Hence

$$u_2 = -6x^6$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^5$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 x^2 + c_2}{x}\right) + (3x^5) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 x^2 + c_2}{x} + 3x^5 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{c_1 x^2 + c_2}{x} + 3x^5$$

Verified OK.

11.34.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = 72x^5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x - y = 0$$

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= -\frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{-\frac{1}{x^2}}}{c} \\
\tau'' &= \frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{c\sqrt{-\frac{1}{x^2}}\, x^3} + \frac{1}{x}\frac{\sqrt{-\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{-\frac{1}{x^2}}}{c}\right)^2} \\
&= 0
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{-\frac{1}{x^2}} dx}{c} \\
&= \frac{\sqrt{-\frac{1}{x^2}}\, x \ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Now the particular solution to this ODE is found

$$x^2y'' + y'x - y = 72x^5$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{72x^4}{-2x} dx$$

Which simplifies to

$$u_1 = -\int -36x^3 dx$$

Hence

$$u_1 = 9x^4$$

And Eq. (3) becomes

$$u_2 = \int \frac{72x^6}{-2x} dx$$

Which simplifies to

$$u_2 = \int -36x^5 dx$$

Hence

$$u_2 = -6x^6$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^5$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}\right) + (3x^5) \\ &= 3x^5 + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \end{aligned}$$

Which simplifies to

$$y = 3x^5 + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Summary

The solution(s) found are the following

$$y = 3x^5 + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x} \tag{1}$$

Verification of solutions

$$y = 3x^5 + \frac{(ic_2 + c_1)\, x^2 - ic_2 + c_1}{2x}$$

Verified OK.

11.34.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = -1, f(x) = 72x^5$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x - y = 0$$

In normal form the ode

$$x^2y'' + y'x - y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = \frac{1}{x}$$
$$q(x) = -\frac{1}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} - \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{3v'(x)}{x} &= 0 \\ v''(x) + \frac{3v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{3u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{3u}{x} \end{aligned}$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$\frac{1}{u}\,du = -\frac{3}{x}\,dx$$
$$\int \frac{1}{u}\,du = \int -\frac{3}{x}\,dx$$
$$\ln(u) = -3\ln(x) + c_1$$
$$u = e^{-3\ln(x)+c_1}$$
$$= \frac{c_1}{x^3}$$

Now that $u(x)$ is known, then

$$v'(x) = u(x)$$
$$v(x) = \int u(x)\,dx + c_2$$
$$= -\frac{c_1}{2x^2} + c_2$$

Hence

$$y = v(x)\,x^n$$
$$= \left(-\frac{c_1}{2x^2} + c_2\right)x$$
$$= \left(-\frac{c_1}{2x^2} + c_2\right)x$$

Now the particular solution to this ODE is found

$$x^2y'' + y'x - y = 72x^5$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = \frac{1}{x}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{72x^4}{-2x}\,dx$$

Which simplifies to

$$u_1 = -\int -36x^3 dx$$

Hence

$$u_1 = 9x^4$$

And Eq. (3) becomes

$$u_2 = \int \frac{72x^6}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int -36x^5 dx$$

Hence

$$u_2 = -6x^6$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^5$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + \left(3x^5\right) \\ &= 3x^5 + \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

Which simplifies to

$$y = 3x^5 + \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Summary

The solution(s) found are the following

$$y = 3x^5 + \left(-\frac{c_1}{2x^2} + c_2\right) x \tag{1}$$

Verification of solutions

$$y = 3x^5 + \left(-\frac{c_1}{2x^2} + c_2\right) x$$

Verified OK.

11.34.5 Solving as second order integrable as is ode

Integrating both sides of the ODE w.r.t x gives

$$
\begin{aligned}
\int \left(x^2 y'' + y' x - y\right) dx &= \int 72 x^5 dx \\
y' x^2 - y x &= 12 x^6 + c_1
\end{aligned}
$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$
y' + p(x) y = q(x)
$$

Where here

$$
\begin{aligned}
p(x) &= -\frac{1}{x} \\
q(x) &= \frac{12 x^6 + c_1}{x^2}
\end{aligned}
$$

Hence the ode is

$$
y' - \frac{y}{x} = \frac{12 x^6 + c_1}{x^2}
$$

The integrating factor μ is

$$
\begin{aligned}
\mu &= e^{\int -\frac{1}{x} dx} \\
&= \frac{1}{x}
\end{aligned}
$$

The ode becomes

$$
\begin{aligned}
\frac{d}{dx}(\mu y) &= (\mu) \left(\frac{12 x^6 + c_1}{x^2}\right) \\
\frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right) \left(\frac{12 x^6 + c_1}{x^2}\right) \\
d\left(\frac{y}{x}\right) &= \left(\frac{12 x^6 + c_1}{x^3}\right) dx
\end{aligned}
$$

Integrating gives

$$
\begin{aligned}
\frac{y}{x} &= \int \frac{12 x^6 + c_1}{x^3} dx \\
\frac{y}{x} &= 3 x^4 - \frac{c_1}{2 x^2} + c_2
\end{aligned}
$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2x$$

Summary

The solution(s) found are the following

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2x \tag{1}$$

Verification of solutions

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2x$$

Verified OK.

11.34.6 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$Ay'' + By' + Cy = F(x)$$

This method reduces the order ode the ODE by one by applying the transformation

$$y = Bv$$

This results in

$$\begin{aligned} y' &= B'v + v'B \\ y'' &= B''v + B'v' + v''B + v'B' \\ &= v''B + 2v' + B' + B''v \end{aligned}$$

And now the original ode becomes

$$\begin{aligned} A(v''B + 2v'B' + B''v) + B(B'v + v'B) + CBv &= 0 \\ ABv'' + \left(2AB' + B^2\right)v' + \left(AB'' + BB' + CB\right)v &= 0 \end{aligned} \tag{1}$$

If the term $AB'' + BB' + CB$ is zero, then this method works and can be used to solve

$$ABv'' + \left(2AB' + B^2\right)v' = 0$$

By Using $u = v'$ which reduces the order of the above ode to one. The new ode is

$$ABu' + \left(2AB' + B^2\right)u = 0$$

The above ode is first order ode which is solved for u. Now a new ode $v' = u$ is solved for v as first order ode. Then the final solution is obtain from $y = Bv$.

This method works only if the term $AB'' + BB' + CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A &= x^2 \\
B &= x \\
C &= -1 \\
F &= 72x^5
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB'' + BB' + CB &= \left(x^2\right)(0) + (x)(1) + (-1)(x) \\
&= 0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
x^3 v'' + \left(3x^2\right) v' = 0
$$

Now by applying $v' = u$ the above becomes

$$
x^2 (u'(x)\, x + 3u(x)) = 0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u' &= F(x, u) \\
&= f(x) g(u) \\
&= -\frac{3u}{x}
\end{aligned}
$$

Where $f(x) = -\frac{3}{x}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= -\frac{3}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{3}{x}\, dx \\
\ln(u) &= -3\ln(x) + c_1 \\
u &= \mathrm{e}^{-3\ln(x) + c_1} \\
&= \frac{c_1}{x^3}
\end{aligned}
$$

The ode for v now becomes

$$\begin{aligned} v' &= u \\ &= \frac{c_1}{x^3} \end{aligned}$$

Which is now solved for v. Integrating both sides gives

$$\begin{aligned} v(x) &= \int \frac{c_1}{x^3} \, \mathrm{d}x \\ &= -\frac{c_1}{2x^2} + c_2 \end{aligned}$$

Therefore the homogeneous solution is

$$\begin{aligned} y_h(x) &= Bv \\ &= (x)\left(-\frac{c_1}{2x^2} + c_2\right) \\ &= \left(-\frac{c_1}{2x^2} + c_2\right) x \end{aligned}$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= \frac{1}{x} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & \frac{1}{x} \\ \frac{d}{dx}(x) & \frac{d}{dx}\left(\frac{1}{x}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix}$$

Therefore

$$W = (x)\left(-\frac{1}{x^2}\right) - \left(\frac{1}{x}\right)(1)$$

Which simplifies to

$$W = -\frac{2}{x}$$

Which simplifies to

$$W = -\frac{2}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{72x^4}{-2x}\, dx$$

Which simplifies to

$$u_1 = -\int -36x^3 dx$$

Hence

$$u_1 = 9x^4$$

And Eq. (3) becomes

$$u_2 = \int \frac{72x^6}{-2x}\, dx$$

Which simplifies to

$$u_2 = \int -36x^5 dx$$

Hence

$$u_2 = -6x^6$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^5$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{2x^2} + c_2\right) x\right) + (3x^5) \\ &= \frac{6x^6 + 2c_2x^2 - c_1}{2x} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{6x^6 + 2c_2x^2 - c_1}{2x} \tag{1}$$

Verification of solutions

$$y = \frac{6x^6 + 2c_2x^2 - c_1}{2x}$$

Verified OK.

11.34.7 Solving as type second_order_integrable_as_is (not using ABC version)

Writing the ode as

$$x^2y'' + y'x - y = 72x^5$$

Integrating both sides of the ODE w.r.t x gives

$$\begin{aligned} \int (x^2y'' + y'x - y)\, dx &= \int 72x^5 dx \\ y'x^2 - yx &= 12x^6 + c_1 \end{aligned}$$

Which is now solved for y.

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y' + p(x)y = q(x)$$

Where here

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{12x^6 + c_1}{x^2} \end{aligned}$$

Hence the ode is

$$y' - \frac{y}{x} = \frac{12x^6 + c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned} \mu &= e^{\int -\frac{1}{x}dx} \\ &= \frac{1}{x} \end{aligned}$$

The ode becomes

$$\begin{aligned} \frac{d}{dx}(\mu y) &= (\mu)\left(\frac{12x^6 + c_1}{x^2}\right) \\ \frac{d}{dx}\left(\frac{y}{x}\right) &= \left(\frac{1}{x}\right)\left(\frac{12x^6 + c_1}{x^2}\right) \\ d\left(\frac{y}{x}\right) &= \left(\frac{12x^6 + c_1}{x^3}\right) dx \end{aligned}$$

Integrating gives

$$\begin{aligned} \frac{y}{x} &= \int \frac{12x^6 + c_1}{x^3}\, dx \\ \frac{y}{x} &= 3x^4 - \frac{c_1}{2x^2} + c_2 \end{aligned}$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2 x$$

Summary
The solution(s) found are the following

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2 x \tag{1}$$

Verification of solutions

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2 x$$

Verified OK.

11.34.8 Solving using Kovacic algorithm

Writing the ode as

$$x^2 y'' + y'x - y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= -1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 513: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = \frac{3}{4x^2}$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$r = \frac{s}{t} = \frac{3}{4x^2}$$

Since the $\gcd(s,t) = 1$. This gives $b = \frac{3}{4}$. Hence

$$\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}
\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{4x^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^- = -\frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - \left(\alpha_{c_1}^-\right) \\ &= -\frac{1}{2} - \left(-\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma}\left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x-c_1}\right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2x} + (-)\,(0) \\ &= -\frac{1}{2x} \\ &= -\frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2x}\right)(0) + \left(\left(\frac{1}{2x^2}\right) + \left(-\frac{1}{2x}\right)^2 - \left(\frac{3}{4x^2}\right)\right) = 0$$

$$0 = 0$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega\,dx} \\ &= \mathrm{e}^{\int -\frac{1}{2x} dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \frac{1}{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(\frac{x^2}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(\frac{1}{x}\right) + c_2 \left(\frac{1}{x}\left(\frac{x^2}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + y'x - y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \frac{c_1}{x} + \frac{c_2 x}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \frac{1}{x}$$
$$y_2 = \frac{x}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ \frac{d}{dx}\left(\frac{1}{x}\right) & \frac{d}{dx}\left(\frac{x}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \frac{1}{x} & \frac{x}{2} \\ -\frac{1}{x^2} & \frac{1}{2} \end{vmatrix}$$

Therefore

$$W = \left(\frac{1}{x}\right)\left(\frac{1}{2}\right) - \left(\frac{x}{2}\right)\left(-\frac{1}{x^2}\right)$$

Which simplifies to

$$W = \frac{1}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{36x^6}{x} \, dx$$

Which simplifies to

$$u_1 = -\int 36x^5 dx$$

Hence

$$u_1 = -6x^6$$

And Eq. (3) becomes

$$u_2 = \int \frac{72x^4}{x} \, dx$$

Which simplifies to

$$u_2 = \int 72x^3 dx$$

Hence

$$u_2 = 18x^4$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = 3x^5$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1}{x} + \frac{c_2 x}{2}\right) + (3x^5) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + 3x^5 \tag{1}$$

Verification of solutions

$$y = \frac{c_1}{x} + \frac{c_2 x}{2} + 3x^5$$

Verified OK.

11.34.9 Solving as exact linear second order ode ode

An ode of the form

$$p(x)\,y'' + q(x)\,y' + r(x)\,y = s(x)$$

is exact if

$$p''(x) - q'(x) + r(x) = 0 \tag{1}$$

For the given ode we have

$$\begin{aligned} p(x) &= x^2 \\ q(x) &= x \\ r(x) &= -1 \\ s(x) &= 72x^5 \end{aligned}$$

Hence

$$\begin{aligned} p''(x) &= 2 \\ q'(x) &= 1 \end{aligned}$$

Therefore (1) becomes

$$2-(1)+(-1)=0$$

Hence the ode is exact. Since we now know the ode is exact, it can be written as

$$(p(x)\,y'+(q(x)-p'(x))\,y)'=s(x)$$

Integrating gives

$$p(x)\,y'+(q(x)-p'(x))\,y=\int s(x)\;dx$$

Substituting the above values for p, q, r, s gives

$$y'x^2-yx=\int 72x^5\;dx$$

We now have a first order ode to solve which is

$$y'x^2-yx=12x^6+c_1$$

Entering Linear first order ODE solver. In canonical form a linear first order is

$$y'+p(x)y=q(x)$$

Where here

$$\begin{aligned}p(x)&=-\frac{1}{x}\\ q(x)&=\frac{12x^6+c_1}{x^2}\end{aligned}$$

Hence the ode is

$$y'-\frac{y}{x}=\frac{12x^6+c_1}{x^2}$$

The integrating factor μ is

$$\begin{aligned}\mu&=\mathrm{e}^{\int -\frac{1}{x}dx}\\ &=\frac{1}{x}\end{aligned}$$

The ode becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\mu y)&=(\mu)\left(\frac{12x^6+c_1}{x^2}\right)\\ \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{y}{x}\right)&=\left(\frac{1}{x}\right)\left(\frac{12x^6+c_1}{x^2}\right)\\ \mathrm{d}\left(\frac{y}{x}\right)&=\left(\frac{12x^6+c_1}{x^3}\right)\,\mathrm{d}x\end{aligned}$$

Integrating gives

$$\frac{y}{x} = \int \frac{12x^6 + c_1}{x^3}\, dx$$
$$\frac{y}{x} = 3x^4 - \frac{c_1}{2x^2} + c_2$$

Dividing both sides by the integrating factor $\mu = \frac{1}{x}$ results in

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2 x$$

Summary

The solution(s) found are the following

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2 x \tag{1}$$

Verification of solutions

$$y = x\left(3x^4 - \frac{c_1}{2x^2}\right) + c_2 x$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
<- high order exact linear fully integrable successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)-y(x)=72*x^5,y(x), singsol=all)
```

$$y(x) = \frac{3x^6 + c_2 x^2 + c_1}{x}$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 21

```
DSolve[x^2*y''[x]+x*y'[x]-y[x]==72*x^5,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to 3x^5 + c_2 x + \frac{c_1}{x}$$

11.35 problem 58

Internal problem ID [253]
Internal file name [`OUTPUT/253_Sunday_June_05_2022_01_37_30_AM_97347265/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 58.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 4y'x + 6y = x^3}$$

11.35.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$x^2y''-4y'x+6y=0$$

This is Euler second order ODE. Let the solution be $y=x^r$, then $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2}-4xrx^{r-1}+6x^r=0$$

Simplifying gives

$$r(r-1)\,x^r-4r\,x^r+6x^r=0$$

Since $x^r\neq 0$ then dividing throughout by x^r gives

$$r(r-1)-4r+6=0$$

Or

$$r^2-5r+6=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=2$$
$$r_2=3$$

Since the roots are real and distinct, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^{r_1}$ and $y_2=x^{r_2}$. Hence

$$y=c_2x^3+c_1x^2$$

Next, we find the particular solution to the ODE

$$x^2y''-4y'x+6y=x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^6}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int 1dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^5}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln (x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3 + \ln (x)\, x^3$$

Which simplifies to

$$y_p(x) = x^3(-1 + \ln (x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= x^3(-1 + \ln (x)) + c_2 x^3 + c_1 x^2 \end{aligned}$$

Summary
The solution(s) found are the following

$$y = x^3(-1 + \ln (x)) + c_2 x^3 + c_1 x^2 \tag{1}$$

Verification of solutions

$$y = x^3(-1 + \ln (x)) + c_2 x^3 + c_1 x^2$$

Verified OK.

11.35.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{4}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned}
M(x) &= e^{\frac{1}{2}\int p\,dx} \\
&= e^{\int -\frac{4}{x}\,dx} \\
&= \frac{1}{x^2}
\end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned}
(M(x)y)'' &= \frac{1}{x} \\
\left(\frac{y}{x^2}\right)'' &= \frac{1}{x}
\end{aligned}$$

Integrating once gives

$$\left(\frac{y}{x^2}\right)' = \ln(x) + c_1$$

Integrating again gives

$$\left(\frac{y}{x^2}\right) = x(\ln(x) + c_1 - 1) + c_2$$

Hence the solution is

$$y = \frac{x(\ln(x) + c_1 - 1) + c_2}{\frac{1}{x^2}}$$

Or

$$y = c_1 x^3 + x^3 \ln(x) + c_2 x^2 - x^3$$

Summary

The solution(s) found are the following

$$y = c_1 x^3 + x^3 \ln(x) + c_2 x^2 - x^3 \tag{1}$$

Verification of solutions

$$y = c_1 x^3 + x^3 \ln(x) + c_2 x^2 - x^3$$

Verified OK.

11.35.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{4}{x}\\ q(x) &= \frac{6}{x^2}\end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$
\begin{aligned}
\tau &= \int e^{-\left(\int p(x)dx\right)} dx \\
&= \int e^{-\left(\int -\frac{4}{x}dx\right)} dx \\
&= \int e^{4\ln(x)}\, dx \\
&= \int x^4 dx \\
&= \frac{x^5}{5} \qquad (6)
\end{aligned}
$$

Using (6) to evaluate q_1 from (5) gives

$$
\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{6}{x^2}}{x^8} \\
&= \frac{6}{x^{10}} \qquad (7)
\end{aligned}
$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$
\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{x^{10}} &= 0
\end{aligned}
$$

But in terms of τ

$$
\frac{6}{x^{10}} = \frac{6}{25\tau^2}
$$

Hence the above ode becomes

$$
\frac{d^2}{d\tau^2}y(\tau) + \frac{6y(\tau)}{25\tau^2} = 0
$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$
25\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 6y(\tau) = 0
$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$
25\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 6\tau^r = 0
$$

Simplifying gives

$$25r(r-1)\,\tau^r+0\,\tau^r+6\tau^r=0$$

Since $\tau^r\neq 0$ then dividing throughout by τ^r gives

$$25r(r-1)+0+6=0$$

Or

$$25r^2-25r+6=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned}r_1&=\frac{2}{5}\\ r_2&=\frac{3}{5}\end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau)=c_1y_1+c_2y_2$$

Where $y_1=\tau^{r_1}$ and $y_2=\tau^{r_2}$. Hence

$$y(\tau)=c_1\tau^{\frac{2}{5}}+c_2\tau^{\frac{3}{5}}$$

The above solution is now transformed back to y using (6) which results in

$$y=\frac{c_1 5^{\frac{3}{5}}(x^5)^{\frac{2}{5}}}{5}+\frac{c_2 5^{\frac{2}{5}}(x^5)^{\frac{3}{5}}}{5}$$

Therefore the homogeneous solution y_h is

$$y_h=\frac{c_1 5^{\frac{3}{5}}(x^5)^{\frac{2}{5}}}{5}+\frac{c_2 5^{\frac{2}{5}}(x^5)^{\frac{3}{5}}}{5}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$

$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} x^3}{x^6} dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}}{x^3} dx$$

Hence

$$u_1 = -\frac{(x^5)^{\frac{3}{5}}}{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} x^3}{x^6} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}}{x^3} dx$$

Hence

$$u_2 = \frac{(x^5)^{\frac{2}{5}} \ln(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3 + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3(-1 + \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5}\right) + (x^3(-1 + \ln(x))) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} + x^3(-1 + \ln(x)) \tag{1}$$

Verification of solutions

$$y = \frac{c_1 5^{\frac{3}{5}} (x^5)^{\frac{2}{5}}}{5} + \frac{c_2 5^{\frac{2}{5}} (x^5)^{\frac{3}{5}}}{5} + x^3(-1 + \ln(x))$$

Verified OK.

11.35.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}\tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3}\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{\sqrt{6}}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{4}{x}\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= -\frac{5c\sqrt{6}}{6}\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) - \frac{5c\sqrt{6}\left(\frac{d}{d\tau}y(\tau)\right)}{6} + c^2 y(\tau) &= 0\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{5\sqrt{6}\,c\tau}{12}}\left(c_1 \cosh\left(\frac{\sqrt{6}\,c\tau}{12}\right) + i c_2 \sinh\left(\frac{\sqrt{6}\,c\tau}{12}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \sqrt{6}\,\sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{6}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x^{\frac{5}{2}}\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
x^2y'' - 4y'x + 6y = x^3
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^5\right)^{\frac{2}{5}} \\
y_2 &= \left(x^5\right)^{\frac{3}{5}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}} x^3}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{\left(x^5\right)^{\frac{3}{5}}}{x^3} dx$$

Hence

$$u_1 = -\frac{\left(x^5\right)^{\frac{3}{5}}}{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\left(x^5\right)^{\frac{2}{5}} x^3}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}}{x^3} dx$$

Hence

$$u_2 = \frac{(x^5)^{\frac{2}{5}} \ln(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3 + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3(-1 + \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + \left(x^3(-1 + \ln(x))\right) \\ &= x^3(-1 + \ln(x)) + x^{\frac{5}{2}}\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + i c_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 + x^3 \ln(x) - x^3$$

Summary
The solution(s) found are the following

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 + x^3 \ln(x) - x^3 \tag{1}$$

Verification of solutions

$$y = i \sinh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_2 + \cosh\left(\frac{\ln(x)}{2}\right) x^{\frac{5}{2}} c_1 + x^3 \ln(x) - x^3$$

Verified OK.

11.35.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{4}{x} \\ q(x) &= \frac{6}{x^2} \end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned} Q &= q - \frac{p'}{2} - \frac{p^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(-\frac{4}{x}\right)'}{2} - \frac{\left(-\frac{4}{x}\right)^2}{4} \\ &= \frac{6}{x^2} - \frac{\left(\frac{4}{x^2}\right)}{2} - \frac{\left(\frac{16}{x^2}\right)}{4} \\ &= \frac{6}{x^2} - \left(\frac{2}{x^2}\right) - \frac{4}{x^2} \\ &= 0 \end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\, z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned} z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)} \\ &= e^{-\int \frac{-\frac{4}{x}}{2}} \\ &= x^2 \end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\, x^2 \tag{4}$$

Applying this change of variable to the original ode results in

$$xv''(x) = 1$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) = \frac{1}{x}$$

Integrating once gives

$$v'(x) = \ln(x) + c_1$$

Integrating again gives

$$v(x) = x\ln(x) - x + c_1x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= (c_1x + x\ln(x) - x + c_2)\,(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = x^2$$

Hence (7) becomes

$$y = x^2(c_1x + x\ln(x) - x + c_2)$$

Therefore the homogeneous solution y_h is

$$y_h = x^2(c_1x + x\ln(x) - x + c_2)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^5\right)^{\frac{2}{5}}$$

$$y_2 = \left(x^5\right)^{\frac{3}{5}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{d}{dx}\left(\left(x^5\right)^{\frac{2}{5}}\right) & \frac{d}{dx}\left(\left(x^5\right)^{\frac{3}{5}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^5\right)^{\frac{2}{5}} & \left(x^5\right)^{\frac{3}{5}} \\ \frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}} & \frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^5\right)^{\frac{2}{5}}\right)\left(\frac{3x^4}{\left(x^5\right)^{\frac{2}{5}}}\right) - \left(\left(x^5\right)^{\frac{3}{5}}\right)\left(\frac{2x^4}{\left(x^5\right)^{\frac{3}{5}}}\right)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}} x^3}{x^6} dx$$

Which simplifies to

$$u_1 = -\int \frac{(x^5)^{\frac{3}{5}}}{x^3} dx$$

Hence

$$u_1 = -\frac{(x^5)^{\frac{3}{5}}}{x^2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}} x^3}{x^6} dx$$

Which simplifies to

$$u_2 = \int \frac{(x^5)^{\frac{2}{5}}}{x^3} dx$$

Hence

$$u_2 = \frac{(x^5)^{\frac{2}{5}} \ln(x)}{x^2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3 + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3(-1 + \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^2(c_1 x + x\ln(x) - x + c_2)\right) + \left(x^3(-1 + \ln(x))\right) \end{aligned}$$

Which simplifies to

$$y = (x \ln(x) + (c_1 - 1)x + c_2)x^2 + x^3(-1 + \ln(x))$$

Summary

The solution(s) found are the following

$$y = (x \ln(x) + (c_1 - 1)x + c_2)x^2 + x^3(-1 + \ln(x)) \tag{1}$$

Verification of solutions

$$y = (x \ln(x) + (c_1 - 1)x + c_2)x^2 + x^3(-1 + \ln(x))$$

Verified OK.

11.35.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -4x, C = 6, f(x) = x^3$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 4y'x + 6y = 0$$

In normal form the ode

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$

Becomes

$$y'' + p(x)y' + q(x)y = 0 \tag{2}$$

Where

$$p(x) = -\frac{4}{x}$$
$$q(x) = \frac{6}{x^2}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{4n}{x^2} + \frac{6}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 3 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\,du &= -\frac{2}{x}\,dx \\
\int \frac{1}{u}\,du &= \int -\frac{2}{x}\,dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\,dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\,x^n \\
&= \left(-\frac{c_1}{x} + c_2\right)x^3 \\
&= (c_2 x - c_1)\,x^2
\end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' - 4y'x + 6y = x^3$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= x^2 \\
y_2 &= x^3
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^6}{x^6} dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^5}{x^6}\,dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x}dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3 + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3(-1 + \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{x} + c_2\right)x^3\right) + \left(x^3(-1 + \ln(x))\right) \\ &= x^3(-1 + \ln(x)) + \left(-\frac{c_1}{x} + c_2\right)x^3 \end{aligned}$$

Which simplifies to

$$y = x^2(x\ln(x) + c_2 x - c_1 - x)$$

Summary

The solution(s) found are the following

$$y = x^2(x\ln(x) + c_2 x - c_1 - x) \tag{1}$$

Verification of solutions

$$y = x^2(x\ln(x) + c_2 x - c_1 - x)$$

Verified OK.

11.35.7 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 4y'x + 6y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -4x \\ C &= 6 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 514: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(x) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{x^2}\,dx}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{2\ln(x)} \\
&= z_1 \left(x^2\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\, dx}}{y_1^2}\, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{x^2}\, dx}}{\left(y_1\right)^2}\, dx \\
&= y_1 \int \frac{e^{4\ln(x)}}{\left(y_1\right)^2}\, dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^2\right) + c_2 \left(x^2(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
x^2 y'' - 4y'x + 6y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
y_h = c_2 x^3 + c_1 x^2
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = x^3$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & x^3 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(x^3) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$$

Therefore

$$W = (x^2)(3x^2) - (x^3)(2x)$$

Which simplifies to

$$W = x^4$$

Which simplifies to

$$W = x^4$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^6}{x^6}\, dx$$

Which simplifies to

$$u_1 = -\int 1 dx$$

Hence

$$u_1 = -x$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^5}{x^6}\, dx$$

Which simplifies to

$$u_2 = \int \frac{1}{x} dx$$

Hence

$$u_2 = \ln(x)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -x^3 + x^3 \ln(x)$$

Which simplifies to

$$y_p(x) = x^3(-1 + \ln(x))$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_2 x^3 + c_1 x^2\right) + \left(x^3(-1 + \ln(x))\right) \end{aligned}$$

Which simplifies to

$$y = x^2(c_2x + c_1) + x^3(-1 + \ln(x))$$

Summary

The solution(s) found are the following

$$y = x^2(c_2x + c_1) + x^3(-1 + \ln(x)) \tag{1}$$

Verification of solutions

$$y = x^2(c_2x + c_1) + x^3(-1 + \ln(x))$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 19

```
dsolve(x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+6*y(x)=x^3,y(x), singsol=all)
```

$$y(x) = x^2(x\ln(x) + (c_1 - 1)x + c_2)$$

✓ Solution by Mathematica

Time used: 0.013 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(x\log(x) + (-1 + c_2)x + c_1)$$

11.36 problem 59

Internal problem ID [254]
Internal file name [`OUTPUT/254_Sunday_June_05_2022_01_37_31_AM_11178115/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 59.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' - 3y'x + 4y = x^4}$$

11.36.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} - 3xrx^{r-1} + 4x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r - 3r\,x^r + 4x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) - 3r + 4 = 0$$

Or

$$r^2 - 4r + 4 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1 = 2$$
$$r_2 = 2$$

Since the roots are equal, then the general solution is

$$y = c_1y_1 + c_2y_2$$

Where $y_1 = x^r$ and $y_2 = x^r \ln(x)$. Hence

$$y = c_1x^2 + c_2x^2 \ln(x)$$

Next, we find the particular solution to the ODE

$$x^2y'' - 3y'x + 4y = x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\,x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^6}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int x\ln(x)\, dx$$

Hence

$$u_1 = -\frac{\ln(x)\, x^2}{2} + \frac{x^2}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^6}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int x dx$$

Hence

$$u_2 = \frac{x^2}{2}$$

Which simplifies to

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$

$$u_2 = \frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4(-1 + 2\ln(x))}{4} + \frac{x^4 \ln(x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x^4}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \frac{x^4}{4} + c_1 x^2 + c_2 x^2 \ln(x) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \frac{x^4}{4} + c_1 x^2 + c_2 x^2 \ln(x) \tag{1}$$

Verification of solutions

$$y = \frac{x^4}{4} + c_1 x^2 + c_2 x^2 \ln(x)$$

Verified OK.

11.36.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{3}{x}dx\right)} dx \\
&= \int e^{3\ln(x)}\, dx \\
&= \int x^3 dx \\
&= \frac{x^4}{4} \qquad (6)
\end{aligned}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{4}{x^2}}{x^6} \\
&= \frac{4}{x^8} \qquad (7)
\end{aligned}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{4y(\tau)}{x^8} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{4}{x^8} = \frac{1}{4\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{y(\tau)}{4\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$4\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$4\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + \tau^r = 0$$

Simplifying gives

$$4r(r-1)\,\tau^{r}+0\,\tau^{r}+\tau^{r}=0$$

Since $\tau^{r}\neq 0$ then dividing throughout by τ^{r} gives

$$4r(r-1)+0+1=0$$

Or

$$4r^{2}-4r+1=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_{1}=\frac{1}{2}$$
$$r_{2}=\frac{1}{2}$$

Since the roots are equal, then the general solution is

$$y(\tau)=c_{1}y_{1}+c_{2}y_{2}$$

Where $y_{1}=\tau^{r}$ and $y_{2}=\tau^{r}\ln\left(\tau\right)$. Hence

$$y(\tau)=c_{1}\sqrt{\tau}+c_{2}\sqrt{\tau}\,\ln\left(\tau\right)$$

The above solution is now transformed back to y using (6) which results in

$$y=\frac{\left(c_{2}\ln\left(x^{4}\right)-2c_{2}\ln\left(2\right)+c_{1}\right)\sqrt{x^{4}}}{2}$$

Therefore the homogeneous solution y_h is

$$y_{h}=\frac{\left(c_{2}\ln\left(x^{4}\right)-2c_{2}\ln\left(2\right)+c_{1}\right)\sqrt{x^{4}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_{p}(x)=u_{1}y_{1}+u_{2}y_{2} \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^4}$$

$$y_2 = -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)x^4}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int x\left(-\frac{\ln(2)}{2} + \ln(x)\right)dx$$

Hence

$$u_1 = \frac{x^2(1 + \ln(2) - 2\ln(x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\,x^4}{2x^5}\,dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{x}{2}dx$$

Hence

$$u_2 = \frac{x^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2(1 + \ln(2) - 2\ln(x))\sqrt{x^4}}{4} + \frac{x^2\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{x^4}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{(c_2\ln(x^4) - 2c_2\ln(2) + c_1)\sqrt{x^4}}{2}\right) + \left(\frac{x^4}{4}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{\left(c_2 \ln \left(x^4\right)-2 c_2 \ln \left(2\right)+c_1\right) \sqrt{x^4}}{2}+\frac{x^4}{4} \tag{1}$$

Verification of solutions

$$y = \frac{\left(c_2 \ln \left(x^4\right)-2 c_2 \ln \left(2\right)+c_1\right) \sqrt{x^4}}{2}+\frac{x^4}{4}$$

Verified OK. `{0 < x}`

11.36.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$A y''(x) + B y'(x) + C y(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2 y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2 y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{2\sqrt{\frac{1}{x^2}}}{c} \qquad (6)\\
\tau'' &= -\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{2}{c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{3}{x}\frac{2\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{2\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\
&= -2c
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) - 2c\left(\frac{d}{d\tau} y(\tau)\right) + c^2 y(\tau) &= 0 \qquad (7)
\end{aligned}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{c\tau} c_1$$

Now from (6)

$$\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int 2\sqrt{\frac{1}{x^2}}\,dx}{c} \\
&= \frac{2\sqrt{\frac{1}{x^2}}\,x\ln(x)}{c}
\end{aligned}$$

Substituting the above into the solution obtained gives

$$y = c_1 x^2$$

Now the particular solution to this ODE is found

$$x^2 y'' - 3y'x + 4y = x^4$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x^4}$$

$$y_2 = -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{d}{dx}\left(\sqrt{x^4}\right) & \frac{d}{dx}\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x^4} & -\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2} \\ \frac{2x^3}{\sqrt{x^4}} & -\frac{2\ln(2)x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)x^3}{\sqrt{x^4}} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x^4}\right)\left(-\frac{2\ln(2)\,x^3}{\sqrt{x^4}} + \frac{2\sqrt{x^4}}{x} + \frac{\ln(x^4)\,x^3}{\sqrt{x^4}}\right) - \left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right)\left(\frac{2x^3}{\sqrt{x^4}}\right)$$

Which simplifies to

$$W = 2x^3$$

Which simplifies to

$$W = 2x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\left(-\ln(2)\sqrt{x^4} + \frac{\ln(x^4)\sqrt{x^4}}{2}\right) x^4}{2x^5}\, dx$$

Which simplifies for $0 < x$ to

$$u_1 = -\int x\left(-\frac{\ln(2)}{2} + \ln(x)\right) dx$$

Hence

$$u_1 = \frac{x^2(1 + \ln(2) - 2\ln(x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\sqrt{x^4}\, x^4}{2x^5}\, dx$$

Which simplifies for $0 < x$ to

$$u_2 = \int \frac{x}{2} dx$$

Hence

$$u_2 = \frac{x^2}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \frac{x^2(1+\ln(2)-2\ln(x))\sqrt{x^4}}{4} + \frac{x^2\left(-\ln(2)\sqrt{x^4}+\frac{\ln(x^4)\sqrt{x^4}}{2}\right)}{4}$$

Which simplifies to

$$y_p(x) = \frac{x^4}{4}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 x^2) + \left(\frac{x^4}{4}\right) \\ &= \frac{1}{4}x^4 + c_1 x^2 \end{aligned}$$

Which simplifies to

$$y = \frac{1}{4}x^4 + c_1 x^2$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{1}{4}x^4 + c_1 x^2 \tag{1}$$

<u>Verification of solutions</u>

$$y = \frac{1}{4}x^4 + c_1 x^2$$

Verified OK. `{0 < x}`

11.36.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = -3x, C = 4, f(x) = x^4$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' - 3y'x + 4y = 0$$

In normal form the ode

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{3}{x} \\ q(x) &= \frac{4}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{3n}{x^2} + \frac{4}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = 2 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{v'(x)}{x} &= 0 \\ v''(x) + \frac{v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{u}{x} \end{aligned}$$

Where $f(x) = -\frac{1}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= -\frac{1}{x}\, dx \\ \int \frac{1}{u}\, du &= \int -\frac{1}{x}\, dx \\ \ln(u) &= -\ln(x) + c_1 \\ u &= e^{-\ln(x) + c_1} \\ &= \frac{c_1}{x} \end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\, dx + c_2 \\ &= c_1 \ln(x) + c_2 \end{aligned}$$

Hence

$$
\begin{aligned}
y &= v(x)\, x^n \\
&= (c_1 \ln(x) + c_2)\, x^2 \\
&= (c_1 \ln(x) + c_2)\, x^2
\end{aligned}
$$

Now the particular solution to this ODE is found

$$
x^2 y'' - 3y'x + 4y = x^4
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= x^2 \\
y_2 &= \ln(x)\, x^2
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$
W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}
$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^6}{x^5}\, dx$$

Which simplifies to

$$u_1 = -\int x\ln(x)\, dx$$

Hence

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^6}{x^5}\, dx$$

Which simplifies to

$$u_2 = \int x dx$$

Hence

$$u_2 = \frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4(-1+2\ln(x))}{4} + \frac{x^4\ln(x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x^4}{4}$$

Therefore the general solution is

$$\begin{aligned}y &= y_h + y_p \\ &= \left((c_1 \ln(x) + c_2)\, x^2\right) + \left(\frac{x^4}{4}\right) \\ &= \frac{x^4}{4} + (c_1 \ln(x) + c_2)\, x^2\end{aligned}$$

Which simplifies to

$$y = \frac{x^4}{4} + (c_1 \ln(x) + c_2)\, x^2$$

Summary

The solution(s) found are the following

$$y = \frac{x^4}{4} + (c_1 \ln(x) + c_2)\, x^2 \tag{1}$$

Verification of solutions

$$y = \frac{x^4}{4} + (c_1 \ln(x) + c_2)\, x^2$$

Verified OK. {0 < x}

11.36.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' - 3y'x + 4y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= -3x \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{1}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 515: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{1}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_c^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{1}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{1}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^{+} &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} \\
\alpha_\infty^{-} &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2}
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{1}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^{+}	α_c^{-}
0	2	0	$\frac{1}{2}$	$\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^{+}	α_∞^{-}
2	0	$\frac{1}{2}$	$\frac{1}{2}$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_\infty^- = \frac{1}{2}$ then

$$\begin{aligned} d &= \alpha_\infty^- - (\alpha_{c_1}^-) \\ &= \frac{1}{2} - \left(\frac{1}{2}\right) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^-}{x - c_1} \right) + (-)[\sqrt{r}]_\infty \\ &= \frac{1}{2x} + (-)(0) \\ &= \frac{1}{2x} \\ &= \frac{1}{2x} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned} (0) + 2\left(\frac{1}{2x}\right)(0) + \left(\left(-\frac{1}{2x^2}\right) + \left(\frac{1}{2x}\right)^2 - \left(-\frac{1}{4x^2}\right)\right) &= 0 \\ 0 &= 0 \end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned} z_1(x) &= p e^{\int \omega \, dx} \\ &= e^{\int \frac{1}{2x} dx} \\ &= \sqrt{x} \end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-3x}{x^2}\,dx} \\
&= z_1 e^{\frac{3\ln(x)}{2}} \\
&= z_1\left(x^{\frac{3}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-3x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{3\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(\ln(x))
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(x^2\right) + c_2\left(x^2(\ln(x))\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$
x^2y'' - 3y'x + 4y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1x^2 + c_2x^2 \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^2$$
$$y_2 = \ln(x)\, x^2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ \frac{d}{dx}(x^2) & \frac{d}{dx}(\ln(x)\, x^2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^2 & \ln(x)\, x^2 \\ 2x & x + 2x\ln(x) \end{vmatrix}$$

Therefore

$$W = (x^2)(x + 2x\ln(x)) - (\ln(x)\, x^2)(2x)$$

Which simplifies to

$$W = x^3$$

Which simplifies to

$$W = x^3$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\ln(x)\, x^6}{x^5} dx$$

Which simplifies to

$$u_1 = -\int x \ln(x)\, dx$$

Hence

$$u_1 = -\frac{x^2(-1 + 2\ln(x))}{4}$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^6}{x^5} dx$$

Which simplifies to

$$u_2 = \int x dx$$

Hence

$$u_2 = \frac{x^2}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{x^4(-1 + 2\ln(x))}{4} + \frac{x^4 \ln(x)}{2}$$

Which simplifies to

$$y_p(x) = \frac{x^4}{4}$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= (c_1 x^2 + c_2 x^2 \ln(x)) + \left(\frac{x^4}{4}\right)
\end{aligned}
$$

Which simplifies to

$$
y = x^2(c_2 \ln(x) + c_1) + \frac{x^4}{4}
$$

Summary

The solution(s) found are the following

$$
y = x^2(c_2 \ln(x) + c_1) + \frac{x^4}{4} \tag{1}
$$

Verification of solutions

$$
y = x^2(c_2 \ln(x) + c_1) + \frac{x^4}{4}
$$

Verified OK. {0 < x}

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 21

```
dsolve(x^2*diff(y(x),x$2)-3*x*diff(y(x),x)+4*y(x)=x^4,y(x), singsol=all)
```

$$y(x) = \frac{x^2(4\ln(x)\,c_1 + x^2 + 4c_2)}{4}$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 26

```
DSolve[x^2*y''[x]-3*x*y'[x]+4*y[x]==x^4,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{4}x^2\left(x^2 + 8c_2\log(x) + 4c_1\right)$$

11.37 problem 60

Internal problem ID [255]
Internal file name [`OUTPUT/255_Sunday_June_05_2022_01_37_32_AM_45537403/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 60.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_1", "second_order_change_of_variable_on_y_method_2", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{4x^2y'' - 4y'x + 3y = 8x^{\frac{4}{3}}}$$

11.37.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x^2, B = -4x, C = 3, f(x) = 8x^{\frac{4}{3}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$4x^2y''-4y'x+3y=0$$

This is Euler second order ODE. Let the solution be $y=x^r$, then $y'=rx^{r-1}$ and $y''=r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$4x^2(r(r-1))x^{r-2}-4xrx^{r-1}+3x^r=0$$

Simplifying gives

$$4r(r-1)\,x^r-4r\,x^r+3x^r=0$$

Since $x^r\neq 0$ then dividing throughout by x^r gives

$$4r(r-1)-4r+3=0$$

Or

$$4r^2-8r+3=0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$r_1=\frac{1}{2}$$
$$r_2=\frac{3}{2}$$

Since the roots are real and distinct, then the general solution is

$$y=c_1y_1+c_2y_2$$

Where $y_1=x^{r_1}$ and $y_2=x^{r_2}$. Hence

$$y=\sqrt{x}\,c_1+c_2x^{\frac{3}{2}}$$

Next, we find the particular solution to the ODE

$$4x^2y''-4y'x+3y=8x^{\frac{4}{3}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}$$
$$y_2 = x^{\frac{3}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & x^{\frac{3}{2}} \\ \frac{d}{dx}(\sqrt{x}) & \frac{d}{dx}\left(x^{\frac{3}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & x^{\frac{3}{2}} \\ \frac{1}{2\sqrt{x}} & \frac{3\sqrt{x}}{2} \end{vmatrix}$$

Therefore

$$W = (\sqrt{x})\left(\frac{3\sqrt{x}}{2}\right) - \left(x^{\frac{3}{2}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^{\frac{17}{6}}}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{x^{\frac{1}{6}}} dx$$

Hence

$$u_1 = -\frac{12x^{\frac{5}{6}}}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^{\frac{11}{6}}}{4x^3} dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^{\frac{7}{6}}} dx$$

Hence

$$u_2 = -\frac{12}{x^{\frac{1}{6}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{72x^{\frac{4}{3}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= -\frac{72x^{\frac{4}{3}}}{5} + \sqrt{x}\, c_1 + c_2 x^{\frac{3}{2}} \end{aligned}$$

Summary

The solution(s) found are the following

$$y = -\frac{72x^{\frac{4}{3}}}{5} + \sqrt{x}\, c_1 + c_2 x^{\frac{3}{2}} \tag{1}$$

Verification of solutions

$$y = -\frac{72x^{\frac{4}{3}}}{5} + \sqrt{x}\, c_1 + c_2 x^{\frac{3}{2}}$$

Verified OK.

11.37.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$y'' + p(x)\, y' + \frac{\left(p(x)^2 + p'(x)\right) y}{2} = f(x)$$

Where $p(x) = -\frac{1}{x}$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2} \int p\, dx} \\ &= e^{\int -\frac{1}{x}\, dx} \\ &= \frac{1}{\sqrt{x}} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$\begin{aligned} (M(x)y)'' &= \frac{2}{x^{\frac{7}{6}}} \\ \left(\frac{y}{\sqrt{x}}\right)'' &= \frac{2}{x^{\frac{7}{6}}} \end{aligned}$$

Integrating once gives

$$\left(\frac{y}{\sqrt{x}}\right)' = -\frac{12}{x^{\frac{1}{6}}} + c_1$$

Integrating again gives

$$\left(\frac{y}{\sqrt{x}}\right) = c_1 x - \frac{72 x^{\frac{5}{6}}}{5} + c_2$$

Hence the solution is

$$y = \frac{c_1 x - \frac{72 x^{\frac{5}{6}}}{5} + c_2}{\frac{1}{\sqrt{x}}}$$

Or

$$y = c_1 x^{\frac{3}{2}} + c_2 \sqrt{x} - \frac{72 x^{\frac{4}{3}}}{5}$$

Summary

The solution(s) found are the following

$$y = c_1 x^{\frac{3}{2}} + c_2 \sqrt{x} - \frac{72 x^{\frac{4}{3}}}{5} \tag{1}$$

Verification of solutions

$$y = c_1 x^{\frac{3}{2}} + c_2\sqrt{x} - \frac{72x^{\frac{4}{3}}}{5}$$

Verified OK.

11.37.3 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' - 4y'x + 3y = 0$$

In normal form the ode

$$4x^2y'' - 4y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{3}{4x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)}dx \\
&= \int \mathrm{e}^{-\left(\int -\frac{1}{x}dx\right)}dx \\
&= \int e^{\ln(x)}\,dx \\
&= \int x dx \\
&= \frac{x^2}{2}
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{3}{4x^2}}{x^2} \\
&= \frac{3}{4x^4}
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2}y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{4x^4} &= 0
\end{aligned}$$

But in terms of τ

$$\frac{3}{4x^4} = \frac{3}{16\tau^2}$$

Hence the above ode becomes

$$\frac{d^2}{d\tau^2}y(\tau) + \frac{3y(\tau)}{16\tau^2} = 0$$

The above ode is now solved for $y(\tau)$. The ode can be written as

$$16\left(\frac{d^2}{d\tau^2}y(\tau)\right)\tau^2 + 3y(\tau) = 0$$

Which shows it is a Euler ODE. This is Euler second order ODE. Let the solution be $y(\tau) = \tau^r$, then $y' = r\tau^{r-1}$ and $y'' = r(r-1)\tau^{r-2}$. Substituting these back into the given ODE gives

$$16\tau^2(r(r-1))\tau^{r-2} + 0r\tau^{r-1} + 3\tau^r = 0$$

Simplifying gives

$$16r(r-1)\,\tau^r + 0\,\tau^r + 3\tau^r = 0$$

Since $\tau^r \neq 0$ then dividing throughout by τ^r gives

$$16r(r-1) + 0 + 3 = 0$$

Or

$$16r^2 - 16r + 3 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= \frac{1}{4} \\ r_2 &= \frac{3}{4} \end{aligned}$$

Since the roots are real and distinct, then the general solution is

$$y(\tau) = c_1 y_1 + c_2 y_2$$

Where $y_1 = \tau^{r_1}$ and $y_2 = \tau^{r_2}$. Hence

$$y(\tau) = c_1 \tau^{\frac{1}{4}} + c_2 \tau^{\frac{3}{4}}$$

The above solution is now transformed back to y using (6) which results in

$$y = \frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2}$$

Therefore the homogeneous solution y_h is

$$y_h = \frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^2\right)^{\frac{1}{4}}$$

$$y_2 = \left(x^2\right)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^2\right)^{\frac{1}{4}} & \left(x^2\right)^{\frac{3}{4}} \\ \frac{d}{dx}\left(\left(x^2\right)^{\frac{1}{4}}\right) & \frac{d}{dx}\left(\left(x^2\right)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^2\right)^{\frac{1}{4}} & \left(x^2\right)^{\frac{3}{4}} \\ \frac{x}{2(x^2)^{\frac{3}{4}}} & \frac{3x}{2(x^2)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^2\right)^{\frac{1}{4}}\right)\left(\frac{3x}{2\left(x^2\right)^{\frac{1}{4}}}\right) - \left(\left(x^2\right)^{\frac{3}{4}}\right)\left(\frac{x}{2\left(x^2\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8(x^2)^{\frac{3}{4}}\, x^{\frac{4}{3}}}{4x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2(x^2)^{\frac{3}{4}}}{x^{\frac{5}{3}}} dx$$

Hence

$$u_1 = -\frac{12(x^2)^{\frac{3}{4}}}{5x^{\frac{2}{3}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8(x^2)^{\frac{1}{4}}\, x^{\frac{4}{3}}}{4x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2(x^2)^{\frac{1}{4}}}{x^{\frac{5}{3}}} dx$$

Hence

$$u_2 = -\frac{12(x^2)^{\frac{1}{4}}}{x^{\frac{2}{3}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{72x^{\frac{4}{3}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2}\right) + \left(-\frac{72x^{\frac{4}{3}}}{5}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2} - \frac{72 x^{\frac{4}{3}}}{5} \tag{1}$$

Verification of solutions

$$y = \frac{c_1 2^{\frac{3}{4}} (x^2)^{\frac{1}{4}}}{2} + \frac{c_2 2^{\frac{1}{4}} (x^2)^{\frac{3}{4}}}{2} - \frac{72 x^{\frac{4}{3}}}{5}$$

Verified OK.

11.37.4 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x^2, B = -4x, C = 3, f(x) = 8x^{\frac{4}{3}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$4x^2y'' - 4y'x + 3y = 0$$

In normal form the ode

$$4x^2y'' - 4y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$p(x) = -\frac{1}{x}$$
$$q(x) = \frac{3}{4x^2}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2} y(\tau) + p_1 \left(\frac{d}{d\tau} y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned}
\tau' &= \frac{1}{c}\sqrt{q} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{2c} \\
\tau'' &= -\frac{\sqrt{3}}{2c\sqrt{\frac{1}{x^2}}\,x^3}
\end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned}
p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\
&= \frac{-\frac{\sqrt{3}}{2c\sqrt{\frac{1}{x^2}}\,x^3} - \frac{1}{x}\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{2c}}{\left(\frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{2c}\right)^2} \\
&= -\frac{4c\sqrt{3}}{3}
\end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned}
y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2}y(\tau) - \frac{4c\sqrt{3}\left(\frac{d}{d\tau}y(\tau)\right)}{3} + c^2 y(\tau) &= 0
\end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = \mathrm{e}^{\frac{2\sqrt{3}\,c\tau}{3}}\left(c_1\cosh\left(\frac{\sqrt{3}\,c\tau}{3}\right) + ic_2\sinh\left(\frac{\sqrt{3}\,c\tau}{3}\right)\right)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\,dx \\
&= \frac{\int \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}}{2}dx}{c} \\
&= \frac{\sqrt{3}\,\sqrt{\frac{1}{x^2}}\,x\ln(x)}{2c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = x\left(c_1\cosh\left(\frac{\ln(x)}{2}\right) + ic_2\sinh\left(\frac{\ln(x)}{2}\right)\right)
$$

Now the particular solution to this ODE is found

$$
4x^2y'' - 4y'x + 3y = 8x^{\frac{4}{3}}
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \left(x^2\right)^{\frac{1}{4}} \\
y_2 &= \left(x^2\right)^{\frac{3}{4}}
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x^2)^{\frac{1}{4}} & (x^2)^{\frac{3}{4}} \\ \frac{d}{dx}\left((x^2)^{\frac{1}{4}}\right) & \frac{d}{dx}\left((x^2)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x^2)^{\frac{1}{4}} & (x^2)^{\frac{3}{4}} \\ \frac{x}{2(x^2)^{\frac{3}{4}}} & \frac{3x}{2(x^2)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left((x^2)^{\frac{1}{4}}\right)\left(\frac{3x}{2\,(x^2)^{\frac{1}{4}}}\right) - \left((x^2)^{\frac{3}{4}}\right)\left(\frac{x}{2\,(x^2)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8(x^2)^{\frac{3}{4}}\, x^{\frac{4}{3}}}{4x^3}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{2(x^2)^{\frac{3}{4}}}{x^{\frac{5}{3}}} dx$$

Hence

$$u_1 = -\frac{12(x^2)^{\frac{3}{4}}}{5x^{\frac{2}{3}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8(x^2)^{\frac{1}{4}}\, x^{\frac{4}{3}}}{4x^3}\, dx$$

Which simplifies to

$$u_2 = \int \frac{2(x^2)^{\frac{1}{4}}}{x^{\frac{5}{3}}} dx$$

Hence

$$u_2 = -\frac{12(x^2)^{\frac{1}{4}}}{x^{\frac{2}{3}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{72x^{\frac{4}{3}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right)\right) + \left(-\frac{72x^{\frac{4}{3}}}{5}\right) \\ &= -\frac{72x^{\frac{4}{3}}}{5} + x\left(c_1 \cosh\left(\frac{\ln(x)}{2}\right) + ic_2 \sinh\left(\frac{\ln(x)}{2}\right)\right) \end{aligned}$$

Which simplifies to

$$y = -\frac{72x^{\frac{4}{3}}}{5} + i \sinh\left(\frac{\ln(x)}{2}\right) c_2 x + \cosh\left(\frac{\ln(x)}{2}\right) c_1 x$$

Summary

The solution(s) found are the following

$$y = -\frac{72x^{\frac{4}{3}}}{5} + i \sinh\left(\frac{\ln(x)}{2}\right) c_2 x + \cosh\left(\frac{\ln(x)}{2}\right) c_1 x \tag{1}$$

Verification of solutions

$$y = -\frac{72x^{\frac{4}{3}}}{5} + i \sinh\left(\frac{\ln(x)}{2}\right) c_2 x + \cosh\left(\frac{\ln(x)}{2}\right) c_1 x$$

Verified OK.

11.37.5 Solving as second order change of variable on y method 1 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$4x^2y'' - 4y'x + 3y = 0$$

In normal form the given ode is written as

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned}p(x) &= -\frac{1}{x}\\ q(x) &= \frac{3}{4x^2}\end{aligned}$$

Calculating the Liouville ode invariant Q given by

$$\begin{aligned}Q &= q - \frac{p'}{2} - \frac{p^2}{4}\\ &= \frac{3}{4x^2} - \frac{\left(-\frac{1}{x}\right)'}{2} - \frac{\left(-\frac{1}{x}\right)^2}{4}\\ &= \frac{3}{4x^2} - \frac{\left(\frac{1}{x^2}\right)}{2} - \frac{\left(\frac{1}{x^2}\right)}{4}\\ &= \frac{3}{4x^2} - \left(\frac{1}{2x^2}\right) - \frac{1}{4x^2}\\ &= 0\end{aligned}$$

Since the Liouville ode invariant does not depend on the independent variable x then the transformation

$$y = v(x)\,z(x) \tag{3}$$

is used to change the original ode to a constant coefficients ode in v. In (3) the term $z(x)$ is given by

$$\begin{aligned}z(x) &= \mathrm{e}^{-\left(\int \frac{p(x)}{2}dx\right)}\\ &= e^{-\int \frac{-\frac{1}{x}}{2}}\\ &= \sqrt{x}\end{aligned} \tag{5}$$

Hence (3) becomes

$$y = v(x)\sqrt{x} \tag{4}$$

Applying this change of variable to the original ode results in

$$x^{\frac{7}{6}} v''(x) = 2$$

Which is now solved for $v(x)$ Simplyfing the ode gives

$$v''(x) = \frac{2}{x^{\frac{7}{6}}}$$

Integrating once gives

$$v'(x) = -\frac{12}{x^{\frac{1}{6}}} + c_1$$

Integrating again gives

$$v(x) = -\frac{72x^{\frac{5}{6}}}{5} + c_1 x + c_2$$

Now that $v(x)$ is known, then

$$\begin{aligned} y &= v(x)\, z(x) \\ &= \left(c_1 x - \frac{72x^{\frac{5}{6}}}{5} + c_2\right)(z(x)) \end{aligned} \tag{7}$$

But from (5)

$$z(x) = \sqrt{x}$$

Hence (7) becomes

$$y = \left(c_1 x - \frac{72x^{\frac{5}{6}}}{5} + c_2\right)\sqrt{x}$$

Therefore the homogeneous solution y_h is

$$y_h = \left(c_1 x - \frac{72x^{\frac{5}{6}}}{5} + c_2\right)\sqrt{x}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of

parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \left(x^2\right)^{\frac{1}{4}}$$
$$y_2 = \left(x^2\right)^{\frac{3}{4}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \left(x^2\right)^{\frac{1}{4}} & \left(x^2\right)^{\frac{3}{4}} \\ \frac{d}{dx}\left(\left(x^2\right)^{\frac{1}{4}}\right) & \frac{d}{dx}\left(\left(x^2\right)^{\frac{3}{4}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \left(x^2\right)^{\frac{1}{4}} & \left(x^2\right)^{\frac{3}{4}} \\ \frac{x}{2(x^2)^{\frac{3}{4}}} & \frac{3x}{2(x^2)^{\frac{1}{4}}} \end{vmatrix}$$

Therefore

$$W = \left(\left(x^2\right)^{\frac{1}{4}}\right)\left(\frac{3x}{2\left(x^2\right)^{\frac{1}{4}}}\right) - \left(\left(x^2\right)^{\frac{3}{4}}\right)\left(\frac{x}{2\left(x^2\right)^{\frac{3}{4}}}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8(x^2)^{\frac{3}{4}} x^{\frac{4}{3}}}{4x^3} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{2(x^2)^{\frac{3}{4}}}{x^{\frac{5}{3}}} dx$$

Hence

$$u_1 = -\frac{12(x^2)^{\frac{3}{4}}}{5x^{\frac{2}{3}}}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8(x^2)^{\frac{1}{4}} x^{\frac{4}{3}}}{4x^3} \, dx$$

Which simplifies to

$$u_2 = \int \frac{2(x^2)^{\frac{1}{4}}}{x^{\frac{5}{3}}} dx$$

Hence

$$u_2 = -\frac{12(x^2)^{\frac{1}{4}}}{x^{\frac{2}{3}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{72x^{\frac{4}{3}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(c_1 x - \frac{72x^{\frac{5}{6}}}{5} + c_2\right)\sqrt{x}\right) + \left(-\frac{72x^{\frac{4}{3}}}{5}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \left(c_1 x - \frac{72x^{\frac{5}{6}}}{5} + c_2\right)\sqrt{x} - \frac{72x^{\frac{4}{3}}}{5} \tag{1}$$

Verification of solutions

$$y = \left(c_1 x - \frac{72x^{\frac{5}{6}}}{5} + c_2\right)\sqrt{x} - \frac{72x^{\frac{4}{3}}}{5}$$

Verified OK.

11.37.6 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = 4x^2, B = -4x, C = 3, f(x) = 8x^{\frac{4}{3}}$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$4x^2y'' - 4y'x + 3y = 0$$

In normal form the ode

$$4x^2y'' - 4y'x + 3y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{1}{x} \\ q(x) &= \frac{3}{4x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} - \frac{n}{x^2} + \frac{3}{4x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = \frac{3}{2} \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \frac{2v'(x)}{x} &= 0 \\ v''(x) + \frac{2v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{2u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= -\frac{2u}{x} \end{aligned}$$

Where $f(x) = -\frac{2}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned}
\frac{1}{u}\, du &= -\frac{2}{x}\, dx \\
\int \frac{1}{u}\, du &= \int -\frac{2}{x}\, dx \\
\ln(u) &= -2\ln(x) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+c_1} \\
&= \frac{c_1}{x^2}
\end{aligned}$$

Now that $u(x)$ is known, then

$$\begin{aligned}
v'(x) &= u(x) \\
v(x) &= \int u(x)\, dx + c_2 \\
&= -\frac{c_1}{x} + c_2
\end{aligned}$$

Hence

$$\begin{aligned}
y &= v(x)\, x^n \\
&= \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}} \\
&= (c_2 x - c_1)\sqrt{x}
\end{aligned}$$

Now the particular solution to this ODE is found

$$4x^2 y'' - 4y'x + 3y = 8x^{\frac{4}{3}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned}
y_1 &= \sqrt{x} \\
y_2 &= x^{\frac{3}{2}}
\end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & x^{\frac{3}{2}} \\ \frac{d}{dx}\left(\sqrt{x}\right) & \frac{d}{dx}\left(x^{\frac{3}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & x^{\frac{3}{2}} \\ \frac{1}{2\sqrt{x}} & \frac{3\sqrt{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\right)\left(\frac{3\sqrt{x}}{2}\right) - \left(x^{\frac{3}{2}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^{\frac{17}{6}}}{4x^3}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{x^{\frac{1}{6}}}\,dx$$

Hence

$$u_1 = -\frac{12x^{\frac{5}{6}}}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^{\frac{11}{6}}}{4x^3}\,dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^{\frac{7}{6}}}dx$$

Hence

$$u_2 = -\frac{12}{x^{\frac{1}{6}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{72x^{\frac{4}{3}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}}\right) + \left(-\frac{72x^{\frac{4}{3}}}{5}\right) \\ &= -\frac{72x^{\frac{4}{3}}}{5} + \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}} \end{aligned}$$

Which simplifies to

$$y = -\frac{72x^{\frac{4}{3}}}{5} + \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}}$$

Summary

The solution(s) found are the following

$$y = -\frac{72x^{\frac{4}{3}}}{5} + \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}} \tag{1}$$

Verification of solutions

$$y = -\frac{72x^{\frac{4}{3}}}{5} + \left(-\frac{c_1}{x} + c_2\right) x^{\frac{3}{2}}$$

Verified OK.

11.37.7 Solving using Kovacic algorithm

Writing the ode as

$$4x^2y'' - 4y'x + 3y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 4x^2 \\ B &= -4x \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A} dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 516: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - -\infty \\ &= \infty \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = 0$ is not a function of x, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(x) = 1$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-4x}{4x^2}\,dx} \\
&= z_1 e^{\frac{\ln(x)}{2}} \\
&= z_1\left(\sqrt{x}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = \sqrt{x}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-4x}{4x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{\ln(x)}}{(y_1)^2}\,dx \\
&= y_1(x)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left(\sqrt{x}\right) + c_2\left(\sqrt{x}\,(x)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$
4x^2y'' - 4y'x + 3y = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = \sqrt{x}\, c_1 + c_2 x^{\frac{3}{2}}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \sqrt{x}$$
$$y_2 = x^{\frac{3}{2}}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{a W(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{a W(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \sqrt{x} & x^{\frac{3}{2}} \\ \frac{d}{dx}\left(\sqrt{x}\right) & \frac{d}{dx}\left(x^{\frac{3}{2}}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \sqrt{x} & x^{\frac{3}{2}} \\ \frac{1}{2\sqrt{x}} & \frac{3\sqrt{x}}{2} \end{vmatrix}$$

Therefore

$$W = \left(\sqrt{x}\right)\left(\frac{3\sqrt{x}}{2}\right) - \left(x^{\frac{3}{2}}\right)\left(\frac{1}{2\sqrt{x}}\right)$$

Which simplifies to

$$W = x$$

Which simplifies to

$$W = x$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{8x^{\frac{17}{6}}}{4x^3} \,dx$$

Which simplifies to

$$u_1 = -\int \frac{2}{x^{\frac{1}{6}}} dx$$

Hence

$$u_1 = -\frac{12x^{\frac{5}{6}}}{5}$$

And Eq. (3) becomes

$$u_2 = \int \frac{8x^{\frac{11}{6}}}{4x^3} \,dx$$

Which simplifies to

$$u_2 = \int \frac{2}{x^{\frac{7}{6}}} dx$$

Hence

$$u_2 = -\frac{12}{x^{\frac{1}{6}}}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = -\frac{72x^{\frac{4}{3}}}{5}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\sqrt{x}\, c_1 + c_2 x^{\frac{3}{2}}\right) + \left(-\frac{72x^{\frac{4}{3}}}{5}\right) \end{aligned}$$

Which simplifies to

$$y = \sqrt{x}\,(c_2 x + c_1) - \frac{72x^{\frac{4}{3}}}{5}$$

Summary

The solution(s) found are the following

$$y = \sqrt{x}\,(c_2 x + c_1) - \frac{72x^{\frac{4}{3}}}{5} \tag{1}$$

Verification of solutions

$$y = \sqrt{x}\,(c_2 x + c_1) - \frac{72x^{\frac{4}{3}}}{5}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 20

```
dsolve(4*x^2*diff(y(x),x$2)-4*x*diff(y(x),x)+3*y(x)=8*x^(4/3),y(x), singsol=all)
```

$$y(x) = x^{\frac{3}{2}} c_2 + c_1 \sqrt{x} - \frac{72x^{\frac{4}{3}}}{5}$$

✓ Solution by Mathematica

Time used: 0.016 (sec). Leaf size: 31

```
DSolve[4*x^2*y''[x]-4*x*y'[x]+3*y[x]==8*x^(4/3),y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \frac{1}{5}\sqrt{x}\left(-72x^{5/6} + 5c_2x + 5c_1\right)$$

11.38 problem 61

Internal problem ID [256]
Internal file name [`OUTPUT/256_Sunday_June_05_2022_01_37_33_AM_73416235/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 61.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_euler_ode", "second_order_change_of_variable_on_x_method_1", "second_order_change_of_variable_on_x_method_2", "second_order_change_of_variable_on_y_method_2"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{x^2y'' + y'x + y = \ln(x)}$$

11.38.1 Solving as second order euler ode ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + y = 0$$

This is Euler second order ODE. Let the solution be $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$. Substituting these back into the given ODE gives

$$x^2(r(r-1))x^{r-2} + xrx^{r-1} + x^r = 0$$

Simplifying gives

$$r(r-1)\,x^r + r\,x^r + x^r = 0$$

Since $x^r \neq 0$ then dividing throughout by x^r gives

$$r(r-1) + r + 1 = 0$$

Or

$$r^2 + 1 = 0 \tag{1}$$

Equation (1) is the characteristic equation. Its roots determine the form of the general solution. Using the quadratic equation the roots are

$$\begin{aligned} r_1 &= -i \\ r_2 &= i \end{aligned}$$

The roots are complex conjugate of each others. Let the roots be

$$\begin{aligned} r_1 &= \alpha + i\beta \\ r_2 &= \alpha - i\beta \end{aligned}$$

Where in this case $\alpha = 0$ and $\beta = -1$. Hence the solution becomes

$$\begin{aligned} y &= c_1 x^{r_1} + c_2 x^{r_2} \\ &= c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta} \\ &= x^\alpha \left(c_1 x^{i\beta} + c_2 x^{-i\beta}\right) \\ &= x^\alpha \left(c_1 e^{\ln\left(x^{i\beta}\right)} + c_2 e^{\ln\left(x^{-i\beta}\right)}\right) \\ &= x^\alpha \left(c_1 e^{i(\beta \ln x)} + c_2 e^{-i(\beta \ln x)}\right) \end{aligned}$$

Using the values for $\alpha = 0, \beta = -1$, the above becomes

$$y = x^0 \left(c_1 e^{-i\ln(x)} + c_2 e^{i\ln(x)}\right)$$

Using Euler relation, the expression $c_1 e^{iA} + c_2 e^{-iA}$ is transformed to $c_1 \cos A + c_1 \sin A$ where the constants are free to change. Applying this to the above result gives

$$y = c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Next, we find the particular solution to the ODE

$$x^2y'' + y'x + y = \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = \cos(\ln(x))$$
$$y_2 = -\sin(\ln(x))$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & -\sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(-\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & -\sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & -\frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(-\frac{\cos(\ln(x))}{x}\right) - (-\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = -\frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = -\frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\sin(\ln(x))\ln(x)}{-x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(\ln(x))\ln(x)}{x}dx$$

Hence

$$u_1 = \cos(\ln(x))\ln(x) - \sin(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(\ln(x))\ln(x)}{-x}\,dx$$

Which simplifies to

$$u_2 = \int -\frac{\cos(\ln(x))\ln(x)}{x}dx$$

Hence

$$u_2 = -\cos(\ln(x)) - \sin(\ln(x))\ln(x)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned}y_p(x) = &(\cos(\ln(x))\ln(x) - \sin(\ln(x)))\cos(\ln(x)) \\ &- (-\cos(\ln(x)) - \sin(\ln(x))\ln(x))\sin(\ln(x))\end{aligned}$$

Which simplifies to

$$y_p(x) = \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \ln(x) + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = \ln(x) + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}$$

Verification of solutions

$$y = \ln(x) + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Verified OK.

11.38.2 Solving as second order change of variable on x method 2 ode

This is second order non-homogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. y_h is the solution to

$$x^2y'' + y'x + y = 0$$

In normal form the ode

$$x^2y'' + y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\, y' + q(x)\, y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) gives

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $p_1 = 0$. Eq (4) simplifies to

$$\tau''(x) + p(x)\,\tau'(x) = 0$$

This ode is solved resulting in

$$\begin{aligned}
\tau &= \int \mathrm{e}^{-\left(\int p(x)dx\right)} dx \\
&= \int \mathrm{e}^{-\left(\int \frac{1}{x}dx\right)} dx \\
&= \int e^{-\ln(x)}\, dx \\
&= \int \frac{1}{x} dx \\
&= \ln(x)
\end{aligned} \tag{6}$$

Using (6) to evaluate q_1 from (5) gives

$$\begin{aligned}
q_1(\tau) &= \frac{q(x)}{\tau'(x)^2} \\
&= \frac{\frac{1}{x^2}}{\frac{1}{x^2}} \\
&= 1
\end{aligned} \tag{7}$$

Substituting the above in (3) and noting that now $p_1 = 0$ results in

$$\begin{aligned}
\frac{d^2}{d\tau^2} y(\tau) + q_1 y(\tau) &= 0 \\
\frac{d^2}{d\tau^2} y(\tau) + y(\tau) &= 0
\end{aligned}$$

The above ode is now solved for $y(\tau)$.This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ay''(\tau) + By'(\tau) + Cy(\tau) = 0$$

Where in the above $A=1,B=0,C=1$. Let the solution be $y(\tau)=e^{\lambda\tau}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda\tau}+\mathrm{e}^{\lambda\tau}=0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda\tau}$ gives

$$\lambda^2+1=0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2}=\frac{-B}{2A}\pm\frac{1}{2A}\sqrt{B^2-4AC}$$

Substituting $A=1,B=0,C=1$ into the above gives

$$\begin{aligned}\lambda_{1,2}&=\frac{0}{(2)(1)}\pm\frac{1}{(2)(1)}\sqrt{0^2-(4)(1)(1)}\\&=\pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1&=+i\\\lambda_2&=-i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1&=i\\\lambda_2&=-i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2}=\alpha\pm i\beta$$

Where $\alpha=0$ and $\beta=1$. Therefore the final solution, when using Euler relation, can be written as

$$y(\tau)=e^{\alpha\tau}(c_1\cos(\beta\tau)+c_2\sin(\beta\tau))$$

Which becomes

$$y(\tau)=e^0(c_1\cos(\tau)+c_2\sin(\tau))$$

Or

$$y(\tau)=c_1\cos(\tau)+c_2\sin(\tau)$$

The above solution is now transformed back to y using (6) which results in

$$y = c_1 \cos (\ln (x)) + c_2 \sin (\ln (x))$$

Therefore the homogeneous solution y_h is

$$y_h = c_1 \cos (\ln (x)) + c_2 \sin (\ln (x))$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= \cos (\ln (x)) \\ y_2 &= \sin (\ln (x)) \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos (\ln (x)) & \sin (\ln (x)) \\ \frac{d}{dx}(\cos (\ln (x))) & \frac{d}{dx}(\sin (\ln (x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos (\ln (x)) & \sin (\ln (x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(\frac{\cos(\ln(x))}{x}\right) - (\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = \frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(\ln(x))\ln(x)}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(\ln(x))\ln(x)}{x}dx$$

Hence

$$u_1 = \cos(\ln(x))\ln(x) - \sin(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(\ln(x))\ln(x)}{x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(\ln(x))\ln(x)}{x}dx$$

Hence

$$u_2 = \cos(\ln(x)) + \sin(\ln(x))\ln(x)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = {} & (\cos(\ln(x))\ln(x) - \sin(\ln(x)))\cos(\ln(x)) \\ & + (\cos(\ln(x)) + \sin(\ln(x))\ln(x))\sin(\ln(x)) \end{aligned}$$

Which simplifies to

$$y_p(x) = \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))) + (\ln(x)) \end{aligned}$$

Summary
The solution(s) found are the following

$$y = \ln(x) + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x)) \tag{1}$$

Verification of solutions

$$y = \ln(x) + c_1 \cos(\ln(x)) + c_2 \sin(\ln(x))$$

Verified OK.

11.38.3 Solving as second order change of variable on x method 1 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + y = 0$$

In normal form the ode

$$x^2y'' + y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables $\tau = g(x)$ to (2) results

$$\frac{d^2}{d\tau^2}y(\tau) + p_1\left(\frac{d}{d\tau}y(\tau)\right) + q_1 y(\tau) = 0 \tag{3}$$

Where τ is the new independent variable, and

$$p_1(\tau) = \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \tag{4}$$

$$q_1(\tau) = \frac{q(x)}{\tau'(x)^2} \tag{5}$$

Let $q_1 = c^2$ where c is some constant. Therefore from (5)

$$\begin{aligned} \tau' &= \frac{1}{c}\sqrt{q} \\ &= \frac{\sqrt{\frac{1}{x^2}}}{c} \\ \tau'' &= -\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} \end{aligned} \tag{6}$$

Substituting the above into (4) results in

$$\begin{aligned} p_1(\tau) &= \frac{\tau''(x) + p(x)\,\tau'(x)}{\tau'(x)^2} \\ &= \frac{-\frac{1}{c\sqrt{\frac{1}{x^2}}\,x^3} + \frac{1}{x}\frac{\sqrt{\frac{1}{x^2}}}{c}}{\left(\frac{\sqrt{\frac{1}{x^2}}}{c}\right)^2} \\ &= 0 \end{aligned}$$

Therefore ode (3) now becomes

$$\begin{aligned} y(\tau)'' + p_1 y(\tau)' + q_1 y(\tau) &= 0 \\ \frac{d^2}{d\tau^2}y(\tau) + c^2 y(\tau) &= 0 \end{aligned} \tag{7}$$

The above ode is now solved for $y(\tau)$. Since the ode is now constant coefficients, it can be easily solved to give

$$y(\tau) = c_1 \cos(c\tau) + c_2 \sin(c\tau)$$

Now from (6)

$$
\begin{aligned}
\tau &= \int \frac{1}{c}\sqrt{q}\, dx \\
&= \frac{\int \sqrt{\frac{1}{x^2}}dx}{c} \\
&= \frac{\sqrt{\frac{1}{x^2}}\, x \ln (x)}{c}
\end{aligned}
$$

Substituting the above into the solution obtained gives

$$
y = c_1 \cos (\ln (x)) + c_2 \sin (\ln (x))
$$

Now the particular solution to this ODE is found

$$
x^2 y'' + y'x + y = \ln (x)
$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$
\begin{aligned}
y_1 &= \cos (\ln (x)) \\
y_2 &= \sin (\ln (x))
\end{aligned}
$$

In the Variation of parameters u_1, u_2 are found using

$$
u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}
$$

$$
u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}
$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ \frac{d}{dx}(\cos(\ln(x))) & \frac{d}{dx}(\sin(\ln(x))) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} \cos(\ln(x)) & \sin(\ln(x)) \\ -\frac{\sin(\ln(x))}{x} & \frac{\cos(\ln(x))}{x} \end{vmatrix}$$

Therefore

$$W = (\cos(\ln(x)))\left(\frac{\cos(\ln(x))}{x}\right) - (\sin(\ln(x)))\left(-\frac{\sin(\ln(x))}{x}\right)$$

Which simplifies to

$$W = \frac{\cos(\ln(x))^2 + \sin(\ln(x))^2}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{\sin(\ln(x))\ln(x)}{x}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{\sin(\ln(x))\ln(x)}{x}dx$$

Hence

$$u_1 = \cos(\ln(x))\ln(x) - \sin(\ln(x))$$

And Eq. (3) becomes

$$u_2 = \int \frac{\cos(\ln(x))\ln(x)}{x}\,dx$$

Which simplifies to

$$u_2 = \int \frac{\cos(\ln(x))\ln(x)}{x} dx$$

Hence

$$u_2 = \cos(\ln(x)) + \sin(\ln(x))\ln(x)$$

Therefore the particular solution, from equation (1) is

$$\begin{aligned} y_p(x) = &(\cos(\ln(x))\ln(x) - \sin(\ln(x)))\cos(\ln(x)) \\ &+ (\cos(\ln(x)) + \sin(\ln(x))\ln(x))\sin(\ln(x)) \end{aligned}$$

Which simplifies to

$$y_p(x) = \ln(x)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= (c_1\cos(\ln(x)) + c_2\sin(\ln(x))) + (\ln(x)) \\ &= \ln(x) + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) \end{aligned}$$

Which simplifies to

$$y = \ln(x) + c_1\cos(\ln(x)) + c_2\sin(\ln(x))$$

Summary

The solution(s) found are the following

$$y = \ln(x) + c_1\cos(\ln(x)) + c_2\sin(\ln(x)) \tag{1}$$

Verification of solutions

$$y = \ln(x) + c_1\cos(\ln(x)) + c_2\sin(\ln(x))$$

Verified OK.

11.38.4 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x) + By'(x) + Cy(x) = f(x)$$

Where $A = x^2, B = x, C = 1, f(x) = \ln(x)$. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = f(x)$. Solving for y_h from

$$x^2y'' + y'x + y = 0$$

In normal form the ode

$$x^2y'' + y'x + y = 0 \tag{1}$$

Becomes

$$y'' + p(x)\,y' + q(x)\,y = 0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= \frac{1}{x} \\ q(x) &= \frac{1}{x^2} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\,x^n$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x) + \left(\frac{2n}{x} + p\right) v'(x) + \left(\frac{n(n-1)}{x^2} + \frac{np}{x} + q\right) v(x) = 0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^2} + \frac{np}{x} + q = 0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^2} + \frac{n}{x^2} + \frac{1}{x^2} = 0 \tag{5}$$

Solving (5) for n gives

$$n = i \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x) + \left(\frac{2i}{x} + \frac{1}{x}\right) v'(x) &= 0 \\ v''(x) + \frac{(1+2i)\, v'(x)}{x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x) = v'(x)$$

Then (7) becomes

$$u'(x) + \frac{(1+2i)\, u(x)}{x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x,u) \\ &= f(x)g(u) \\ &= \frac{(-1-2i)\, u}{x} \end{aligned}$$

Where $f(x) = \frac{-1-2i}{x}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u}\, du &= \frac{-1-2i}{x}\, dx \\ \int \frac{1}{u}\, du &= \int \frac{-1-2i}{x}\, dx \\ \ln(u) &= (-1-2i)\ln(x) + c_1 \\ u &= \mathrm{e}^{(-1-2i)\ln(x)+c_1} \\ &= c_1 \mathrm{e}^{(-1-2i)\ln(x)} \end{aligned}$$

Which simplifies to

$$u(x) = \frac{c_1 x^{-2i}}{x}$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x)\,dx + c_2 \\ &= \frac{ic_1x^{-2i}}{2} + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x)\,x^n \\ &= \left(\frac{ic_1x^{-2i}}{2} + c_2\right) x^i \\ &= x^i c_2 + \frac{ix^{-i}c_1}{2} \end{aligned}$$

Now the particular solution to this ODE is found

$$x^2y'' + y'x + y = \ln(x)$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x^i \\ y_2 &= x^{-i} \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^i & x^{-i} \\ \frac{d}{dx}(x^i) & \frac{d}{dx}(x^{-i}) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^i & x^{-i} \\ \frac{ix^i}{x} & -\frac{ix^{-i}}{x} \end{vmatrix}$$

Therefore

$$W = (x^i)\left(-\frac{ix^{-i}}{x}\right) - (x^{-i})\left(\frac{ix^i}{x}\right)$$

Which simplifies to

$$W = -\frac{2i}{x}$$

Which simplifies to

$$W = -\frac{2i}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{x^{-i}\ln(x)}{-2ix}\,dx$$

Which simplifies to

$$u_1 = -\int \frac{ix^{-1-i}\ln(x)}{2}dx$$

Hence

$$u_1 = \mathit{undefined}$$

And Eq. (3) becomes

$$u_2 = \int \frac{\ln(x)\,x^i}{-2ix}\,dx$$

Which simplifies to

$$u_2 = \int \frac{i x^{-1+i} \ln (x)}{2} dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^i + \text{undefined } x^{-i}$$

Which simplifies to

$$y_p(x) = \text{undefined } \left(x^i + x^{-i}\right)$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(\left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i\right) + \left(\text{undefined } \left(x^i + x^{-i}\right)\right) \\ &= \text{undefined } \left(x^i + x^{-i}\right) + \left(\frac{i c_1 x^{-2i}}{2} + c_2\right) x^i \end{aligned}$$

Which simplifies to

$$y = \frac{x^{-i}((\text{undefined } + 2c_2) x^{2i} + i c_1 + \text{undefined})}{2}$$

Summary

The solution(s) found are the following

$$y = \frac{x^{-i}((\text{undefined } + 2c_2) x^{2i} + i c_1 + \text{undefined})}{2} \tag{1}$$

Verification of solutions

$$y = \frac{x^{-i}((\text{undefined } + 2c_2) x^{2i} + i c_1 + \text{undefined})}{2}$$

Verified OK.

11.38.5 Solving using Kovacic algorithm

Writing the ode as

$$x^2y'' + y'x + y = 0 \tag{1}$$

$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 \\ B &= x \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = ye^{\int \frac{B}{2A}\,dx}$$

Then (2) becomes

$$z''(x) = rz(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-5}{4x^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -5 \\ t &= 4x^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(-\frac{5}{4x^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\,dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 517: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 2 - 0 \\
&= 2
\end{aligned}
$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = 4x^2$. There is a pole at $x = 0$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 2 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 2 then the necessary conditions for case three are met. Therefore

$$
L = [1, 2, 4, 6, 12]
$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$
r = -\frac{5}{4x^2}
$$

For the pole at $x = 0$ let b be the coefficient of $\frac{1}{x^2}$ in the partial fractions decomposition of r given above. Therefore $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_c &= 0 \\
\alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

Since the order of r at ∞ is 2 then $[\sqrt{r}]_\infty = 0$. Let b be the coefficient of $\frac{1}{x^2}$ in the Laurent series expansion of r at ∞. which can be found by dividing the leading coefficient of s by the leading coefficient of t from

$$
r = \frac{s}{t} = -\frac{5}{4x^2}
$$

Since the $\gcd(s,t) = 1$. This gives $b = -\frac{5}{4}$. Hence

$$
\begin{aligned}
[\sqrt{r}]_\infty &= 0 \\
\alpha_\infty^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{1}{2} + i \\
\alpha_\infty^- &= \frac{1}{2} - \sqrt{1+4b} = \frac{1}{2} - i
\end{aligned}
$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$
r = -\frac{5}{4x^2}
$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
0	2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
2	0	$\frac{1}{2} + i$	$\frac{1}{2} - i$

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^\pm$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^\pm$ have also been found, the next step is to determine possible non negative integer d from these using

$$
d = \alpha_\infty^{s(\infty)} - \sum_{c \in \Gamma} \alpha_c^{s(c)}
$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^\pm$. This is done by trial over all set of families $s = (s(c))_{c \in \Gamma \cup \infty}$ until such d is found to work in finding candidate ω.

Trying $\alpha_{\infty}^{-} = \frac{1}{2} - i$ then

$$\begin{aligned}
d &= \alpha_{\infty}^{-} - (\alpha_{c_1}^{-}) \\
&= \frac{1}{2} - i - \left(\frac{1}{2} - i\right) \\
&= 0
\end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c \in \Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x - c} \right) + s(\infty)[\sqrt{r}]_{\infty}$$

The above gives

$$\begin{aligned}
\omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x - c_1} \right) + (-)[\sqrt{r}]_{\infty} \\
&= \frac{\frac{1}{2} - i}{x} + (-)\,(0) \\
&= \frac{\frac{1}{2} - i}{x} \\
&= \frac{\frac{1}{2} - i}{x}
\end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$\begin{aligned}
(0) + 2\left(\frac{\frac{1}{2} - i}{x}\right)(0) + \left(\left(\frac{-\frac{1}{2} + i}{x^2}\right) + \left(\frac{\frac{1}{2} - i}{x}\right)^2 - \left(-\frac{5}{4x^2}\right) \right) &= 0 \\
0 &= 0
\end{aligned}$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z'' = rz$ is

$$\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= e^{\int \frac{\frac{1}{2} - i}{x} dx} \\
&= x^{\frac{1}{2} - i}
\end{aligned}$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{x}{x^2}\,dx} \\
&= z_1 e^{-\frac{\ln(x)}{2}} \\
&= z_1 \left(\frac{1}{\sqrt{x}}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = x^{-i}
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A}\,dx}}{y_1^2}\,dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{x}{x^2}\,dx}}{(y_1)^2}\,dx \\
&= y_1 \int \frac{e^{-\ln(x)}}{(y_1)^2}\,dx \\
&= y_1 \left(-\frac{i x^{2i}}{2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1 \left(x^{-i}\right) + c_2 \left(x^{-i}\left(-\frac{i x^{2i}}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
y = y_h + y_p
$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. y_h is the solution to

$$x^2y''+y'x+y=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = x^{-i}c_1 - \frac{ic_2x^i}{2}$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1y_1 + u_2y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x^{-i}$$
$$y_2 = -\frac{ix^i}{2}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x^{-i} & -\frac{ix^i}{2} \\ \frac{d}{dx}\left(x^{-i}\right) & \frac{d}{dx}\left(-\frac{ix^i}{2}\right) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x^{-i} & -\frac{i x^{i}}{2} \\ -\frac{i x^{-i}}{x} & \frac{x^{i}}{2x} \end{vmatrix}$$

Therefore

$$W = \left(x^{-i}\right)\left(\frac{x^{i}}{2x}\right) - \left(-\frac{i x^{i}}{2}\right)\left(-\frac{i x^{-i}}{x}\right)$$

Which simplifies to

$$W = \frac{x^{i} x^{-i}}{x}$$

Which simplifies to

$$W = \frac{1}{x}$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-\frac{i x^{i} \ln(x)}{2}}{x}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{i x^{-1+i} \ln(x)}{2} dx$$

Hence

$$u_1 = undefined$$

And Eq. (3) becomes

$$u_2 = \int \frac{x^{-i} \ln(x)}{x}\, dx$$

Which simplifies to

$$u_2 = \int x^{-1-i} \ln(x)\, dx$$

Hence

$$u_2 = undefined$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \text{undefined } x^{-i} - i \,\text{undefined } x^i$$

Which simplifies to

$$y_p(x) = \left(i x^i + x^{-i}\right) \text{undefined}$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(x^{-i} c_1 - \frac{i c_2 x^i}{2}\right) + \left(\left(i x^i + x^{-i}\right) \text{undefined}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = x^{-i} c_1 - \frac{i c_2 x^i}{2} + \left(i x^i + x^{-i}\right) \text{undefined} \tag{1}$$

Verification of solutions

$$y = x^{-i} c_1 - \frac{i c_2 x^i}{2} + \left(i x^i + x^{-i}\right) \text{undefined}$$

Verified OK.

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   <- LODE of Euler type successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 17

```
dsolve(x^2*diff(y(x),x$2)+x*diff(y(x),x)+y(x)=ln(x),y(x), singsol=all)
```

$$y(x) = \sin(\ln(x))\, c_2 + \cos(\ln(x))\, c_1 + \ln(x)$$

✓ Solution by Mathematica

Time used: 0.073 (sec). Leaf size: 20

```
DSolve[x^2*y''[x]+x*y'[x]+y[x]==Log[x],y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to \log(x) + c_1 \cos(\log(x)) + c_2 \sin(\log(x))$$

11.39 problem 62

Internal problem ID [257]
Internal file name [`OUTPUT/257_Sunday_June_05_2022_01_37_34_AM_29204451/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.5, Nonhomogeneous equations and undetermined coefficients Page 351
Problem number: 62.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_change_of_variable_on_y_method_2", "second_order_ode_non_constant_coeff_transformation_on_B"**

Maple gives the following as the ode type

```
[[_2nd_order, _with_linear_symmetries]]
```

$$\boxed{\left(x^2-1\right) y''-2y'x+2y=x^2-1}$$

11.39.1 Solving as second order change of variable on y method 2 ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ay''(x)+By'(x)+Cy(x)=f(x)$$

Where $A=x^2-1, B=-2x, C=2, f(x)=x^2-1$. Let the solution be

$$y=y_h+y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=0$, and y_p is a particular solution to the non-homogeneous ODE $Ay''(x)+By'(x)+Cy(x)=f(x)$. Solving for y_h from

$$\left(x^2-1\right) y''-2y'x+2y=0$$

In normal form the ode

$$\left(x^{2}-1\right) y''-2y'x+2y=0 \tag{1}$$

Becomes

$$y''+p(x)\, y'+q(x)\, y=0 \tag{2}$$

Where

$$\begin{aligned} p(x) &= -\frac{2x}{x^{2}-1} \\ q(x) &= \frac{2}{x^{2}-1} \end{aligned}$$

Applying change of variables on the depndent variable $y = v(x)\, x^{n}$ to (2) gives the following ode where the dependent variables is $v(x)$ and not y.

$$v''(x)+\left(\frac{2n}{x}+p\right) v'(x)+\left(\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q\right) v(x)=0 \tag{3}$$

Let the coefficient of $v(x)$ above be zero. Hence

$$\frac{n(n-1)}{x^{2}}+\frac{np}{x}+q=0 \tag{4}$$

Substituting the earlier values found for $p(x)$ and $q(x)$ into (4) gives

$$\frac{n(n-1)}{x^{2}}-\frac{2n}{x^{2}-1}+\frac{2}{x^{2}-1}=0 \tag{5}$$

Solving (5) for n gives

$$n=1 \tag{6}$$

Substituting this value in (3) gives

$$\begin{aligned} v''(x)+\left(\frac{2}{x}-\frac{2x}{x^{2}-1}\right) v'(x) &= 0 \\ v''(x)-\frac{2v'(x)}{x^{3}-x} &= 0 \end{aligned} \tag{7}$$

Using the substitution

$$u(x)=v'(x)$$

Then (7) becomes

$$u'(x) - \frac{2u(x)}{x^3 - x} = 0 \tag{8}$$

The above is now solved for $u(x)$. In canonical form the ODE is

$$\begin{aligned} u' &= F(x, u) \\ &= f(x)g(u) \\ &= \frac{2u}{x(x^2-1)} \end{aligned}$$

Where $f(x) = \frac{2}{x(x^2-1)}$ and $g(u) = u$. Integrating both sides gives

$$\begin{aligned} \frac{1}{u} \, du &= \frac{2}{x(x^2-1)} \, dx \\ \int \frac{1}{u} \, du &= \int \frac{2}{x(x^2-1)} \, dx \\ \ln(u) &= -2\ln(x) + \ln(x+1) + \ln(x-1) + c_1 \\ u &= \mathrm{e}^{-2\ln(x)+\ln(x+1)+\ln(x-1)+c_1} \\ &= c_1 \mathrm{e}^{-2\ln(x)+\ln(x+1)+\ln(x-1)} \end{aligned}$$

Which simplifies to

$$u(x) = c_1\left(1 - \frac{1}{x^2}\right)$$

Now that $u(x)$ is known, then

$$\begin{aligned} v'(x) &= u(x) \\ v(x) &= \int u(x) \, dx + c_2 \\ &= c_1\left(x + \frac{1}{x}\right) + c_2 \end{aligned}$$

Hence

$$\begin{aligned} y &= v(x) \, x^n \\ &= \left(c_1\left(x + \frac{1}{x}\right) + c_2\right) x \\ &= c_1 x^2 + c_2 x + c_1 \end{aligned}$$

Now the particular solution to this ODE is found

$$(x^2-1)\,y''-2y'x+2y=x^2-1$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x)=u_1y_1+u_2y_2 \tag{1}$$

Where u_1,u_2 to be determined, and y_1,y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= x \\ y_2 &= x^2+1 \end{aligned}$$

In the Variation of parameters u_1,u_2 are found using

$$u_1=-\int \frac{y_2f(x)}{aW(x)} \tag{2}$$

$$u_2=\int \frac{y_1f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W=\begin{vmatrix} x & x^2+1 \\ \frac{d}{dx}(x) & \frac{d}{dx}(x^2+1) \end{vmatrix}$$

Which gives

$$W=\begin{vmatrix} x & x^2+1 \\ 1 & 2x \end{vmatrix}$$

Therefore

$$W=(x)\,(2x)-(x^2+1)\,(1)$$

Which simplifies to

$$W = x^2 - 1$$

Which simplifies to

$$W = x^2 - 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(x^2+1)(x^2-1)}{(x^2-1)^2}\, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2+1}{x^2-1} dx$$

Hence

$$u_1 = -x - \ln(x-1) + \ln(x+1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2-1)}{(x^2-1)^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{x}{x^2-1} dx$$

Hence

$$u_2 = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-x - \ln(x-1) + \ln(x+1))\, x + \left(\frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}\right)(x^2+1)$$

Which simplifies to

$$y_p(x) = \frac{(x-1)^2 \ln(x-1)}{2} + \frac{(x+1)^2 \ln(x+1)}{2} - x^2$$

Therefore the general solution is

$$
\begin{aligned}
y &= y_h + y_p \\
&= \left(\left(c_1\left(x+\frac{1}{x}\right)+c_2\right)x\right)+\left(\frac{(x-1)^2\ln(x-1)}{2}+\frac{(x+1)^2\ln(x+1)}{2}-x^2\right) \\
&= \frac{(x-1)^2\ln(x-1)}{2}+\frac{(x+1)^2\ln(x+1)}{2}-x^2+\left(c_1\left(x+\frac{1}{x}\right)+c_2\right)x
\end{aligned}
$$

Which simplifies to

$$
y = \frac{(x-1)^2\ln(x-1)}{2}+\frac{(x+1)^2\ln(x+1)}{2}+(c_1-1)x^2+c_2x+c_1
$$

Summary

The solution(s) found are the following

$$
y = \frac{(x-1)^2\ln(x-1)}{2}+\frac{(x+1)^2\ln(x+1)}{2}+(c_1-1)x^2+c_2x+c_1 \tag{1}
$$

Verification of solutions

$$
y = \frac{(x-1)^2\ln(x-1)}{2}+\frac{(x+1)^2\ln(x+1)}{2}+(c_1-1)x^2+c_2x+c_1
$$

Verified OK.

11.39.2 Solving as second order ode non constant coeff transformation on B ode

Given an ode of the form

$$
Ay'' + By' + Cy = F(x)
$$

This method reduces the order ode the ODE by one by applying the transformation

$$
y = Bv
$$

This results in

$$
\begin{aligned}
y' &= B'v + v'B \\
y'' &= B''v + B'v' + v''B + v'B' \\
&= v''B + 2v' + B' + B''v
\end{aligned}
$$

And now the original ode becomes

$$
\begin{aligned}
A(v''B+2v'B'+B''v)+B(B'v+v'B)+CBv&=0\\
ABv''+\left(2AB'+B^2\right)v'+\left(AB''+BB'+CB\right)v&=0 \qquad (1)
\end{aligned}
$$

If the term $AB''+BB'+CB$ is zero, then this method works and can be used to solve

$$
ABv''+\left(2AB'+B^2\right)v'=0
$$

By Using $u=v'$ which reduces the order of the above ode to one. The new ode is

$$
ABu'+\left(2AB'+B^2\right)u=0
$$

The above ode is first order ode which is solved for u. Now a new ode $v'=u$ is solved for v as first order ode. Then the final solution is obtain from $y=Bv$.

This method works only if the term $AB''+BB'+CB$ is zero. The given ODE shows that

$$
\begin{aligned}
A&=x^2-1\\
B&=-2x\\
C&=2\\
F&=x^2-1
\end{aligned}
$$

The above shows that for this ode

$$
\begin{aligned}
AB''+BB'+CB&=\left(x^2-1\right)(0)+(-2x)(-2)+(2)(-2x)\\
&=0
\end{aligned}
$$

Hence the ode in v given in (1) now simplifies to

$$
-2x^3+2xv''+(4)v'=0
$$

Now by applying $v'=u$ the above becomes

$$
\left(-2x^3+2x\right)u'(x)+4u(x)=0
$$

Which is now solved for u. In canonical form the ODE is

$$
\begin{aligned}
u'&=F(x,u)\\
&=f(x)g(u)\\
&=\frac{2u}{x\left(x^2-1\right)}
\end{aligned}
$$

Where $f(x) = \frac{2}{x(x^2-1)}$ and $g(u) = u$. Integrating both sides gives

$$
\begin{aligned}
\frac{1}{u}\, du &= \frac{2}{x\,(x^2-1)}\, dx \\
\int \frac{1}{u}\, du &= \int \frac{2}{x\,(x^2-1)}\, dx \\
\ln\left(u\right) &= -2\ln\left(x\right) + \ln\left(x+1\right) + \ln\left(x-1\right) + c_1 \\
u &= \mathrm{e}^{-2\ln(x)+\ln(x+1)+\ln(x-1)+c_1} \\
&= c_1 \mathrm{e}^{-2\ln(x)+\ln(x+1)+\ln(x-1)}
\end{aligned}
$$

Which simplifies to

$$
u(x) = c_1\left(1 - \frac{1}{x^2}\right)
$$

The ode for v now becomes

$$
\begin{aligned}
v' &= u \\
&= c_1\left(1 - \frac{1}{x^2}\right)
\end{aligned}
$$

Which is now solved for v. Integrating both sides gives

$$
\begin{aligned}
v(x) &= \int \frac{c_1(x^2-1)}{x^2}\, \mathrm{d}x \\
&= c_1\left(x + \frac{1}{x}\right) + c_2
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
y_h(x) &= Bv \\
&= (-2x)\left(c_1\left(x + \frac{1}{x}\right) + c_2\right) \\
&= -2c_1x^2 - 2c_2x - 2c_1
\end{aligned}
$$

And now the particular solution $y_p(x)$ will be found. The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$
y_p(x) = u_1y_1 + u_2y_2 \tag{1}
$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$y_1 = x$$
$$y_2 = -2x^2 - 2$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$
$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} x & -2x^2 - 2 \\ \frac{d}{dx}(x) & \frac{d}{dx}(-2x^2 - 2) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} x & -2x^2 - 2 \\ 1 & -4x \end{vmatrix}$$

Therefore

$$W = (x)(-4x) - (-2x^2 - 2)(1)$$

Which simplifies to

$$W = -2x^2 + 2$$

Which simplifies to

$$W = -2x^2 + 2$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{(-2x^2 - 2)(x^2 - 1)}{(x^2 - 1)(-2x^2 + 2)} \, dx$$

Which simplifies to

$$u_1 = -\int \frac{x^2+1}{x^2-1} dx$$

Hence

$$u_1 = -x - \ln(x-1) + \ln(x+1)$$

And Eq. (3) becomes

$$u_2 = \int \frac{x(x^2-1)}{(x^2-1)(-2x^2+2)} dx$$

Which simplifies to

$$u_2 = \int -\frac{x}{2x^2-2} dx$$

Hence

$$u_2 = -\frac{\ln(x-1)}{4} - \frac{\ln(x+1)}{4}$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = (-x - \ln(x-1) + \ln(x+1))\, x + \left(-\frac{\ln(x-1)}{4} - \frac{\ln(x+1)}{4}\right)(-2x^2-2)$$

Which simplifies to

$$y_p(x) = \frac{(x-1)^2 \ln(x-1)}{2} + \frac{(x+1)^2 \ln(x+1)}{2} - x^2$$

Hence the complete solution is

$$\begin{aligned} y(x) &= y_h + y_p \\ &= (-2c_1x^2 - 2c_2x - 2c_1) + \left(\frac{(x-1)^2 \ln(x-1)}{2} + \frac{(x+1)^2 \ln(x+1)}{2} - x^2\right) \\ &= \frac{(x-1)^2 \ln(x-1)}{2} + \frac{(x+1)^2 \ln(x+1)}{2} + (-1-2c_1)\, x^2 - 2c_2x - 2c_1 \end{aligned}$$

<u>Summary</u>
The solution(s) found are the following

$$y = \frac{(x-1)^2 \ln(x-1)}{2} + \frac{(x+1)^2 \ln(x+1)}{2} + (-1-2c_1)\, x^2 - 2c_2x - 2c_1 \quad (1)$$

Verification of solutions

$$y = \frac{(x-1)^2 \ln(x-1)}{2} + \frac{(x+1)^2 \ln(x+1)}{2} + (-1-2c_1)\, x^2 - 2c_2 x - 2c_1$$

Verified OK.

11.39.3 Solving using Kovacic algorithm

Writing the ode as

$$\left(x^2-1\right) y'' - 2y'x + 2y = 0 \tag{1}$$
$$Ay'' + By' + Cy = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= x^2 - 1 \\ B &= -2x \\ C &= 2 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(x) = y e^{\int \frac{B}{2A}\, dx}$$

Then (2) becomes

$$z''(x) = r z(x) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{3}{(x^2-1)^2} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 3 \\ t &= \left(x^2-1\right)^2 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(x) = \left(\frac{3}{\left(x^2-1\right)^2}\right) z(x) \tag{7}$$

Equation (7) is now solved. After finding $z(x)$ then y is found using the inverse transformation

$$y = z(x)\, e^{-\int \frac{B}{2A}\, dx}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 518: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 4 - 0 \\ &= 4 \end{aligned}$$

The poles of r in eq. (7) and the order of each pole are determined by solving for the roots of $t = \left(x^2-1\right)^2$. There is a pole at $x = 1$ of order 2. There is a pole at $x = -1$ of order 2. Since there is no odd order pole larger than 2 and the order at ∞ is 4 then the necessary conditions for case one are met. Since there is a pole of order 2 then necessary conditions for case two are met. Since pole order is not larger than 2 and the order at ∞ is 4 then the necessary conditions for case three are met. Therefore

$$L = [1, 2, 4, 6, 12]$$

Attempting to find a solution using case $n = 1$.

Looking at poles of order 2. The partial fractions decomposition of r is

$$r = -\frac{3}{4(x-1)} + \frac{3}{4(x+1)} + \frac{3}{4(x+1)^2} + \frac{3}{4(x-1)^2}$$

For the pole at $x = 1$ let b be the coefficient of $\frac{1}{(x-1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

For the pole at $x = -1$ let b be the coefficient of $\frac{1}{(x+1)^2}$ in the partial fractions decomposition of r given above. Therefore $b = \frac{3}{4}$. Hence

$$\begin{aligned}[\sqrt{r}]_c &= 0 \\ \alpha_c^+ &= \frac{1}{2} + \sqrt{1+4b} = \frac{3}{2} \\ \alpha_c^- &= \frac{1}{2} - \sqrt{1+4b} = -\frac{1}{2}\end{aligned}$$

Since the order of r at ∞ is $4 > 2$ then

$$\begin{aligned}[\sqrt{r}]_\infty &= 0 \\ \alpha_\infty^+ &= 0 \\ \alpha_\infty^- &= 1\end{aligned}$$

The following table summarizes the findings so far for poles and for the order of r at ∞ where r is

$$r = \frac{3}{(x^2-1)^2}$$

pole c location	pole order	$[\sqrt{r}]_c$	α_c^+	α_c^-
1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$
-1	2	0	$\frac{3}{2}$	$-\frac{1}{2}$

Order of r at ∞	$[\sqrt{r}]_\infty$	α_∞^+	α_∞^-
4	0	0	1

Now that the all $[\sqrt{r}]_c$ and its associated $\alpha_c^{\pm}$ have been determined for all the poles in the set Γ and $[\sqrt{r}]_\infty$ and its associated $\alpha_\infty^{\pm}$ have also been found, the next step is to determine possible non negative integer d from these using

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma} \alpha_c^{s(c)}$$

Where $s(c)$ is either $+$ or $-$ and $s(\infty)$ is the sign of $\alpha_\infty^{\pm}$. This is done by trial over all set of families $s = (s(c))_{c\in\Gamma\cup\infty}$ until such d is found to work in finding candidate ω. Trying $\alpha_\infty^{-} = 1$ then

$$\begin{aligned} d &= \alpha_\infty^{-} - \left(\alpha_{c_1}^{-} + \alpha_{c_2}^{+}\right) \\ &= 1 - (1) \\ &= 0 \end{aligned}$$

Since d an integer and $d \geq 0$ then it can be used to find ω using

$$\omega = \sum_{c\in\Gamma} \left(s(c)[\sqrt{r}]_c + \frac{\alpha_c^{s(c)}}{x-c} \right) + s(\infty)[\sqrt{r}]_\infty$$

The above gives

$$\begin{aligned} \omega &= \left((-)[\sqrt{r}]_{c_1} + \frac{\alpha_{c_1}^{-}}{x-c_1} \right) + \left((+)[\sqrt{r}]_{c_2} + \frac{\alpha_{c_2}^{+}}{x-c_2} \right) + (-)[\sqrt{r}]_\infty \\ &= -\frac{1}{2(x-1)} + \frac{3}{2(x+1)} + (-)(0) \\ &= -\frac{1}{2(x-1)} + \frac{3}{2(x+1)} \\ &= \frac{-2+x}{x^2-1} \end{aligned}$$

Now that ω is determined, the next step is find a corresponding minimal polynomial $p(x)$ of degree $d = 0$ to solve the ode. The polynomial $p(x)$ needs to satisfy the equation

$$p'' + 2\omega p' + \left(\omega' + \omega^2 - r\right) p = 0 \tag{1A}$$

Let

$$p(x) = 1 \tag{2A}$$

Substituting the above in eq. (1A) gives

$$(0) + 2\left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right)(0) + \left(\left(\frac{1}{2(x-1)^2} - \frac{3}{2(x+1)^2}\right) + \left(-\frac{1}{2(x-1)} + \frac{3}{2(x+1)}\right)^2 - \left(\right.\right.$$

The equation is satisfied since both sides are zero. Therefore the first solution to the ode $z''=rz$ is

$$
\begin{aligned}
z_1(x) &= p e^{\int \omega \, dx} \\
&= \mathrm{e}^{\int \left(-\frac{1}{2(x-1)}+\frac{3}{2(x+1)}\right) dx} \\
&= \frac{(x+1)^{\frac{3}{2}}}{\sqrt{x-1}}
\end{aligned}
$$

The first solution to the original ode in y is found from

$$
\begin{aligned}
y_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A} \, dx} \\
&= z_1 e^{-\int \frac{1}{2}\frac{-2x}{x^2-1} \, dx} \\
&= z_1 e^{\frac{\ln(x-1)}{2}+\frac{\ln(x+1)}{2}} \\
&= z_1\left(\sqrt{x-1}\,\sqrt{x+1}\right)
\end{aligned}
$$

Which simplifies to

$$
y_1 = (x+1)^2
$$

The second solution y_2 to the original ode is found using reduction of order

$$
y_2 = y_1 \int \frac{e^{\int -\frac{B}{A} \, dx}}{y_1^2} \, dx
$$

Substituting gives

$$
\begin{aligned}
y_2 &= y_1 \int \frac{e^{\int -\frac{-2x}{x^2-1} \, dx}}{(y_1)^2} \, dx \\
&= y_1 \int \frac{e^{\ln(x-1)+\ln(x+1)}}{(y_1)^2} \, dx \\
&= y_1\left(-\frac{x}{(x+1)^2}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
y &= c_1 y_1 + c_2 y_2 \\
&= c_1\left((x+1)^2\right) + c_2\left((x+1)^2\left(-\frac{x}{(x+1)^2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$y = y_h + y_p$$

Where y_h is the solution to the homogeneous ODE $Ay''(x)+By'(x)+Cy(x) = 0$, and y_p is a particular solution to the nonhomogeneous ODE $Ay''(x) + By'(x) + Cy(x) = f(x)$. y_h is the solution to

$$\left(x^2 - 1\right) y'' - 2y'x + 2y = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$y_h = c_1(x+1)^2 - c_2 x$$

The particular solution y_p can be found using either the method of undetermined coefficients, or the method of variation of parameters. The method of variation of parameters will be used as it is more general and can be used when the coefficients of the ODE depend on x as well. Let

$$y_p(x) = u_1 y_1 + u_2 y_2 \tag{1}$$

Where u_1, u_2 to be determined, and y_1, y_2 are the two basis solutions (the two linearly independent solutions of the homogeneous ODE) found earlier when solving the homogeneous ODE as

$$\begin{aligned} y_1 &= (x+1)^2 \\ y_2 &= -x \end{aligned}$$

In the Variation of parameters u_1, u_2 are found using

$$u_1 = -\int \frac{y_2 f(x)}{aW(x)} \tag{2}$$

$$u_2 = \int \frac{y_1 f(x)}{aW(x)} \tag{3}$$

Where $W(x)$ is the Wronskian and a is the coefficient in front of y'' in the given ODE. The Wronskian is given by $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$. Hence

$$W = \begin{vmatrix} (x+1)^2 & -x \\ \frac{d}{dx}\left((x+1)^2\right) & \frac{d}{dx}(-x) \end{vmatrix}$$

Which gives

$$W = \begin{vmatrix} (x+1)^2 & -x \\ 2+2x & -1 \end{vmatrix}$$

Therefore

$$W = \left((x+1)^2\right)(-1) - (-x)(2+2x)$$

Which simplifies to

$$W = x^2 - 1$$

Which simplifies to

$$W = x^2 - 1$$

Therefore Eq. (2) becomes

$$u_1 = -\int \frac{-x(x^2-1)}{(x^2-1)^2}\, dx$$

Which simplifies to

$$u_1 = -\int -\frac{x}{x^2-1} dx$$

Hence

$$u_1 = \frac{\ln(x-1)}{2} + \frac{\ln(x+1)}{2}$$

And Eq. (3) becomes

$$u_2 = \int \frac{(x+1)^2 (x^2-1)}{(x^2-1)^2}\, dx$$

Which simplifies to

$$u_2 = \int \frac{x+1}{x-1} dx$$

Hence

$$u_2 = x + 2\ln(x-1)$$

Therefore the particular solution, from equation (1) is

$$y_p(x) = \left(\frac{\ln (x-1)}{2} + \frac{\ln (x+1)}{2}\right) (x+1)^2 - (x + 2\ln (x-1))\, x$$

Which simplifies to

$$y_p(x) = \frac{(x-1)^2 \ln (x-1)}{2} + \frac{(x+1)^2 \ln (x+1)}{2} - x^2$$

Therefore the general solution is

$$\begin{aligned} y &= y_h + y_p \\ &= \left(c_1 (x+1)^2 - c_2 x\right) + \left(\frac{(x-1)^2 \ln (x-1)}{2} + \frac{(x+1)^2 \ln (x+1)}{2} - x^2\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$y = c_1 (x+1)^2 - c_2 x + \frac{(x-1)^2 \ln (x-1)}{2} + \frac{(x+1)^2 \ln (x+1)}{2} - x^2 \tag{1}$$

Verification of solutions

$$y = c_1 (x+1)^2 - c_2 x + \frac{(x-1)^2 \ln (x-1)}{2} + \frac{(x+1)^2 \ln (x+1)}{2} - x^2$$

Verified OK.

Maple trace **Kovacic algorithm successful**

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   checking if the LODE is of Euler type
   trying a symmetry of the form [xi=0, eta=F(x)]
   checking if the LODE is missing y
   -> Trying a Liouvillian solution using Kovacics algorithm
      A Liouvillian solution exists
      Reducible group (found an exponential solution)
   <- Kovacics algorithm successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 38

```
dsolve((x^2-1)*diff(y(x),x$2)-2*x*diff(y(x),x)+2*y(x)=x^2-1,y(x), singsol=all)
```

$$y(x) = \frac{(x-1)^2 \ln (x-1)}{2} + \frac{(x+1)^2 \ln (x+1)}{2} + (c_1 - 1) x^2 + c_2 x + c_1$$

✓ Solution by Mathematica

Time used: 0.012 (sec). Leaf size: 22

```
DSolve[x^2*y''[x]-4*x*y'[x]+6*y[x]==x^3,y[x],x,IncludeSingularSolutions -> True]
```

$$y(x) \to x^2(x \log(x) + (-1 + c_2)x + c_1)$$

12 Section 5.6, Forced Oscillations and Resonance. Page 362

12.1 problem 1

Internal problem ID [258]
Internal file name [`OUTPUT/258_Sunday_June_05_2022_01_37_36_AM_66724802/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 1.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 9x = 10\cos(2t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

12.1.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned}p(t) &= 0\\ q(t) &= 9\\ F &= 10\cos(2t)\end{aligned}$$

Hence the ode is

$$x'' + 9x = 10\cos(2t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 9$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\cos(2t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.1.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 9, f(t) = 10\cos(2t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 9x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 9$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 9\,e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 9 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 9$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(9)} \\ &= \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +3i \\ \lambda_2 &= -3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 3i \\ \lambda_2 &= -3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(3t) + c_2 \sin(3t))$$

Or

$$x = c_1 \cos(3t) + c_2 \sin(3t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(3t) + c_2 \sin(3t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(3t), \sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2t) + 5A_2 \sin(2t) = 10 \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 2 \cos(2t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(3t) + c_2 \sin(3t)) + (2 \cos(2t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(3t) + c_2 \sin(3t) + 2 \cos(2t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \sin(3t) + 3c_2 \cos(3t) - 4\sin(2t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 3c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 0$$

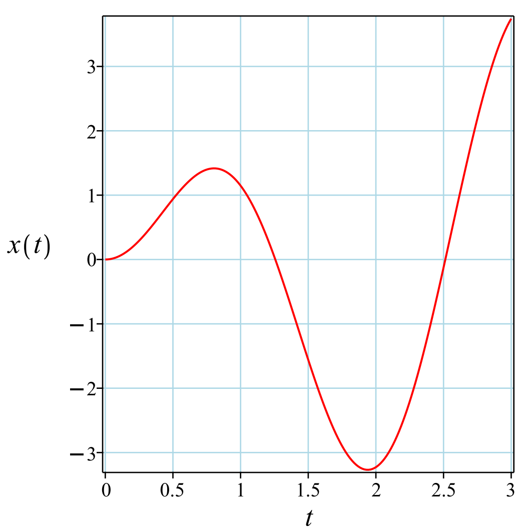
Substituting these values back in above solution results in

$$x = 2\cos(2t) - 2\cos(3t)$$

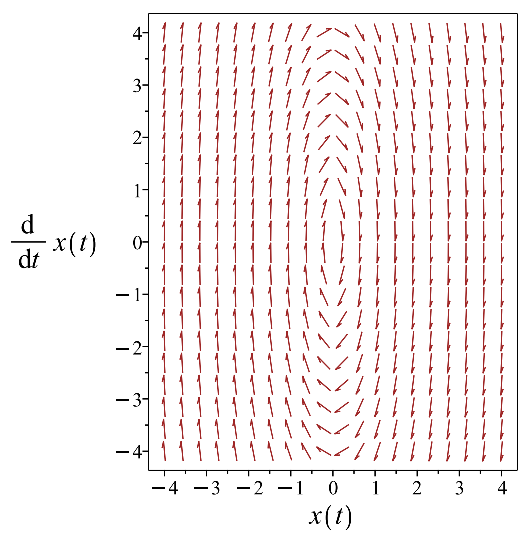
Summary

The solution(s) found are the following

$$x = 2\cos(2t) - 2\cos(3t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 2\cos(2t) - 2\cos(3t)$$

Verified OK.

12.1.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 9x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 9 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 519: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(3t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$
\begin{aligned}
x_1 &= z_1 \\
&= \cos(3t)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \cos(3t)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Since $B = 0$ then the above becomes

$$
\begin{aligned}
x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\
&= \cos(3t) \int \frac{1}{\cos(3t)^2}\, dt \\
&= \cos(3t) \left(\frac{\tan(3t)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1(\cos(3t)) + c_2\left(\cos(3t) \left(\frac{\tan(3t)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 9x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \cos(2t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(2t), \sin(2t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(3t)}{3}, \cos(3t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(2t) + A_2 \sin(2t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$5A_1 \cos(2t) + 5A_2 \sin(2t) = 10 \cos(2t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 2, A_2 = 0]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 2 \cos(2t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3}\right) + (2 \cos(2t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(3t) + \frac{c_2 \sin(3t)}{3} + 2\cos(2t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 + 2 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 \sin(3t) + c_2 \cos(3t) - 4\sin(2t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -2$$
$$c_2 = 0$$

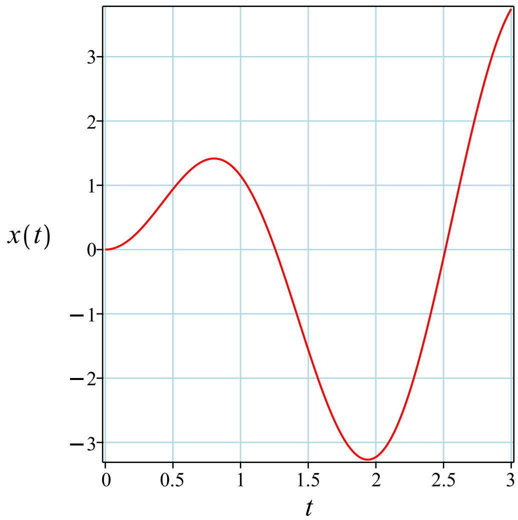
Substituting these values back in above solution results in

$$x = 2\cos(2t) - 2\cos(3t)$$

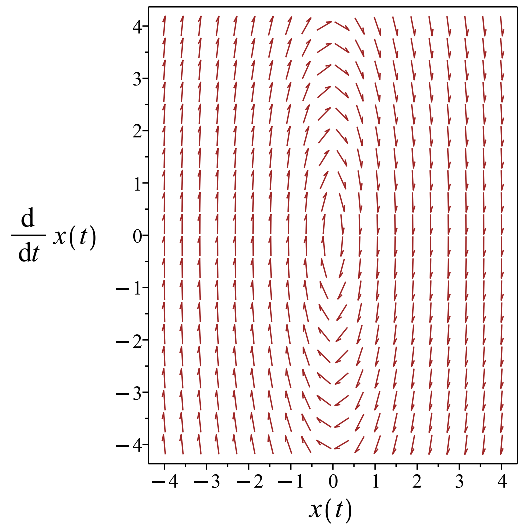
Summary

The solution(s) found are the following

$$x = 2\cos(2t) - 2\cos(3t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = 2\cos(2t) - 2\cos(3t)$$

Verified OK.

12.1.4 Maple step by step solution

Let's solve

$$\left[x'' + 9x = 10\cos(2t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 9 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-3\,\mathrm{I}, 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$x_1(t) = \cos(3t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(3t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(3t) + c_2 \sin(3t) + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 10\cos(2t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 3$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{5\cos(3t)\left(\int(\sin(5t)+\sin(t))dt\right)}{3} + \frac{5\sin(3t)\left(\int(\cos(t)+\cos(5t))dt\right)}{3}$

  - Compute integrals

    $x_p(t) = 2\cos(2t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(3t) + c_2 \sin(3t) + 2\cos(2t)$

☐ Check validity of solution $x = c_1 \cos(3t) + c_2 \sin(3t) + 2\cos(2t)$

  - Use initial condition $x(0) = 0$

    $0 = c_1 + 2$

  - Compute derivative of the solution

    $x' = -3c_1 \sin(3t) + 3c_2 \cos(3t) - 4\sin(2t)$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = 3c_2$

- Solve for c_1 and c_2

  $$\{c_1 = -2, c_2 = 0\}$$

- Substitute constant values into general solution and simplify

  $$x = 4\cos(t)^2 - 2 - 8\cos(t)^3 + 6\cos(t)$$

- Solution to the IVP

  $$x = 4\cos(t)^2 - 2 - 8\cos(t)^3 + 6\cos(t)$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([diff(x(t),t$2)+9*x(t)=10*cos(2*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = -8\cos(t)^3 + 6\cos(t) + 4\cos(t)^2 - 2$$

✓ Solution by Mathematica

Time used: 0.02 (sec). Leaf size: 18

```
DSolve[{x''[t]+9*x[t]==10*Cos[2*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> Tr
```

$$x(t) \to 2(\cos(2t) - \cos(3t))$$

12.2 problem 2

Internal problem ID [259]
Internal file name [`OUTPUT/259_Sunday_June_05_2022_01_37_37_AM_67793643/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 2.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 4x = 5 \sin(3t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

12.2.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 4 \\ F &= 5 \sin(3t) \end{aligned}$$

Hence the ode is

$$x'' + 4x = 5 \sin(3t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 5 \sin(3t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.2.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 4, f(t) = 5 \sin(3t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(4)}\\ &= \pm 2i\end{aligned}$$

Hence

$$\lambda_1 = +2i$$
$$\lambda_2 = -2i$$

Which simplifies to

$$\lambda_1 = 2i$$
$$\lambda_2 = -2i$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{0}(c_1 \cos(2t) + c_2 \sin(2t))$$

Or

$$x = c_1 \cos(2t) + c_2 \sin(2t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(2t) + c_2 \sin(2t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(2t), \sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(3t) - 5A_2 \sin(3t) = 5\sin(3t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\sin(3t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(2t) + c_2 \sin(2t)) + (-\sin(3t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(2t) + c_2 \sin(2t) - \sin(3t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - 3\cos(3t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3 + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = \frac{3}{2}$$

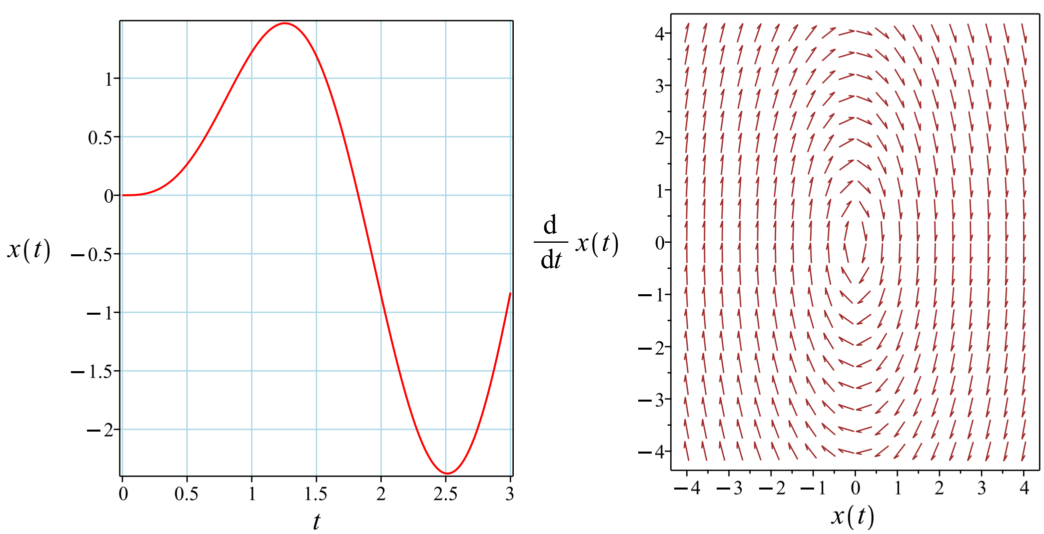
Substituting these values back in above solution results in

$$x = \frac{3\sin(2t)}{2} - \sin(3t)$$

Summary

The solution(s) found are the following

$$x = \frac{3\sin(2t)}{2} - \sin(3t) \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{3\sin(2t)}{2} - \sin(3t)$$

Verified OK.

12.2.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -4 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 521: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(2t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}
$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(2t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(2t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= \cos(2t) \int \frac{1}{\cos(2t)^2}\, dt \\ &= \cos(2t) \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1(\cos(2t)) + c_2\left(\cos(2t)\left(\frac{\tan(2t)}{2}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$5 \sin(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(2t)}{2}, \cos(2t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(3t) - 5A_2 \sin(3t) = 5 \sin(3t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 0, A_2 = -1]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\sin(3t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2}\right) + (-\sin(3t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(2t) + \frac{c_2 \sin(2t)}{2} - \sin(3t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2c_1 \sin(2t) + c_2 \cos(2t) - 3\cos(3t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = 0$$
$$c_2 = 3$$

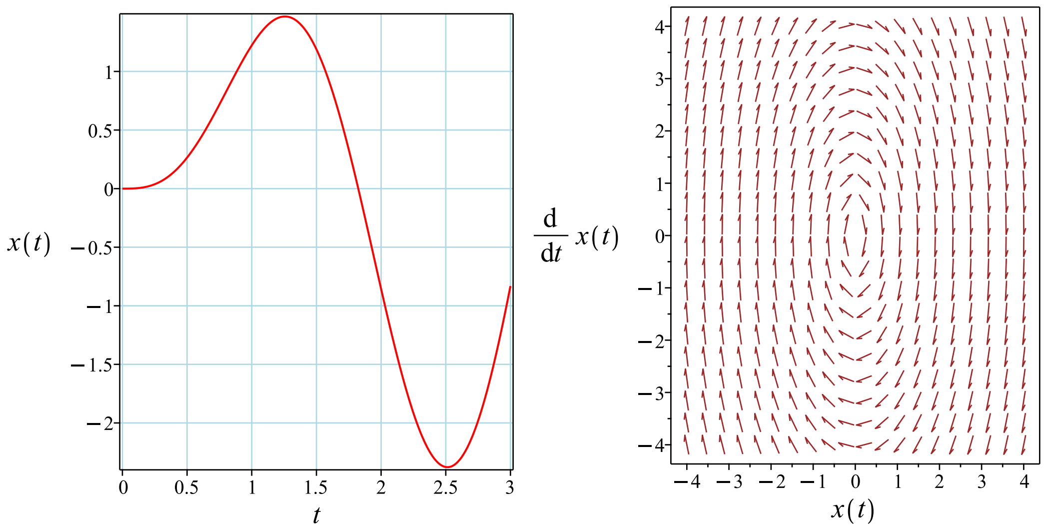
Substituting these values back in above solution results in

$$x = \frac{3\sin(2t)}{2} - \sin(3t)$$

Summary

The solution(s) found are the following

$$x = \frac{3\sin(2t)}{2} - \sin(3t) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = \frac{3\sin(2t)}{2} - \sin(3t)$$

Verified OK.

12.2.4 Maple step by step solution

Let's solve

$$\left[x'' + 4x = 5\sin(3t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-2\,\mathrm{I}, 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$x_1(t) = \cos(2t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(2t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(2t) + c_2 \sin(2t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 5\sin(3t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{bmatrix}$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 2$

  - Substitute functions into equation for $x_p(t)$

    $x_p(t) = -\frac{5\cos(2t)\left(\int(\cos(t)-\cos(5t))dt\right)}{4} + \frac{5\sin(2t)\left(\int(\sin(5t)+\sin(t))dt\right)}{4}$

  - Compute integrals

    $x_p(t) = -\sin(3t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(2t) + c_2 \sin(2t) - \sin(3t)$

□ Check validity of solution $x = c_1 \cos(2t) + c_2 \sin(2t) - \sin(3t)$

  - Use initial condition $x(0) = 0$

    $0 = c_1$

  - Compute derivative of the solution

    $x' = -2c_1 \sin(2t) + 2c_2 \cos(2t) - 3\cos(3t)$

  - Use the initial condition $x'\Big|_{\{t=0\}} = 0$

    $0 = -3 + 2c_2$

- Solve for c_1 and c_2

  $\{c_1 = 0, c_2 = \frac{3}{2}\}$
- Substitute constant values into general solution and simplify

  $x = \frac{3\sin(2t)}{2} - \sin(3t)$

- Solution to the IVP

  $x = \frac{3\sin(2t)}{2} - \sin(3t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.015 (sec). Leaf size: 17

```
dsolve([diff(x(t),t$2)+4*x(t)=5*sin(3*t),x(0) = 0, D(x)(0) = 0],x(t), singsol=all)
```

$$x(t) = \frac{3\sin(2t)}{2} - \sin(3t)$$

✓ Solution by Mathematica

Time used: 0.018 (sec). Leaf size: 18

```
DSolve[{x''[t]+4*x[t]==5*Sin[3*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to 3\sin(t)\cos(t) - \sin(3t)$$

12.3 problem 3

Internal problem ID [260]
Internal file name [`OUTPUT/260_Sunday_June_05_2022_01_37_38_AM_54155777/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 3.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 100x = 225\cos(5t) + 300\sin(5t)}$$

With initial conditions

$$[x(0) = 375, x'(0) = 0]$$

12.3.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 100 \\ F &= 225\cos(5t) + 300\sin(5t) \end{aligned}$$

Hence the ode is

$$x'' + 100x = 225\cos(5t) + 300\sin(5t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 100$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 225\cos(5t) + 300\sin(5t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.3.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 100, f(t) = 225\cos(5t) + 300\sin(5t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 100x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 100$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 100\,e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 100 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 100$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(100)} \\ &= \pm 10i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +10i \\ \lambda_2 &= -10i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 10i \\ \lambda_2 &= -10i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 10$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(10t) + c_2 \sin(10t))$$

Or

$$x = c_1 \cos(10t) + c_2 \sin(10t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(10t) + c_2 \sin(10t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$225 \cos(5t) + 300 \sin(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t), \sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(10t), \sin(10t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(5t) + A_2 \sin(5t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$75A_1 \cos(5t) + 75A_2 \sin(5t) = 225 \cos(5t) + 300 \sin(5t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 4]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 3 \cos(5t) + 4 \sin(5t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(10t) + c_2 \sin(10t)) + (3 \cos(5t) + 4 \sin(5t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(10t) + c_2 \sin(10t) + 3 \cos(5t) + 4 \sin(5t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 375$ and $t = 0$ in the above gives

$$375 = 3 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -10c_1 \sin(10t) + 10c_2 \cos(10t) - 15\sin(5t) + 20\cos(5t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 20 + 10c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

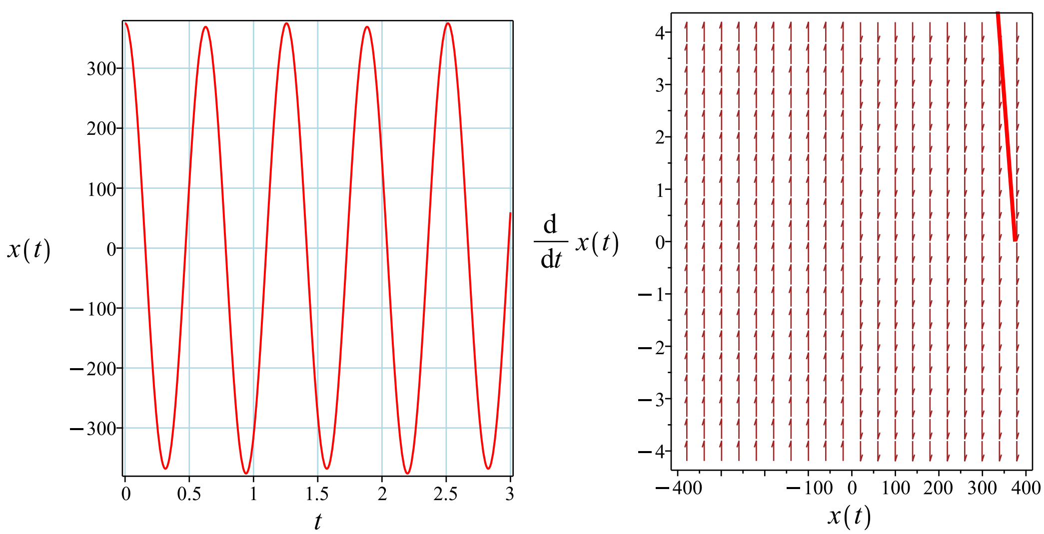
$$c_1 = 372$$
$$c_2 = -2$$

Substituting these values back in above solution results in

$$x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t)$$

<u>Summary</u>
The solution(s) found are the following

$$x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t) \tag{1}$$



(a) Solution plot

(b) Slope field plot

<u>Verification of solutions</u>

$$x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t)$$

Verified OK.

12.3.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 100x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 100 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-100}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -100 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -100z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 523: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -100$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(10t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(10t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(10t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A} \, dt}}{x_1^2} \, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2} \, dt \\ &= \cos(10t) \int \frac{1}{\cos(10t)^2} \, dt \\ &= \cos(10t) \left(\frac{\tan(10t)}{10} \right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1(\cos(10t)) + c_2 \left(\cos(10t) \left(\frac{\tan(10t)}{10} \right) \right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 100x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(10t) + \frac{c_2 \sin(10t)}{10}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$225 \cos(5t) + 300 \sin(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t), \sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(10t)}{10}, \cos(10t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(5t) + A_2 \sin(5t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$75A_1 \cos(5t) + 75A_2 \sin(5t) = 225 \cos(5t) + 300 \sin(5t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 3, A_2 = 4]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 3 \cos(5t) + 4 \sin(5t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos(10t) + \frac{c_2 \sin(10t)}{10}\right) + (3 \cos(5t) + 4 \sin(5t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(10t) + \frac{c_2 \sin(10t)}{10} + 3\cos(5t) + 4\sin(5t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 375$ and $t = 0$ in the above gives

$$375 = 3 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -10c_1 \sin(10t) + c_2 \cos(10t) - 15\sin(5t) + 20\cos(5t)$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = 20 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= 372 \\ c_2 &= -20 \end{aligned}$$

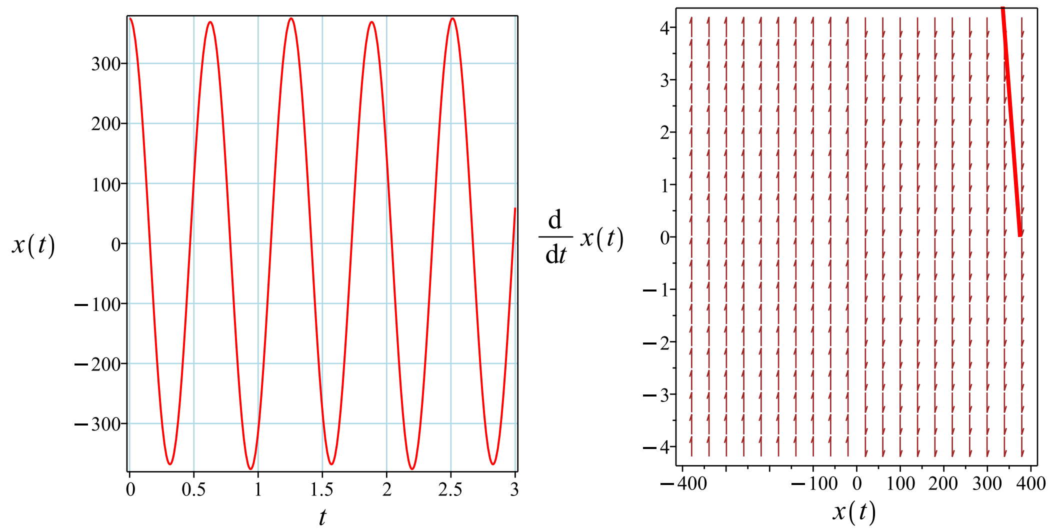
Substituting these values back in above solution results in

$$x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t)$$

Summary

The solution(s) found are the following

$$x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t) \tag{1}$$



(a) Solution plot (b) Slope field plot

<u>Verification of solutions</u>

$$x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t)$$

Verified OK.

12.3.4 Maple step by step solution

Let's solve

$$\left[x'' + 100x = 225\cos(5t) + 300\sin(5t)\,, x(0) = 375, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 100 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0\pm(\sqrt{-400})}{2}$

- Roots of the characteristic polynomial

  $r = (-10\,\mathrm{I}, 10\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$x_1(t) = \cos(10t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(10t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(10t) + c_2 \sin(10t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 225\cos(5t) + 300\sin(5t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(10t) & \sin(10t) \\ -10\sin(10t) & 10\cos(10t) \end{bmatrix}$

  ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = 10$

  ◦ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -\frac{15\cos(10t)\left(\int \sin(10t)(3\cos(5t)+4\sin(5t))dt\right)}{2} + \frac{15\sin(10t)\left(\int \cos(10t)(3\cos(5t)+4\sin(5t))dt\right)}{2}$

  ◦ Compute integrals

  $x_p(t) = 3\cos(5t) + 4\sin(5t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(10t) + c_2 \sin(10t) + 3\cos(5t) + 4\sin(5t)$

□ Check validity of solution $x = c_1 \cos(10t) + c_2 \sin(10t) + 3\cos(5t) + 4\sin(5t)$

  ◦ Use initial condition $x(0) = 375$

  $375 = 3 + c_1$

  ◦ Compute derivative of the solution

  $x' = -10c_1 \sin(10t) + 10c_2 \cos(10t) - 15\sin(5t) + 20\cos(5t)$

  ◦ Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = 20 + 10c_2$

- Solve for c_1 and c_2

  $\{c_1 = 372, c_2 = -2\}$
- Substitute constant values into general solution and simplify

  $x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t)$

- Solution to the IVP

  $x = 4\sin(5t) + 3\cos(5t) + 372\cos(10t) - 2\sin(10t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 29

```
dsolve([diff(x(t),t$2)+100*x(t)=225*cos(5*t)+300*sin(5*t),x(0) = 375, D(x)(0) = 0],x(t), sin
```

$$x(t) = -2\sin(10t) + 372\cos(10t) + 3\cos(5t) + 4\sin(5t)$$

✓ Solution by Mathematica

Time used: 0.022 (sec). Leaf size: 30

```
DSolve[{x''[t]+100*x[t]==225*Cos[5*t]+300*Sin[5*t],{x[0]==375,x'[0]==0}},x[t],t,IncludeSingu
```

$$x(t) \to 4\sin(5t) - 2\sin(10t) + 3\cos(5t) + 372\cos(10t)$$

12.4 problem 4

Internal problem ID [261]
Internal file name [`OUTPUT/261_Sunday_June_05_2022_01_37_40_AM_36557915/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 4.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 25x = 90\cos(4t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 90]$$

12.4.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 0 \\ q(t) &= 25 \\ F &= 90\cos(4t) \end{aligned}$$

Hence the ode is

$$x'' + 25x = 90\cos(4t)$$

The domain of $p(t) = 0$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 25$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 90\cos(4t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.4.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 0, C = 25, f(t) = 90\cos(4t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 25x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 0, C = 25$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 25\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 0, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{0^2 - (4)(1)(25)}\\ &= \pm 5i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +5i\\ \lambda_2 &= -5i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= 5i\\ \lambda_2 &= -5i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = 0$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^0(c_1 \cos(5t) + c_2 \sin(5t))$$

Or

$$x = c_1 \cos(5t) + c_2 \sin(5t)$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \cos(5t) + c_2 \sin(5t)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$90 \cos(4t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4t), \sin(4t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{\cos(5t), \sin(5t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(4t) + A_2 \sin(4t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 \cos(4t) + 9A_2 \sin(4t) = 90 \cos(4t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 10, A_2 = 0]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 10 \cos(4t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= (c_1 \cos(5t) + c_2 \sin(5t)) + (10 \cos(4t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(5t) + c_2 \sin(5t) + 10 \cos(4t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = 10 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -5c_1 \sin(5t) + 5c_2 \cos(5t) - 40\sin(4t)$$

substituting $x' = 90$ and $t = 0$ in the above gives

$$90 = 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -10$$
$$c_2 = 18$$

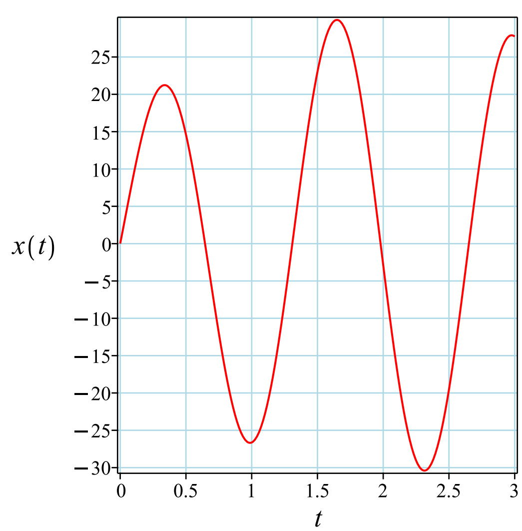
Substituting these values back in above solution results in

$$x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t)$$

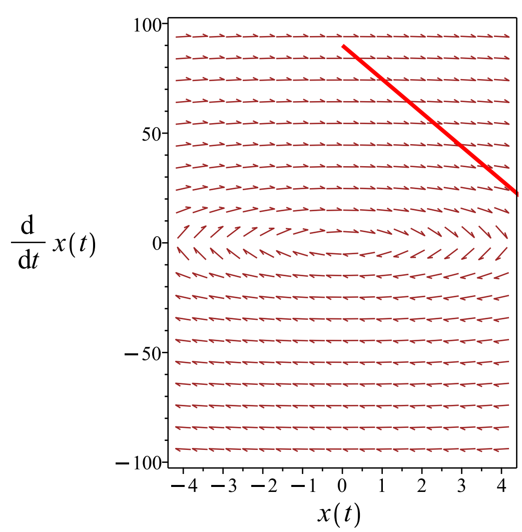
Summary

The solution(s) found are the following

$$x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t)$$

Verified OK.

12.4.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 25x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 0 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -25 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -25z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 525: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -25$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(5t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2} \frac{B}{A} \, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= \cos(5t) \end{aligned}$$

Which simplifies to

$$x_1 = \cos(5t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\,dt \\ &= \cos(5t) \int \frac{1}{\cos(5t)^2}\,dt \\ &= \cos(5t)\left(\frac{\tan(5t)}{5}\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1(\cos(5t)) + c_2\left(\cos(5t)\left(\frac{\tan(5t)}{5}\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 25x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = c_1 \cos(5t) + \frac{c_2 \sin(5t)}{5}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$90 \cos(4t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(4t), \sin(4t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\frac{\sin(5t)}{5}, \cos(5t)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(4t) + A_2 \sin(4t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$9A_1 \cos(4t) + 9A_2 \sin(4t) = 90 \cos(4t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 10, A_2 = 0]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = 10 \cos(4t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \cos(5t) + \frac{c_2 \sin(5t)}{5}\right) + (10 \cos(4t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \cos(5t) + \frac{c_2 \sin(5t)}{5} + 10\cos(4t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = 10 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -5c_1 \sin(5t) + c_2 \cos(5t) - 40\sin(4t)$$

substituting $x' = 90$ and $t = 0$ in the above gives

$$90 = c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$\begin{aligned} c_1 &= -10 \\ c_2 &= 90 \end{aligned}$$

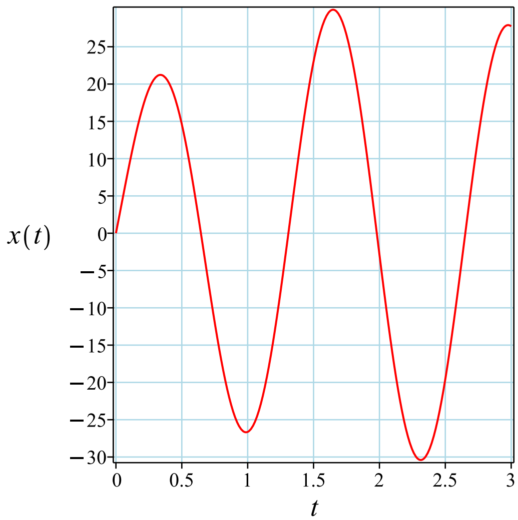
Substituting these values back in above solution results in

$$x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t)$$

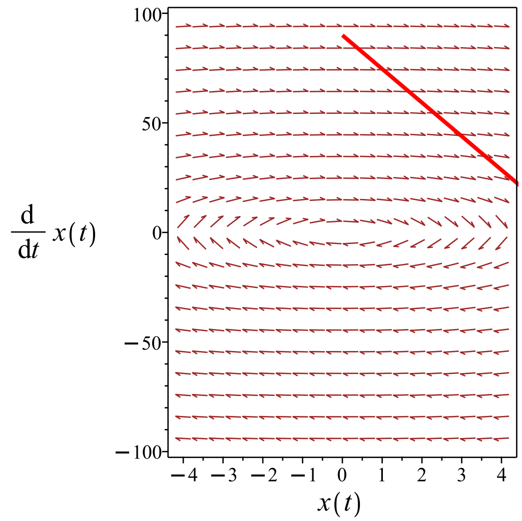
Summary

The solution(s) found are the following

$$x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t)$$

Verified OK.

12.4.4 Maple step by step solution

Let's solve

$$\left[x'' + 25x = 90\cos(4t), x(0) = 0, x'\Big|_{\{t=0\}} = 90\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{0 \pm (\sqrt{-100})}{2}$

- Roots of the characteristic polynomial

  $r = (-5\,\mathrm{I}, 5\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$x_1(t) = \cos(5t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \sin(5t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \cos(5t) + c_2 \sin(5t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t),x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t),x_2(t))} dt\right), f(t) = 90\cos(4t)\right]$

- ◦ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \cos(5t) & \sin(5t) \\ -5\sin(5t) & 5\cos(5t) \end{bmatrix}$

- ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = 5$

- ◦ Substitute functions into equation for $x_p(t)$

  $x_p(t) = -9\cos(5t)\left(\int (\sin(9t) + \sin(t))\, dt\right) + 9\sin(5t)\left(\int (\cos(t) + \cos(9t))\, dt\right)$

- ◦ Compute integrals

  $x_p(t) = 10\cos(4t)$

- Substitute particular solution into general solution to ODE

  $x = c_1 \cos(5t) + c_2 \sin(5t) + 10\cos(4t)$

□ Check validity of solution $x = c_1 \cos(5t) + c_2 \sin(5t) + 10\cos(4t)$

- ◦ Use initial condition $x(0) = 0$

  $0 = 10 + c_1$

- ◦ Compute derivative of the solution

  $x' = -5c_1 \sin(5t) + 5c_2 \cos(5t) - 40\sin(4t)$

- ◦ Use the initial condition $x'\Big|_{\{t=0\}} = 90$

  $90 = 5c_2$

- Solve for c_1 and c_2

  $\{c_1 = -10, c_2 = 18\}$
- Substitute constant values into general solution and simplify

  $x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t)$

- Solution to the IVP

  $x = 10\cos(4t) - 10\cos(5t) + 18\sin(5t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 23

```
dsolve([diff(x(t),t$2)+25*x(t)=90*cos(4*t),x(0) = 0, D(x)(0) = 90],x(t), singsol=all)
```

$$x(t) = 18\sin(5t) - 10\cos(5t) + 10\cos(4t)$$

✓ Solution by Mathematica

Time used: 0.019 (sec). Leaf size: 26

```
DSolve[{x''[t]+25*x[t]==90*Cos[4*t],{x[0]==0,x'[0]==90}},x[t],t,IncludeSingularSolutions ->
```

$$x(t) \to 2(9\sin(5t) + 5\cos(4t) - 5\cos(5t))$$

12.5 problem 5

Internal problem ID [262]
Internal file name [`OUTPUT/262_Sunday_June_05_2022_01_37_41_AM_45300286/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 5.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{mx'' + kx = F_0 \cos(\omega t)}$$

12.5.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = m, B = 0, C = k, f(t) = F_0 \cos(\omega t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$mx'' + kx = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = m, B = 0, C = k$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$m\,\lambda^2 \mathrm{e}^{\lambda t} + k\,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$m\,\lambda^2 + k = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = m, B = 0, C = k$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{0}{(2)\,(m)} \pm \frac{1}{(2)\,(m)}\sqrt{0^2 - (4)\,(m)\,(k)}\\ &= \pm \frac{\sqrt{-mk}}{m}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= +\frac{\sqrt{-mk}}{m}\\ \lambda_2 &= -\frac{\sqrt{-mk}}{m}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= \frac{\sqrt{-mk}}{m}\\ \lambda_2 &= -\frac{\sqrt{-mk}}{m}\end{aligned}$$

Since roots are real and distinct, then the solution is

$$\begin{aligned}x &= c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}\\ x &= c_1 e^{\left(\frac{\sqrt{-mk}}{m}\right)t} + c_2 e^{\left(-\frac{\sqrt{-mk}}{m}\right)t}\end{aligned}$$

Or

$$x = c_1 \mathrm{e}^{\frac{\sqrt{-mk}\,t}{m}} + c_2 \mathrm{e}^{-\frac{\sqrt{-mk}\,t}{m}}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 e^{\frac{\sqrt{-mk}\, t}{m}} + c_2 e^{-\frac{\sqrt{-mk}\, t}{m}}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$F_0 \cos(\omega t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(\omega t), \sin(\omega t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{ e^{\frac{\sqrt{-mk}\, t}{m}}, e^{-\frac{\sqrt{-mk}\, t}{m}} \right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(\omega t) + A_2 \sin(\omega t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$m\left(-A_1\omega^2 \cos(\omega t) - A_2\omega^2 \sin(\omega t)\right) + k(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = F_0 \cos(\omega t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{F_0}{-m\,\omega^2 + k}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{\frac{\sqrt{-mk}\, t}{m}} + c_2 e^{-\frac{\sqrt{-mk}\, t}{m}}\right) + \left(\frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 \mathrm{e}^{\frac{\sqrt{-mk}\,t}{m}} + c_2 \mathrm{e}^{-\frac{\sqrt{-mk}\,t}{m}} + \frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k} \tag{1}$$

Verification of solutions

$$x = c_1 \mathrm{e}^{\frac{\sqrt{-mk}\,t}{m}} + c_2 \mathrm{e}^{-\frac{\sqrt{-mk}\,t}{m}} + \frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k}$$

Verified OK.

12.5.2 Solving using Kovacic algorithm

Writing the ode as

$$mx'' + kx = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= m \\ B &= 0 \\ C &= k \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-k}{m} \tag{6}$$

Comparing the above to (5) shows that

$$s = -k$$
$$t = m$$

Therefore eq. (4) becomes

$$z''(t) = \left(-\frac{k}{m}\right) z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 527: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{k}{m}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = e^{\sqrt{-\frac{k}{m}}\, t}$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$x_1 = z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt}$$

Since $B = 0$ then the above reduces to

$$\begin{aligned} x_1 &= z_1 \\ &= e^{\sqrt{-\frac{k}{m}}\, t} \end{aligned}$$

Which simplifies to

$$x_1 = e^{\sqrt{-\frac{k}{m}}\, t}$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Since $B = 0$ then the above becomes

$$\begin{aligned} x_2 &= x_1 \int \frac{1}{x_1^2}\, dt \\ &= e^{\sqrt{-\frac{k}{m}}\, t} \int \frac{1}{e^{2\sqrt{-\frac{k}{m}}\, t}}\, dt \\ &= e^{\sqrt{-\frac{k}{m}}\, t} \left(\frac{\sqrt{-\frac{k}{m}}\, m\, e^{-2\sqrt{-\frac{k}{m}}\, t}}{2k} \right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2\\
&= c_1\left(\mathrm{e}^{\sqrt{-\frac{k}{m}}\, t}\right) + c_2\left(\mathrm{e}^{\sqrt{-\frac{k}{m}}\, t}\left(\frac{\sqrt{-\frac{k}{m}}\, m\, \mathrm{e}^{-2\sqrt{-\frac{k}{m}}\, t}}{2k}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
mx'' + kx = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1\mathrm{e}^{\sqrt{-\frac{k}{m}}\, t} + \frac{c_2 m\sqrt{-\frac{k}{m}}\, \mathrm{e}^{-\sqrt{-\frac{k}{m}}\, t}}{2k}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
F_0 \cos(\omega t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(\omega t), \sin(\omega t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\frac{m\sqrt{-\frac{k}{m}}\, \mathrm{e}^{-\sqrt{-\frac{k}{m}}\, t}}{2k}, \mathrm{e}^{\sqrt{-\frac{k}{m}}\, t}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \cos(\omega t) + A_2 \sin(\omega t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$m\left(-A_1\omega^2 \cos(\omega t) - A_2\omega^2 \sin(\omega t)\right) + k(A_1 \cos(\omega t) + A_2 \sin(\omega t)) = F_0 \cos(\omega t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{F_0}{-m\,\omega^2 + k}, A_2 = 0\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{\sqrt{-\frac{k}{m}}\,t} + \frac{c_2 m \sqrt{-\frac{k}{m}}\,\mathrm{e}^{-\sqrt{-\frac{k}{m}}\,t}}{2k}\right) + \left(\frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k}\right) \end{aligned}$$

Summary
The solution(s) found are the following

$$x = c_1 \mathrm{e}^{\sqrt{-\frac{k}{m}}\,t} + \frac{c_2 m \sqrt{-\frac{k}{m}}\,\mathrm{e}^{-\sqrt{-\frac{k}{m}}\,t}}{2k} + \frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k} \tag{1}$$

Verification of solutions

$$x = c_1 \mathrm{e}^{\sqrt{-\frac{k}{m}}\,t} + \frac{c_2 m \sqrt{-\frac{k}{m}}\,\mathrm{e}^{-\sqrt{-\frac{k}{m}}\,t}}{2k} + \frac{F_0 \cos(\omega t)}{-m\,\omega^2 + k}$$

Verified OK.

12.5.3 Maple step by step solution

Let's solve

$mx'' + kx = F_0 \cos(\omega t)$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = \frac{F_0 \cos(\omega t)}{m} - \frac{kx}{m}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + \frac{kx}{m} = \frac{F_0 \cos(\omega t)}{m}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + \frac{k}{m} = 0$

- Factor the characteristic polynomial

  $\frac{r^2 m + k}{m} = 0$

- Roots of the characteristic polynomial

  $r = \left(\frac{\sqrt{-mk}}{m}, -\frac{\sqrt{-mk}}{m}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{\frac{\sqrt{-mk}\,t}{m}}$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{\sqrt{-mk}\,t}{m}}$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 e^{\frac{\sqrt{-mk}\,t}{m}} + c_2 e^{-\frac{\sqrt{-mk}\,t}{m}} + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \frac{F_0 \cos(\omega t)}{m}\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{\frac{\sqrt{-mk}\, t}{m}} & e^{-\frac{\sqrt{-mk}\, t}{m}} \\ \frac{\sqrt{-mk}\, e^{\frac{\sqrt{-mk}\, t}{m}}}{m} & -\frac{\sqrt{-mk}\, e^{-\frac{\sqrt{-mk}\, t}{m}}}{m} \end{bmatrix}$$

- Compute Wronskian

  $W(x_1(t), x_2(t)) = -\frac{2\sqrt{-mk}}{m}$

- Substitute functions into equation for $x_p(t)$

  $x_p(t) = \frac{F_0\left(e^{\frac{\sqrt{-mk}\, t}{m}}\left(\int e^{-\frac{\sqrt{-mk}\, t}{m}} \cos(\omega t)dt\right) - e^{-\frac{\sqrt{-mk}\, t}{m}}\left(\int e^{\frac{\sqrt{-mk}\, t}{m}} \cos(\omega t)dt\right)\right)}{2\sqrt{-mk}}$

- Compute integrals

  $x_p(t) = \frac{F_0 \cos(\omega t)}{-m\,\omega^2+k}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{\frac{\sqrt{-mk}\, t}{m}} + c_2 e^{-\frac{\sqrt{-mk}\, t}{m}} + \frac{F_0 \cos(\omega t)}{-m\,\omega^2+k}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 60

```
dsolve(m*diff(x(t),t$2)+k*x(t)=F__0*cos(omega*t),x(t), singsol=all)
```

$$x(t) = \frac{c_1(-m\,\omega^2+k)\cos\left(\frac{\sqrt{k}\, t}{\sqrt{m}}\right) + c_2(-m\,\omega^2+k)\sin\left(\frac{\sqrt{k}\, t}{\sqrt{m}}\right) + F_0\cos(\omega t)}{-m\,\omega^2+k}$$

✓ Solution by Mathematica

Time used: 0.044 (sec). Leaf size: 54

```
DSolve[m*x''[t]+k*x[t]==F0*Cos[omega*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{\text{F0}\cos(\omega t)}{k - m\omega^2} + c_1 \cos\left(\frac{\sqrt{k}t}{\sqrt{m}}\right) + c_2 \sin\left(\frac{\sqrt{k}t}{\sqrt{m}}\right)$$

12.6 problem 7

Internal problem ID [263]
Internal file name [`OUTPUT/263_Sunday_June_05_2022_01_37_43_AM_58046521/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 7.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff", "linear_second_order_ode_solved_by_an_integrating_factor"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 4x' + 4x = 10 \cos(3t)}$$

12.6.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 4, C = 4, f(t) = 10 \cos(3t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x' + 4x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 4, C = 4$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 4\lambda\, \mathrm{e}^{\lambda t} + 4\, \mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 4 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 4$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)\,(1)} \pm \frac{1}{(2)\,(1)}\sqrt{(4)^2 - (4)\,(1)\,(4)} \\ &= -2\end{aligned}$$

Hence this is the case of a double root $\lambda_{1,2} = 2$. Therefore the solution is

$$x = c_1 \mathrm{e}^{-2t} + c_2 t\, \mathrm{e}^{-2t} \tag{1}$$

Therefore the homogeneous solution x_h is

$$x_h = c_1 \mathrm{e}^{-2t} + c_2 t\, \mathrm{e}^{-2t}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10\cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{t\, \mathrm{e}^{-2t}, \mathrm{e}^{-2t}\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(3t) + A_2 \sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-5A_1 \cos(3t) - 5A_2 \sin(3t) - 12A_1 \sin(3t) + 12A_2 \cos(3t) = 10\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{50}{169}, A_2 = \frac{120}{169}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-2t} + c_2 t\,\mathrm{e}^{-2t}\right) + \left(-\frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}\right) \end{aligned}$$

Which simplifies to

$$x = \mathrm{e}^{-2t}(c_2 t + c_1) - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$$

Summary
The solution(s) found are the following

$$x = \mathrm{e}^{-2t}(c_2 t + c_1) - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169} \tag{1}$$

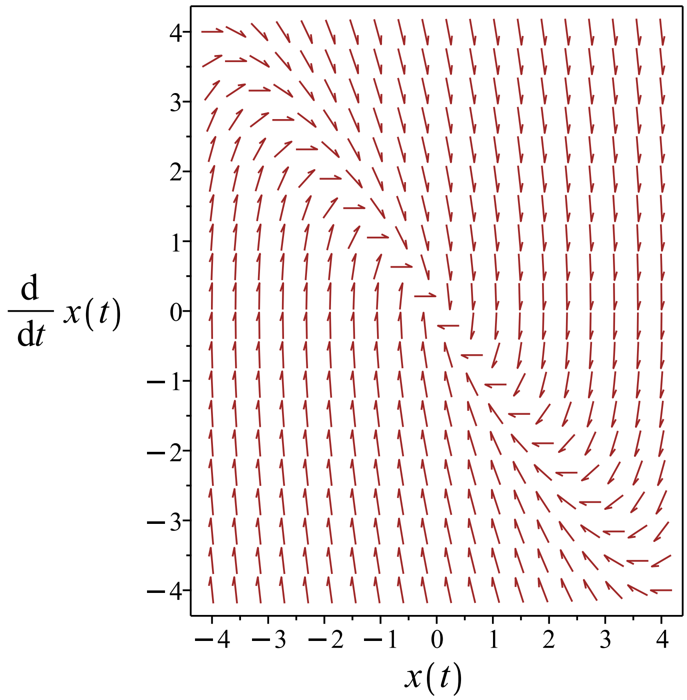


Figure 677: Slope field plot

Verification of solutions

$$x = e^{-2t}(c_2 t + c_1) - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$$

Verified OK.

12.6.2 Solving as linear second order ode solved by an integrating factor ode

The ode satisfies this form

$$x'' + p(t)\,x' + \frac{\left(p(t)^2 + p'(t)\right)x}{2} = f(t)$$

Where $p(t) = 4$. Therefore, there is an integrating factor given by

$$\begin{aligned} M(x) &= e^{\frac{1}{2}\int p\,dx} \\ &= e^{\int 4\,dx} \\ &= e^{2t} \end{aligned}$$

Multiplying both sides of the ODE by the integrating factor $M(x)$ makes the left side of the ODE a complete differential

$$
\begin{aligned}
(M(x)x)'' &= 10\,\mathrm{e}^{2t}\cos(3t) \\
\left(\mathrm{e}^{2t}x\right)'' &= 10\,\mathrm{e}^{2t}\cos(3t)
\end{aligned}
$$

Integrating once gives

$$
\left(\mathrm{e}^{2t}x\right)' = \frac{10\,\mathrm{e}^{2t}(2\cos(3t) + 3\sin(3t))}{13} + c_1
$$

Integrating again gives

$$
\left(\mathrm{e}^{2t}x\right) = \frac{10\,\mathrm{e}^{2t}(-5\cos(3t) + 12\sin(3t))}{169} + c_1 t + c_2
$$

Hence the solution is

$$
x = \frac{\frac{10\,\mathrm{e}^{2t}(-5\cos(3t)+12\sin(3t))}{169} + c_1 t + c_2}{\mathrm{e}^{2t}}
$$

Or

$$
x = -\frac{200\cos(t)^3}{169} + \frac{480\cos(t)^2\sin(t)}{169} + \frac{150\cos(t)}{169} - \frac{120\sin(t)}{169} + c_1 t\,\mathrm{e}^{-2t} + \mathrm{e}^{-2t}c_2
$$

Summary

The solution(s) found are the following

$$
x = -\frac{200\cos(t)^3}{169} + \frac{480\cos(t)^2\sin(t)}{169} + \frac{150\cos(t)}{169} - \frac{120\sin(t)}{169} + c_1 t\,\mathrm{e}^{-2t} + \mathrm{e}^{-2t}c_2 \tag{1}
$$

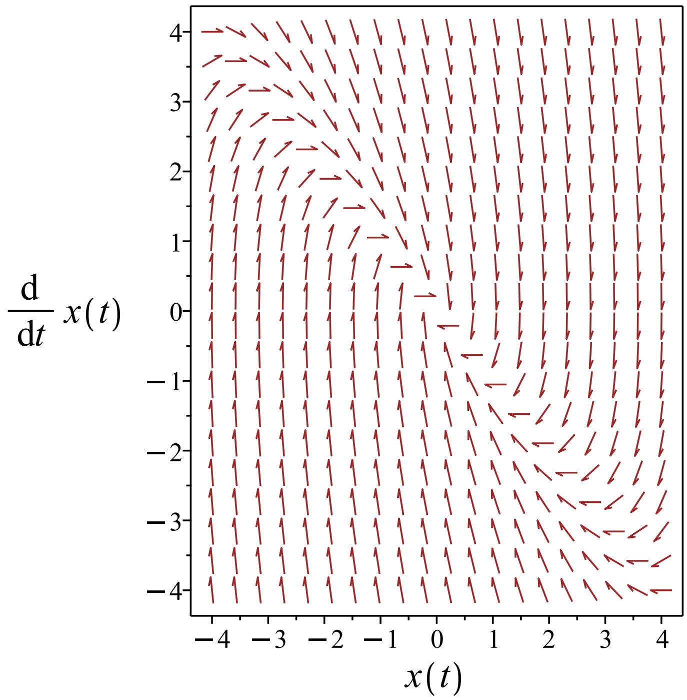


Figure 678: Slope field plot

Verification of solutions

$$x = -\frac{200\cos(t)^3}{169} + \frac{480\cos(t)^2\sin(t)}{169} + \frac{150\cos(t)}{169} - \frac{120\sin(t)}{169} + c_1 t\,e^{-2t} + e^{-2t}c_2$$

Verified OK.

12.6.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x' + 4x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 4 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{0}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= 0 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = 0 \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 529: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - -\infty \\
&= \infty
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is $infinity$ then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = 0$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = 1
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\
&= z_1 e^{-2t} \\
&= z_1 \left(\mathrm{e}^{-2t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-2t}
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-4t}}{(x_1)^2}\,dt \\
&= x_1(t)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-2t}\right) + c_2 \left(\mathrm{e}^{-2t}(t)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 4x' + 4x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-2t} + c_2 t \, \mathrm{e}^{-2t}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
10 \cos(3t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(3t), \sin(3t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{t \, \mathrm{e}^{-2t}, \mathrm{e}^{-2t}\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \cos(3t) + A_2 \sin(3t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-5A_1 \cos(3t) - 5A_2 \sin(3t) - 12A_1 \sin(3t) + 12A_2 \cos(3t) = 10 \cos(3t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{50}{169}, A_2 = \frac{120}{169}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 \mathrm{e}^{-2t} + c_2 t\, \mathrm{e}^{-2t}\right) + \left(-\frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}\right) \end{aligned}$$

Which simplifies to

$$x = \mathrm{e}^{-2t}(c_2 t + c_1) - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$$

Summary

The solution(s) found are the following

$$x = \mathrm{e}^{-2t}(c_2 t + c_1) - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169} \tag{1}$$

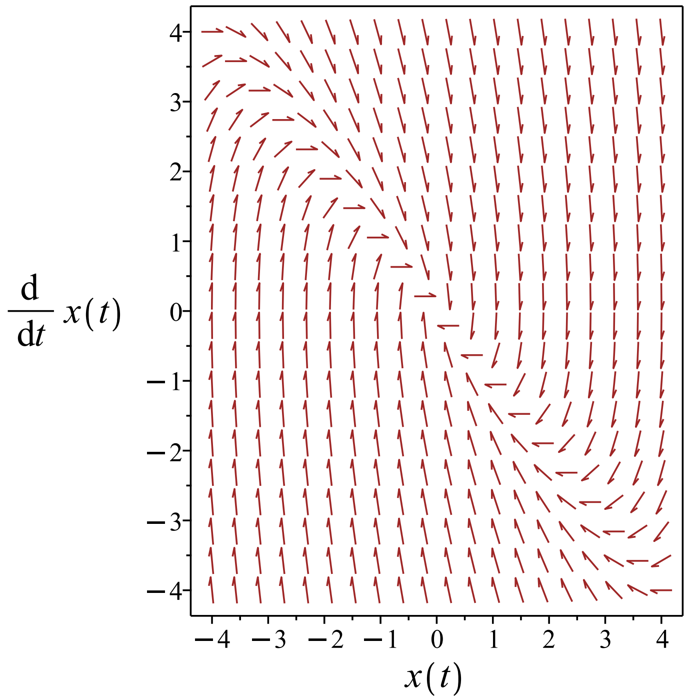


Figure 679: Slope field plot

Verification of solutions

$$x = e^{-2t}(c_2 t + c_1) - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$$

Verified OK.

12.6.4 Maple step by step solution

Let's solve

$x'' + 4x' + 4x = 10\cos(3t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 4r + 4 = 0$

- Factor the characteristic polynomial

  $(r+2)^2 = 0$

- Root of the characteristic polynomial

$r = -2$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-2t}$

- Repeated root, multiply $x_1(t)$ by t to ensure linear independence

  $x_2(t) = t\,e^{-2t}$

- General solution of the ODE

  $x = c_1x_1(t) + c_2x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1e^{-2t} + c_2t\,e^{-2t} + x_p(t)$

☐ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 10\cos(3t)\right]$

  ◦ Wronskian of solutions of the homogeneous equation

  $$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-2t} & t\,e^{-2t} \\ -2\,e^{-2t} & e^{-2t} - 2t\,e^{-2t} \end{bmatrix}$$

  ◦ Compute Wronskian

  $W(x_1(t), x_2(t)) = e^{-4t}$

  ◦ Substitute functions into equation for $x_p(t)$

  $x_p(t) = 10\,e^{-2t}\left(-\left(\int \cos(3t)\,t\,e^{2t}dt\right) + t\left(\int e^{2t}\cos(3t)\,dt\right)\right)$

  ◦ Compute integrals

  $x_p(t) = -\frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$

- Substitute particular solution into general solution to ODE

  $x = c_2t\,e^{-2t} + c_1e^{-2t} - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 27

```
dsolve(diff(x(t),t$2)+4*diff(x(t),t)+4*x(t)=10*cos(3*t),x(t), singsol=all)
```

$$x(t) = (c_1 t + c_2)\,\mathrm{e}^{-2t} - \frac{50\cos(3t)}{169} + \frac{120\sin(3t)}{169}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 35

```
DSolve[x''[t]+4*x'[t]+4*x[t]==10*Cos[3*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{120}{169}\sin(3t) - \frac{50}{169}\cos(3t) + e^{-2t}(c_2 t + c_1)$$

12.7 problem 8

Internal problem ID [264]
Internal file name [`OUTPUT/264_Sunday_June_05_2022_01_37_44_AM_45998882/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 8.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 3x' + 5x = -4\cos(5t)}$$

12.7.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 3, C = 5, f(t) = -4\cos(5t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 3x' + 5x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 3, C = 5$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 5 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(5)} \\ &= -\frac{3}{2} \pm \frac{i\sqrt{11}}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{3}{2} + \frac{i\sqrt{11}}{2} \\ \lambda_2 &= -\frac{3}{2} - \frac{i\sqrt{11}}{2}\end{aligned}$$

Which simplifies to

$$\lambda_1 = -\frac{3}{2} + \frac{i\sqrt{11}}{2}$$

$$\lambda_2 = -\frac{3}{2} - \frac{i\sqrt{11}}{2}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{3}{2}$ and $\beta = \frac{\sqrt{11}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{3t}{2}}\left(c_1 \cos\left(\frac{\sqrt{11}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{3t}{2}} \left(c_1 \cos\left(\frac{\sqrt{11}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{11}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-4 \cos(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t), \sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right), e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(5t) + A_2 \sin(5t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-20A_1 \cos(5t) - 20A_2 \sin(5t) - 15A_1 \sin(5t) + 15A_2 \cos(5t) = -4 \cos(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = \frac{16}{125}, A_2 = -\frac{12}{125}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{16 \cos(5t)}{125} - \frac{12 \sin(5t)}{125}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(e^{-\frac{3t}{2}}\left(c_1 \cos\left(\frac{\sqrt{11}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right)\right)\right) + \left(\frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125}\right)
\end{aligned}
$$

<u>Summary</u>

The solution(s) found are the following

$$
x = e^{-\frac{3t}{2}}\left(c_1 \cos\left(\frac{\sqrt{11}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right)\right) + \frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125} \tag{1}
$$

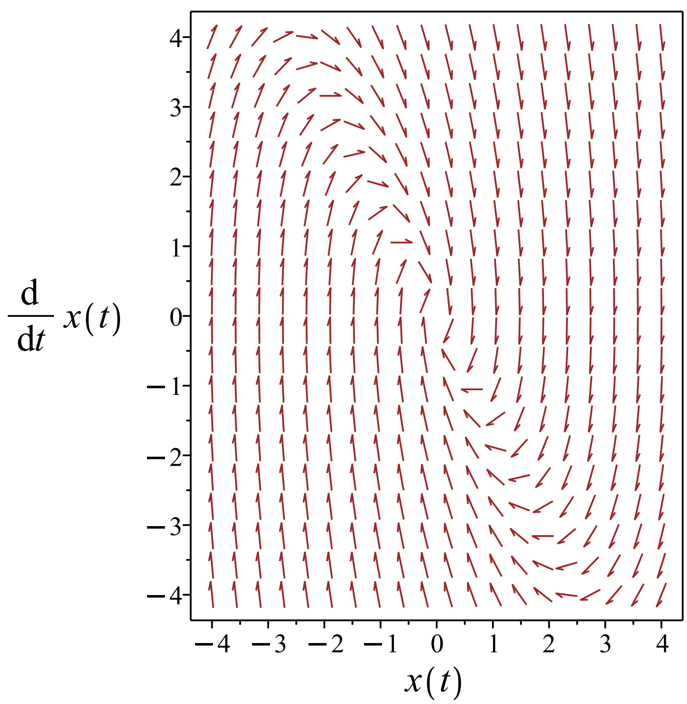


Figure 680: Slope field plot

<u>Verification of solutions</u>

$$
x = e^{-\frac{3t}{2}}\left(c_1 \cos\left(\frac{\sqrt{11}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right)\right) + \frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125}
$$

Verified OK.

12.7.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 3x' + 5x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A} dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-11}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -11 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{11z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A} dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 531: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{11}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{\sqrt{11}\, t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-3t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{11}\,\tan\left(\frac{\sqrt{11}\,t}{2}\right)}{11}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\,t}{2}\right) \left(\frac{2\sqrt{11}\,\tan\left(\frac{\sqrt{11}\,t}{2}\right)}{11}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+3x'+5x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h=\cos\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{3t}{2}}c_1+\frac{2c_2\sin\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{3t}{2}}\sqrt{11}}{11}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$-4\cos(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t),\sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{3t}{2}}\cos\left(\frac{\sqrt{11}\,t}{2}\right),\frac{2\sin\left(\frac{\sqrt{11}\,t}{2}\right)\mathrm{e}^{-\frac{3t}{2}}\sqrt{11}}{11}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p=A_1\cos(5t)+A_2\sin(5t)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-20A_1\cos(5t)-20A_2\sin(5t)-15A_1\sin(5t)+15A_2\cos(5t)=-4\cos(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=\frac{16}{125},A_2=-\frac{12}{125}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{3t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{3t}{2}} \sqrt{11}}{11}\right) + \left(\frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \cos\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{3t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{11}\,t}{2}\right) e^{-\frac{3t}{2}} \sqrt{11}}{11} + \frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125} \tag{1}$$

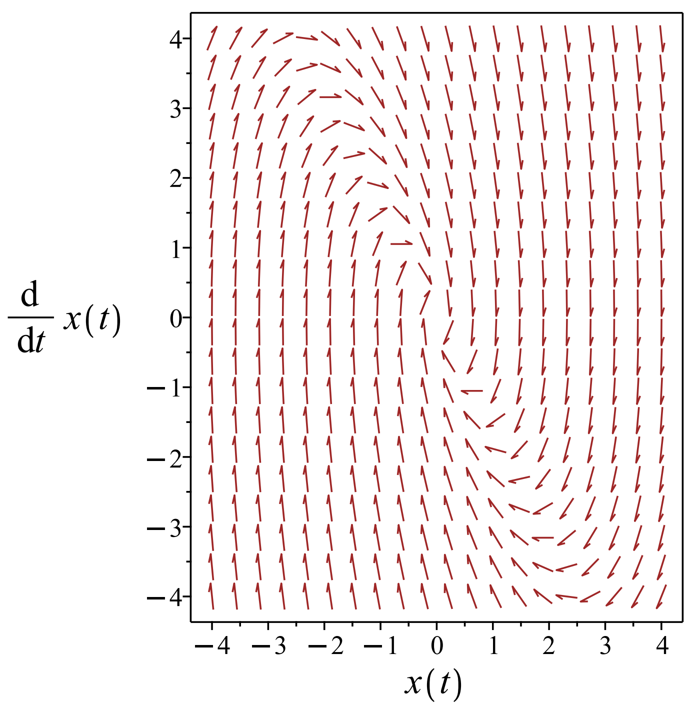


Figure 681: Slope field plot

Verification of solutions

$$x = \cos\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} \sqrt{11}}{11} + \frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125}$$

Verified OK.

12.7.3 Maple step by step solution

Let's solve

$x'' + 3x' + 5x = -4\cos(5t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 5 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-3) \pm (\sqrt{-11})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2} - \frac{\mathrm{I}\sqrt{11}}{2}, -\frac{3}{2} + \frac{\mathrm{I}\sqrt{11}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = \cos\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} c_1 + \sin\left(\frac{\sqrt{11}\, t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = -4\cos(5t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right) & e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right) \\ -\frac{3\,e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{11}\, t}{2}\right) e^{-\frac{3t}{2}} \sqrt{11}}{2} & -\frac{3\,e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right)}{2} + \frac{e^{-\frac{3t}{2}} \sqrt{11} \cos\left(\frac{\sqrt{11}\, t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

$$W(x_1(t), x_2(t)) = \frac{\sqrt{11}\, e^{-3t}}{2}$$

- Substitute functions into equation for $x_p(t)$

$$x_p(t) = \frac{8\,e^{-\frac{3t}{2}} \sqrt{11} \left(\cos\left(\frac{\sqrt{11}\, t}{2}\right)\left(\int \cos(5t) e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right) dt\right) - \sin\left(\frac{\sqrt{11}\, t}{2}\right)\left(\int \cos(5t) e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right) dt\right)\right)}{11}$$

- Compute integrals

$$x_p(t) = \frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125}$$

- Substitute particular solution into general solution to ODE

$$x = \cos\left(\frac{\sqrt{11}\, t}{2}\right) e^{-\frac{3t}{2}} c_1 + \sin\left(\frac{\sqrt{11}\, t}{2}\right) e^{-\frac{3t}{2}} c_2 + \frac{16\cos(5t)}{125} - \frac{12\sin(5t)}{125}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(x(t),t$2)+3*diff(x(t),t)+5*x(t)=-4*cos(5*t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{11}\, t}{2}\right) c_2 + e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{11}\, t}{2}\right) c_1 - \frac{12\sin(5t)}{125} + \frac{16\cos(5t)}{125}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 65

```
DSolve[x''[t]+3*x'[t]+5*x[t]==-4*Cos[5*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{4}{125}(4\cos(5t) - 3\sin(5t)) + c_2 e^{-3t/2}\cos\left(\frac{\sqrt{11}t}{2}\right) + c_1 e^{-3t/2}\sin\left(\frac{\sqrt{11}t}{2}\right)$$

12.8 problem 9

Internal problem ID [265]
Internal file name [`OUTPUT/265_Sunday_June_05_2022_01_37_45_AM_13586252/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 9.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{2x'' + 2x' + x = 3\sin(10t)}$$

12.8.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 2, B = 2, C = 1, f(t) = 3\sin(10t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$2x'' + 2x' + x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 2, B = 2, C = 1$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$2\lambda^2 e^{\lambda t} + 2\lambda\, e^{\lambda t} + e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$2\lambda^2 + 2\lambda + 1 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 2, B = 2, C = 1$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)\,(2)} \pm \frac{1}{(2)\,(2)}\sqrt{2^2 - (4)\,(2)\,(1)}\\ &= -\frac{1}{2} \pm \frac{i}{2}\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i}{2}\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -\frac{1}{2} + \frac{i}{2}\\ \lambda_2 &= -\frac{1}{2} - \frac{i}{2}\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{1}{2}$ and $\beta = \frac{1}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{t}{2}}\left(c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\sin(10t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(10t), \sin(10t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right), e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(10t) + A_2\sin(10t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-199A_1\cos(10t) - 199A_2\sin(10t) - 20A_1\sin(10t) + 20A_2\cos(10t) = 3\sin(10t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{60}{40001}, A_2 = -\frac{597}{40001}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{60\cos(10t)}{40001} - \frac{597\sin(10t)}{40001}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-\frac{t}{2}}\left(c_1\cos\left(\frac{t}{2}\right) + c_2\sin\left(\frac{t}{2}\right)\right)\right) + \left(-\frac{60\cos(10t)}{40001} - \frac{597\sin(10t)}{40001}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right)\right) - \frac{60 \cos(10t)}{40001} - \frac{597 \sin(10t)}{40001} \tag{1}$$

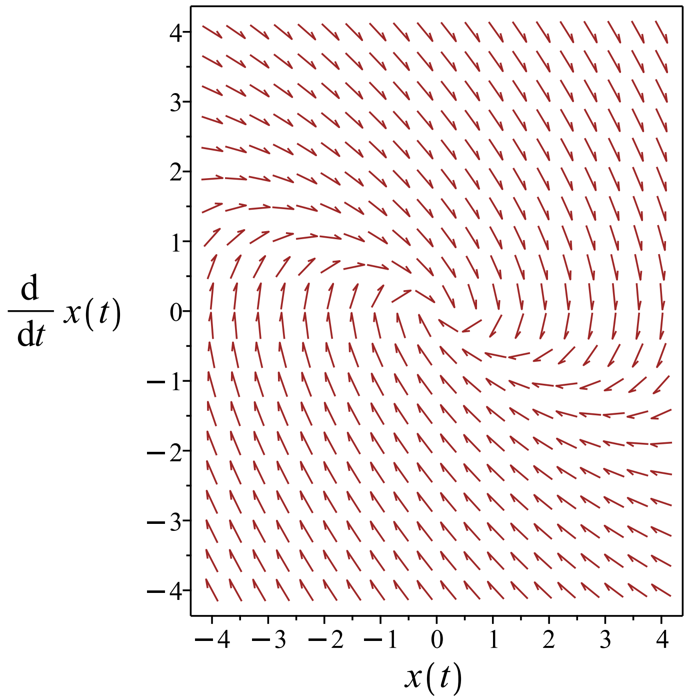


Figure 682: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right)\right) - \frac{60 \cos(10t)}{40001} - \frac{597 \sin(10t)}{40001}$$

Verified OK.

12.8.2 Solving using Kovacic algorithm

Writing the ode as

$$2x'' + 2x' + x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 2 \\ B &= 2 \\ C &= 1 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0,1,2,4,6,8,\cdots\}$	$\{\cdots,-6,-4,-2,0,2,3,4,5,6,\cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1,2\}$	$\{2,3,4,5,6,7,\cdots\}$

Table 533: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -\frac{1}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos\left(\frac{t}{2}\right)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{2}\, dt} \\
&= z_1 e^{-\frac{t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{2}{2}\, dt}}{(x_1)^2}\, dt \\ &= x_1 \int \frac{e^{-t}}{(x_1)^2}\, dt \\ &= x_1 \left(2\tan\left(\frac{t}{2}\right)\right) \end{aligned}$$

Therefore the solution is

$$\begin{aligned} x &= c_1 x_1 + c_2 x_2 \\ &= c_1 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) \left(2\tan\left(\frac{t}{2}\right)\right)\right) \end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$2x'' + 2x' + x = 0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h = \mathrm{e}^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) c_1 + 2\,\mathrm{e}^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) c_2$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$3\sin(10t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(10t), \sin(10t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right), 2\,e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(10t) + A_2\sin(10t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-199A_1\cos(10t) - 199A_2\sin(10t) - 20A_1\sin(10t) + 20A_2\cos(10t) = 3\sin(10t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{60}{40001}, A_2 = -\frac{597}{40001}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{60\cos(10t)}{40001} - \frac{597\sin(10t)}{40001}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)c_1 + 2\,e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)c_2\right) + \left(-\frac{60\cos(10t)}{40001} - \frac{597\sin(10t)}{40001}\right) \end{aligned}$$

Which simplifies to

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{t}{2}\right) + 2c_2 \sin\left(\frac{t}{2}\right) \right) - \frac{60\cos(10t)}{40001} - \frac{597\sin(10t)}{40001}$$

Summary

The solution(s) found are the following

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{t}{2}\right) + 2c_2 \sin\left(\frac{t}{2}\right) \right) - \frac{60\cos(10t)}{40001} - \frac{597\sin(10t)}{40001} \tag{1}$$

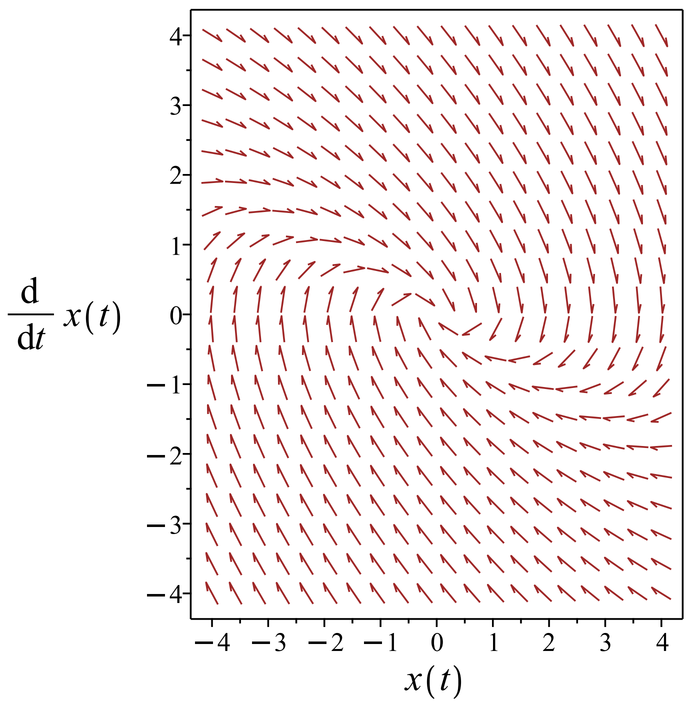


Figure 683: Slope field plot

Verification of solutions

$$x = e^{-\frac{t}{2}} \left(c_1 \cos\left(\frac{t}{2}\right) + 2c_2 \sin\left(\frac{t}{2}\right) \right) - \frac{60\cos(10t)}{40001} - \frac{597\sin(10t)}{40001}$$

Verified OK.

12.8.3 Maple step by step solution

Let's solve

$2x'' + 2x' + x = 3\sin(10t)$

- Highest derivative means the order of the ODE is 2

  x''

- Isolate 2nd derivative

  $x'' = -x' - \frac{x}{2} + \frac{3\sin(10t)}{2}$

- Group terms with x on the lhs of the ODE and the rest on the rhs of the ODE; ODE is linear

  $x'' + x' + \frac{x}{2} = \frac{3\sin(10t)}{2}$

- Characteristic polynomial of homogeneous ODE

  $r^2 + r + \frac{1}{2} = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-1)\pm(\sqrt{-1})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{1}{2} - \frac{I}{2}, -\frac{1}{2} + \frac{I}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) c_1 + e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  ◦ Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = \frac{3\sin(10t)}{2}\right]$

  ◦ Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right) & e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right) \\ -\frac{e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right)}{2} - \frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)}{2} & -\frac{e^{-\frac{t}{2}} \sin\left(\frac{t}{2}\right)}{2} + \frac{e^{-\frac{t}{2}} \cos\left(\frac{t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(x_1(t), x_2(t)) = \frac{e^{-t}}{2}$$

- Substitute functions into equation for $x_p(t)$

  $$x_p(t) = -3\,e^{-\frac{t}{2}}\left(\cos\left(\frac{t}{2}\right)\left(\int \sin\left(\frac{t}{2}\right)\sin(10t)\,e^{\frac{t}{2}}dt\right) - \sin\left(\frac{t}{2}\right)\left(\int \cos\left(\frac{t}{2}\right)\sin(10t)\,e^{\frac{t}{2}}dt\right)\right)$$

- Compute integrals

  $$x_p(t) = \frac{3\left(-221\cos\left(\frac{19t}{2}\right) - 4199\sin\left(\frac{19t}{2}\right) + 181\cos\left(\frac{21t}{2}\right) + 3801\sin\left(\frac{21t}{2}\right)\right)\cos\left(\frac{t}{2}\right)}{80002} - \frac{57\sin\left(\frac{t}{2}\right)\left(\cos\left(\frac{19t}{2}\right) + \frac{3801\cos\left(\frac{21t}{2}\right)}{4199} - \frac{\sin\left(\frac{19t}{2}\right)}{19}\right.}{362}$$

- Substitute particular solution into general solution to ODE

  $$x = e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)c_1 + e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)c_2 + \frac{3\left(-221\cos\left(\frac{19t}{2}\right) - 4199\sin\left(\frac{19t}{2}\right) + 181\cos\left(\frac{21t}{2}\right) + 3801\sin\left(\frac{21t}{2}\right)\right)\cos\left(\frac{t}{2}\right)}{80002} - \frac{57\sin(}{}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 37

```
dsolve(2*diff(x(t),t$2)+2*diff(x(t),t)+x(t)=3*sin(10*t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{t}{2}}\sin\left(\frac{t}{2}\right)c_2 + e^{-\frac{t}{2}}\cos\left(\frac{t}{2}\right)c_1 - \frac{597\sin(10t)}{40001} - \frac{60\cos(10t)}{40001}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 55

```
DSolve[2*x''[t]+2*x'[t]+x[t]==3*Sin[10*t],x[t],t,IncludeSingularSolutions -> True]
```

$$x(t) \to -\frac{3(199 \sin(10t) + 20 \cos(10t))}{40001} + c_2 e^{-t/2} \cos\left(\frac{t}{2}\right) + c_1 e^{-t/2} \sin\left(\frac{t}{2}\right)$$

12.9 problem 10

Internal problem ID [266]
Internal file name [`OUTPUT/266_Sunday_June_05_2022_01_37_48_AM_17526350/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 10.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 3x' + 3x = 8\cos(10t) + 6\sin(10t)}$$

12.9.1 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 3, C = 3, f(t) = 8\cos(10t) + 6\sin(10t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 3x' + 3x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 3, C = 3$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 3\lambda e^{\lambda t} + 3 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 3\lambda + 3 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 3, C = 3$ into the above gives

$$\begin{aligned}
\lambda_{1,2} &= \frac{-3}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{3^2 - (4)(1)(3)} \\
&= -\frac{3}{2} \pm \frac{i\sqrt{3}}{2}
\end{aligned}$$

Hence

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Which simplifies to

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2} + \frac{i\sqrt{3}}{2} \\
\lambda_2 &= -\frac{3}{2} - \frac{i\sqrt{3}}{2}
\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -\frac{3}{2}$ and $\beta = \frac{\sqrt{3}}{2}$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-\frac{3t}{2}}\left(c_1 \cos\left(\frac{\sqrt{3}\,t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)$$

Therefore the homogeneous solution x_h is

$$x_h = e^{-\frac{3t}{2}} \left(c_1 \cos\left(\frac{\sqrt{3}\, t}{2}\right) + c_2 \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right)$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\cos(10t) + 6\sin(10t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(10t), \sin(10t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right), e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(10t) + A_2 \sin(10t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-97A_1 \cos(10t) - 97A_2 \sin(10t) - 30A_1 \sin(10t) + 30A_2 \cos(10t) = 8\cos(10t) + 6\sin(10t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{956}{10309}, A_2 = -\frac{342}{10309}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{956\cos(10t)}{10309} - \frac{342\sin(10t)}{10309}$$

Therefore the general solution is

$$
\begin{aligned}
x &= x_h + x_p \\
&= \left(e^{-\frac{3t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)\right)+\left(-\frac{956\cos(10t)}{10309}-\frac{342\sin(10t)}{10309}\right)
\end{aligned}
$$

Summary

The solution(s) found are the following

$$
x = e^{-\frac{3t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)-\frac{956\cos(10t)}{10309}-\frac{342\sin(10t)}{10309} \tag{1}
$$

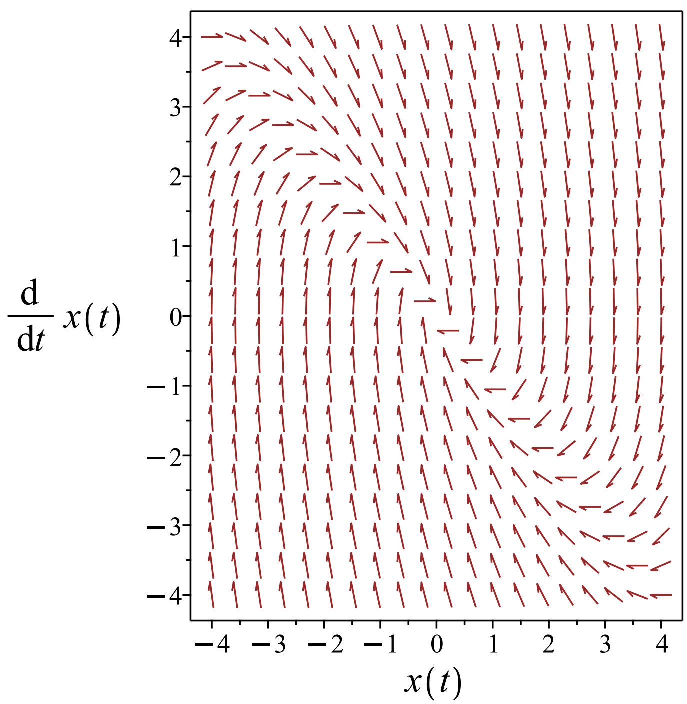


Figure 684: Slope field plot

Verification of solutions

$$
x = e^{-\frac{3t}{2}}\left(c_1\cos\left(\frac{\sqrt{3}\,t}{2}\right)+c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\right)-\frac{956\cos(10t)}{10309}-\frac{342\sin(10t)}{10309}
$$

Verified OK.

12.9.2 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 3x' + 3x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 3 \\ C &= 3 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-3}{4} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -3 \\ t &= 4 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -\frac{3z(t)}{4} \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\,e^{-\int \frac{B}{2A}\,dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 535: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -\frac{3}{4}$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos\left(\frac{\sqrt{3}\,t}{2}\right)
$$

Using the above, the solution for the original ode can now be found. The first solution

to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{3}{1}\,dt} \\
&= z_1 e^{-\frac{3t}{2}} \\
&= z_1 \left(\mathrm{e}^{-\frac{3t}{2}}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{3}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-3t}}{(x_1)^2}\,dt \\
&= x_1 \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)
\end{aligned}$$

Therefore the solution is

$$\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)\right) + c_2 \left(\mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right) \left(\frac{2\sqrt{3}\,\tan\left(\frac{\sqrt{3}\,t}{2}\right)}{3}\right)\right)
\end{aligned}$$

This is second order nonhomogeneous ODE. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$x''+3x'+3x=0$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$x_h=\cos\left(\frac{\sqrt{3}\,t}{2}\right)\mathrm{e}^{-\frac{3t}{2}}c_1+\frac{2c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\mathrm{e}^{-\frac{3t}{2}}\sqrt{3}}{3}$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$8\cos(10t)+6\sin(10t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(10t),\sin(10t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{\mathrm{e}^{-\frac{3t}{2}}\cos\left(\frac{\sqrt{3}\,t}{2}\right),\frac{2\sin\left(\frac{\sqrt{3}\,t}{2}\right)\mathrm{e}^{-\frac{3t}{2}}\sqrt{3}}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p=A_1\cos(10t)+A_2\sin(10t)$$

The unknowns $\{A_1,A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-97A_1\cos(10t)-97A_2\sin(10t)-30A_1\sin(10t)+30A_2\cos(10t)=8\cos(10t)+6\sin(10t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1=-\frac{956}{10309},A_2=-\frac{342}{10309}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{956\cos(10t)}{10309} - \frac{342\sin(10t)}{10309}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(\cos\left(\frac{\sqrt{3}\,t}{2}\right)e^{-\frac{3t}{2}}c_1 + \frac{2c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)e^{-\frac{3t}{2}}\sqrt{3}}{3}\right) + \left(-\frac{956\cos(10t)}{10309} - \frac{342\sin(10t)}{10309}\right) \end{aligned}$$

Summary

The solution(s) found are the following

$$x = \cos\left(\frac{\sqrt{3}\,t}{2}\right)e^{-\frac{3t}{2}}c_1 + \frac{2c_2\sin\left(\frac{\sqrt{3}\,t}{2}\right)e^{-\frac{3t}{2}}\sqrt{3}}{3} - \frac{956\cos(10t)}{10309} - \frac{342\sin(10t)}{10309} \tag{1}$$

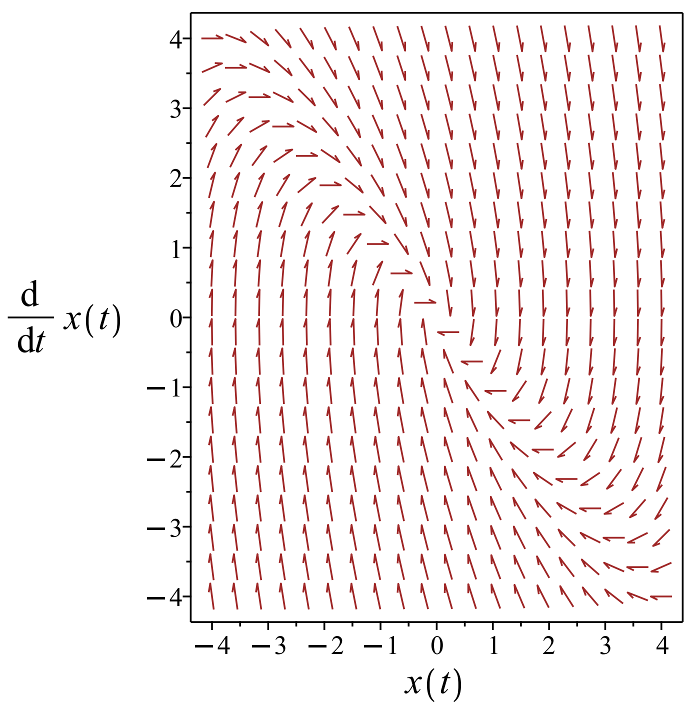


Figure 685: Slope field plot

Verification of solutions

$$x = \cos\left(\frac{\sqrt{3}\,t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} c_1 + \frac{2c_2 \sin\left(\frac{\sqrt{3}\,t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} \sqrt{3}}{3} - \frac{956 \cos(10t)}{10309} - \frac{342 \sin(10t)}{10309}$$

Verified OK.

12.9.3 Maple step by step solution

Let's solve

$x'' + 3x' + 3x = 8\cos(10t) + 6\sin(10t)$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 3r + 3 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-3)\pm(\sqrt{-3})}{2}$

- Roots of the characteristic polynomial

  $r = \left(-\frac{3}{2} - \frac{\mathrm{I}\sqrt{3}}{2}, -\frac{3}{2} + \frac{\mathrm{I}\sqrt{3}}{2}\right)$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\,t}{2}\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\,t}{2}\right)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = \cos\left(\frac{\sqrt{3}\,t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} c_1 + \sin\left(\frac{\sqrt{3}\,t}{2}\right) \mathrm{e}^{-\frac{3t}{2}} c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 8\cos(10t) + 6\sin(10t)\right]$

  - Wronskian of solutions of the homogeneous equation

$$W(x_1(t), x_2(t)) = \begin{bmatrix} e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) & e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) \\ -\frac{3\, e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} - \frac{\sin\left(\frac{\sqrt{3}\, t}{2}\right) e^{-\frac{3t}{2}} \sqrt{3}}{2} & -\frac{3\, e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right)}{2} + \frac{e^{-\frac{3t}{2}} \sqrt{3} \cos\left(\frac{\sqrt{3}\, t}{2}\right)}{2} \end{bmatrix}$$

- Compute Wronskian

  $$W(x_1(t), x_2(t)) = \frac{\sqrt{3}\, e^{-3t}}{2}$$

- Substitute functions into equation for $x_p(t)$

  $$x_p(t) = -\frac{4\, e^{-\frac{3t}{2}} \sqrt{3} \left(\cos\left(\frac{\sqrt{3}\, t}{2}\right) \left(\int e^{\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) (4\cos(10t) + 3\sin(10t)) dt\right) - \sin\left(\frac{\sqrt{3}\, t}{2}\right) \left(\int e^{\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) (4\cos(10t) + 3\sin(10t))\right.\right.}{3}$$

- Compute integrals

  $$x_p(t) = -\frac{956\cos(10t)}{10309} - \frac{342\sin(10t)}{10309}$$

- Substitute particular solution into general solution to ODE

  $$x = \cos\left(\frac{\sqrt{3}\, t}{2}\right) e^{-\frac{3t}{2}} c_1 + \sin\left(\frac{\sqrt{3}\, t}{2}\right) e^{-\frac{3t}{2}} c_2 - \frac{342\sin(10t)}{10309} - \frac{956\cos(10t)}{10309}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.0 (sec). Leaf size: 43

```
dsolve(diff(x(t),t$2)+3*diff(x(t),t)+3*x(t)=8*cos(10*t)+6*sin(10*t),x(t), singsol=all)
```

$$x(t) = e^{-\frac{3t}{2}} \sin\left(\frac{\sqrt{3}\, t}{2}\right) c_2 + e^{-\frac{3t}{2}} \cos\left(\frac{\sqrt{3}\, t}{2}\right) c_1 - \frac{342\sin(10t)}{10309} - \frac{956\cos(10t)}{10309}$$

✓ Solution by Mathematica

Time used: 0.031 (sec). Leaf size: 65

```
DSolve[x''[t]+3*x'[t]+3*x[t]==8*Cos[10*t]+6*Sin[10*t],x[t],t,IncludeSingularSolutions -> Tru
```

$$x(t) \to -\frac{2(171\sin(10t)+478\cos(10t))}{10309}+c_2e^{-3t/2}\cos\left(\frac{\sqrt{3}t}{2}\right)+c_1e^{-3t/2}\sin\left(\frac{\sqrt{3}t}{2}\right)$$

12.10 problem 11

Internal problem ID [267]
Internal file name [`OUTPUT/267_Sunday_June_05_2022_01_37_49_AM_78390413/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 11.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 4x' + 5x = 10\cos(3t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

12.10.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 4 \\ q(t) &= 5 \\ F &= 10\cos(3t) \end{aligned}$$

Hence the ode is

$$x'' + 4x' + 5x = 10\cos(3t)$$

The domain of $p(t) = 4$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 5$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\cos(3t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.10.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 4, C = 5, f(t) = 10\cos(3t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 4x' + 5x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 4, C = 5$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 5 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 4\lambda + 5 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 4, C = 5$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-4}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{4^2 - (4)(1)(5)}\\ &= -2 \pm i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -2 + i\\ \lambda_2 &= -2 - i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -2$ and $\beta = 1$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-2t}(c_1 \cos(t) + c_2 \sin(t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-2t}(c_1 \cos(t) + c_2 \sin(t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \cos(3t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(3t), \sin(3t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-2t}\cos(t), e^{-2t}\sin(t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(3t) + A_2\sin(3t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-4A_1\cos(3t) - 4A_2\sin(3t) - 12A_1\sin(3t) + 12A_2\cos(3t) = 10\cos(3t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-2t}(c_1\cos(t) + c_2\sin(t))\right) + \left(-\frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-2t}(c_1\cos(t) + c_2\sin(t)) - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{1}{4} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2\,\mathrm{e}^{-2t}(c_1 \cos(t) + c_2 \sin(t)) + \mathrm{e}^{-2t}(-c_1 \sin(t) + c_2 \cos(t)) + \frac{3\sin(3t)}{4} + \frac{9\cos(3t)}{4}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 + \frac{9}{4} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{4}$$
$$c_2 = -\frac{7}{4}$$

Substituting these values back in above solution results in

$$x = \frac{\mathrm{e}^{-2t}\cos(t)}{4} - \frac{7\,\mathrm{e}^{-2t}\sin(t)}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

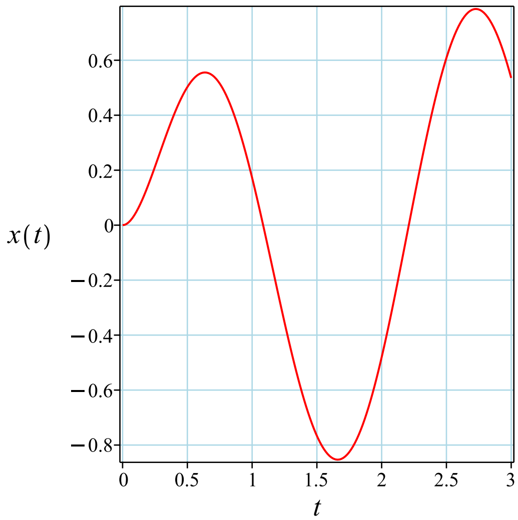
Which simplifies to

$$x = \frac{(\cos(t) - 7\sin(t))\,\mathrm{e}^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

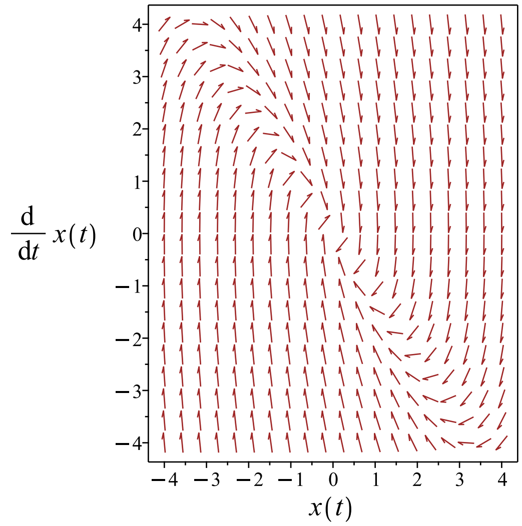
Summary

The solution(s) found are the following

$$x = \frac{(\cos(t) - 7\sin(t))\,\mathrm{e}^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{(\cos(t) - 7\sin(t))\,\mathrm{e}^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Verified OK.

12.10.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 4x' + 5x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 4 \\ C &= 5 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-1}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -1 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 537: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -1$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{4}{1}\,dt} \\ &= z_1 e^{-2t} \\ &= z_1\left(\mathrm{e}^{-2t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-2t}\cos(t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{4}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-4t}}{(x_1)^2}\,dt \\ &= x_1(\tan(t)) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1x_1 + c_2x_2 \\
&= c_1\left(\mathrm{e}^{-2t}\cos(t)\right) + c_2\left(\mathrm{e}^{-2t}\cos(t)\left(\tan(t)\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 4x' + 5x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1\mathrm{e}^{-2t}\cos(t) + \mathrm{e}^{-2t}c_2\sin(t)
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
10\cos(3t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(3t), \sin(3t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\{\mathrm{e}^{-2t}\cos(t), \mathrm{e}^{-2t}\sin(t)\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1\cos(3t) + A_2\sin(3t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$
-4A_1\cos(3t) - 4A_2\sin(3t) - 12A_1\sin(3t) + 12A_2\cos(3t) = 10\cos(3t)
$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{1}{4}, A_2 = \frac{3}{4}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Therefore the general solution is

$$\begin{aligned}x &= x_h + x_p \\ &= \left(c_1 e^{-2t}\cos(t) + e^{-2t} c_2 \sin(t)\right) + \left(-\frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}\right)\end{aligned}$$

Which simplifies to

$$x = e^{-2t}(c_1\cos(t) + c_2\sin(t)) - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-2t}(c_1\cos(t) + c_2\sin(t)) - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{1}{4} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -2\,e^{-2t}(c_1\cos(t) + c_2\sin(t)) + e^{-2t}(-c_1\sin(t) + c_2\cos(t)) + \frac{3\sin(3t)}{4} + \frac{9\cos(3t)}{4}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -2c_1 + \frac{9}{4} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{1}{4}$$
$$c_2 = -\frac{7}{4}$$

Substituting these values back in above solution results in

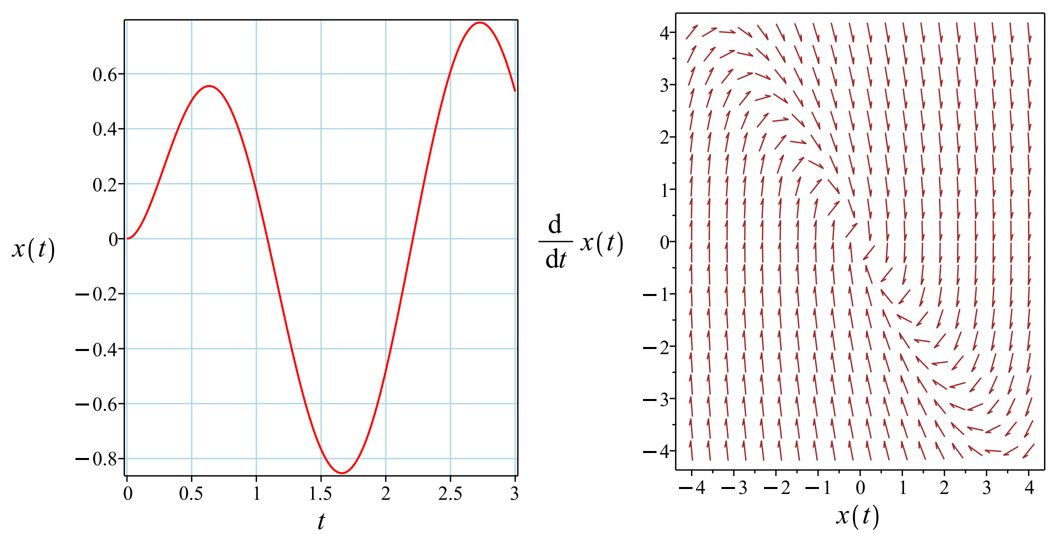
$$x = \frac{e^{-2t}\cos(t)}{4} - \frac{7\,e^{-2t}\sin(t)}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Which simplifies to

$$x = \frac{(\cos(t) - 7\sin(t))\,e^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Summary
The solution(s) found are the following

$$x = \frac{(\cos(t) - 7\sin(t))\,e^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{(\cos(t) - 7\sin(t))\,e^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Verified OK.

12.10.4 Maple step by step solution

Let's solve

$$\left[x''+4x'+5x=10\cos(3t), x(0)=0, x'\Big|_{\{t=0\}}=0\right]$$

- Highest derivative means the order of the ODE is 2

  x''
- Characteristic polynomial of homogeneous ODE

  $r^2+4r+5=0$
- Use quadratic formula to solve for r

  $r=\frac{(-4)\pm(\sqrt{-4})}{2}$
- Roots of the characteristic polynomial

  $r=(-2-\mathrm{I}, -2+\mathrm{I})$
- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{-2t}\cos(t)$
- 2nd solution of the homogeneous ODE

  $x_2(t)=\mathrm{e}^{-2t}\sin(t)$
- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$
- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{-2t}\cos(t)+\mathrm{e}^{-2t}c_2\sin(t)+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=10\cos(3t)\right]$
- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t))=\left[\begin{array}{cc}\mathrm{e}^{-2t}\cos(t) & \mathrm{e}^{-2t}\sin(t)\\ -2\,\mathrm{e}^{-2t}\cos(t)-\mathrm{e}^{-2t}\sin(t) & -2\,\mathrm{e}^{-2t}\sin(t)+\mathrm{e}^{-2t}\cos(t)\end{array}\right]$
- ○ Compute Wronskian

  $W(x_1(t), x_2(t))=\mathrm{e}^{-4t}$
- ○ Substitute functions into equation for $x_p(t)$

$$x_p(t) = 10\,\mathrm{e}^{-2t}\left(-\cos(t)\left(\int \sin(t)\cos(3t)\,\mathrm{e}^{2t}dt\right) + \sin(t)\left(\int \cos(t)\cos(3t)\,\mathrm{e}^{2t}dt\right)\right)$$

- Compute integrals

$$x_p(t) = -\frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

- Substitute particular solution into general solution to ODE

$$x = c_1\mathrm{e}^{-2t}\cos(t) + \mathrm{e}^{-2t}c_2\sin(t) - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

□ Check validity of solution $x = c_1\mathrm{e}^{-2t}\cos(t) + \mathrm{e}^{-2t}c_2\sin(t) - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$

- Use initial condition $x(0) = 0$

$$0 = c_1 - \frac{1}{4}$$

- Compute derivative of the solution

$$x' = -2c_1\mathrm{e}^{-2t}\cos(t) - c_1\mathrm{e}^{-2t}\sin(t) - 2\,\mathrm{e}^{-2t}c_2\sin(t) + \mathrm{e}^{-2t}c_2\cos(t) + \frac{3\sin(3t)}{4} + \frac{9\cos(3t)}{4}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

$$0 = -2c_1 + \frac{9}{4} + c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{1}{4}, c_2 = -\frac{7}{4}\right\}$$

- Substitute constant values into general solution and simplify

$$x = \frac{(\cos(t) - 7\sin(t))\,\mathrm{e}^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

- Solution to the IVP

$$x = \frac{(\cos(t) - 7\sin(t))\,\mathrm{e}^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 30

```
dsolve([diff(x(t),t$2)+4*diff(x(t),t)+5*x(t)=10*cos(3*t),x(0) = 0, D(x)(0) = 0],x(t), singso
```

$$x(t) = \frac{(\cos(t) - 7\sin(t))\,e^{-2t}}{4} - \frac{\cos(3t)}{4} + \frac{3\sin(3t)}{4}$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 43

```
DSolve[{x''[t]+4*x'[t]+5*x[t]==10*Cos[3*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSolutio
```

$$x(t) \to \frac{1}{4}e^{-2t}\left(-7\sin(t) + 3e^{2t}\sin(3t) + \cos(t) - e^{2t}\cos(3t)\right)$$

12.11 problem 12

Internal problem ID [268]
Internal file name [`OUTPUT/268_Sunday_June_05_2022_01_37_50_AM_73786296/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 6x' + 13x = 10\sin(5t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

12.11.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 13 \\ F &= 10\sin(5t) \end{aligned}$$

Hence the ode is

$$x'' + 6x' + 13x = 10\sin(5t)$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\sin(5t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.11.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 6, C = 13, f(t) = 10\sin(5t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 6x' + 13x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda\, e^{\lambda t} + 13\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(13)} \\ &= -3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-3t}(c_1 \cos(2t) + c_2 \sin(2t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-3t}(c_1 \cos(2t) + c_2 \sin(2t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \sin(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t), \sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}\cos(2t), e^{-3t}\sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(5t) + A_2\sin(5t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1\cos(5t) - 12A_2\sin(5t) - 30A_1\sin(5t) + 30A_2\cos(5t) = 10\sin(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{25}{87}, A_2 = -\frac{10}{87}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-3t}(c_1\cos(2t) + c_2\sin(2t))\right) + \left(-\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-3t}(c_1\cos(2t) + c_2\sin(2t)) - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{25}{87} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3\,\mathrm{e}^{-3t}(c_1 \cos(2t) + c_2 \sin(2t)) + \mathrm{e}^{-3t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)) + \frac{125 \sin(5t)}{87} - \frac{50 \cos(5t)}{87}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{50}{87} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{25}{87}$$
$$c_2 = \frac{125}{174}$$

Substituting these values back in above solution results in

$$x = \frac{25\,\mathrm{e}^{-3t} \cos(2t)}{87} + \frac{125\,\mathrm{e}^{-3t} \sin(2t)}{174} - \frac{25 \cos(5t)}{87} - \frac{10 \sin(5t)}{87}$$

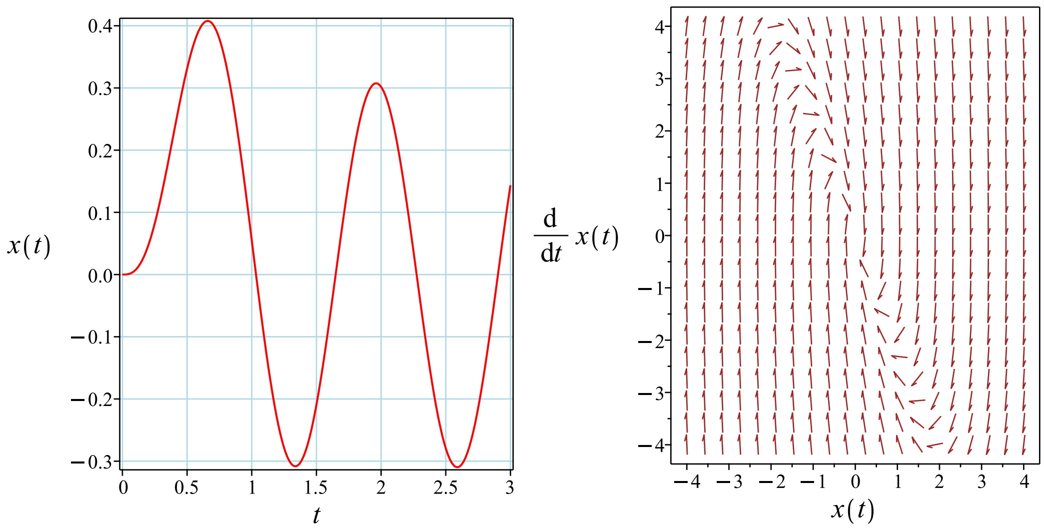
Which simplifies to

$$x = \frac{25(2 \cos(2t) + 5 \sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25 \cos(5t)}{87} - \frac{10 \sin(5t)}{87}$$

Summary

The solution(s) found are the following

$$x = \frac{25(2 \cos(2t) + 5 \sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25 \cos(5t)}{87} - \frac{10 \sin(5t)}{87} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Verified OK.

12.11.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 6x' + 13x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 539: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned} O(\infty) &= \deg(t) - \deg(s) \\ &= 0 - 0 \\ &= 0 \end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned} x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\ &= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\,dt} \\ &= z_1 e^{-3t} \\ &= z_1 \left(\mathrm{e}^{-3t}\right) \end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t} \cos(2t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned} x_2 &= x_1 \int \frac{e^{\int -\frac{6}{1}\,dt}}{(x_1)^2}\,dt \\ &= x_1 \int \frac{e^{-6t}}{(x_1)^2}\,dt \\ &= x_1 \left(\frac{\tan(2t)}{2}\right) \end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-3t} \cos (2t)\right) + c_2 \left(\mathrm{e}^{-3t} \cos (2t) \left(\frac{\tan (2t)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 6x' + 13x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-3t} \cos (2t) + \frac{\mathrm{e}^{-3t} c_2 \sin (2t)}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
10 \sin (5t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos (5t), \sin (5t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-3t} \cos (2t), \frac{\mathrm{e}^{-3t} \sin (2t)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \cos (5t) + A_2 \sin (5t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \cos(5t) - 12A_2 \sin(5t) - 30A_1 \sin(5t) + 30A_2 \cos(5t) = 10 \sin(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{25}{87}, A_2 = -\frac{10}{87}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-3t}\cos(2t) + \frac{e^{-3t}c_2\sin(2t)}{2}\right) + \left(-\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-3t}\cos(2t) + \frac{e^{-3t}c_2\sin(2t)}{2} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{25}{87} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 e^{-3t}\cos(2t) - 2c_1 e^{-3t}\sin(2t) - \frac{3\,e^{-3t}c_2\sin(2t)}{2} + e^{-3t}c_2\cos(2t) + \frac{125\sin(5t)}{87} - \frac{50\cos(5t)}{87}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{50}{87} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{25}{87}$$
$$c_2 = \frac{125}{87}$$

Substituting these values back in above solution results in

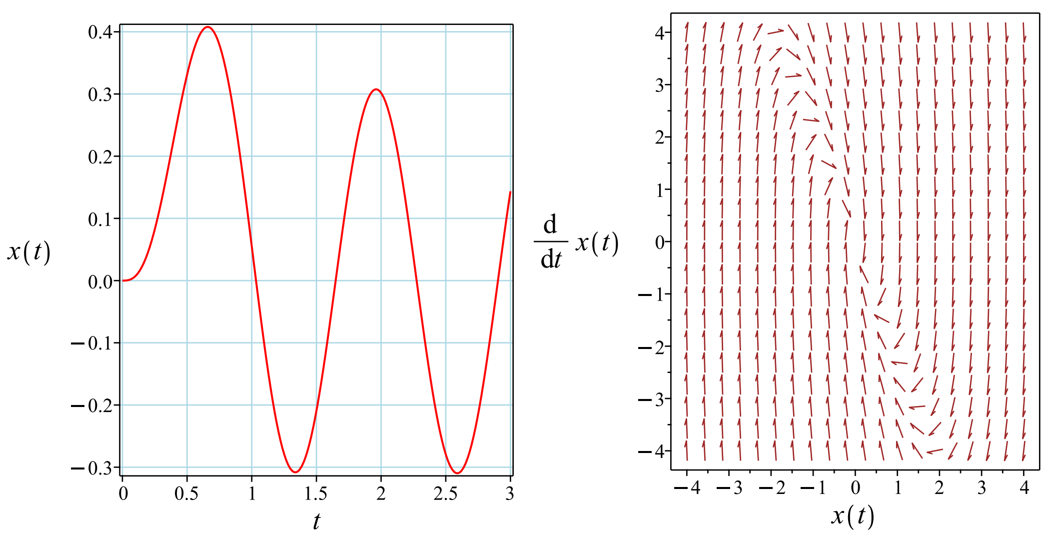
$$x = \frac{25\,\mathrm{e}^{-3t}\cos(2t)}{87} + \frac{125\,\mathrm{e}^{-3t}\sin(2t)}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Which simplifies to

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Summary
The solution(s) found are the following

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Verified OK.

12.11.4 Maple step by step solution

Let's solve

$$\left[x'' + 6x' + 13x = 10\sin(5t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-3t}\cos(2t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-3t}\sin(2t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1 \mathrm{e}^{-3t}\cos(2t) + \mathrm{e}^{-3t} c_2 \sin(2t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 10\sin(5t)\right]$

  - Wronskian of solutions of the homogeneous equation

    $$W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t}\cos(2t) & \mathrm{e}^{-3t}\sin(2t) \\ -3\,\mathrm{e}^{-3t}\cos(2t) - 2\,\mathrm{e}^{-3t}\sin(2t) & -3\,\mathrm{e}^{-3t}\sin(2t) + 2\,\mathrm{e}^{-3t}\cos(2t) \end{bmatrix}$$

  - Compute Wronskian

    $W(x_1(t), x_2(t)) = 2\,\mathrm{e}^{-6t}$

  - Substitute functions into equation for $x_p(t)$

$x_p(t) = -5\,e^{-3t}\left(\cos(2t)\left(\int \sin(2t)\sin(5t)\,e^{3t}dt\right) - \sin(2t)\left(\int \cos(2t)\sin(5t)\,e^{3t}dt\right)\right)$

- Compute integrals

  $x_p(t) = -\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$

- Substitute particular solution into general solution to ODE

  $x = c_1 e^{-3t}\cos(2t) + e^{-3t}c_2\sin(2t) - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$

□ Check validity of solution $x = c_1 e^{-3t}\cos(2t) + e^{-3t}c_2\sin(2t) - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$

- Use initial condition $x(0) = 0$

  $0 = c_1 - \frac{25}{87}$

- Compute derivative of the solution

  $x' = -3c_1 e^{-3t}\cos(2t) - 2c_1 e^{-3t}\sin(2t) - 3\,e^{-3t}c_2\sin(2t) + 2\,e^{-3t}c_2\cos(2t) + \frac{125\sin(5t)}{87} - \frac{50\cos(5t)}{87}$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

  $0 = -3c_1 - \frac{50}{87} + 2c_2$

- Solve for c_1 and c_2

  $\left\{c_1 = \frac{25}{87}, c_2 = \frac{125}{174}\right\}$

- Substitute constant values into general solution and simplify

  $x = \frac{25(2\cos(2t)+5\sin(2t))e^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$

- Solution to the IVP

  $x = \frac{25(2\cos(2t)+5\sin(2t))e^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 37

```
dsolve([diff(x(t),t$2)+6*diff(x(t),t)+13*x(t)=10*sin(5*t),x(0) = 0, D(x)(0) = 0],x(t), sings
```

$$x(t) = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 49

```
DSolve[{x''[t]+6*x'[t]+13*x[t]==10*Sin[5*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSoluti
```

$$x(t) \to \frac{5}{174}e^{-3t}\left(25\sin(2t) - 4e^{3t}\sin(5t) + 10\cos(2t) - 10e^{3t}\cos(5t)\right)$$

12.12 problem 12

Internal problem ID [269]
Internal file name [`OUTPUT/269_Sunday_June_05_2022_01_37_52_AM_83086868/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 12.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 6x' + 13x = 10 \sin (5t)}$$

With initial conditions

$$[x(0) = 0, x'(0) = 0]$$

12.12.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 6 \\ q(t) &= 13 \\ F &= 10 \sin (5t) \end{aligned}$$

Hence the ode is

$$x'' + 6x' + 13x = 10\sin(5t)$$

The domain of $p(t) = 6$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 13$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 10\sin(5t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.12.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 6, C = 13, f(t) = 10\sin(5t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 6x' + 13x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 6, C = 13$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 6\lambda\, e^{\lambda t} + 13\, e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 6\lambda + 13 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 6, C = 13$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-6}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{6^2 - (4)(1)(13)} \\ &= -3 \pm 2i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -3 + 2i \\ \lambda_2 &= -3 - 2i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -3$ and $\beta = 2$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-3t}(c_1 \cos(2t) + c_2 \sin(2t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-3t}(c_1 \cos(2t) + c_2 \sin(2t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$10 \sin(5t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(5t), \sin(5t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-3t}\cos(2t), e^{-3t}\sin(2t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1 \cos(5t) + A_2 \sin(5t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \cos(5t) - 12A_2 \sin(5t) - 30A_1 \sin(5t) + 30A_2 \cos(5t) = 10\sin(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{25}{87}, A_2 = -\frac{10}{87}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-3t}(c_1 \cos(2t) + c_2 \sin(2t))\right) + \left(-\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-3t}(c_1 \cos(2t) + c_2 \sin(2t)) - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{25}{87} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3\,\mathrm{e}^{-3t}(c_1 \cos(2t) + c_2 \sin(2t)) + \mathrm{e}^{-3t}(-2c_1 \sin(2t) + 2c_2 \cos(2t)) + \frac{125 \sin(5t)}{87} - \frac{50 \cos(5t)}{87}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{50}{87} + 2c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{25}{87}$$
$$c_2 = \frac{125}{174}$$

Substituting these values back in above solution results in

$$x = \frac{25\,\mathrm{e}^{-3t} \cos(2t)}{87} + \frac{125\,\mathrm{e}^{-3t} \sin(2t)}{174} - \frac{25 \cos(5t)}{87} - \frac{10 \sin(5t)}{87}$$

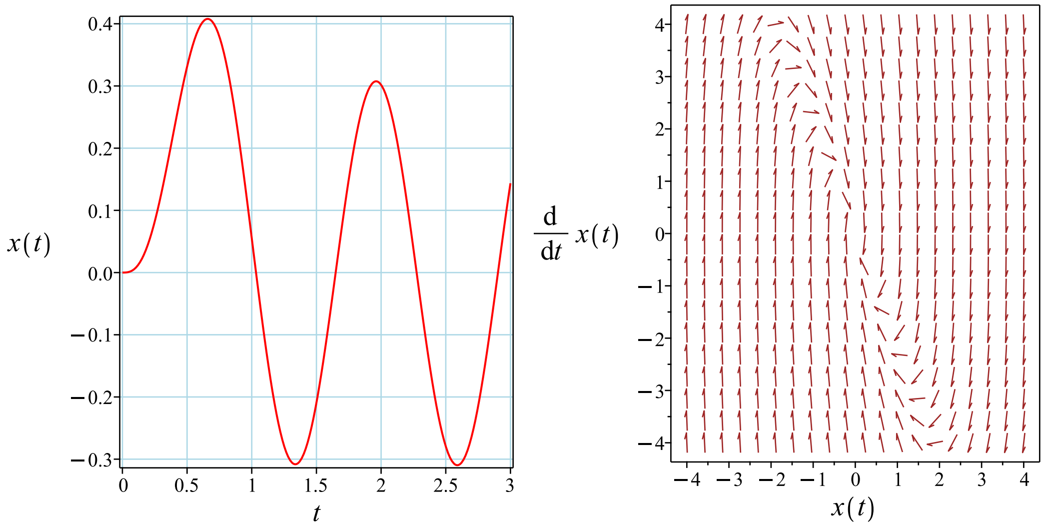
Which simplifies to

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Summary

The solution(s) found are the following

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Verified OK.

12.12.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 6x' + 13x = 0 \tag{1}$$

$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 6 \\ C &= 13 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-4}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -4$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -4z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 541: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -4$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(2t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{6}{1}\, dt} \\
&= z_1 e^{-3t} \\
&= z_1 \left(\mathrm{e}^{-3t}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-3t} \cos(2t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{6}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-6t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{\tan(2t)}{2}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-3t} \cos\left(2t\right)\right) + c_2 \left(\mathrm{e}^{-3t} \cos\left(2t\right) \left(\frac{\tan\left(2t\right)}{2}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t)+Bx'(t)+Cx(t)=f(t)$. x_h is the solution to

$$
x'' + 6x' + 13x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-3t} \cos\left(2t\right) + \frac{\mathrm{e}^{-3t} c_2 \sin\left(2t\right)}{2}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
10 \sin\left(5t\right)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos\left(5t\right), \sin\left(5t\right)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-3t} \cos\left(2t\right), \frac{\mathrm{e}^{-3t} \sin\left(2t\right)}{2}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \cos\left(5t\right) + A_2 \sin\left(5t\right)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-12A_1 \cos(5t) - 12A_2 \sin(5t) - 30A_1 \sin(5t) + 30A_2 \cos(5t) = 10 \sin(5t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{25}{87}, A_2 = -\frac{10}{87}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-3t}\cos(2t) + \frac{e^{-3t}c_2\sin(2t)}{2}\right) + \left(-\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-3t}\cos(2t) + \frac{e^{-3t}c_2\sin(2t)}{2} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 0$ and $t = 0$ in the above gives

$$0 = c_1 - \frac{25}{87} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -3c_1 e^{-3t}\cos(2t) - 2c_1 e^{-3t}\sin(2t) - \frac{3\,e^{-3t}c_2\sin(2t)}{2} + e^{-3t}c_2\cos(2t) + \frac{125\sin(5t)}{87} - \frac{50\cos(5t)}{87}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -3c_1 - \frac{50}{87} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{25}{87}$$
$$c_2 = \frac{125}{87}$$

Substituting these values back in above solution results in

$$x = \frac{25\,\mathrm{e}^{-3t}\cos(2t)}{87} + \frac{125\,\mathrm{e}^{-3t}\sin(2t)}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

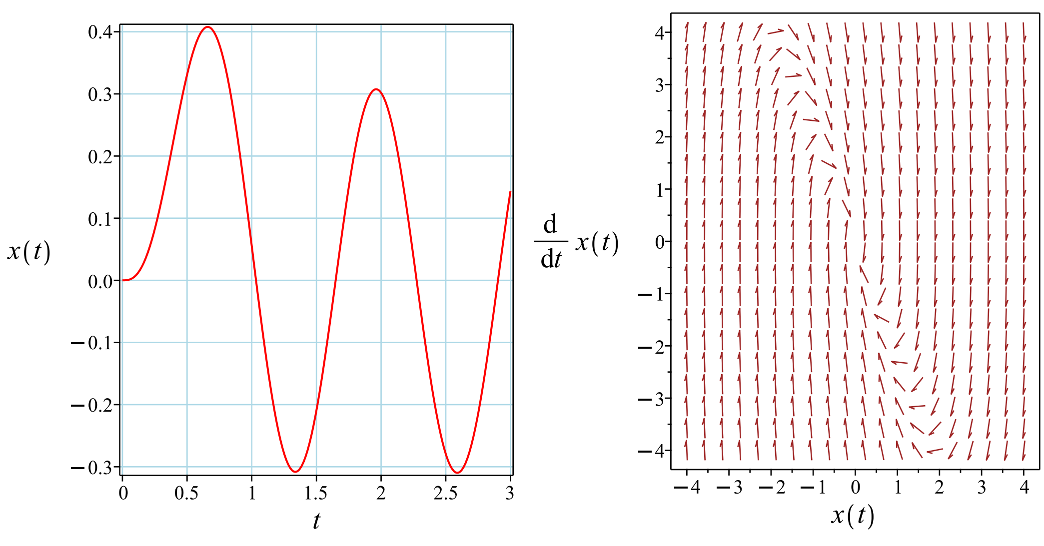
Which simplifies to

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Summary

The solution(s) found are the following

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87} \tag{1}$$



(a) Solution plot

(b) Slope field plot

Verification of solutions

$$x = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Verified OK.

12.12.4 Maple step by step solution

Let's solve

$$\left[x'' + 6x' + 13x = 10\sin(5t), x(0) = 0, x'\Big|_{\{t=0\}} = 0\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 6r + 13 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-6)\pm(\sqrt{-16})}{2}$

- Roots of the characteristic polynomial

  $r = (-3 - 2\,\mathrm{I}, -3 + 2\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t) = \mathrm{e}^{-3t}\cos(2t)$

- 2nd solution of the homogeneous ODE

  $x_2(t) = \mathrm{e}^{-3t}\sin(2t)$

- General solution of the ODE

  $x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x = c_1\mathrm{e}^{-3t}\cos(2t) + \mathrm{e}^{-3t}c_2\sin(2t) + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- ○ Use variation of parameters to find x_p here $f(t)$ is the forcing function

  $\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right) + x_2(t)\left(\int \frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t) = 10\sin(5t)\right]$

- ○ Wronskian of solutions of the homogeneous equation

  $W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-3t}\cos(2t) & \mathrm{e}^{-3t}\sin(2t) \\ -3\,\mathrm{e}^{-3t}\cos(2t) - 2\,\mathrm{e}^{-3t}\sin(2t) & -3\,\mathrm{e}^{-3t}\sin(2t) + 2\,\mathrm{e}^{-3t}\cos(2t) \end{bmatrix}$

- ○ Compute Wronskian

  $W(x_1(t), x_2(t)) = 2\,\mathrm{e}^{-6t}$

- ○ Substitute functions into equation for $x_p(t)$

$$x_p(t) = -5\,\mathrm{e}^{-3t}\left(\cos(2t)\left(\int \sin(2t)\sin(5t)\,\mathrm{e}^{3t}dt\right) - \sin(2t)\left(\int \cos(2t)\sin(5t)\,\mathrm{e}^{3t}dt\right)\right)$$

- Compute integrals

$$x_p(t) = -\frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

- Substitute particular solution into general solution to ODE

$$x = c_1\mathrm{e}^{-3t}\cos(2t) + \mathrm{e}^{-3t}c_2\sin(2t) - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

□ Check validity of solution $x = c_1\mathrm{e}^{-3t}\cos(2t) + \mathrm{e}^{-3t}c_2\sin(2t) - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$

- Use initial condition $x(0) = 0$

$$0 = c_1 - \frac{25}{87}$$

- Compute derivative of the solution

$$x' = -3c_1\mathrm{e}^{-3t}\cos(2t) - 2c_1\mathrm{e}^{-3t}\sin(2t) - 3\,\mathrm{e}^{-3t}c_2\sin(2t) + 2\,\mathrm{e}^{-3t}c_2\cos(2t) + \frac{125\sin(5t)}{87} - \frac{50\cos(5t)}{87}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

$$0 = -3c_1 - \frac{50}{87} + 2c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{25}{87}, c_2 = \frac{125}{174}\right\}$$

- Substitute constant values into general solution and simplify

$$x = \frac{25(2\cos(2t)+5\sin(2t))\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

- Solution to the IVP

$$x = \frac{25(2\cos(2t)+5\sin(2t))\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve([diff(x(t),t$2)+6*diff(x(t),t)+13*x(t)=10*sin(5*t),x(0) = 0, D(x)(0) = 0],x(t), sings
```

$$x(t) = \frac{25(2\cos(2t) + 5\sin(2t))\,\mathrm{e}^{-3t}}{174} - \frac{25\cos(5t)}{87} - \frac{10\sin(5t)}{87}$$

✓ Solution by Mathematica

Time used: 0.024 (sec). Leaf size: 49

```
DSolve[{x''[t]+6*x'[t]+13*x[t]==10*Sin[5*t],{x[0]==0,x'[0]==0}},x[t],t,IncludeSingularSoluti
```

$$x(t) \to \frac{5}{174}e^{-3t}\left(25\sin(2t) - 4e^{3t}\sin(5t) + 10\cos(2t) - 10e^{3t}\cos(5t)\right)$$

12.13 problem 13

Internal problem ID [270]
Internal file name [`OUTPUT/270_Sunday_June_05_2022_01_37_53_AM_76490170/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 13.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

```
[[_2nd_order, _linear, _nonhomogeneous]]
```

$$\boxed{x'' + 2x' + 26x = 600 \cos(10t)}$$

With initial conditions

$$[x(0) = 10, x'(0) = 0]$$

12.13.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 2 \\ q(t) &= 26 \\ F &= 600 \cos(10t) \end{aligned}$$

Hence the ode is

$$x'' + 2x' + 26x = 600\cos(10t)$$

The domain of $p(t) = 2$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 26$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 600\cos(10t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.13.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 2, C = 26, f(t) = 600\cos(10t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t)+Bx'(t)+Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 2x' + 26x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 2, C = 26$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 26 e^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 2\lambda + 26 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 2, C = 26$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-2}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{2^2 - (4)(1)(26)} \\ &= -1 \pm 5i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -1 + 5i \\ \lambda_2 &= -1 - 5i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -1 + 5i \\ \lambda_2 &= -1 - 5i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -1$ and $\beta = 5$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-t}(c_1 \cos(5t) + c_2 \sin(5t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-t}(c_1 \cos(5t) + c_2 \sin(5t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$600 \cos(10t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(10t), \sin(10t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-t}\cos(5t), e^{-t}\sin(5t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(10t) + A_2\sin(10t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-74A_1\cos(10t) - 74A_2\sin(10t) - 20A_1\sin(10t) + 20A_2\cos(10t) = 600\cos(10t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{11100}{1469}, A_2 = \frac{3000}{1469}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-t}(c_1\cos(5t) + c_2\sin(5t))\right) + \left(-\frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-t}(c_1\cos(5t) + c_2\sin(5t)) - \frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 10$ and $t = 0$ in the above gives

$$10 = c_1 - \frac{11100}{1469} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -\mathrm{e}^{-t}(c_1 \cos(5t) + c_2 \sin(5t)) + \mathrm{e}^{-t}(-5c_1 \sin(5t) + 5c_2 \cos(5t)) + \frac{111000 \sin(10t)}{1469} + \frac{30000 \cos(10t)}{1469}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + \frac{30000}{1469} + 5c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{25790}{1469}$$
$$c_2 = -\frac{842}{1469}$$

Substituting these values back in above solution results in

$$x = \frac{25790\,\mathrm{e}^{-t} \cos(5t)}{1469} - \frac{842\,\mathrm{e}^{-t} \sin(5t)}{1469} - \frac{11100 \cos(10t)}{1469} + \frac{3000 \sin(10t)}{1469}$$

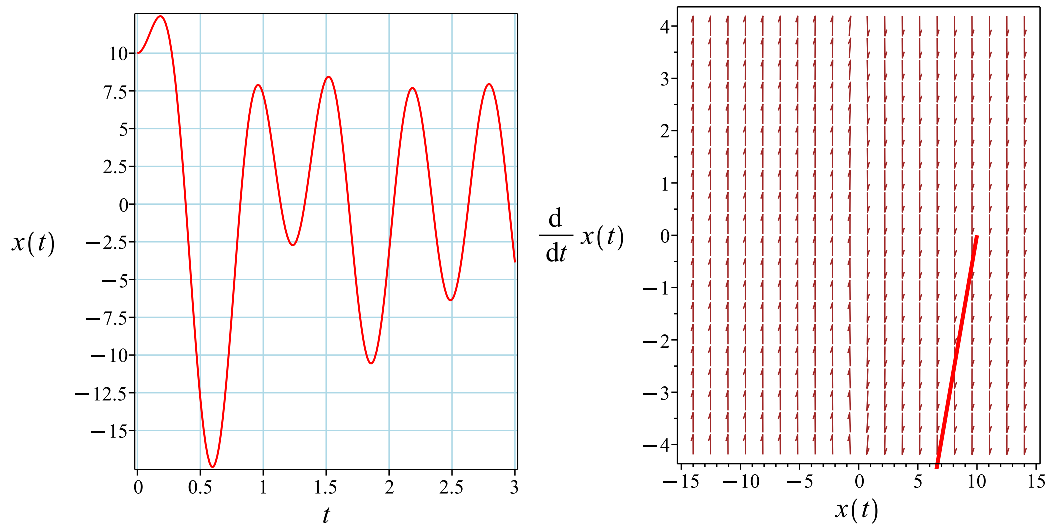
Which simplifies to

$$x = \frac{(25790 \cos(5t) - 842 \sin(5t))\,\mathrm{e}^{-t}}{1469} - \frac{11100 \cos(10t)}{1469} + \frac{3000 \sin(10t)}{1469}$$

Summary

The solution(s) found are the following

$$x = \frac{(25790 \cos(5t) - 842 \sin(5t))\,\mathrm{e}^{-t}}{1469} - \frac{11100 \cos(10t)}{1469} + \frac{3000 \sin(10t)}{1469} \tag{1}$$



(a) Solution plot (b) Slope field plot

Verification of solutions

$$x = \frac{(25790\cos(5t) - 842\sin(5t))\,e^{-t}}{1469} - \frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469}$$

Verified OK.

12.13.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 2x' + 26x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 2 \\ C &= 26 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\,dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$r = \frac{s}{t} = \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-25}{1} \tag{6}$$

Comparing the above to (5) shows that

$$s = -25$$
$$t = 1$$

Therefore eq. (4) becomes

$$z''(t) = -25z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 543: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$L = [1]$$

Since $r = -25$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$z_1(t) = \cos(5t)$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\,dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{2}{1}\,dt} \\
&= z_1 e^{-t} \\
&= z_1\left(\mathrm{e}^{-t}\right)
\end{aligned}$$

Which simplifies to

$$x_1 = \mathrm{e}^{-t}\cos(5t)$$

The second solution x_2 to the original ode is found using reduction of order

$$x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\,dt}}{x_1^2}\,dt$$

Substituting gives

$$\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{2}{1}\,dt}}{(x_1)^2}\,dt \\
&= x_1 \int \frac{e^{-2t}}{(x_1)^2}\,dt \\
&= x_1\left(\frac{\tan(5t)}{5}\right)
\end{aligned}$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-t} \cos(5t)\right) + c_2 \left(\mathrm{e}^{-t} \cos(5t) \left(\frac{\tan(5t)}{5}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 2x' + 26x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-t} \cos(5t) + \frac{\mathrm{e}^{-t} c_2 \sin(5t)}{5}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$
600 \cos(10t)
$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$
[\{\cos(10t), \sin(10t)\}]
$$

While the set of the basis functions for the homogeneous solution found earlier is

$$
\left\{\mathrm{e}^{-t} \cos(5t), \frac{\mathrm{e}^{-t} \sin(5t)}{5}\right\}
$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$
x_p = A_1 \cos(10t) + A_2 \sin(10t)
$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$-74A_1 \cos(10t) - 74A_2 \sin(10t) - 20A_1 \sin(10t) + 20A_2 \cos(10t) = 600 \cos(10t)$$

Solving for the unknowns by comparing coefficients results in

$$\left[A_1 = -\frac{11100}{1469}, A_2 = \frac{3000}{1469}\right]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = -\frac{11100 \cos(10t)}{1469} + \frac{3000 \sin(10t)}{1469}$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-t} \cos(5t) + \frac{e^{-t} c_2 \sin(5t)}{5}\right) + \left(-\frac{11100 \cos(10t)}{1469} + \frac{3000 \sin(10t)}{1469}\right) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 e^{-t} \cos(5t) + \frac{e^{-t} c_2 \sin(5t)}{5} - \frac{11100 \cos(10t)}{1469} + \frac{3000 \sin(10t)}{1469} \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = 10$ and $t = 0$ in the above gives

$$10 = c_1 - \frac{11100}{1469} \tag{1A}$$

Taking derivative of the solution gives

$$x' = -c_1 e^{-t} \cos(5t) - 5c_1 e^{-t} \sin(5t) - \frac{e^{-t} c_2 \sin(5t)}{5} + e^{-t} c_2 \cos(5t) + \frac{111000 \sin(10t)}{1469} + \frac{30000 \cos(10t)}{1469}$$

substituting $x' = 0$ and $t = 0$ in the above gives

$$0 = -c_1 + \frac{30000}{1469} + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = \frac{25790}{1469}$$
$$c_2 = -\frac{4210}{1469}$$

Substituting these values back in above solution results in

$$x = \frac{25790\,\mathrm{e}^{-t}\cos(5t)}{1469} - \frac{842\,\mathrm{e}^{-t}\sin(5t)}{1469} - \frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469}$$

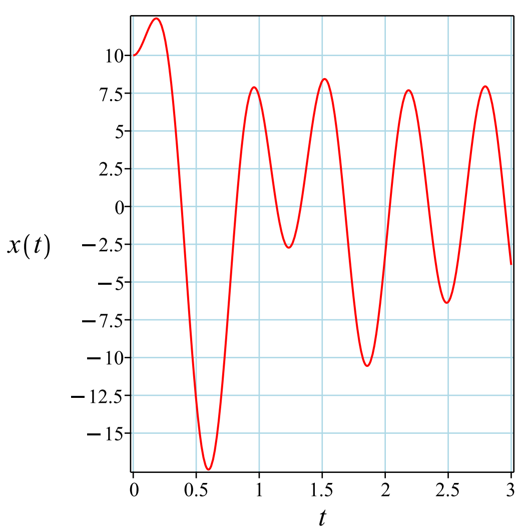
Which simplifies to

$$x = \frac{(25790\cos(5t) - 842\sin(5t))\,\mathrm{e}^{-t}}{1469} - \frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469}$$

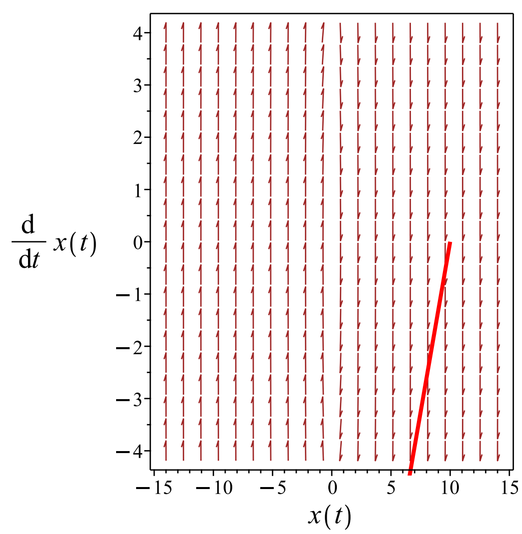
Summary

The solution(s) found are the following

$$x = \frac{(25790\cos(5t) - 842\sin(5t))\,\mathrm{e}^{-t}}{1469} - \frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469} \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = \frac{(25790\cos(5t) - 842\sin(5t))\,\mathrm{e}^{-t}}{1469} - \frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469}$$

Verified OK.

12.13.4 Maple step by step solution

Let's solve

$\left[x''+2x'+26x=600\cos\left(10t\right), x(0)=10, x'\Big|_{\{t=0\}}=0\right]$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2+2r+26=0$

- Use quadratic formula to solve for r

  $r=\frac{(-2)\pm\left(\sqrt{-100}\right)}{2}$

- Roots of the characteristic polynomial

  $r=(-1-5\,\mathrm{I}, -1+5\,\mathrm{I})$

- 1st solution of the homogeneous ODE

  $x_1(t)=\mathrm{e}^{-t}\cos\left(5t\right)$

- 2nd solution of the homogeneous ODE

  $x_2(t)=\mathrm{e}^{-t}\sin\left(5t\right)$

- General solution of the ODE

  $x=c_1x_1(t)+c_2x_2(t)+x_p(t)$

- Substitute in solutions of the homogeneous ODE

  $x=c_1\mathrm{e}^{-t}\cos\left(5t\right)+\mathrm{e}^{-t}c_2\sin\left(5t\right)+x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

  - Use variation of parameters to find x_p here $f(t)$ is the forcing function

    $\left[x_p(t)=-x_1(t)\left(\int\frac{x_2(t)f(t)}{W(x_1(t),x_2(t))}dt\right)+x_2(t)\left(\int\frac{x_1(t)f(t)}{W(x_1(t),x_2(t))}dt\right), f(t)=600\cos\left(10t\right)\right]$

  - Wronskian of solutions of the homogeneous equation

    $W\left(x_1(t), x_2(t)\right)=\left[\begin{array}{cc}\mathrm{e}^{-t}\cos\left(5t\right) & \mathrm{e}^{-t}\sin\left(5t\right)\\ -\mathrm{e}^{-t}\cos\left(5t\right)-5\,\mathrm{e}^{-t}\sin\left(5t\right) & -\mathrm{e}^{-t}\sin\left(5t\right)+5\,\mathrm{e}^{-t}\cos\left(5t\right)\end{array}\right]$

  - Compute Wronskian

    $W\left(x_1(t), x_2(t)\right)=5\,\mathrm{e}^{-2t}$

  - Substitute functions into equation for $x_p(t)$

$$x_p(t) = 60\,\mathrm{e}^{-t}\left(\cos(5t)\left(\int(-\sin(15t)+\sin(5t))\,\mathrm{e}^{t}dt\right)+\sin(5t)\left(\int(\cos(5t)+\cos(15t))\,\mathrm{e}^{t}dt\right)\right)$$

- Compute integrals

$$x_p(t) = -\frac{11100\cos(10t)}{1469}+\frac{3000\sin(10t)}{1469}$$

- Substitute particular solution into general solution to ODE

$$x = c_1\mathrm{e}^{-t}\cos(5t)+\mathrm{e}^{-t}c_2\sin(5t)-\frac{11100\cos(10t)}{1469}+\frac{3000\sin(10t)}{1469}$$

□ Check validity of solution $x = c_1\mathrm{e}^{-t}\cos(5t)+\mathrm{e}^{-t}c_2\sin(5t)-\frac{11100\cos(10t)}{1469}+\frac{3000\sin(10t)}{1469}$

- Use initial condition $x(0) = 10$

$$10 = c_1-\frac{11100}{1469}$$

- Compute derivative of the solution

$$x' = -c_1\mathrm{e}^{-t}\cos(5t)-5c_1\mathrm{e}^{-t}\sin(5t)-\mathrm{e}^{-t}c_2\sin(5t)+5\,\mathrm{e}^{-t}c_2\cos(5t)+\frac{111000\sin(10t)}{1469}+\frac{30000\cos(10t)}{1469}$$

- Use the initial condition $x'\Big|_{\{t=0\}} = 0$

$$0 = -c_1+\frac{30000}{1469}+5c_2$$

- Solve for c_1 and c_2

$$\left\{c_1 = \frac{25790}{1469}, c_2 = -\frac{842}{1469}\right\}$$

- Substitute constant values into general solution and simplify

$$x = \frac{(25790\cos(5t)-842\sin(5t))\,\mathrm{e}^{-t}}{1469}-\frac{11100\cos(10t)}{1469}+\frac{3000\sin(10t)}{1469}$$

- Solution to the IVP

$$x = \frac{(25790\cos(5t)-842\sin(5t))\,\mathrm{e}^{-t}}{1469}-\frac{11100\cos(10t)}{1469}+\frac{3000\sin(10t)}{1469}$$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.016 (sec). Leaf size: 37

```
dsolve([diff(x(t),t$2)+2*diff(x(t),t)+26*x(t)=600*cos(10*t),x(0) = 10, D(x)(0) = 0],x(t), si
```

$$x(t) = \frac{(25790\cos(5t) - 842\sin(5t))\,\mathrm{e}^{-t}}{1469} - \frac{11100\cos(10t)}{1469} + \frac{3000\sin(10t)}{1469}$$

✓ Solution by Mathematica

Time used: 0.027 (sec). Leaf size: 45

```
DSolve[{x''[t]+2*x'[t]+26*x[t]==600*Cos[10*t],{x[0]==10,x'[0]==0}},x[t],t,IncludeSingularSol
```

$$x(t) \to -\frac{2e^{-t}(421\sin(5t) - 1500e^{t}\sin(10t) - 12895\cos(5t) + 5550e^{t}\cos(10t))}{1469}$$

12.14 problem 14

Internal problem ID [271]
Internal file name [`OUTPUT/271_Sunday_June_05_2022_01_37_55_AM_36694649/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 5.6, Forced Oscillations and Resonance. Page 362
Problem number: 14.
ODE order: 2.
ODE degree: 1.

The type(s) of ODE detected by this program : **"kovacic", "second_order_linear_constant_coeff"**

Maple gives the following as the ode type

`[[_2nd_order, _linear, _nonhomogeneous]]`

$$\boxed{x'' + 8x' + 25x = 200\cos(t) + 520\sin(t)}$$

With initial conditions

$$[x(0) = -30, x'(0) = -10]$$

12.14.1 Existence and uniqueness analysis

This is a linear ODE. In canonical form it is written as

$$x'' + p(t)x' + q(t)x = F$$

Where here

$$\begin{aligned} p(t) &= 8 \\ q(t) &= 25 \\ F &= 200\cos(t) + 520\sin(t) \end{aligned}$$

Hence the ode is

$$x'' + 8x' + 25x = 200 \cos(t) + 520 \sin(t)$$

The domain of $p(t) = 8$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is inside this domain. The domain of $q(t) = 25$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. The domain of $F = 200 \cos(t) + 520 \sin(t)$ is

$$\{-\infty < t < \infty\}$$

And the point $t_0 = 0$ is also inside this domain. Hence solution exists and is unique.

12.14.2 Solving as second order linear constant coeff ode

This is second order non-homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = f(t)$$

Where $A = 1, B = 8, C = 25, f(t) = 200 \cos(t) + 520 \sin(t)$. Let the solution be

$$x = x_h + x_p$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the non-homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$x'' + 8x' + 25x = 0$$

This is second order with constant coefficients homogeneous ODE. In standard form the ODE is

$$Ax''(t) + Bx'(t) + Cx(t) = 0$$

Where in the above $A = 1, B = 8, C = 25$. Let the solution be $x = e^{\lambda t}$. Substituting this into the ODE gives

$$\lambda^2 \mathrm{e}^{\lambda t} + 8\lambda \,\mathrm{e}^{\lambda t} + 25 \,\mathrm{e}^{\lambda t} = 0 \tag{1}$$

Since exponential function is never zero, then dividing Eq(2) throughout by $e^{\lambda t}$ gives

$$\lambda^2 + 8\lambda + 25 = 0 \tag{2}$$

Equation (2) is the characteristic equation of the ODE. Its roots determine the general solution form.Using the quadratic formula

$$\lambda_{1,2} = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

Substituting $A = 1, B = 8, C = 25$ into the above gives

$$\begin{aligned}\lambda_{1,2} &= \frac{-8}{(2)(1)} \pm \frac{1}{(2)(1)}\sqrt{8^2 - (4)(1)(25)}\\ &= -4 \pm 3i\end{aligned}$$

Hence

$$\begin{aligned}\lambda_1 &= -4 + 3i\\ \lambda_2 &= -4 - 3i\end{aligned}$$

Which simplifies to

$$\begin{aligned}\lambda_1 &= -4 + 3i\\ \lambda_2 &= -4 - 3i\end{aligned}$$

Since roots are complex conjugate of each others, then let the roots be

$$\lambda_{1,2} = \alpha \pm i\beta$$

Where $\alpha = -4$ and $\beta = 3$. Therefore the final solution, when using Euler relation, can be written as

$$x = e^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t))$$

Which becomes

$$x = e^{-4t}(c_1 \cos(3t) + c_2 \sin(3t))$$

Therefore the homogeneous solution x_h is

$$x_h = \mathrm{e}^{-4t}(c_1 \cos(3t) + c_2 \sin(3t))$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$200\cos(t) + 520\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\{e^{-4t}\cos(3t), e^{-4t}\sin(3t)\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(t) + A_2\sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_1\cos(t) + 24A_2\sin(t) - 8A_1\sin(t) + 8A_2\cos(t) = 200\cos(t) + 520\sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 22]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \cos(t) + 22\sin(t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(e^{-4t}(c_1\cos(3t) + c_2\sin(3t))\right) + (\cos(t) + 22\sin(t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = e^{-4t}(c_1\cos(3t) + c_2\sin(3t)) + \cos(t) + 22\sin(t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -30$ and $t = 0$ in the above gives

$$-30 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4\,e^{-4t}(c_1\cos(3t) + c_2\sin(3t)) + e^{-4t}(-3c_1\sin(3t) + 3c_2\cos(3t)) - \sin(t) + 22\cos(t)$$

substituting $x' = -10$ and $t = 0$ in the above gives

$$-10 = -4c_1 + 3c_2 + 22 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -31$$
$$c_2 = -52$$

Substituting these values back in above solution results in

$$x = -31\,e^{-4t}\cos(3t) - 52\,e^{-4t}\sin(3t) + 22\sin(t) + \cos(t)$$

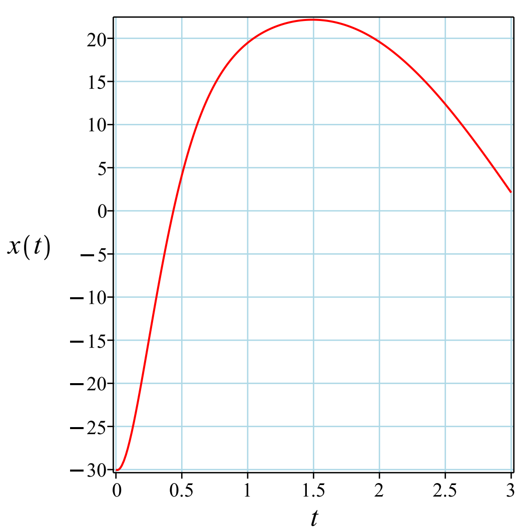
Which simplifies to

$$x = (-31\cos(3t) - 52\sin(3t))\,e^{-4t} + \cos(t) + 22\sin(t)$$

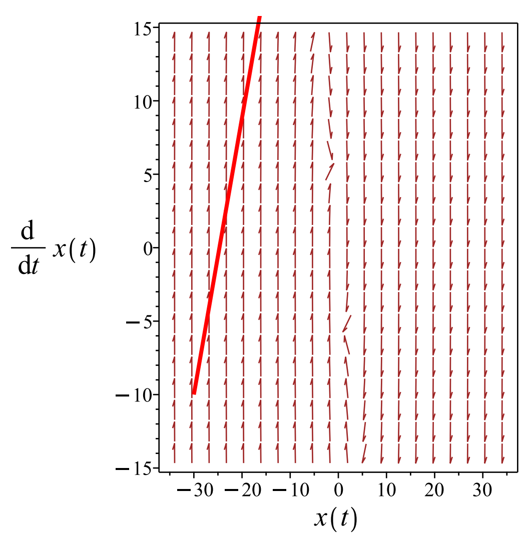
Summary

The solution(s) found are the following

$$x = (-31\cos(3t) - 52\sin(3t))\,e^{-4t} + \cos(t) + 22\sin(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

Verification of solutions

$$x = (-31\cos(3t) - 52\sin(3t))\,e^{-4t} + \cos(t) + 22\sin(t)$$

Verified OK.

12.14.3 Solving using Kovacic algorithm

Writing the ode as

$$x'' + 8x' + 25x = 0 \tag{1}$$
$$Ax'' + Bx' + Cx = 0 \tag{2}$$

Comparing (1) and (2) shows that

$$\begin{aligned} A &= 1 \\ B &= 8 \\ C &= 25 \end{aligned} \tag{3}$$

Applying the Liouville transformation on the dependent variable gives

$$z(t) = xe^{\int \frac{B}{2A}\, dt}$$

Then (2) becomes

$$z''(t) = rz(t) \tag{4}$$

Where r is given by

$$\begin{aligned} r &= \frac{s}{t} \\ &= \frac{2AB' - 2BA' + B^2 - 4AC}{4A^2} \end{aligned} \tag{5}$$

Substituting the values of A, B, C from (3) in the above and simplifying gives

$$r = \frac{-9}{1} \tag{6}$$

Comparing the above to (5) shows that

$$\begin{aligned} s &= -9 \\ t &= 1 \end{aligned}$$

Therefore eq. (4) becomes

$$z''(t) = -9z(t) \tag{7}$$

Equation (7) is now solved. After finding $z(t)$ then x is found using the inverse transformation

$$x = z(t)\, e^{-\int \frac{B}{2A}\, dt}$$

The first step is to determine the case of Kovacic algorithm this ode belongs to. There are 3 cases depending on the order of poles of r and the order of r at ∞. The following table summarizes these cases.

Case	Allowed pole order for r	Allowed value for $\mathcal{O}(\infty)$
1	$\{0, 1, 2, 4, 6, 8, \cdots\}$	$\{\cdots, -6, -4, -2, 0, 2, 3, 4, 5, 6, \cdots\}$
2	Need to have at least one pole that is either order 2 or odd order greater than 2. Any other pole order is allowed as long as the above condition is satisfied. Hence the following set of pole orders are all allowed. $\{1,2\}$,$\{1,3\}$,$\{2\}$,$\{3\}$,$\{3,4\}$,$\{1,2,5\}$.	no condition
3	$\{1, 2\}$	$\{2, 3, 4, 5, 6, 7, \cdots\}$

Table 545: Necessary conditions for each Kovacic case

The order of r at ∞ is the degree of t minus the degree of s. Therefore

$$
\begin{aligned}
O(\infty) &= \deg(t) - \deg(s) \\
&= 0 - 0 \\
&= 0
\end{aligned}
$$

There are no poles in r. Therefore the set of poles Γ is empty. Since there is no odd order pole larger than 2 and the order at ∞ is 0 then the necessary conditions for case one are met. Therefore

$$
L = [1]
$$

Since $r = -9$ is not a function of t, then there is no need run Kovacic algorithm to obtain a solution for transformed ode $z'' = rz$ as one solution is

$$
z_1(t) = \cos(3t)
$$

Using the above, the solution for the original ode can now be found. The first solution to the original ode in x is found from

$$
\begin{aligned}
x_1 &= z_1 e^{\int -\frac{1}{2}\frac{B}{A}\, dt} \\
&= z_1 e^{-\int \frac{1}{2}\frac{8}{1}\, dt}
\end{aligned}
$$

$$
\begin{aligned}
&= z_1 e^{-4t} \\
&= z_1 \left(\mathrm{e}^{-4t}\right)
\end{aligned}
$$

Which simplifies to

$$
x_1 = \mathrm{e}^{-4t} \cos(3t)
$$

The second solution x_2 to the original ode is found using reduction of order

$$
x_2 = x_1 \int \frac{e^{\int -\frac{B}{A}\, dt}}{x_1^2}\, dt
$$

Substituting gives

$$
\begin{aligned}
x_2 &= x_1 \int \frac{e^{\int -\frac{8}{1}\, dt}}{(x_1)^2}\, dt \\
&= x_1 \int \frac{e^{-8t}}{(x_1)^2}\, dt \\
&= x_1 \left(\frac{\tan(3t)}{3}\right)
\end{aligned}
$$

Therefore the solution is

$$
\begin{aligned}
x &= c_1 x_1 + c_2 x_2 \\
&= c_1 \left(\mathrm{e}^{-4t} \cos(3t)\right) + c_2 \left(\mathrm{e}^{-4t} \cos(3t) \left(\frac{\tan(3t)}{3}\right)\right)
\end{aligned}
$$

This is second order nonhomogeneous ODE. Let the solution be

$$
x = x_h + x_p
$$

Where x_h is the solution to the homogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = 0$, and x_p is a particular solution to the nonhomogeneous ODE $Ax''(t) + Bx'(t) + Cx(t) = f(t)$. x_h is the solution to

$$
x'' + 8x' + 25x = 0
$$

The homogeneous solution is found using the Kovacic algorithm which results in

$$
x_h = c_1 \mathrm{e}^{-4t} \cos(3t) + \frac{\mathrm{e}^{-4t} \sin(3t)\, c_2}{3}
$$

The particular solution is now found using the method of undetermined coefficients. Looking at the RHS of the ode, which is

$$200\cos(t) + 520\sin(t)$$

Shows that the corresponding undetermined set of the basis functions (UC_set) for the trial solution is

$$[\{\cos(t), \sin(t)\}]$$

While the set of the basis functions for the homogeneous solution found earlier is

$$\left\{e^{-4t}\cos(3t), \frac{e^{-4t}\sin(3t)}{3}\right\}$$

Since there is no duplication between the basis function in the UC_set and the basis functions of the homogeneous solution, the trial solution is a linear combination of all the basis in the UC_set.

$$x_p = A_1\cos(t) + A_2\sin(t)$$

The unknowns $\{A_1, A_2\}$ are found by substituting the above trial solution x_p into the ODE and comparing coefficients. Substituting the trial solution into the ODE and simplifying gives

$$24A_1\cos(t) + 24A_2\sin(t) - 8A_1\sin(t) + 8A_2\cos(t) = 200\cos(t) + 520\sin(t)$$

Solving for the unknowns by comparing coefficients results in

$$[A_1 = 1, A_2 = 22]$$

Substituting the above back in the above trial solution x_p, gives the particular solution

$$x_p = \cos(t) + 22\sin(t)$$

Therefore the general solution is

$$\begin{aligned} x &= x_h + x_p \\ &= \left(c_1 e^{-4t}\cos(3t) + \frac{e^{-4t}\sin(3t)\, c_2}{3}\right) + (\cos(t) + 22\sin(t)) \end{aligned}$$

Initial conditions are used to solve for the constants of integration.

Looking at the above solution

$$x = c_1 \mathrm{e}^{-4t} \cos(3t) + \frac{\mathrm{e}^{-4t} \sin(3t) c_2}{3} + \cos(t) + 22 \sin(t) \tag{1}$$

Initial conditions are now substituted in the above solution. This will generate the required equations to solve for the integration constants. substituting $x = -30$ and $t = 0$ in the above gives

$$-30 = 1 + c_1 \tag{1A}$$

Taking derivative of the solution gives

$$x' = -4c_1 \mathrm{e}^{-4t} \cos(3t) - 3c_1 \mathrm{e}^{-4t} \sin(3t) - \frac{4\,\mathrm{e}^{-4t} \sin(3t) c_2}{3} + \mathrm{e}^{-4t} \cos(3t) c_2 - \sin(t) + 22 \cos(t)$$

substituting $x' = -10$ and $t = 0$ in the above gives

$$-10 = -4c_1 + 22 + c_2 \tag{2A}$$

Equations {1A,2A} are now solved for $\{c_1, c_2\}$. Solving for the constants gives

$$c_1 = -31$$
$$c_2 = -156$$

Substituting these values back in above solution results in

$$x = -31\,\mathrm{e}^{-4t} \cos(3t) - 52\,\mathrm{e}^{-4t} \sin(3t) + 22 \sin(t) + \cos(t)$$

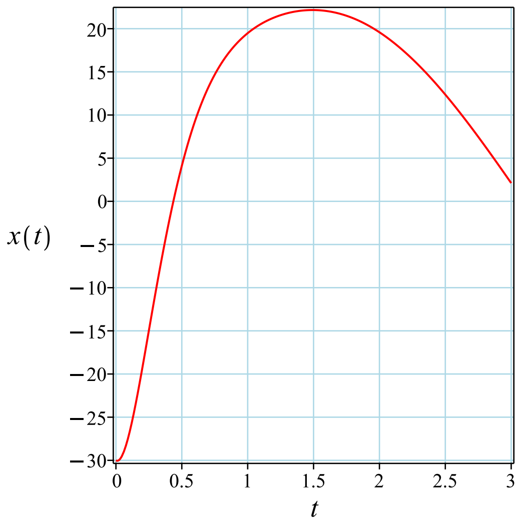
Which simplifies to

$$x = (-31 \cos(3t) - 52 \sin(3t))\,\mathrm{e}^{-4t} + \cos(t) + 22 \sin(t)$$

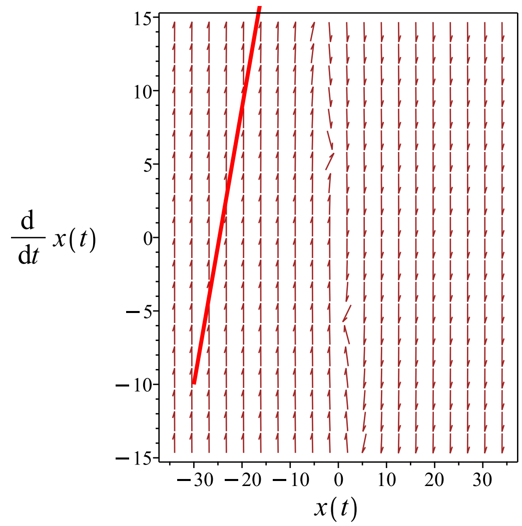
Summary

The solution(s) found are the following

$$x = (-31 \cos(3t) - 52 \sin(3t))\,\mathrm{e}^{-4t} + \cos(t) + 22 \sin(t) \tag{1}$$



(a) Solution plot



(b) Slope field plot

<u>Verification of solutions</u>

$$x = (-31\cos(3t) - 52\sin(3t))\,\mathrm{e}^{-4t} + \cos(t) + 22\sin(t)$$

Verified OK.

12.14.4 Maple step by step solution

Let's solve

$$\left[x'' + 8x' + 25x = 200\cos(t) + 520\sin(t), x(0) = -30, x'\Big|_{\{t=0\}} = -10\right]$$

- Highest derivative means the order of the ODE is 2

  x''

- Characteristic polynomial of homogeneous ODE

  $r^2 + 8r + 25 = 0$

- Use quadratic formula to solve for r

  $r = \frac{(-8)\pm(\sqrt{-36})}{2}$

- Roots of the characteristic polynomial

  $r = (-4 - 3\,\mathrm{I}, -4 + 3\,\mathrm{I})$

- 1st solution of the homogeneous ODE

$x_1(t) = \mathrm{e}^{-4t} \cos(3t)$

- 2nd solution of the homogeneous ODE

$x_2(t) = \mathrm{e}^{-4t} \sin(3t)$

- General solution of the ODE

$x = c_1 x_1(t) + c_2 x_2(t) + x_p(t)$

- Substitute in solutions of the homogeneous ODE

$x = c_1 \mathrm{e}^{-4t} \cos(3t) + \mathrm{e}^{-4t} \sin(3t) c_2 + x_p(t)$

□ Find a particular solution $x_p(t)$ of the ODE

- Use variation of parameters to find x_p here $f(t)$ is the forcing function

$\left[x_p(t) = -x_1(t)\left(\int \frac{x_2(t) f(t)}{W(x_1(t), x_2(t))} dt\right) + x_2(t)\left(\int \frac{x_1(t) f(t)}{W(x_1(t), x_2(t))} dt\right), f(t) = 200 \cos(t) + 520 \sin(t)\right]$

- Wronskian of solutions of the homogeneous equation

$W(x_1(t), x_2(t)) = \begin{bmatrix} \mathrm{e}^{-4t} \cos(3t) & \mathrm{e}^{-4t} \sin(3t) \\ -4\,\mathrm{e}^{-4t} \cos(3t) - 3\,\mathrm{e}^{-4t} \sin(3t) & -4\,\mathrm{e}^{-4t} \sin(3t) + 3\,\mathrm{e}^{-4t} \cos(3t) \end{bmatrix}$

- Compute Wronskian

$W(x_1(t), x_2(t)) = 3\,\mathrm{e}^{-8t}$

- Substitute functions into equation for $x_p(t)$

$x_p(t) = -\frac{40\,\mathrm{e}^{-4t}\left(\cos(3t)\left(\int \sin(3t)(5\cos(t)+13\sin(t))\mathrm{e}^{4t} dt\right) - \sin(3t)\left(\int \cos(3t)(5\cos(t)+13\sin(t))\mathrm{e}^{4t} dt\right)\right)}{3}$

- Compute integrals

$x_p(t) = \cos(t) + 22 \sin(t)$

- Substitute particular solution into general solution to ODE

$x = c_1 \mathrm{e}^{-4t} \cos(3t) + \mathrm{e}^{-4t} \sin(3t) c_2 + \cos(t) + 22 \sin(t)$

□ Check validity of solution $x = c_1 \mathrm{e}^{-4t} \cos(3t) + \mathrm{e}^{-4t} \sin(3t) c_2 + \cos(t) + 22 \sin(t)$

- Use initial condition $x(0) = -30$

$-30 = 1 + c_1$

- Compute derivative of the solution

$x' = -4 c_1 \mathrm{e}^{-4t} \cos(3t) - 3 c_1 \mathrm{e}^{-4t} \sin(3t) - 4\,\mathrm{e}^{-4t} \sin(3t) c_2 + 3\,\mathrm{e}^{-4t} \cos(3t) c_2 - \sin(t) + 22 \cos(t)$

- Use the initial condition $x'\Big|_{\{t=0\}} = -10$

$-10 = -4 c_1 + 3 c_2 + 22$

- Solve for c_1 and c_2

  $\{c_1 = -31, c_2 = -52\}$

- Substitute constant values into general solution and simplify

  $x = (-31\cos(3t) - 52\sin(3t))\,\mathrm{e}^{-4t} + \cos(t) + 22\sin(t)$

- Solution to the IVP

  $x = (-31\cos(3t) - 52\sin(3t))\,\mathrm{e}^{-4t} + \cos(t) + 22\sin(t)$

Maple trace

```
`Methods for second order ODEs:
--- Trying classification methods ---
trying a quadrature
trying high order exact linear fully integrable
trying differential order: 2; linear nonhomogeneous with symmetry [0,1]
trying a double symmetry of the form [xi=0, eta=F(x)]
-> Try solving first the homogeneous part of the ODE
   checking if the LODE has constant coefficients
   <- constant coefficients successful
<- solving first the homogeneous part of the ODE successful`
```

✓ Solution by Maple

Time used: 0.032 (sec). Leaf size: 31

```
dsolve([diff(x(t),t$2)+8*diff(x(t),t)+25*x(t)=200*cos(t)+520*sin(t),x(0) = -30, D(x)(0) = -1
```

$$x(t) = (-31\cos(3t) - 52\sin(3t))\,\mathrm{e}^{-4t} + 22\sin(t) + \cos(t)$$

✓ Solution by Mathematica

Time used: 0.025 (sec). Leaf size: 34

```
DSolve[{x''[t]+8*x'[t]+25*x[t]==200*Cos[t]+520*Sin[t],{x[0]==-30,x'[0]==-10}},x[t],t,Include
```

$$x(t) \to 22\sin(t) - 52e^{-4t}\sin(3t) + \cos(t) - 31e^{-4t}\cos(3t)$$

13 Section 7.2, Matrices and Linear systems. Page 417

13.1 problem problem 3

Internal problem ID [272]
Internal file name [`OUTPUT/272_Sunday_June_05_2022_01_37_56_AM_33254347/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 417
Problem number: problem 3.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= -3y(t) \\
y'(t) &= 3x
\end{aligned}
$$

13.1.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
e^{At} = \begin{bmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix}
$$

Therefore the homogeneous solution is

$$\begin{aligned}\vec{x}_h(t) &= e^{At}\vec{c} \\ &= \begin{bmatrix} \cos(3t) & -\sin(3t) \\ \sin(3t) & \cos(3t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\ &= \begin{bmatrix} \cos(3t)\, c_1 - \sin(3t)\, c_2 \\ \sin(3t)\, c_1 + \cos(3t)\, c_2 \end{bmatrix}\end{aligned}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

13.1.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & -3 \\ 3 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^2 + 9 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 3i$$
$$\lambda_2 = -3i$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-3i$	1	complex eigenvalue
$3i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} - (-3i)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 3i & -3 \\ 3 & 3i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 3i & -3 & 0 \\ 3 & 3i & 0 \end{array}\right]$$

$$R_2 = iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} 3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 3i & -3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 3i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} - (3i) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array}\right]$$

$$R_2 = -iR_1 + R_2 \Longrightarrow \left[\begin{array}{cc|c} -3i & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -3i & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it\}$

Hence the solution is

$$\begin{bmatrix} It \\ t \end{bmatrix} = \begin{bmatrix} it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} It \\ t \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} It \\ t \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$3i$	1	1	No	$\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-3i$	1	1	No	$\begin{bmatrix} -i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} i e^{3it} \\ e^{3it} \end{bmatrix} + c_2 \begin{bmatrix} -i e^{-3it} \\ e^{-3it} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -i(c_2 e^{-3it} - c_1 e^{3it}) \\ c_1 e^{3it} + c_2 e^{-3it} \end{bmatrix}$$

The following is the phase plot of the system.

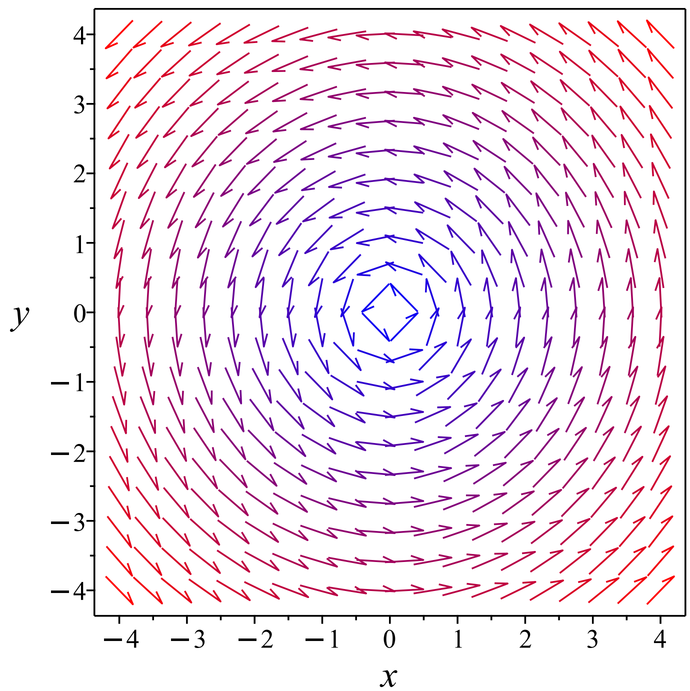


Figure 696: Phase plot

13.1.3 Maple step by step solution

Let's solve

$[x' = -3y(t), y'(t) = 3x]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\vec{x}'(t) = A \cdot \vec{x}(t)$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-3\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right], \left[3\,\mathrm{I}, \begin{bmatrix} \mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-3\,\mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-3\,\mathrm{I}t} \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$(\cos(3t) - \mathrm{I}\sin(3t)) \cdot \begin{bmatrix} -\mathrm{I} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\begin{bmatrix} -\mathrm{I}(\cos(3t) - \mathrm{I}\sin(3t)) \\ \cos(3t) - \mathrm{I}\sin(3t) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_1(t) = \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix}, \vec{x}_2(t) = \begin{bmatrix} -\cos(3t) \\ -\sin(3t) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\vec{x} = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)$

- Substitute solutions into the general solution

  $$\vec{x} = \begin{bmatrix} -c_2\cos(3t) - c_1\sin(3t) \\ -c_2\sin(3t) + c_1\cos(3t) \end{bmatrix}$$

- Substitute in vector of dependent variables

  $$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -c_2\cos(3t) - c_1\sin(3t) \\ -c_2\sin(3t) + c_1\cos(3t) \end{bmatrix}$$

- Solution to the system of ODEs

  $\{x = -c_2\cos(3t) - c_1\sin(3t), y(t) = -c_2\sin(3t) + c_1\cos(3t)\}$

✓ Solution by Maple

Time used: 0.063 (sec). Leaf size: 35

```
dsolve([diff(x(t),t)=-3*y(t),diff(y(t),t)=3*x(t)],singsol=all)
```

$$x(t) = c_1\sin(3t) + c_2\cos(3t)$$
$$y(t) = -c_1\cos(3t) + c_2\sin(3t)$$

✓ Solution by Mathematica

Time used: 0.017 (sec). Leaf size: 68

```
DSolve[{x'[t]==3*y[t],y'[t]==3*x[t]},{x[t],y[t]},t,IncludeSingularSolutions -> True]
```

$$x(t) \to \frac{1}{2}e^{-3t}\left(c_1\left(e^{6t}+1\right) + c_2\left(e^{6t}-1\right)\right)$$
$$y(t) \to \frac{1}{2}e^{-3t}\left(c_1\left(e^{6t}-1\right) + c_2\left(e^{6t}+1\right)\right)$$

13.2 problem problem 4

Internal problem ID [273]
Internal file name [`OUTPUT/273_Sunday_June_05_2022_01_37_57_AM_31748718/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 417
Problem number: problem 4.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 3x - 2y(t) \\
y'(t) &= 2x + y(t)
\end{aligned}
$$

13.2.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

For the above matrix A, the matrix exponential can be found to be

$$
\begin{aligned}
e^{At} &= \begin{bmatrix} e^{2t}\cos\left(\sqrt{3}\,t\right) + \frac{e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & -\frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \\ \frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & e^{2t}\cos\left(\sqrt{3}\,t\right) - \frac{e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}+3\cos\left(\sqrt{3}\,t\right)\right)}{3} & -\frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \\ \frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & -\frac{e^{2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)}{3} \end{bmatrix}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{e^{2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}+3\cos\left(\sqrt{3}\,t\right)\right)}{3} & -\frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} \\ \frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{3} & -\frac{e^{2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{e^{2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}+3\cos\left(\sqrt{3}\,t\right)\right)c_1}{3} - \frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}\,c_2}{3} \\ \frac{2\,e^{2t}\sin\left(\sqrt{3}\,t\right)\sqrt{3}\,c_1}{3} - \frac{e^{2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}-3\cos\left(\sqrt{3}\,t\right)\right)c_2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{\left(\sqrt{3}\,(c_1-2c_2)\sin\left(\sqrt{3}\,t\right)+3\cos\left(\sqrt{3}\,t\right)c_1\right)e^{2t}}{3} \\ \frac{2\left(\sqrt{3}\,(c_1-\frac{c_2}{2})\sin\left(\sqrt{3}\,t\right)+\frac{3\cos\left(\sqrt{3}\,t\right)c_2}{2}\right)e^{2t}}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

13.2.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$
\vec{x}'(t) = A\,\vec{x}(t)
$$

Or

$$
\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix}
$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$
\det\left(A - \lambda I\right) = 0
$$

Expanding gives

$$
\det\left(\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0
$$

Therefore

$$
\det\left(\begin{bmatrix} 3-\lambda & -2 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0
$$

Which gives the characteristic equation

$$\lambda^2 - 4\lambda + 7 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = 2 + i\sqrt{3}$$
$$\lambda_2 = 2 - i\sqrt{3}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$2 - i\sqrt{3}$	1	complex eigenvalue
$2 + i\sqrt{3}$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 2 - i\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} - \left(2 - i\sqrt{3}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} 1 + i\sqrt{3} & -2 \\ 2 & i\sqrt{3} - 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 + i\sqrt{3} & -2 & 0 \\ 2 & i\sqrt{3} - 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{1 + i\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 1 + i\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 + i\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{2t}{1+i\sqrt{3}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2t}{1+i\sqrt{3}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{2t}{1+\mathrm{I}\sqrt{3}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{2}{1+i\sqrt{3}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2 + i\sqrt{3}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} - \left(2 + i\sqrt{3}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 - i\sqrt{3} & -2 \\ 2 & -i\sqrt{3} - 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} 1 - i\sqrt{3} & -2 & 0 \\ 2 & -i\sqrt{3} - 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{2R_1}{1 - i\sqrt{3}} \Longrightarrow \left[\begin{array}{cc|c} 1 - i\sqrt{3} & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i\sqrt{3} & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{2t}{i\sqrt{3}-1}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2t}{i\sqrt{3}-1} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{2t}{\mathrm{I}\sqrt{3}-1} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$2+i\sqrt{3}$	1	1	No	$\left[\begin{array}{c} -\frac{2}{i\sqrt{3}-1} \\ 1 \end{array}\right]$
$2-i\sqrt{3}$	1	1	No	$\left[\begin{array}{c} -\frac{2}{-i\sqrt{3}-1} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Therefore the final solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\left[\begin{array}{c} x \\ y(t) \end{array}\right] = c_1 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{\left(2+i\sqrt{3}\right)t}}{i\sqrt{3}-1} \\ \mathrm{e}^{\left(2+i\sqrt{3}\right)t} \end{array}\right] + c_2 \left[\begin{array}{c} -\frac{2\,\mathrm{e}^{\left(2-i\sqrt{3}\right)t}}{-i\sqrt{3}-1} \\ \mathrm{e}^{\left(2-i\sqrt{3}\right)t} \end{array}\right]
$$

Which becomes

$$
\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{i\left(\sqrt{3}+i\right)c_2\,\mathrm{e}^{-\left(i\sqrt{3}-2\right)t}}{2} - \frac{i\left(i-\sqrt{3}\right)c_1\,\mathrm{e}^{\left(2+i\sqrt{3}\right)t}}{2} \\ c_1\mathrm{e}^{\left(2+i\sqrt{3}\right)t} + c_2\mathrm{e}^{-\left(i\sqrt{3}-2\right)t} \end{array}\right]
$$

The following is the phase plot of the system.

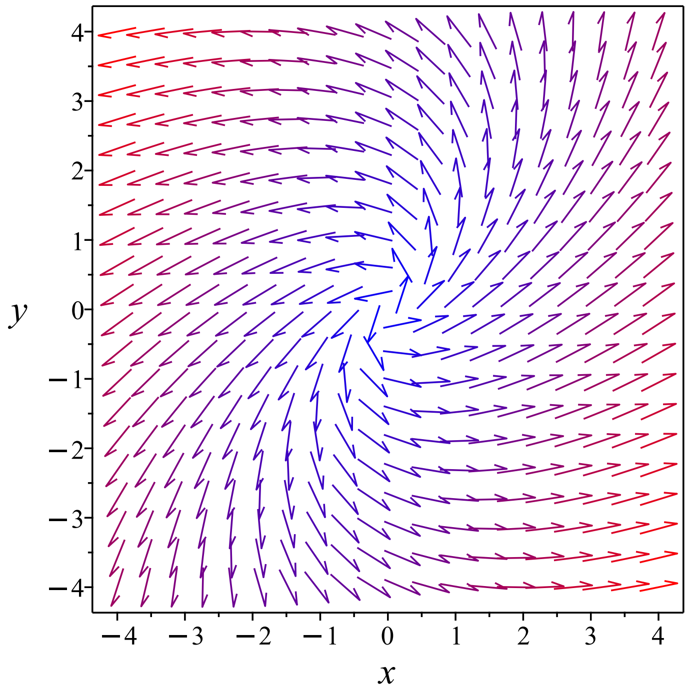


Figure 697: Phase plot

13.2.3 Maple step by step solution

Let's solve

$[x' = 3x - 2y(t), y'(t) = 2x + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$$

- Rewrite the system as

  $\overrightarrow{x}'(t) = A \cdot \overrightarrow{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[2 - \mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right], \left[2 + \mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{2}{\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[2 - \mathrm{I}\sqrt{3}, \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$\mathrm{e}^{\left(2-\mathrm{I}\sqrt{3}\right)t} \cdot \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}$$

- Use Euler identity to write solution in terms of $\sin$ and $\cos$

$$\mathrm{e}^{2t} \cdot \left(\cos\left(\sqrt{3}\,t\right) - \mathrm{I}\sin\left(\sqrt{3}\,t\right)\right) \cdot \begin{bmatrix} -\frac{2}{-\mathrm{I}\sqrt{3}-1} \\ 1 \end{bmatrix}$$

- Simplify expression

$$\mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{2\left(\cos\left(\sqrt{3}\,t\right)-\mathrm{I}\sin\left(\sqrt{3}\,t\right)\right)}{-\mathrm{I}\sqrt{3}-1} \\ \cos\left(\sqrt{3}\,t\right) - \mathrm{I}\sin\left(\sqrt{3}\,t\right) \end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\overrightarrow{x}_1(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} \frac{\cos\left(\sqrt{3}\,t\right)}{2} - \frac{\sin\left(\sqrt{3}\,t\right)\sqrt{3}}{2} \\ \cos\left(\sqrt{3}\,t\right) \end{bmatrix}, \overrightarrow{x}_2(t) = \mathrm{e}^{2t} \cdot \begin{bmatrix} -\frac{\cos\left(\sqrt{3}\,t\right)\sqrt{3}}{2} - \frac{\sin\left(\sqrt{3}\,t\right)}{2} \\ -\sin\left(\sqrt{3}\,t\right) \end{bmatrix}\right]$$

- General solution to the system of ODEs

  $\overrightarrow{x} = c_1\overrightarrow{x}_1(t) + c_2\overrightarrow{x}_2(t)$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{2t} \cdot \begin{bmatrix} \frac{\cos(\sqrt{3}\,t)}{2} - \frac{\sin(\sqrt{3}\,t)\sqrt{3}}{2} \\ \cos\left(\sqrt{3}\,t\right) \end{bmatrix} + c_2 e^{2t} \cdot \begin{bmatrix} -\frac{\cos(\sqrt{3}\,t)\sqrt{3}}{2} - \frac{\sin(\sqrt{3}\,t)}{2} \\ -\sin\left(\sqrt{3}\,t\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \end{bmatrix} = \begin{bmatrix} -\frac{\left(\left(\sqrt{3}\,c_2 - c_1\right)\cos\left(\sqrt{3}\,t\right) + \sin\left(\sqrt{3}\,t\right)\left(\sqrt{3}\,c_1 + c_2\right)\right)e^{2t}}{2} \\ e^{2t}\left(\cos\left(\sqrt{3}\,t\right)c_1 - \sin\left(\sqrt{3}\,t\right)c_2\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x = -\frac{\left(\left(\sqrt{3}\,c_2 - c_1\right)\cos\left(\sqrt{3}\,t\right) + \sin\left(\sqrt{3}\,t\right)\left(\sqrt{3}\,c_1 + c_2\right)\right)e^{2t}}{2}, y(t) = e^{2t}\left(\cos\left(\sqrt{3}\,t\right)c_1 - \sin\left(\sqrt{3}\,t\right)c_2\right)\right\}$$

✓ Solution by Maple

Time used: 0.031 (sec). Leaf size: 76

```
dsolve([diff(x(t),t)=3*x(t)-2*y(t),diff(y(t),t)=2*x(t)+y(t)],singsol=all)
```

$$x(t) = e^{2t}\left(\sin\left(\sqrt{3}\,t\right)c_1 + \cos\left(\sqrt{3}\,t\right)c_2\right)$$

$$y(t) = \frac{e^{2t}\left(\sin\left(\sqrt{3}\,t\right)\sqrt{3}\,c_2 - \cos\left(\sqrt{3}\,t\right)\sqrt{3}\,c_1 + \sin\left(\sqrt{3}\,t\right)c_1 + \cos\left(\sqrt{3}\,t\right)c_2\right)}{2}$$

✓ Solution by Mathematica

Time used: 0.015 (sec). Leaf size: 96

```
DSolve[{x'[t]==3*x[t]-2*y[t],y'[t]==2*x[t]+y[t]},{x[t],y[t]},t,IncludeSingularSolutions -> T
```

$$x(t) \to \frac{1}{3}e^{2t}\left(3c_1\cos\left(\sqrt{3}t\right) + \sqrt{3}(c_1 - 2c_2)\sin\left(\sqrt{3}t\right)\right)$$

$$y(t) \to \frac{1}{3}e^{2t}\left(3c_2\cos\left(\sqrt{3}t\right) + \sqrt{3}(2c_1 - c_2)\sin\left(\sqrt{3}t\right)\right)$$

13.3 problem problem 5

Internal problem ID [274]
Internal file name [`OUTPUT/274_Sunday_June_05_2022_01_37_59_AM_8563329/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 417
Problem number: problem 5.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= 2x + 4y(t) + 3\,\mathrm{e}^{t} \\
y'(t) &= 5x - y(t) - t^{2}
\end{aligned}
$$

13.3.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 3\,\mathrm{e}^{t} \\ -t^{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{cc} \frac{\left(-3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{e^{\frac{\left(1+\sqrt{89}\right) t}{2}} \left(3\sqrt{89}+89\right)}{178} & -\frac{4 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}}{89} \\ -\frac{5 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}}{89} & \frac{\left(3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{\left(-3\sqrt{89}+89\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}}{178} \end{array}\right]$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{cc} \frac{\left(-3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{e^{\frac{\left(1+\sqrt{89}\right) t}{2}} \left(3\sqrt{89}+89\right)}{178} & -\frac{4 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}}{89} \\ -\frac{5 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}}{89} & \frac{\left(3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{\left(-3\sqrt{89}+89\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}}{178} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \left(\frac{\left(-3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{e^{\frac{\left(1+\sqrt{89}\right) t}{2}} \left(3\sqrt{89}+89\right)}{178}\right) c_1 - \frac{4 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}\, c_2}{89} \\ -\frac{5 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}\, c_1}{89} + \left(\frac{\left(3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{\left(-3\sqrt{89}+89\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}}{178}\right) c_2 \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\left(\left(-3c_1-8c_2\right)\sqrt{89}+89c_1\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{3\left(\left(c_1+\frac{8c_2}{3}\right)\sqrt{89}+\frac{89c_1}{3}\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}}{178} \\ \frac{\left(\left(-10c_1+3c_2\right)\sqrt{89}+89c_2\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{5\left(\left(c_1-\frac{3c_2}{10}\right)\sqrt{89}+\frac{89c_2}{10}\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}}{89} \end{array}\right]
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At} \int e^{-At}\vec{G}(t)\, dt$$

But

$$e^{-At} = (e^{At})^{-1}$$

$$= \left[\begin{array}{cc} \frac{e^{-t}\left(3\sqrt{89}\, e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}} - 3\sqrt{89}\, e^{\frac{\left(1+\sqrt{89}\right)t}{2}} + 89\, e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}} + 89\, e^{\frac{\left(1+\sqrt{89}\right)t}{2}}\right)}{178} & \frac{4\sqrt{89}\, e^{-t}\left(-e^{\frac{\left(1+\sqrt{89}\right)t}{2}} + e^{-\frac{\left(-1+\sqrt{\cdots}\right)t}{2}} \cdots\right.}{89} \\ \frac{5\sqrt{89}\, e^{-t}\left(-e^{\frac{\left(1+\sqrt{89}\right)t}{2}} + e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}\right)}{89} & \frac{e^{-t}\left(\left(-3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}} + 3\, e^{\frac{\left(1+\sqrt{89}\right)t}{2}} \cdots\right.}{178} \end{array}\right.$$

Hence

$$\vec{x}_p(t) = \left[\begin{array}{cc} \frac{\left(-3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} + \frac{e^{\frac{\left(1+\sqrt{89}\right)t}{2}}\left(3\sqrt{89}+89\right)}{178} & -\frac{4\left(-e^{\frac{\left(1+\sqrt{89}\right)t}{2}} + e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}\right)\sqrt{89}}{89} \\ -\frac{5\left(-e^{\frac{\left(1+\sqrt{89}\right)t}{2}} + e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}\right)\sqrt{89}}{89} & \frac{\left(3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} + \frac{\left(-3\sqrt{89}+89\right) e^{\frac{\left(1+\sqrt{89}\right)t}{2}}}{178} \end{array}\right] \int \left[\begin{array}{c} e \cdots \\ \cdots \end{array}\right.$$

$$= \left[\begin{array}{cc} \frac{\left(-3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} + \frac{e^{\frac{\left(1+\sqrt{89}\right)t}{2}}\left(3\sqrt{89}+89\right)}{178} & -\frac{4\left(-e^{\frac{\left(1+\sqrt{89}\right)t}{2}} + e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}\right)\sqrt{89}}{89} \\ -\frac{5\left(-e^{\frac{\left(1+\sqrt{89}\right)t}{2}} + e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}\right)\sqrt{89}}{89} & \frac{\left(3\sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} + \frac{\left(-3\sqrt{89}+89\right) e^{\frac{\left(1+\sqrt{89}\right)t}{2}}}{178} \end{array}\right] \left[\begin{array}{c} \frac{((6 \cdots}{\cdots} \\ \frac{((-7 \cdots}{\cdots} \end{array}\right.$$

$$= \left[\begin{array}{c} \frac{2t^2}{11} - \frac{3\, e^t}{11} - \frac{2t}{121} + \frac{23}{1331} \\ -\frac{t^2}{11} - \frac{15\, e^t}{22} + \frac{12t}{121} - \frac{17}{1331} \end{array}\right]$$

Hence the complete solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$= \left[\begin{array}{c} \frac{23}{1331} + \frac{\left((-3c_1-8c_2)\sqrt{89}+89c_1\right) e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} + \frac{\left((3c_1+8c_2)\sqrt{89}+89c_1\right) e^{\frac{\left(1+\sqrt{89}\right)t}{2}}}{178} + \frac{2t^2}{11} - \frac{2t}{121} - \frac{3\, e^t}{11} \\ -\frac{17}{1331} + \frac{\left((-10c_1+3c_2)\sqrt{89}+89c_2\right) e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} + \frac{\left((10c_1-3c_2)\sqrt{89}+89c_2\right) e^{\frac{\left(1+\sqrt{89}\right)t}{2}}}{178} - \frac{t^2}{11} + \frac{12t}{121} - \frac{15\, e^t}{22} \end{array}\right]$$

13.3.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y(t) \end{bmatrix} + \begin{bmatrix} 3\,\mathrm{e}^{t} \\ -t^{2} \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det\left(A - \lambda I\right) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} 2-\lambda & 4 \\ 5 & -1-\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^{2} - \lambda - 22 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \frac{1}{2} + \frac{\sqrt{89}}{2}$$
$$\lambda_2 = \frac{1}{2} - \frac{\sqrt{89}}{2}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$\frac{1}{2}+\frac{\sqrt{89}}{2}$	1	real eigenvalue
$\frac{1}{2}-\frac{\sqrt{89}}{2}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \frac{1}{2} - \frac{\sqrt{89}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} - \left(\frac{1}{2} - \frac{\sqrt{89}}{2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2}+\frac{\sqrt{89}}{2} & 4 \\ 5 & -\frac{3}{2}+\frac{\sqrt{89}}{2} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2}+\frac{\sqrt{89}}{2} & 4 & 0 \\ 5 & -\frac{3}{2}+\frac{\sqrt{89}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{\frac{3}{2}+\frac{\sqrt{89}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2}+\frac{\sqrt{89}}{2} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2}+\frac{\sqrt{89}}{2} & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{8t}{3+\sqrt{89}}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{8t}{3+\sqrt{89}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8t}{3+\sqrt{89}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{8t}{3+\sqrt{89}} \\ t \end{bmatrix} = t\begin{bmatrix} -\frac{8}{3+\sqrt{89}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{8t}{3+\sqrt{89}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8}{3+\sqrt{89}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{8t}{3+\sqrt{89}} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{8}{3+\sqrt{89}} \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = \frac{1}{2} + \frac{\sqrt{89}}{2}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} - \left(\frac{1}{2} + \frac{\sqrt{89}}{2} \right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} - \frac{\sqrt{89}}{2} & 4 \\ 5 & -\frac{3}{2} - \frac{\sqrt{89}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cc|c} \frac{3}{2} - \frac{\sqrt{89}}{2} & 4 & 0 \\ 5 & -\frac{3}{2} - \frac{\sqrt{89}}{2} & 0 \end{array}\right]$$

$$R_2 = R_2 - \frac{5R_1}{\frac{3}{2} - \frac{\sqrt{89}}{2}} \Longrightarrow \left[\begin{array}{cc|c} \frac{3}{2} - \frac{\sqrt{89}}{2} & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} \frac{3}{2} - \frac{\sqrt{89}}{2} & 4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{8t}{-3+\sqrt{89}}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{8t}{-3+\sqrt{89}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8t}{-3+\sqrt{89}} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{8t}{-3+\sqrt{89}} \\ t \end{bmatrix} = t \begin{bmatrix} \frac{8}{-3+\sqrt{89}} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{8t}{-3+\sqrt{89}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8}{-3+\sqrt{89}} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{8t}{-3+\sqrt{89}} \\ t \end{bmatrix} = \begin{bmatrix} \frac{8}{-3+\sqrt{89}} \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
$\frac{1}{2} + \frac{\sqrt{89}}{2}$	1	1	No	$\begin{bmatrix} \frac{4}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ 1 \end{bmatrix}$
$\frac{1}{2} - \frac{\sqrt{89}}{2}$	1	1	No	$\begin{bmatrix} \frac{4}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\frac{1}{2} + \frac{\sqrt{89}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} \\
&= \begin{bmatrix} \frac{4}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ 1 \end{bmatrix} e^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}
\end{aligned}$$

Since eigenvalue $\frac{1}{2}-\frac{\sqrt{89}}{2}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \\
&= \begin{bmatrix} \frac{4}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ 1 \end{bmatrix} e^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}
\end{aligned}
$$

Therefore the homogeneous solution is

$$
\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t)
$$

Which is written as

$$
\begin{bmatrix} x \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{4\,\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} \end{bmatrix} + c_2 \begin{bmatrix} \frac{4\,\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \end{bmatrix}
$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$
\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}
$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$
\Phi(t) = \begin{bmatrix} \frac{4\,\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} & \frac{4\,\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \end{bmatrix}
$$

The particular solution is then given by

$$
\vec{x}_p(t) = \Phi \int \Phi^{-1}\vec{G}(t)\,dt
$$

But

$$
\Phi^{-1} = \begin{bmatrix} \frac{5\sqrt{89}\,\mathrm{e}^{-\frac{\left(1+\sqrt{89}\right)t}{2}}}{89} & \frac{\sqrt{89}\left(-3+\sqrt{89}\right)\mathrm{e}^{-\frac{\left(1+\sqrt{89}\right)t}{2}}}{178} \\ -\frac{5\sqrt{89}\,\mathrm{e}^{\frac{\left(-1+\sqrt{89}\right)t}{2}}}{89} & \frac{\sqrt{89}\left(3+\sqrt{89}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} \end{bmatrix}
$$

Hence

$$\vec{x}_p(t) = \left[\begin{array}{cc} \frac{4\,\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} & \frac{4\,\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{cc} \frac{5\sqrt{89}\,\mathrm{e}^{-\frac{\left(1+\sqrt{89}\right)t}{2}}}{89} & \frac{\sqrt{89}\left(-3+\sqrt{89}\right)\mathrm{e}^{-\frac{\left(1+\sqrt{89}\right)t}{2}}}{178} \\ -\frac{5\sqrt{89}\,\mathrm{e}^{\frac{\left(-1+\sqrt{89}\right)t}{2}}}{89} & \frac{\sqrt{89}\left(3+\sqrt{89}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} \end{array}\right] \left[\begin{array}{c} 3\,\mathrm{e}^{t} \\ -t^{2} \end{array}\right] dt$$

$$= \left[\begin{array}{cc} \frac{4\,\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} & \frac{4\,\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \end{array}\right] \int \left[\begin{array}{c} \frac{3\left(\sqrt{89}-\frac{89}{3}\right)t^{2}\mathrm{e}^{-\frac{\left(1+\sqrt{89}\right)t}{2}}}{178} + \frac{15\sqrt{89}\,\mathrm{e}^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{89} \\ -\frac{3t^{2}\left(\sqrt{89}+\frac{89}{3}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{89}\right)t}{2}}}{178} - \frac{15\sqrt{89}\,\mathrm{e}^{\frac{\left(1+\sqrt{89}\right)t}{2}}}{89} \end{array}\right] dt$$

$$= \left[\begin{array}{cc} \frac{4\,\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} & \frac{4\,\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \end{array}\right] \left[\begin{array}{c} -\frac{176\sqrt{89}\left(\left(\left(t^{2}-2t+\frac{2}{11}\right)\sqrt{89}-43t^{2}+6t-\frac{46}{11}\right)\mathrm{e}^{-\frac{\left(1+\sqrt{89}\right)t}{2}}+\frac{345\left(\sqrt{89}+\frac{67}{23}\right)\mathrm{e}^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{22}\right)}{89\left(-1+\sqrt{89}\right)\left(1+\sqrt{89}\right)^{3}} \\ -\frac{176\left(\left(\left(t^{2}-2t+\frac{2}{11}\right)\sqrt{89}+43t^{2}-6t+\frac{46}{11}\right)\mathrm{e}^{\frac{\left(-1+\sqrt{89}\right)t}{2}}+\frac{345\,\mathrm{e}^{\frac{\left(1+\sqrt{89}\right)t}{2}}\left(\sqrt{89}-\frac{67}{23}\right)}{22}\right)\sqrt{89}}{89\left(1+\sqrt{89}\right)\left(-1+\sqrt{89}\right)^{3}} \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{2t^{2}}{11} - \frac{3\,\mathrm{e}^{t}}{11} - \frac{2t}{121} + \frac{23}{1331} \\ -\frac{t^{2}}{11} - \frac{15\,\mathrm{e}^{t}}{22} + \frac{12t}{121} - \frac{17}{1331} \end{array}\right]$$

Now that we found particular solution, the final solution is

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{4c_1\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ c_1\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{4c_2\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ c_2\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \end{array}\right] + \left[\begin{array}{c} \frac{2t^{2}}{11} - \frac{3\,\mathrm{e}^{t}}{11} - \frac{2t}{121} + \frac{23}{1331} \\ -\frac{t^{2}}{11} - \frac{15\,\mathrm{e}^{t}}{22} + \frac{12t}{121} - \frac{17}{1331} \end{array}\right]$$

Which becomes

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} -\frac{c_2\left(-3+\sqrt{89}\right)\mathrm{e}^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}}{10} + \frac{c_1\left(3+\sqrt{89}\right)\mathrm{e}^{\frac{\left(1+\sqrt{89}\right)t}{2}}}{10} + \frac{2t^{2}}{11} - \frac{2t}{121} - \frac{3\,\mathrm{e}^{t}}{11} + \frac{23}{1331} \\ c_1\mathrm{e}^{\frac{\left(1+\sqrt{89}\right)t}{2}} + c_2\mathrm{e}^{-\frac{\left(-1+\sqrt{89}\right)t}{2}} - \frac{t^{2}}{11} - \frac{15\,\mathrm{e}^{t}}{22} + \frac{12t}{121} - \frac{17}{1331} \end{array}\right]$$

13.3.3 Maple step by step solution

Let's solve

$[x' = 2x + 4y(t) + 3\,\mathrm{e}^{t}, y'(t) = 5x - y(t) - t^2]$

- Define vector

$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \end{bmatrix}$

- Convert system into a vector equation

$\vec{x}'(t) = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3\,\mathrm{e}^{t} \\ -t^2 \end{bmatrix}$

- System to solve

$\vec{x}'(t) = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 3\,\mathrm{e}^{t} \\ -t^2 \end{bmatrix}$

- Define the forcing function

$\vec{f}(t) = \begin{bmatrix} 3\,\mathrm{e}^{t} \\ -t^2 \end{bmatrix}$

- Define the coefficient matrix

$A = \begin{bmatrix} 2 & 4 \\ 5 & -1 \end{bmatrix}$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t) + \vec{f}$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$\left[\left[\frac{1}{2} - \frac{\sqrt{89}}{2}, \begin{bmatrix} \frac{4}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ 1 \end{bmatrix}\right], \left[\frac{1}{2} + \frac{\sqrt{89}}{2}, \begin{bmatrix} \frac{4}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ 1 \end{bmatrix}\right]\right]$

- Consider eigenpair

$$\left[\frac{1}{2}-\frac{\sqrt{89}}{2}, \left[\begin{array}{c}\frac{4}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_1 = \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} \cdot \left[\begin{array}{c}\frac{4}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} \\ 1\end{array}\right]$$

- Consider eigenpair

$$\left[\frac{1}{2}+\frac{\sqrt{89}}{2}, \left[\begin{array}{c}\frac{4}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\overrightarrow{x}_2 = \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t} \cdot \left[\begin{array}{c}\frac{4}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ 1\end{array}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\overrightarrow{x}_p($

$\overrightarrow{x}(t) = c_1\overrightarrow{x}_1 + c_2\overrightarrow{x}_2 + \overrightarrow{x}_p(t)$

☐ Fundamental matrix

◦ Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syste

$$\phi(t) = \left[\begin{array}{cc}\frac{4\,\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} & \frac{4\,\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}\end{array}\right]$$

◦ The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0) = I$ where I is th

$\Phi(t) = \phi(t) \cdot \frac{1}{\phi(0)}$

◦ Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t) = \left[\begin{array}{cc}\frac{4\,\mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} & \frac{4\,\mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ \mathrm{e}^{\left(\frac{1}{2}-\frac{\sqrt{89}}{2}\right)t} & \mathrm{e}^{\left(\frac{1}{2}+\frac{\sqrt{89}}{2}\right)t}\end{array}\right] \cdot \frac{1}{\left[\begin{array}{cc}\frac{4}{-\frac{3}{2}-\frac{\sqrt{89}}{2}} & \frac{4}{-\frac{3}{2}+\frac{\sqrt{89}}{2}} \\ 1 & 1\end{array}\right]}$$

◦ Evaluate and simplify to get the fundamental matrix

$$\Phi(t) = \left[\begin{array}{cc} \frac{\left(\left(-3+\sqrt{89}\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}+e^{\frac{\left(1+\sqrt{89}\right) t}{2}} \left(3+\sqrt{89}\right)\right) \sqrt{89}}{178} & -\frac{4 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}}{89} \\ -\frac{5 \left(-e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}\right) \sqrt{89}}{89} & \frac{\left(3 \sqrt{89}+89\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}}{178}+\frac{\left(-3 \sqrt{89}+89\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}}{178} \end{array}\right]$$

☐ Find a particular solution of the system of ODEs using variation of parameters

- Let the particular solution be the fundamental matrix multiplied by $\overrightarrow{v}(t)$ and solve for $\overrightarrow{v}(t)$

  $\overrightarrow{x}_{p}(t) = \Phi(t) \cdot \overrightarrow{v}(t)$
- Take the derivative of the particular solution

  $\overrightarrow{x}_{p}^{\prime}(t) = \Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t)$
- Substitute particular solution and its derivative into the system of ODEs

  $\Phi'(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- The fundamental matrix has columns that are solutions to the homogeneous system so its der

  $A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = A \cdot \Phi(t) \cdot \overrightarrow{v}(t) + \overrightarrow{f}(t)$
- Cancel like terms

  $\Phi(t) \cdot \overrightarrow{v}^{\prime}(t) = \overrightarrow{f}(t)$
- Multiply by the inverse of the fundamental matrix

  $\overrightarrow{v}^{\prime}(t) = \frac{1}{\Phi(t)} \cdot \overrightarrow{f}(t)$
- Integrate to solve for $\overrightarrow{v}(t)$

  $\overrightarrow{v}(t) = \int_{0}^{t} \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds$
- Plug $\overrightarrow{v}(t)$ into the equation for the particular solution

  $\overrightarrow{x}_{p}(t) = \Phi(t) \cdot \left(\int_{0}^{t} \frac{1}{\Phi(s)} \cdot \overrightarrow{f}(s)\, ds\right)$
- Plug in the fundamental matrix and the forcing function and compute

$$\overrightarrow{x}_{p}(t) = \left[\begin{array}{c} \frac{1024 \left(\left(7565-2104 \sqrt{89}\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}+\left(7565+2104 \sqrt{89}\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}+10769 t^{2}-979 t-\frac{32307 e^{t}}{2}+\frac{2047}{2}\right)}{89 \left(1+\sqrt{89}\right)^{3} \left(-1+\sqrt{89}\right)^{3}} \\ \frac{128 \left(-6052+\left(164561-1253 \sqrt{89}\right) e^{-\frac{\left(-1+\sqrt{89}\right) t}{2}}+\left(164561+1253 \sqrt{89}\right) e^{\frac{\left(1+\sqrt{89}\right) t}{2}}-43076 t^{2}+46992 t-323070 e^{t}\right)}{89 \left(1+\sqrt{89}\right)^{3} \left(-1+\sqrt{89}\right)^{3}} \end{array}\right]$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \left[\begin{array}{c} \frac{1024\left(\left(7565-2104\sqrt{89}\right)e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}+\left(7565+2104\sqrt{89}\right)e^{\frac{\left(1+\sqrt{89}\right)t}{2}}+10769t^2-979t-\frac{32307}{2}\right.}{89\left(1+\sqrt{89}\right)^3\left(-1+\sqrt{89}\right)^3} \\ \frac{128\left(-6052+\left(164561-1253\sqrt{89}\right)e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}+\left(164561+1253\sqrt{89}\right)e^{\frac{\left(1+\sqrt{89}\right)t}{2}}-43076t^2+4699\right.}{89\left(1+\sqrt{89}\right)^3\left(-1+\sqrt{89}\right)^3} \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x \\ y(t) \end{array}\right] = \left[\begin{array}{c} \frac{256\left(20470+\left(151300+(-42080-118459c_1)\sqrt{89}+355377c_1\right)e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}+\left(151300+(42080+118459c_2)\sqrt{89}+\right.\right.}{445\left(1+\sqrt{89}\right)^3\left(-1+\sqrt{89}\right)^3} \\ \frac{128\left(-6052+\left(164561+473836c_1-1253\sqrt{89}\right)e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}+\left(164561+473836c_2+1253\sqrt{89}\right)e^{\frac{(1}{}}\right.}{89\left(1+\sqrt{89}\right)^3\left(-1+\sqrt{89}\right)^3} \end{array}\right.$$

- Solution to the system of ODEs

$$\left\{ x = \frac{256\left(20470+\left(151300+(-42080-118459c_1)\sqrt{89}+355377c_1\right)e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}+\left(151300+(42080+118459c_2)\sqrt{89}+355377c_2\right)e\right.}{445\left(1+\sqrt{89}\right)^3\left(-1+\sqrt{89}\right)^3} \right.$$

✓ Solution by Maple

Time used: 0.094 (sec). Leaf size: 112

```
dsolve([diff(x(t),t)=2*x(t)+4*y(t)+3*exp(t),diff(y(t),t)=5*x(t)-y(t)-t^2],singsol=all)
```

$$x(t) = \frac{e^{\frac{\left(1+\sqrt{89}\right)t}{2}}c_2\sqrt{89}}{10} - \frac{e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}c_1\sqrt{89}}{10} + \frac{3\,e^{\frac{\left(1+\sqrt{89}\right)t}{2}}c_2}{10} + \frac{3\,e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}c_1}{10} + \frac{2t^2}{11} - \frac{3\,e^t}{11} - \frac{2t}{121} + \frac{23}{1331}$$

$$y(t) = e^{\frac{\left(1+\sqrt{89}\right)t}{2}}c_2 + e^{-\frac{\left(-1+\sqrt{89}\right)t}{2}}c_1 - \frac{t^2}{11} - \frac{15\,e^t}{22} + \frac{12t}{121} - \frac{17}{1331}$$

✓ Solution by Mathematica

Time used: 0.711 (sec). Leaf size: 212

```
DSolve[{x'[t]==2*x[t]+4*y[t]+3*Exp[t],y'[t]==5*x[t]-y[t]-t^2},{x[t],y[t]},t,IncludeSingularS
```

$$x(t) \to \frac{242t^2-22t+23}{1331} - \frac{3e^t}{11} + \frac{1}{178}\left(\left(89-3\sqrt{89}\right)c_1 - 8\sqrt{89}c_2\right)e^{-\frac{1}{2}\left(\sqrt{89}-1\right)t} + \frac{1}{178}\left(\left(89+3\sqrt{89}\right)c_1 + 8\sqrt{89}c_2\right)e^{\frac{1}{2}\left(1+\sqrt{89}\right)t}$$

$$y(t) \to \frac{-121t^2+132t-17}{1331} - \frac{15e^t}{22} + \left(\frac{5c_1}{\sqrt{89}} + \frac{1}{178}\left(89-3\sqrt{89}\right)c_2\right)e^{\frac{1}{2}\left(1+\sqrt{89}\right)t} + \left(\frac{1}{178}\left(89+3\sqrt{89}\right)c_2 - \frac{5c_1}{\sqrt{89}}\right)e^{-\frac{1}{2}\left(\sqrt{89}-1\right)t}$$

13.4 problem problem 7

Internal problem ID [275]
Internal file name [`OUTPUT/275_Sunday_June_05_2022_01_38_01_AM_55622945/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 417
Problem number: problem 7.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x' &= y(t) + z(t) \\
y'(t) &= z(t) + x \\
z'(t) &= x + y(t)
\end{aligned}
$$

13.4.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \\ z(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \begin{bmatrix} \frac{2\,e^{-t}}{3} + \frac{e^{2t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{2\,e^{-t}}{3} + \frac{e^{2t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} \\ \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{e^{2t}}{3} - \frac{e^{-t}}{3} & \frac{2\,e^{-t}}{3} + \frac{e^{2t}}{3} \end{bmatrix}$$

Therefore the homogeneous solution is

$$
\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \begin{bmatrix} \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} \\ \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3} & \frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \\
&= \begin{bmatrix} \left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_3 \\ \left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_1+\left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_2+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_3 \\ \left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_1+\left(\frac{\mathrm{e}^{2t}}{3}-\frac{\mathrm{e}^{-t}}{3}\right)c_2+\left(\frac{2\,\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}}{3}\right)c_3 \end{bmatrix} \\
&= \begin{bmatrix} \frac{(2c_1-c_2-c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_1+c_2+c_3)}{3} \\ \frac{(-c_1+2c_2-c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_1+c_2+c_3)}{3} \\ \frac{(-c_1-c_2+2c_3)\mathrm{e}^{-t}}{3}+\frac{\mathrm{e}^{2t}(c_1+c_2+c_3)}{3} \end{bmatrix}
\end{aligned}
$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

13.4.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x' \\ y'(t) \\ z'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y(t) \\ z(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^3 - 3\lambda - 2 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = -1$$
$$\lambda_2 = 2$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
-1	1	real eigenvalue
2	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = -1$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_2 = R_2 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

$$R_3 = R_3 - R_1 \Longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_2, v_3\}$ and the leading variables are $\{v_1\}$. Let $v_2 = t$. Let $v_3 = s$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t - s\}$

Hence the solution is

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix}$$

Since there are two free Variable, we have found two eigenvectors associated with this eigenvalue. The above can be written as

$$\begin{aligned} \begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} &= \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} -s \\ 0 \\ s \end{bmatrix} \\ &= t \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

By letting $t = 1$ and $s = 1$ then the above becomes

$$\begin{bmatrix} -t - s \\ t \\ s \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Hence the two eigenvectors associated with this eigenvalue are

$$\left(\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right)$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix} - (2)\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right]$$

$$R_2 = R_2 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c}-2&1&1&0\\0&-\frac{3}{2}&\frac{3}{2}&0\\1&1&-2&0\end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{ccc|c}-2&1&1&0\\0&-\frac{3}{2}&\frac{3}{2}&0\\0&\frac{3}{2}&-\frac{3}{2}&0\end{array}\right]$$

$$R_3 = R_3 + R_2 \Longrightarrow \left[\begin{array}{ccc|c}-2&1&1&0\\0&-\frac{3}{2}&\frac{3}{2}&0\\0&0&0&0\end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_3\}$ and the leading variables are $\{v_1, v_2\}$. Let $v_3 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity		defective?	eigenvectors
	algebraic m	geometric k		
-1	2	2	No	$\begin{bmatrix} -1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. eigenvalue -1 is real and repated eigenvalue of multiplicity 2.There are two possible cases that can happen. This is illustrated in this diagram

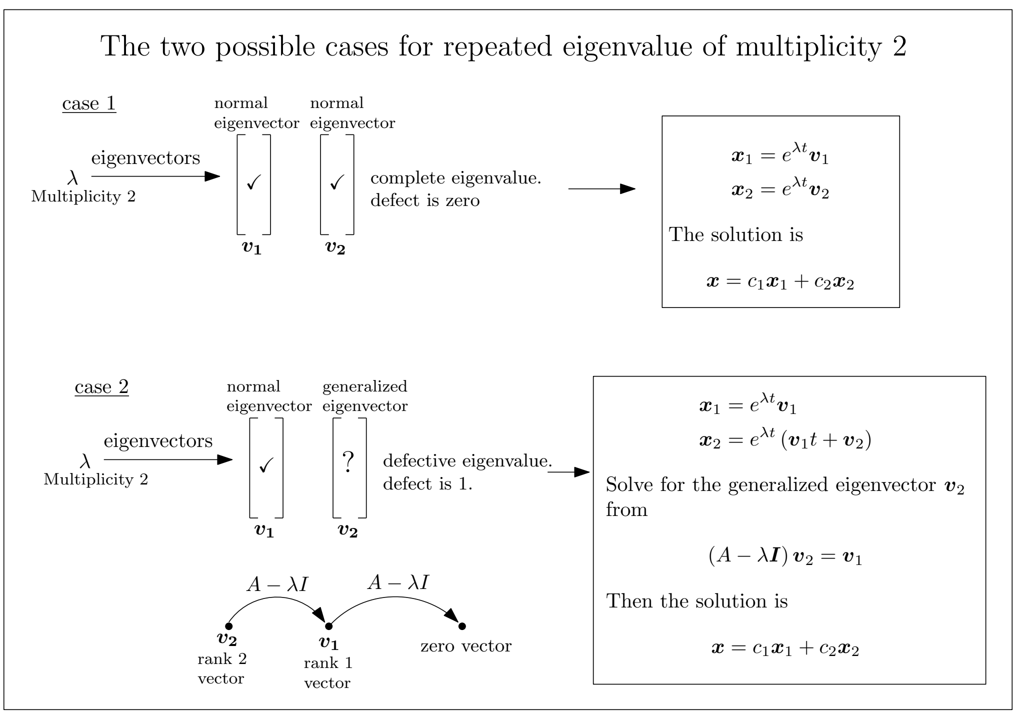


Figure 698: Possible case for repeated λ of multiplicity 2

This eigenvalue has algebraic multiplicity of 2 which is the same as its geometric multiplicity 2, then it is complete eigenvalue and this falls into case 1 shown above. Hence the corresponding eigenvector basis are

$$\vec{x}_1(t) = \vec{v}_1 e^{-t}$$
$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$\vec{x}_2(t) = \vec{v}_2 e^{-t}$$
$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_3(t) = \vec{v}_3 e^{2t}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the final solution is

$$\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

Which is written as

$$\begin{bmatrix} x \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} -\mathrm{e}^{-t} \\ 0 \\ \mathrm{e}^{-t} \end{bmatrix} + c_2 \begin{bmatrix} -\mathrm{e}^{-t} \\ \mathrm{e}^{-t} \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \\ \mathrm{e}^{2t} \end{bmatrix}$$

Which becomes

$$\begin{bmatrix} x \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} (-c_1 - c_2)\,\mathrm{e}^{-t} + c_3 \mathrm{e}^{2t} \\ c_2 \mathrm{e}^{-t} + c_3 \mathrm{e}^{2t} \\ c_1 \mathrm{e}^{-t} + c_3 \mathrm{e}^{2t} \end{bmatrix}$$

13.4.3 Maple step by step solution

Let's solve

$[x' = y(t) + z(t), y'(t) = z(t) + x, z'(t) = x + y(t)]$

- Define vector

$$\vec{x}(t) = \begin{bmatrix} x \\ y(t) \\ z(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\vec{x}'(t) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \vec{x}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Rewrite the system as

$\vec{x}'(t) = A \cdot \vec{x}(t)$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right], \left[-1, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair, with eigenvalue of algebraic multiplicity 2

$$\left[-1, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right]$$

- First solution from eigenvalue -1

$$\vec{x}_1(t) = e^{-t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Form of the 2nd homogeneous solution where $\vec{p}$ is to be solved for, $\lambda = -1$ is the eigenvalue, an

$$\vec{x}_2(t) = e^{\lambda t}\left(t\vec{v} + \vec{p}\right)$$

- Note that the t multiplying $\vec{v}$ makes this solution linearly independent to the 1st solution obtai
- Substitute $\vec{x}_2(t)$ into the homogeneous system

$$\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = \left(e^{\lambda t}A\right) \cdot \left(t\vec{v} + \vec{p}\right)$$

- Use the fact that $\vec{v}$ is an eigenvector of A

$$\lambda\, e^{\lambda t}\left(t\vec{v} + \vec{p}\right) + e^{\lambda t}\vec{v} = e^{\lambda t}\left(\lambda t\vec{v} + A \cdot \vec{p}\right)$$

- Simplify equation

$$\lambda\vec{p} + \vec{v} = A \cdot \vec{p}$$

- Make use of the identity matrix I

$$(\lambda \cdot I) \cdot \vec{p} + \vec{v} = A \cdot \vec{p}$$

- Condition $\vec{p}$ must meet for $\vec{x}_2(t)$ to be a solution to the homogeneous system

$$(A - \lambda \cdot I) \cdot \vec{p} = \vec{v}$$

- Choose $\vec{p}$ to use in the second solution to the homogeneous system from eigenvalue -1

$$\left(\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - (-1) \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \cdot \vec{p} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

- Choice of $\vec{p}$

$$\vec{p} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

- Second solution from eigenvalue -1

$$\vec{x}_2(t) = e^{-t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$$

- Consider eigenpair

$$\left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

- Solution to homogeneous system from eigenpair

$$\vec{x}_3 = e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\vec{x} = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3$$

- Substitute solutions into the general solution

$$\vec{x} = c_1 e^{-t} \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-t} \cdot \left(t \cdot \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right) + c_3 e^{2t} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x \\ y(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \left((-t-1) c_2 - c_1 \right) e^{-t} + c_3 e^{2t} \\ c_3 e^{2t} \\ (t c_2 + c_1) e^{-t} + c_3 e^{2t} \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{ x = \left((-t-1) c_2 - c_1 \right) e^{-t} + c_3 e^{2t}, y(t) = c_3 e^{2t}, z(t) = (t c_2 + c_1) e^{-t} + c_3 e^{2t} \right\}$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 64

```
dsolve([diff(x(t),t)=y(t)+z(t),diff(y(t),t)=z(t)+x(t),diff(z(t),t)=x(t)+y(t)],singsol=all)
```

$$x(t) = c_2 e^{-t} + c_3 e^{2t}$$
$$y(t) = c_2 e^{-t} + c_3 e^{2t} + e^{-t} c_1$$
$$z(t) = -2c_2 e^{-t} + c_3 e^{2t} - e^{-t} c_1$$

✓ Solution by Mathematica

Time used: 0.009 (sec). Leaf size: 124

```
DSolve[{x'[t]==y[t]+z[t],y'[t]==z[t]+x[t],z'[t]==x[t]+y[t]},{x[t],y[t],z[t]},t,IncludeSingul
```

$$x(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} + 2\right) + (c_2 + c_3) \left(e^{3t} - 1\right)\right)$$
$$y(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1\right) + c_2 \left(e^{3t} + 2\right) + c_3 \left(e^{3t} - 1\right)\right)$$
$$z(t) \to \frac{1}{3} e^{-t} \left(c_1 \left(e^{3t} - 1\right) + c_2 \left(e^{3t} - 1\right) + c_3 \left(e^{3t} + 2\right)\right)$$

13.5 problem problem 11

Internal problem ID [276]
Internal file name [`OUTPUT/276_Sunday_June_05_2022_01_38_03_AM_94704504/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 417
Problem number: problem 11.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$\begin{aligned} x_1'(t) &= x_2(t) \\ x_2'(t) &= 2x_3(t) \\ x_3'(t) &= 3x_4(t) \\ x_4'(t) &= 4x_1(t) \end{aligned}$$

13.5.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \text{Expression too large to display}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = e^{At}\vec{c}$$

$$= \text{Expression too large to display} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

$$= \text{Expression too large to display}$$

Since no forcing function is given, then the final solution is $\vec{x}_h(t)$ above.

13.5.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 2 & 0 \\ 0 & 0 & -\lambda & 3 \\ 4 & 0 & 0 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 24 = 0$$

The roots of the above are the eigenvalues.

$$\lambda_1 = \sqrt{2}\, 6^{\frac{1}{4}}$$
$$\lambda_2 = -\sqrt{2}\, 6^{\frac{1}{4}}$$
$$\lambda_3 = i\sqrt{2}\, 6^{\frac{1}{4}}$$
$$\lambda_4 = -i\sqrt{2}\, 6^{\frac{1}{4}}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
$-\sqrt{2}\, 6^{\frac{1}{4}}$	1	real eigenvalue
$i\sqrt{2}\, 6^{\frac{1}{4}}$	1	complex eigenvalue
$-i\sqrt{2}\, 6^{\frac{1}{4}}$	1	complex eigenvalue
$\sqrt{2}\, 6^{\frac{1}{4}}$	1	real eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = \sqrt{2}\, 6^{\frac{1}{4}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} - \left(\sqrt{2}\, 6^{\frac{1}{4}}\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2^{\frac{3}{4}} 3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & -2^{\frac{3}{4}} 3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & -2^{\frac{3}{4}} 3^{\frac{1}{4}} & 3 \\ 4 & 0 & 0 & -2^{\frac{3}{4}} 3^{\frac{1}{4}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -\sqrt{2}\,6^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -\sqrt{2}\,6^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -\sqrt{2}\,6^{\frac{1}{4}} & 3 & 0 \\ 4 & 0 & 0 & -\sqrt{2}\,6^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{\sqrt{2}\,6^{\frac{3}{4}} R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & \frac{2\,2^{\frac{1}{4}}3^{\frac{3}{4}}}{3} & 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{\sqrt{2}\,\sqrt{3}\,R_2}{3} \Longrightarrow \left[\begin{array}{cccc|c} -2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & \frac{2\sqrt{6}}{3} & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{\sqrt{6}\,2^{\frac{1}{4}}3^{\frac{3}{4}} R_3}{9} \Longrightarrow \left[\begin{array}{cccc|c} -2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & -2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{3^{\frac{1}{4}}2^{\frac{3}{4}}t}{4}, v_2 = \frac{t\sqrt{6}}{2}, v_3 = \frac{2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix}\frac{3^{\frac{1}{4}} 2^{\frac{3}{4}} t}{4} \\ \frac{t\sqrt{6}}{2} \\ \frac{2^{\frac{1}{4}} 3^{\frac{3}{4}} t}{2} \\ t\end{bmatrix} = \begin{bmatrix}\frac{3^{\frac{1}{4}} 2^{\frac{3}{4}} t}{4} \\ \frac{t\sqrt{6}}{2} \\ \frac{2^{\frac{1}{4}} 3^{\frac{3}{4}} t}{2} \\ t\end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix}\frac{3^{\frac{1}{4}} 2^{\frac{3}{4}} t}{4} \\ \frac{t\sqrt{6}}{2} \\ \frac{2^{\frac{1}{4}} 3^{\frac{3}{4}} t}{2} \\ t\end{bmatrix} = t \begin{bmatrix}\frac{2^{\frac{3}{4}} 3^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{2^{\frac{1}{4}} 3^{\frac{3}{4}}}{2} \\ 1\end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix}\frac{3^{\frac{1}{4}} 2^{\frac{3}{4}} t}{4} \\ \frac{t\sqrt{6}}{2} \\ \frac{2^{\frac{1}{4}} 3^{\frac{3}{4}} t}{2} \\ t\end{bmatrix} = \begin{bmatrix}\frac{2^{\frac{3}{4}} 3^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{2^{\frac{1}{4}} 3^{\frac{3}{4}}}{2} \\ 1\end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix}\frac{3^{\frac{1}{4}} 2^{\frac{3}{4}} t}{4} \\ \frac{t\sqrt{6}}{2} \\ \frac{2^{\frac{1}{4}} 3^{\frac{3}{4}} t}{2} \\ t\end{bmatrix} = \begin{bmatrix}2^{\frac{3}{4}} 3^{\frac{1}{4}} \\ 2\sqrt{6} \\ 2\, 2^{\frac{1}{4}} 3^{\frac{3}{4}} \\ 4\end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = -i\sqrt{2}\, 6^{\frac{1}{4}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} - \left(-i\sqrt{2}\,6^{\frac{1}{4}}\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 \\ 4 & 0 & 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} i\sqrt{2}\,6^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & i\sqrt{2}\,6^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & i\sqrt{2}\,6^{\frac{1}{4}} & 3 & 0 \\ 4 & 0 & 0 & i\sqrt{2}\,6^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{i\sqrt{2}\,6^{\frac{3}{4}}R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & \frac{2i2^{\frac{1}{4}}3^{\frac{3}{4}}}{3} & 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{\sqrt{2}\,\sqrt{3}\,R_2}{3} \Longrightarrow \left[\begin{array}{cccc|c} i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & -\frac{2\sqrt{6}}{3} & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{i\sqrt{6}\,2^{\frac{1}{4}}3^{\frac{3}{4}}R_3}{9} \Longrightarrow \left[\begin{array}{cccc|c} i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}t}{4}, v_2 = -\frac{t\sqrt{6}}{2}, v_3 = \frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{\mathrm{I}}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ \frac{\mathrm{I}}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}t}{4} \\ -\frac{t\sqrt{6}}{2} \\ \frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{\mathrm{I}}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ \frac{\mathrm{I}}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}}{4} \\ -\frac{\sqrt{6}}{2} \\ \frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{\mathrm{I}}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ \frac{\mathrm{I}}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = \begin{bmatrix} -\frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}}{4} \\ -\frac{\sqrt{6}}{2} \\ \frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{\mathrm{I}}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ \frac{\mathrm{I}}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = \begin{bmatrix} -i2^{\frac{3}{4}}3^{\frac{1}{4}} \\ -2\sqrt{6} \\ 2i2^{\frac{1}{4}}3^{\frac{3}{4}} \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = i\sqrt{2}\,6^{\frac{1}{4}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} - \left(i\sqrt{2}\,6^{\frac{1}{4}}\right)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 \\ 4 & 0 & 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -i\sqrt{2}\,6^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{2}\,6^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -i\sqrt{2}\,6^{\frac{1}{4}} & 3 & 0 \\ 4 & 0 & 0 & -i\sqrt{2}\,6^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{i\sqrt{2}\,6^{\frac{3}{4}}R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & -\frac{2i2^{\frac{1}{4}}3^{\frac{3}{4}}}{3} & 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{\sqrt{2}\sqrt{3}\,R_2}{3} \Longrightarrow \left[\begin{array}{cccc|c} -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & -\frac{2\sqrt{6}}{3} & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{i\sqrt{6}\,2^{\frac{1}{4}}3^{\frac{3}{4}}R_3}{9} \Longrightarrow \left[\begin{array}{cccc|c} -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & -i2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = \frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}t}{4}, v_2 = -\frac{t\sqrt{6}}{2}, v_3 = -\frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} \frac{I}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ -\frac{I}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}t}{4} \\ -\frac{t\sqrt{6}}{2} \\ -\frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \frac{I}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ -\frac{I}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}}{4} \\ -\frac{\sqrt{6}}{2} \\ -\frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \frac{I}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ -\frac{I}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = \begin{bmatrix} \frac{i2^{\frac{3}{4}}3^{\frac{1}{4}}}{4} \\ -\frac{\sqrt{6}}{2} \\ -\frac{i2^{\frac{1}{4}}3^{\frac{3}{4}}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} \frac{I}{4}2^{\frac{3}{4}}3^{\frac{1}{4}}t \\ -\frac{t\sqrt{6}}{2} \\ -\frac{I}{2}2^{\frac{1}{4}}3^{\frac{3}{4}}t \\ t \end{bmatrix} = \begin{bmatrix} i2^{\frac{3}{4}}3^{\frac{1}{4}} \\ -2\sqrt{6} \\ -2i2^{\frac{1}{4}}3^{\frac{3}{4}} \\ 4 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = -\sqrt{2}\,6^{\frac{1}{4}}$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} - \left(-\sqrt{2}\,6^{\frac{1}{4}}\right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 \\ 4 & 0 & 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} \sqrt{2}\,6^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & \sqrt{2}\,6^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & \sqrt{2}\,6^{\frac{1}{4}} & 3 & 0 \\ 4 & 0 & 0 & \sqrt{2}\,6^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{\sqrt{2}\,6^{\frac{3}{4}} R_1}{3} \Longrightarrow \left[\begin{array}{cccc|c} 2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & -\frac{2\,2^{\frac{1}{4}}3^{\frac{3}{4}}}{3} & 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{\sqrt{2}\,\sqrt{3}\,R_2}{3} \Longrightarrow \left[\begin{array}{cccc|c} 2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & \frac{2\sqrt{6}}{3} & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{\sqrt{6}\,2^{\frac{1}{4}}3^{\frac{3}{4}} R_3}{9} \Longrightarrow \left[\begin{array}{cccc|c} 2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 & 0 \\ 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 & 0 \\ 0 & 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 2^{\frac{3}{4}}3^{\frac{1}{4}} & 1 & 0 & 0 \\ 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 2 & 0 \\ 0 & 0 & 2^{\frac{3}{4}}3^{\frac{1}{4}} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\left\{v_1 = -\frac{3^{\frac{1}{4}}2^{\frac{3}{4}}t}{4}, v_2 = \frac{t\sqrt{6}}{2}, v_3 = -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2}\right\}$

Hence the solution is

$$\begin{bmatrix} -\frac{3^{\frac{1}{4}}2^{\frac{3}{4}}t}{4} \\ \frac{t\sqrt{6}}{2} \\ -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{3^{\frac{1}{4}}2^{\frac{3}{4}}t}{4} \\ \frac{t\sqrt{6}}{2} \\ -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2} \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -\frac{3^{\frac{1}{4}}2^{\frac{3}{4}}t}{4} \\ \frac{t\sqrt{6}}{2} \\ -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2} \\ t \end{bmatrix} = t \begin{bmatrix} -\frac{2^{\frac{3}{4}}3^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}}{2} \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -\frac{3^{\frac{1}{4}}2^{\frac{3}{4}}t}{4} \\ \frac{t\sqrt{6}}{2} \\ -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -\frac{2^{\frac{3}{4}}3^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}}{2} \\ 1 \end{bmatrix}$$

Which is normalized to

$$\begin{bmatrix} -\frac{3^{\frac{1}{4}}2^{\frac{3}{4}}t}{4} \\ \frac{t\sqrt{6}}{2} \\ -\frac{2^{\frac{1}{4}}3^{\frac{3}{4}}t}{2} \\ t \end{bmatrix} = \begin{bmatrix} -2^{\frac{3}{4}}3^{\frac{1}{4}} \\ 2\sqrt{6} \\ -2\,2^{\frac{1}{4}}3^{\frac{3}{4}} \\ 4 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

	multiplicity			
eigenvalue	algebraic m	geometric k	defective?	eigenvectors
$\sqrt{2}\,6^{\frac{1}{4}}$	1	1	No	$\left[\begin{array}{c} \frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{array}\right]$
$-\sqrt{2}\,6^{\frac{1}{4}}$	1	1	No	$\left[\begin{array}{c} -\frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{array}\right]$
$i\sqrt{2}\,6^{\frac{1}{4}}$	1	1	No	$\left[\begin{array}{c} \frac{i\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ -\frac{\sqrt{6}}{2} \\ -\frac{i\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{array}\right]$
$-i\sqrt{2}\,6^{\frac{1}{4}}$	1	1	No	$\left[\begin{array}{c} -\frac{i\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ -\frac{\sqrt{6}}{2} \\ \frac{i\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{array}\right]$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue $\sqrt{2}\, 6^{\frac{1}{4}}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_1(t) &= \vec{v}_1 e^{\sqrt{2}\, 6^{\frac{1}{4}} t} \\
&= \begin{bmatrix} \frac{\sqrt{2}\, 6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}\, 6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix} e^{\sqrt{2}\, 6^{\frac{1}{4}} t}
\end{aligned}
$$

Since eigenvalue $-\sqrt{2}\, 6^{\frac{1}{4}}$ is real and distinct then the corresponding eigenvector solution is

$$
\begin{aligned}
\vec{x}_2(t) &= \vec{v}_2 e^{-\sqrt{2}\, 6^{\frac{1}{4}} t} \\
&= \begin{bmatrix} -\frac{\sqrt{2}\, 6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{2}\, 6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix} e^{-\sqrt{2}\, 6^{\frac{1}{4}} t}
\end{aligned}
$$

Therefore the final solution is

$$
\vec{x}_h(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t) + c_4 \vec{x}_4(t)
$$

Which is written as

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} \frac{e^{\sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{1}{4}}}{4} \\ \frac{e^{\sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{6}}{2} \\ \frac{e^{\sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{3}{4}}}{4} \\ e^{\sqrt{2}\, 6^{\frac{1}{4}} t} \end{bmatrix} + c_2 \begin{bmatrix} -\frac{e^{-\sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{1}{4}}}{4} \\ \frac{e^{-\sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{6}}{2} \\ -\frac{e^{-\sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{3}{4}}}{4} \\ e^{-\sqrt{2}\, 6^{\frac{1}{4}} t} \end{bmatrix} + c_3 \begin{bmatrix} \frac{i e^{i \sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{1}{4}}}{4} \\ -\frac{e^{i \sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{6}}{2} \\ -\frac{i e^{i \sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{3}{4}}}{4} \\ e^{i \sqrt{2}\, 6^{\frac{1}{4}} t} \end{bmatrix} + c_4 \begin{bmatrix} -\frac{i e^{-i \sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{1}{4}}}{4} \\ -\frac{e^{-i \sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{6}}{2} \\ \frac{i e^{-i \sqrt{2}\, 6^{\frac{1}{4}} t} \sqrt{2}\, 6^{\frac{3}{4}}}{4} \\ e^{-i \sqrt{2}\, 6^{\frac{1}{4}} t} \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} \frac{2^{\frac{3}{4}} 3^{\frac{1}{4}} \left(i c_3 e^{i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} - i c_4 e^{-i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} + c_1 e^{2^{\frac{3}{4}} 3^{\frac{1}{4}} t} - c_2 e^{-2^{\frac{3}{4}} 3^{\frac{1}{4}} t}\right)}{4} \\ -\frac{\sqrt{6}\left(c_3 e^{i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} + c_4 e^{-i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} - c_1 e^{2^{\frac{3}{4}} 3^{\frac{1}{4}} t} - c_2 e^{-2^{\frac{3}{4}} 3^{\frac{1}{4}} t}\right)}{2} \\ -\frac{2^{\frac{1}{4}} 3^{\frac{3}{4}} \left(i c_3 e^{i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} - i c_4 e^{-i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} - c_1 e^{2^{\frac{3}{4}} 3^{\frac{1}{4}} t} + c_2 e^{-2^{\frac{3}{4}} 3^{\frac{1}{4}} t}\right)}{2} \\ c_1 e^{2^{\frac{3}{4}} 3^{\frac{1}{4}} t} + c_2 e^{-2^{\frac{3}{4}} 3^{\frac{1}{4}} t} + c_3 e^{i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} + c_4 e^{-i 2^{\frac{3}{4}} 3^{\frac{1}{4}} t} \end{bmatrix}$$

13.5.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_2(t), x_2'(t) = 2x_3(t), x_3'(t) = 3x_4(t), x_4'(t) = 4x_1(t)]$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t)$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as
  $\underline{x}^{\rightarrow\prime}(t) = A \cdot \underline{x}^{\rightarrow}(t)$
- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A
  $$\left[\left[\sqrt{2}\,6^{\frac{1}{4}}, \begin{bmatrix} \frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix}\right], \left[-\mathrm{I}\sqrt{2}\,6^{\frac{1}{4}}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2}\,6^{\frac{1}{4}} \\ -\frac{\sqrt{6}}{2} \\ \frac{\mathrm{I}}{4}\sqrt{2}\,6^{\frac{3}{4}} \\ 1 \end{bmatrix}\right], \left[\mathrm{I}\sqrt{2}\,6^{\frac{1}{4}}, \begin{bmatrix} \frac{\mathrm{I}}{4}\sqrt{2}\,6^{\frac{1}{4}} \\ -\frac{\sqrt{6}}{2} \\ -\frac{\mathrm{I}}{4}\sqrt{2}\,6^{\frac{3}{4}} \\ 1 \end{bmatrix}\right], \left[-\sqrt{2}\,6^{\frac{1}{4}}, \begin{bmatrix} -\frac{\sqrt{2}\,6}{4} \\ \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{2}\,6}{4} \\ 1 \end{bmatrix}\right.\right.$$
- Consider eigenpair
  $$\left[\sqrt{2}\,6^{\frac{1}{4}}, \begin{bmatrix} \frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix}\right]$$
- Solution to homogeneous system from eigenpair
  $$\underline{x}^{\rightarrow}_{1} = \mathrm{e}^{\sqrt{2}\,6^{\frac{1}{4}}t} \cdot \begin{bmatrix} \frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix}$$
- Consider complex eigenpair, complex conjugate eigenvalue can be ignored
  $$\left[-\mathrm{I}\sqrt{2}\,6^{\frac{1}{4}}, \begin{bmatrix} -\frac{\mathrm{I}}{4}\sqrt{2}\,6^{\frac{1}{4}} \\ -\frac{\sqrt{6}}{2} \\ \frac{\mathrm{I}}{4}\sqrt{2}\,6^{\frac{3}{4}} \\ 1 \end{bmatrix}\right]$$

- Solution from eigenpair

$$e^{-I\sqrt{2}\,6^{\frac{1}{4}}t}\cdot\left[\begin{array}{c}-\frac{I}{4}\sqrt{2}\,6^{\frac{1}{4}}\\ -\frac{\sqrt{6}}{2}\\ \frac{I}{4}\sqrt{2}\,6^{\frac{3}{4}}\\ 1\end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$\left(\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)-I\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)\right)\cdot\left[\begin{array}{c}-\frac{I}{4}\sqrt{2}\,6^{\frac{1}{4}}\\ -\frac{\sqrt{6}}{2}\\ \frac{I}{4}\sqrt{2}\,6^{\frac{3}{4}}\\ 1\end{array}\right]$$

- Simplify expression

$$\left[\begin{array}{c}-\frac{I}{4}\left(\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)-I\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)\right)\sqrt{2}\,6^{\frac{1}{4}}\\ -\frac{\left(\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)-I\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)\right)\sqrt{6}}{2}\\ \frac{I}{4}\left(\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)-I\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)\right)\sqrt{2}\,6^{\frac{3}{4}}\\ \cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)-I\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)\end{array}\right]$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\vec{x}_{2}(t)=\left[\begin{array}{c}-\frac{\sqrt{2}\,6^{\frac{1}{4}}\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4}\\ -\frac{\sqrt{6}\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{2}\\ \frac{\sqrt{2}\,6^{\frac{3}{4}}\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4}\\ \cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)\end{array}\right],\vec{x}_{3}(t)=\left[\begin{array}{c}-\frac{\sqrt{2}\,6^{\frac{1}{4}}\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4}\\ \frac{\sqrt{6}\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{2}\\ \frac{\sqrt{2}\,6^{\frac{3}{4}}\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4}\\ -\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)\end{array}\right]\right]$$

- Consider eigenpair

$$\left[-\sqrt{2}\,6^{\frac{1}{4}},\left[\begin{array}{c}-\frac{\sqrt{2}\,6^{\frac{1}{4}}}{4}\\ \frac{\sqrt{6}}{2}\\ -\frac{\sqrt{2}\,6^{\frac{3}{4}}}{4}\\ 1\end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{x}^{\rightarrow}{}_4 = \mathrm{e}^{-\sqrt{2}\,6^{\frac{1}{4}}t} \cdot \begin{bmatrix} -\frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix}$$

- General solution to the system of ODEs

$$\underline{x}^{\rightarrow} = c_1 \underline{x}^{\rightarrow}{}_1 + c_2 \underline{x}^{\rightarrow}{}_2(t) + c_3 \underline{x}^{\rightarrow}{}_3(t) + c_4 \underline{x}^{\rightarrow}{}_4$$

- Substitute solutions into the general solution

$$\underline{x}^{\rightarrow} = c_1 \mathrm{e}^{\sqrt{2}\,6^{\frac{1}{4}}t} \cdot \begin{bmatrix} \frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ \frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix} + c_4 \mathrm{e}^{-\sqrt{2}\,6^{\frac{1}{4}}t} \cdot \begin{bmatrix} -\frac{\sqrt{2}\,6^{\frac{1}{4}}}{4} \\ \frac{\sqrt{6}}{2} \\ -\frac{\sqrt{2}\,6^{\frac{3}{4}}}{4} \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{c_2\sqrt{2}\,6^{\frac{1}{4}} \sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4} - \frac{c_3\sqrt{2}\,6^{\frac{1}{4}} \cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4} \\ -\frac{c_2\sqrt{6}\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{2} + \frac{c_3\sqrt{6}\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{2} \\ \frac{c_2\sqrt{2}\,6^{\frac{3}{4}} \sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4} + \frac{c_3\sqrt{2}\,6^{\frac{3}{4}} \cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right)}{4} \\ c_2\cos\left(\sqrt{2}\,6^{\frac{1}{4}}t\right) - c_3\sin\left(\sqrt{2}\,6^{\frac{1}{4}}t\right) \end{bmatrix}$$

- Substitute in vector of dependent variables

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -\frac{2^{\frac{3}{4}}3^{\frac{1}{4}}\left(-c_1\mathrm{e}^{2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_4\mathrm{e}^{-2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_3\cos\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right) + c_2\sin\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right)\right)}{4} \\ \frac{\sqrt{6}\left(c_1\mathrm{e}^{2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_4\mathrm{e}^{-2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_3\sin\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right) - c_2\cos\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right)\right)}{2} \\ \frac{2^{\frac{1}{4}}3^{\frac{3}{4}}\left(c_1\mathrm{e}^{2^{\frac{3}{4}}3^{\frac{1}{4}}t} - c_4\mathrm{e}^{-2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_3\cos\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right) + c_2\sin\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right)\right)}{2} \\ c_1\mathrm{e}^{2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_4\mathrm{e}^{-2^{\frac{3}{4}}3^{\frac{1}{4}}t} - c_3\sin\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right) + c_2\cos\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right) \end{bmatrix}$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{2^{\frac{3}{4}}3^{\frac{1}{4}}\left(-c_1\mathrm{e}^{2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_4\mathrm{e}^{-2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_3\cos\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right) + c_2\sin\left(2^{\frac{3}{4}}3^{\frac{1}{4}}t\right)\right)}{4}, x_2(t) = \frac{\sqrt{6}\left(c_1\mathrm{e}^{2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_4\mathrm{e}^{-2^{\frac{3}{4}}3^{\frac{1}{4}}t} + c_3\sin\left(\right.\right.}{2}\right.$$

✓ Solution by Maple

Time used: 0.047 (sec). Leaf size: 170

```
dsolve([diff(x__1(t),t)=x__2(t),diff(x__2(t),t)=2*x__3(t),diff(x__3(t),t)=3*x__4(t),diff(x__
```

$$x_1(t) = c_1 e^{-24^{\frac{1}{4}}t} + c_2 e^{24^{\frac{1}{4}}t} - c_3 \sin\left(24^{\frac{1}{4}}t\right) + c_4 \cos\left(24^{\frac{1}{4}}t\right)$$

$$x_2(t) = -24^{\frac{1}{4}}\left(c_1 e^{-24^{\frac{1}{4}}t} - c_2 e^{24^{\frac{1}{4}}t} + \cos\left(24^{\frac{1}{4}}t\right) c_3 + \sin\left(24^{\frac{1}{4}}t\right) c_4\right)$$

$$x_3(t) = \sqrt{6}\left(c_1 e^{-24^{\frac{1}{4}}t} + c_2 e^{24^{\frac{1}{4}}t} - c_4 \cos\left(24^{\frac{1}{4}}t\right) + c_3 \sin\left(24^{\frac{1}{4}}t\right)\right)$$

$$x_4(t) = -\frac{24^{\frac{3}{4}}\left(c_1 e^{-24^{\frac{1}{4}}t} - c_2 e^{24^{\frac{1}{4}}t} - \cos\left(24^{\frac{1}{4}}t\right) c_3 - \sin\left(24^{\frac{1}{4}}t\right) c_4\right)}{6}$$

✓ Solution by Mathematica

Time used: 0.042 (sec). Leaf size: 400

```
DSolve[{x1'[t]==x2[t],x2'[t]==2*x3[t],x3'[t]==3*x4[t],x4'[t]==4*x1[t]},{x1[t],x2[t],x3[t],x4
```

$$\begin{aligned}
\text{x1}(t) \to\ & \frac{1}{4}c_1\text{RootSum}\left[\#1^4 - 24\&, e^{\#1t}\&\right] + \frac{1}{4}c_2\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1}\&\right] \\
& + \frac{3}{2}c_4\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^3}\&\right] + \frac{1}{2}c_3\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^2}\&\right] \\
\text{x2}(t) \to\ & \frac{1}{4}c_2\text{RootSum}\left[\#1^4 - 24\&, e^{\#1t}\&\right] + \frac{1}{2}c_3\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1}\&\right] \\
& + 6c_1\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^3}\&\right] + \frac{3}{2}c_4\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^2}\&\right] \\
\text{x3}(t) \to\ & \frac{1}{4}c_3\text{RootSum}\left[\#1^4 - 24\&, e^{\#1t}\&\right] + \frac{3}{4}c_4\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1}\&\right] \\
& + 3c_2\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^3}\&\right] + 3c_1\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^2}\&\right] \\
\text{x4}(t) \to\ & \frac{1}{4}c_4\text{RootSum}\left[\#1^4 - 24\&, e^{\#1t}\&\right] + c_1\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1}\&\right] \\
& + 2c_3\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^3}\&\right] + c_2\text{RootSum}\left[\#1^4 - 24\&, \frac{e^{\#1t}}{\#1^2}\&\right]
\end{aligned}$$

13.6 problem problem 12

Internal problem ID [277]
Internal file name [`OUTPUT/277_Sunday_June_05_2022_01_38_06_AM_13138377/index.tex`]

Book: Differential equations and linear algebra, 3rd ed., Edwards and Penney
Section: Section 7.2, Matrices and Linear systems. Page 417
Problem number: problem 12.
ODE order: 1.
ODE degree: 1.

The type(s) of ODE detected by this program : **"system of linear ODEs"**

Solve

$$
\begin{aligned}
x_1'(t) &= x_2(t) + x_3(t) + 1 \\
x_2'(t) &= x_3(t) + x_4(t) + t \\
x_3'(t) &= x_1(t) + x_4(t) + t^2 \\
x_4'(t) &= x_1(t) + x_2(t) + t^3
\end{aligned}
$$

13.6.1 Solution using Matrix exponential method

In this method, we will assume we have found the matrix exponential e^{At} allready. There are different methods to determine this but will not be shown here. This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$
\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}
$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix. For the above matrix A, the matrix exponential can be found to be

$$e^{At} = \left[\begin{array}{cccc} \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} \\ -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} \\ -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} \\ \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} \end{array}\right]$$

Therefore the homogeneous solution is

$$\begin{aligned}
\vec{x}_h(t) &= e^{At}\vec{c} \\
&= \left[\begin{array}{cccc} \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} \\ -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} \\ -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} \\ \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} \end{array}\right] \left[\begin{array}{c} c_1 \\ c_2 \\ c_3 \\ c_4 \end{array}\right] \\
&= \left[\begin{array}{c} \left(\frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}\right)c_1+\left(\frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2}\right)c_2+\left(-\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4}\right)c_3+\left(-\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\right. \\ \left(-\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4}\right)c_1+\left(\frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}\right)c_2+\left(\frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2}\right)c_3+\left(-\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}\right. \\ \left(-\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4}\right)c_1+\left(-\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4}\right)c_2+\left(\frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}\right)c_3+\left(\frac{e^{2t}}{4}-\frac{1}{4}+\right. \\ \left(\frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2}\right)c_1+\left(-\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4}\right)c_2+\left(-\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4}\right)c_3+\left(\frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\right. \end{array}\right] \\
&= \left[\begin{array}{c} \frac{\left((2c_1-2c_3)\cos(t)+2\sin(t)(c_2-c_4)\right)e^{-t}}{4}+\frac{(c_1+c_2+c_3+c_4)e^{2t}}{4}+\frac{c_1}{4}-\frac{c_2}{4}+\frac{c_3}{4}-\frac{c_4}{4} \\ \frac{\left((2c_2-2c_4)\cos(t)-2\sin(t)(c_1-c_3)\right)e^{-t}}{4}+\frac{(c_1+c_2+c_3+c_4)e^{2t}}{4}-\frac{c_1}{4}+\frac{c_2}{4}-\frac{c_3}{4}+\frac{c_4}{4} \\ \frac{\left((-2c_1+2c_3)\cos(t)-2\sin(t)(c_2-c_4)\right)e^{-t}}{4}+\frac{(c_1+c_2+c_3+c_4)e^{2t}}{4}+\frac{c_1}{4}-\frac{c_2}{4}+\frac{c_3}{4}-\frac{c_4}{4} \\ \frac{\left((-2c_2+2c_4)\cos(t)+2\sin(t)(c_1-c_3)\right)e^{-t}}{4}+\frac{(c_1+c_2+c_3+c_4)e^{2t}}{4}-\frac{c_1}{4}+\frac{c_2}{4}-\frac{c_3}{4}+\frac{c_4}{4} \end{array}\right]
\end{aligned}$$

The particular solution given by

$$\vec{x}_p(t) = e^{At}\int e^{-At}\vec{G}(t)\,dt$$

But

$$
e^{-At} = \left(e^{At}\right)^{-1}
$$

$$
= \left[\begin{array}{cccc}
\frac{\left(2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4} & -\frac{\left(2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} & \frac{\left(-2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4} & -\frac{\left(-2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} \\
-\frac{\left(-2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} & \frac{\left(2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4} & -\frac{\left(2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} & \frac{\left(-2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4} \\
\frac{\left(-2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4} & -\frac{\left(-2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} & \frac{\left(2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4} & -\frac{\left(2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} \\
-\frac{\left(2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} & \frac{\left(-2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4} & -\frac{\left(-2\,\mathrm{e}^{3t}\sin(t)+\mathrm{e}^{2t}-1\right)\mathrm{e}^{-2t}}{4} & \frac{\left(2\,\mathrm{e}^{3t}\cos(t)+\mathrm{e}^{2t}+1\right)\mathrm{e}^{-2t}}{4}
\end{array}\right]
$$

Hence

$$
\vec{x}_p(t) = \left[\begin{array}{cccc}
\frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} & -\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} & -\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} \\
-\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} & \frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} & -\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} \\
-\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} & -\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} & \frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} \\
\frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} & -\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} & -\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} & \frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}
\end{array}\right] \int \left[\cdots\right]
$$

$$
= \left[\begin{array}{cccc}
\frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} & -\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} & -\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} \\
-\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} & \frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} & -\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} \\
-\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} & -\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} & \frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4} & \frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} \\
\frac{\mathrm{e}^{2t}}{4}-\frac{1}{4}+\frac{\mathrm{e}^{-t}\sin(t)}{2} & -\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}+\frac{1}{4} & -\frac{\mathrm{e}^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{\mathrm{e}^{2t}}{4} & \frac{1}{4}+\frac{\mathrm{e}^{-t}\cos(t)}{2}+\frac{\mathrm{e}^{2t}}{4}
\end{array}\right] \left[\cdots\right]
$$

$$
= \left[\begin{array}{c}
-\frac{1}{16}t^4-\frac{7}{24}t^3+\frac{1}{16}t^2-\frac{11}{16}t-\frac{3}{32} \\
\frac{1}{16}t^4-\frac{11}{24}t^3+\frac{1}{16}t^2-\frac{3}{16}t-\frac{35}{32} \\
-\frac{1}{16}t^4+\frac{5}{24}t^3-\frac{15}{16}t^2+\frac{5}{16}t-\frac{19}{32} \\
\frac{1}{16}t^4+\frac{1}{24}t^3-\frac{7}{16}t^2-\frac{19}{16}t+\frac{13}{32}
\end{array}\right]
$$

Hence the complete solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
= \left[\begin{array}{c}
-\frac{3}{32}+\frac{\left(\left(c_1-c_3\right)\cos(t)+\sin(t)\left(c_2-c_4\right)\right)\mathrm{e}^{-t}}{2}+\frac{\left(c_1+c_2+c_3+c_4\right)\mathrm{e}^{2t}}{4}+\frac{c_1}{4}-\frac{c_2}{4}+\frac{c_3}{4}-\frac{c_4}{4}-\frac{t^4}{16}-\frac{7t^3}{24}+\frac{t^2}{16}-\frac{11t}{16} \\
-\frac{35}{32}+\frac{\left(\left(c_2-c_4\right)\cos(t)-\sin(t)\left(c_1-c_3\right)\right)\mathrm{e}^{-t}}{2}+\frac{\left(c_1+c_2+c_3+c_4\right)\mathrm{e}^{2t}}{4}-\frac{c_1}{4}+\frac{c_2}{4}-\frac{c_3}{4}+\frac{c_4}{4}+\frac{t^4}{16}-\frac{11t^3}{24}+\frac{t^2}{16}-\frac{3t}{16} \\
-\frac{19}{32}+\frac{\left(\left(-c_1+c_3\right)\cos(t)-\sin(t)\left(c_2-c_4\right)\right)\mathrm{e}^{-t}}{2}+\frac{\left(c_1+c_2+c_3+c_4\right)\mathrm{e}^{2t}}{4}+\frac{c_1}{4}-\frac{c_2}{4}+\frac{c_3}{4}-\frac{c_4}{4}-\frac{t^4}{16}+\frac{5t^3}{24}-\frac{15t^2}{16}+\frac{5t}{16} \\
\frac{13}{32}+\frac{\left(\left(-c_2+c_4\right)\cos(t)+\sin(t)\left(c_1-c_3\right)\right)\mathrm{e}^{-t}}{2}+\frac{\left(c_1+c_2+c_3+c_4\right)\mathrm{e}^{2t}}{4}-\frac{c_1}{4}+\frac{c_2}{4}-\frac{c_3}{4}+\frac{c_4}{4}+\frac{t^4}{16}+\frac{t^3}{24}-\frac{7t^2}{16}-\frac{19t}{16}
\end{array}\right]
$$

13.6.2 Solution using explicit Eigenvalue and Eigenvector method

This is a system of linear ODE's given as

$$\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$$

Or

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \\ x_4'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

Since the system is nonhomogeneous, then the solution is given by

$$\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

Where $\vec{x}_h(t)$ is the homogeneous solution to $\vec{x}'(t) = A\,\vec{x}(t)$ and $\vec{x}_p(t)$ is a particular solution to $\vec{x}'(t) = A\,\vec{x}(t) + \vec{G}(t)$. The particular solution will be found using variation of parameters method applied to the fundamental matrix.

The first step is find the homogeneous solution. We start by finding the eigenvalues of A. This is done by solving the following equation for the eigenvalues λ

$$\det(A - \lambda I) = 0$$

Expanding gives

$$\det\left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right) = 0$$

Therefore

$$\det\left(\begin{bmatrix} -\lambda & 1 & 1 & 0 \\ 0 & -\lambda & 1 & 1 \\ 1 & 0 & -\lambda & 1 \\ 1 & 1 & 0 & -\lambda \end{bmatrix}\right) = 0$$

Which gives the characteristic equation

$$\lambda^4 - 2\lambda^2 - 4\lambda = 0$$

The roots of the above are the eigenvalues.

$$\begin{aligned}\lambda_1 &= 0\\ \lambda_2 &= 2\\ \lambda_3 &= -1 + i\\ \lambda_4 &= -1 - i\end{aligned}$$

This table summarises the above result

eigenvalue	algebraic multiplicity	type of eigenvalue
0	1	real eigenvalue
$-1 - i$	1	complex eigenvalue
2	1	real eigenvalue
$-1 + i$	1	complex eigenvalue

Now the eigenvector for each eigenvalue are found.

Considering the eigenvalue $\lambda_1 = 0$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0\end{bmatrix} - (0)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & 1 & 0\\ 0 & 0 & 1 & 1\\ 1 & 0 & 0 & 1\\ 1 & 1 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\\ v_4\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\\ 0\end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c}0 & 1 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0\\ 1 & 0 & 0 & 1 & 0\\ 1 & 1 & 0 & 0 & 0\end{array}\right]$$

Since the current pivot $A(1,1)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 1 and row 3 gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

Since the current pivot $A(2,2)$ is zero, then the current pivot row is replaced with a row with a non-zero pivot. Swapping row 2 and row 3 gives

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$R_4 = R_4 - R_2 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right]$$

$$R_4 = R_4 + R_3 \Longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -t, v_2 = t, v_3 = -t\}$

Hence the solution is

$$\begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} -t \\ t \\ -t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_2 = 2$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} - (2)\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 1 & 0 & -2 & 1 & 0 \\ 1 & 1 & 0 & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 1 & 1 & 0 & -2 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{R_1}{2} \Longrightarrow \left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -2 & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{R_2}{4} \Longrightarrow \left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & -\frac{5}{4} & \frac{5}{4} & 0 \\ 0 & \frac{3}{2} & \frac{1}{2} & -2 & 0 \end{array}\right]$$

$$R_4 = R_4 + \frac{3R_2}{4} \Longrightarrow \left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & -\frac{5}{4} & \frac{5}{4} & 0 \\ 0 & 0 & \frac{5}{4} & -\frac{5}{4} & 0 \end{array}\right]$$

$$R_4 = R_4 + R_3 \Longrightarrow \left[\begin{array}{cccc|c} -2 & 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 1 & 0 \\ 0 & 0 & -\frac{5}{4} & \frac{5}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 1 \\ 0 & 0 & -\frac{5}{4} & \frac{5}{4} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = t, v_2 = t, v_3 = t\}$

Hence the solution is

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_3 = -1 - i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} - (-1-i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1+i & 1 & 1 & 0 \\ 0 & 1+i & 1 & 1 \\ 1 & 0 & 1+i & 1 \\ 1 & 1 & 0 & 1+i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1+i & 1 & 1 & 0 & 0 \\ 0 & 1+i & 1 & 1 & 0 \\ 1 & 0 & 1+i & 1 & 0 \\ 1 & 1 & 0 & 1+i & 0 \end{array} \right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} + \frac{i}{2} \right) R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1+i & 1 & 1 & 0 & 0 \\ 0 & 1+i & 1 & 1 & 0 \\ 0 & -\frac{1}{2} + \frac{i}{2} & \frac{1}{2} + \frac{3i}{2} & 1 & 0 \\ 1 & 1 & 0 & 1+i & 0 \end{array} \right]$$

$$R_4 = R_4 + \left(-\frac{1}{2} + \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1+i & 1 & 1 & 0 & 0 \\ 0 & 1+i & 1 & 1 & 0 \\ 0 & -\frac{1}{2}+\frac{i}{2} & \frac{1}{2}+\frac{3i}{2} & 1 & 0 \\ 0 & \frac{1}{2}+\frac{i}{2} & -\frac{1}{2}+\frac{i}{2} & 1+i & 0 \end{array}\right]$$

$$R_3 = R_3 - \frac{iR_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} 1+i & 1 & 1 & 0 & 0 \\ 0 & 1+i & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2}+i & 1-\frac{i}{2} & 0 \\ 0 & \frac{1}{2}+\frac{i}{2} & -\frac{1}{2}+\frac{i}{2} & 1+i & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} 1+i & 1 & 1 & 0 & 0 \\ 0 & 1+i & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2}+i & 1-\frac{i}{2} & 0 \\ 0 & 0 & -1+\frac{i}{2} & \frac{1}{2}+i & 0 \end{array}\right]$$

$$R_4 = -iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} 1+i & 1 & 1 & 0 & 0 \\ 0 & 1+i & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2}+i & 1-\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1+i & 1 & 1 & 0 \\ 0 & 1+i & 1 & 1 \\ 0 & 0 & \frac{1}{2}+i & 1-\frac{i}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = -it, v_2 = -t, v_3 = it\}$

Hence the solution is

$$\begin{bmatrix} \text{-I}t \\ -t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -it \\ -t \\ it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \text{-I}t \\ -t \\ \text{I}t \\ t \end{bmatrix} = t \begin{bmatrix} -i \\ -1 \\ i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \text{-I}t \\ -t \\ \text{I}t \\ t \end{bmatrix} = \begin{bmatrix} -i \\ -1 \\ i \\ 1 \end{bmatrix}$$

Considering the eigenvalue $\lambda_4 = -1 + i$

We need to solve $A\vec{v} = \lambda\vec{v}$ or $(A - \lambda I)\vec{v} = \vec{0}$ which becomes

$$\left(\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} - (-1+i) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1-i & 1 & 1 & 0 \\ 0 & 1-i & 1 & 1 \\ 1 & 0 & 1-i & 1 \\ 1 & 1 & 0 & 1-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now forward elimination is applied to solve for the eigenvector $\vec{v}$. The augmented matrix is

$$\left[\begin{array}{cccc|c} 1-i & 1 & 1 & 0 & 0 \\ 0 & 1-i & 1 & 1 & 0 \\ 1 & 0 & 1-i & 1 & 0 \\ 1 & 1 & 0 & 1-i & 0 \end{array}\right]$$

$$R_3 = R_3 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1-i & 1 & 1 & 0 & 0 \\ 0 & 1-i & 1 & 1 & 0 \\ 0 & -\frac{1}{2}-\frac{i}{2} & \frac{1}{2}-\frac{3i}{2} & 1 & 0 \\ 1 & 1 & 0 & 1-i & 0 \end{array}\right]$$

$$R_4 = R_4 + \left(-\frac{1}{2} - \frac{i}{2}\right) R_1 \Longrightarrow \left[\begin{array}{cccc|c} 1-i & 1 & 1 & 0 & 0 \\ 0 & 1-i & 1 & 1 & 0 \\ 0 & -\frac{1}{2}-\frac{i}{2} & \frac{1}{2}-\frac{3i}{2} & 1 & 0 \\ 0 & \frac{1}{2}-\frac{i}{2} & -\frac{1}{2}-\frac{i}{2} & 1-i & 0 \end{array}\right]$$

$$R_3 = R_3 + \frac{iR_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} 1-i & 1 & 1 & 0 & 0 \\ 0 & 1-i & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2}-i & 1+\frac{i}{2} & 0 \\ 0 & \frac{1}{2}-\frac{i}{2} & -\frac{1}{2}-\frac{i}{2} & 1-i & 0 \end{array}\right]$$

$$R_4 = R_4 - \frac{R_2}{2} \Longrightarrow \left[\begin{array}{cccc|c} 1-i & 1 & 1 & 0 & 0 \\ 0 & 1-i & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2}-i & 1+\frac{i}{2} & 0 \\ 0 & 0 & -1-\frac{i}{2} & \frac{1}{2}-i & 0 \end{array}\right]$$

$$R_4 = iR_3 + R_4 \Longrightarrow \left[\begin{array}{cccc|c} 1-i & 1 & 1 & 0 & 0 \\ 0 & 1-i & 1 & 1 & 0 \\ 0 & 0 & \frac{1}{2}-i & 1+\frac{i}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore the system in Echelon form is

$$\begin{bmatrix} 1-i & 1 & 1 & 0 \\ 0 & 1-i & 1 & 1 \\ 0 & 0 & \frac{1}{2}-i & 1+\frac{i}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The free variables are $\{v_4\}$ and the leading variables are $\{v_1, v_2, v_3\}$. Let $v_4 = t$. Now we start back substitution. Solving the above equation for the leading variables in terms of free variables gives equation $\{v_1 = it, v_2 = -t, v_3 = -it\}$

Hence the solution is

$$\begin{bmatrix} \mathrm{I}t \\ -t \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} it \\ -t \\ -it \\ t \end{bmatrix}$$

Since there is one free Variable, we have found one eigenvector associated with this eigenvalue. The above can be written as

$$\begin{bmatrix} \mathrm{I}t \\ -t \\ \text{-I}t \\ t \end{bmatrix} = t \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}$$

Let $t = 1$ the eigenvector becomes

$$\begin{bmatrix} \mathrm{I}t \\ -t \\ \text{-I}t \\ t \end{bmatrix} = \begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}$$

The following table gives a summary of this result. It shows for each eigenvalue the algebraic multiplicity m, and its geometric multiplicity k and the eigenvectors associated with the eigenvalue. If $m > k$ then the eigenvalue is defective which means the number of normal linearly independent eigenvectors associated with this eigenvalue (called the geometric multiplicity k) does not equal the algebraic multiplicity m, and we need to determine an additional $m - k$ generalized eigenvectors for this eigenvalue.

eigenvalue	multiplicity algebraic m	multiplicity geometric k	defective?	eigenvectors
0	1	1	No	$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$
2	1	1	No	$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$
$-1+i$	1	1	No	$\begin{bmatrix} i \\ -1 \\ -i \\ 1 \end{bmatrix}$
$-1-i$	1	1	No	$\begin{bmatrix} -i \\ -1 \\ i \\ 1 \end{bmatrix}$

Now that we found the eigenvalues and associated eigenvectors, we will go over each eigenvalue and generate the solution basis. The only problem we need to take care of is if the eigenvalue is defective. Since eigenvalue 0 is real and distinct then the corresponding eigenvector solution is

$$\begin{aligned} \vec{x}_1(t) &= \vec{v}_1 e^0 \\ &= \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} e^0 \end{aligned}$$

Since eigenvalue 2 is real and distinct then the corresponding eigenvector solution is

$$\vec{x}_2(t) = \vec{v}_2 e^{2t}$$
$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} e^{2t}$$

Therefore the homogeneous solution is

$$\vec{x}_h(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + c_3\vec{x}_3(t) + c_4\vec{x}_4(t)$$

Which is written as

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = c_1 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^{2t} \\ e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} + c_3 \begin{bmatrix} i e^{(-1+i)t} \\ -e^{(-1+i)t} \\ -i e^{(-1+i)t} \\ e^{(-1+i)t} \end{bmatrix} + c_4 \begin{bmatrix} -i e^{(-1-i)t} \\ -e^{(-1-i)t} \\ i e^{(-1-i)t} \\ e^{(-1-i)t} \end{bmatrix}$$

Now that we found homogeneous solution above, we need to find a particular solution $\vec{x}_p(t)$. We will use Variation of parameters. The fundamental matrix is

$$\Phi = \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots \end{bmatrix}$$

Where $\vec{x}_i$ are the solution basis found above. Therefore the fundamental matrix is

$$\Phi(t) = \begin{bmatrix} -1 & e^{2t} & i e^{(-1+i)t} & -i e^{(-1-i)t} \\ 1 & e^{2t} & -e^{(-1+i)t} & -e^{(-1-i)t} \\ -1 & e^{2t} & -i e^{(-1+i)t} & i e^{(-1-i)t} \\ 1 & e^{2t} & e^{(-1+i)t} & e^{(-1-i)t} \end{bmatrix}$$

The particular solution is then given by

$$\vec{x}_p(t) = \Phi \int \Phi^{-1} \vec{G}(t) \, dt$$

But

$$\Phi^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{e^{-2t}}{4} & \frac{e^{-2t}}{4} & \frac{e^{-2t}}{4} & \frac{e^{-2t}}{4} \\ -\frac{i e^{(1-i)t}}{4} & -\frac{e^{(1-i)t}}{4} & \frac{i e^{(1-i)t}}{4} & \frac{e^{(1-i)t}}{4} \\ \frac{i e^{(1+i)t}}{4} & -\frac{e^{(1+i)t}}{4} & -\frac{i e^{(1+i)t}}{4} & \frac{e^{(1+i)t}}{4} \end{bmatrix}$$

Hence

$$
\vec{x}_p(t) = \begin{bmatrix} -1 & e^{2t} & i e^{(-1+i)t} & -i e^{(-1-i)t} \\ 1 & e^{2t} & -e^{(-1+i)t} & -e^{(-1-i)t} \\ -1 & e^{2t} & -i e^{(-1+i)t} & i e^{(-1-i)t} \\ 1 & e^{2t} & e^{(-1+i)t} & e^{(-1-i)t} \end{bmatrix} \int \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \\ \frac{e^{-2t}}{4} & \frac{e^{-2t}}{4} & \frac{e^{-2t}}{4} & \frac{e^{-2t}}{4} \\ -\frac{i e^{(1-i)t}}{4} & -\frac{e^{(1-i)t}}{4} & \frac{i e^{(1-i)t}}{4} & \frac{e^{(1-i)t}}{4} \\ \frac{i e^{(1+i)t}}{4} & -\frac{e^{(1+i)t}}{4} & -\frac{i e^{(1+i)t}}{4} & \frac{e^{(1+i)t}}{4} \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} dt
$$

$$
= \begin{bmatrix} -1 & e^{2t} & i e^{(-1+i)t} & -i e^{(-1-i)t} \\ 1 & e^{2t} & -e^{(-1+i)t} & -e^{(-1-i)t} \\ -1 & e^{2t} & -i e^{(-1+i)t} & i e^{(-1-i)t} \\ 1 & e^{2t} & e^{(-1+i)t} & e^{(-1-i)t} \end{bmatrix} \int \begin{bmatrix} \frac{(-1+t)(t^2+1)}{4} \\ \frac{e^{-2t}(t+1)(t^2+1)}{4} \\ \frac{(-1+t)(t+i)(t+1)e^{(1-i)t}}{4} \\ -\frac{(-1+t)(t+1)(i-t)e^{(1+i)t}}{4} \end{bmatrix} dt
$$

$$
= \begin{bmatrix} -1 & e^{2t} & i e^{(-1+i)t} & -i e^{(-1-i)t} \\ 1 & e^{2t} & -e^{(-1+i)t} & -e^{(-1-i)t} \\ -1 & e^{2t} & -i e^{(-1+i)t} & i e^{(-1-i)t} \\ 1 & e^{2t} & e^{(-1+i)t} & e^{(-1-i)t} \end{bmatrix} \begin{bmatrix} \frac{1}{16}t^4 - \frac{1}{12}t^3 + \frac{1}{8}t^2 - \frac{1}{4}t \\ -\frac{e^{-2t}(4t^3+10t^2+14t+11)}{32} \\ \frac{e^{(1-i)t}((1+i)t^3+(-1-2i)t^2+(-2+2i)t+3-i)}{8} \\ -\frac{((-1+i)t^3+(1-2i)t^2+(2+2i)t-3-i)e^{(1+i)t}}{8} \end{bmatrix}
$$

$$
= \begin{bmatrix} -\frac{1}{16}t^4 - \frac{7}{24}t^3 + \frac{1}{16}t^2 - \frac{11}{16}t - \frac{3}{32} \\ \frac{1}{16}t^4 - \frac{11}{24}t^3 + \frac{1}{16}t^2 - \frac{3}{16}t - \frac{35}{32} \\ -\frac{1}{16}t^4 + \frac{5}{24}t^3 - \frac{15}{16}t^2 + \frac{5}{16}t - \frac{19}{32} \\ \frac{1}{16}t^4 + \frac{1}{24}t^3 - \frac{7}{16}t^2 - \frac{19}{16}t + \frac{13}{32} \end{bmatrix}
$$

Now that we found particular solution, the final solution is

$$
\vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)
$$

$$
\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -c_1 \\ c_1 \\ -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 e^{2t} \\ c_2 e^{2t} \\ c_2 e^{2t} \\ c_2 e^{2t} \end{bmatrix} + \begin{bmatrix} i c_3 e^{(-1+i)t} \\ -c_3 e^{(-1+i)t} \\ -i c_3 e^{(-1+i)t} \\ c_3 e^{(-1+i)t} \end{bmatrix} + \begin{bmatrix} -i c_4 e^{(-1-i)t} \\ -c_4 e^{(-1-i)t} \\ i c_4 e^{(-1-i)t} \\ c_4 e^{(-1-i)t} \end{bmatrix} + \begin{bmatrix} -\frac{1}{16}t^4 - \frac{7}{24}t^3 + \frac{1}{16}t^2 - \frac{11}{16}t \\ \frac{1}{16}t^4 - \frac{11}{24}t^3 + \frac{1}{16}t^2 - \frac{3}{16}t \\ -\frac{1}{16}t^4 + \frac{5}{24}t^3 - \frac{15}{16}t^2 + \frac{5}{16}t \\ \frac{1}{16}t^4 + \frac{1}{24}t^3 - \frac{7}{16}t^2 - \frac{19}{16}t \end{bmatrix}
$$

Which becomes

$$\begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} = \begin{bmatrix} -c_1 + c_2 e^{2t} + i c_3 e^{(-1+i)t} - i c_4 e^{(-1-i)t} - \frac{t^4}{16} - \frac{7t^3}{24} + \frac{t^2}{16} - \frac{11t}{16} - \frac{3}{32} \\ c_1 + c_2 e^{2t} - c_3 e^{(-1+i)t} - c_4 e^{(-1-i)t} + \frac{t^4}{16} - \frac{11t^3}{24} + \frac{t^2}{16} - \frac{3t}{16} - \frac{35}{32} \\ -c_1 + c_2 e^{2t} - i c_3 e^{(-1+i)t} + i c_4 e^{(-1-i)t} - \frac{t^4}{16} + \frac{5t^3}{24} - \frac{15t^2}{16} + \frac{5t}{16} - \frac{19}{32} \\ c_1 + c_2 e^{2t} + c_3 e^{(-1+i)t} + c_4 e^{(-1-i)t} + \frac{t^4}{16} + \frac{t^3}{24} - \frac{7t^2}{16} - \frac{19t}{16} + \frac{13}{32} \end{bmatrix}$$

13.6.3 Maple step by step solution

Let's solve

$[x_1'(t) = x_2(t) + x_3(t) + 1, x_2'(t) = x_3(t) + x_4(t) + t, x_3'(t) = x_1(t) + x_4(t) + t^2, x_4'(t) = x_1(t) + x$

- Define vector

$$\underline{x}^{\rightarrow}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix}$$

- Convert system into a vector equation

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

- System to solve

$$\underline{x}^{\rightarrow}{}'(t) = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \cdot \underline{x}^{\rightarrow}(t) + \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

- Define the forcing function

$$\vec{f}(t) = \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

- Define the coefficient matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

- Rewrite the system as

$$\underline{\vec{x}}'(t) = A \cdot \underline{\vec{x}}(t) + \vec{f}$$

- To solve the system, find the eigenvalues and eigenvectors of A
- Eigenpairs of A

$$\left[\left[0, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right], \left[2, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right], \left[-1 - \mathrm{I}, \begin{bmatrix} -\mathrm{I} \\ -1 \\ \mathrm{I} \\ 1 \end{bmatrix}\right], \left[-1 + \mathrm{I}, \begin{bmatrix} \mathrm{I} \\ -1 \\ -\mathrm{I} \\ 1 \end{bmatrix}\right]\right]$$

- Consider eigenpair

$$\left[0, \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

- Consider eigenpair

$$\left[2, \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]\right]$$

- Solution to homogeneous system from eigenpair

$$\underline{\vec{x}}_{2} = e^{2t} \cdot \left[\begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array}\right]$$

- Consider complex eigenpair, complex conjugate eigenvalue can be ignored

$$\left[-1 - I, \left[\begin{array}{c} -I \\ -1 \\ I \\ 1 \end{array}\right]\right]$$

- Solution from eigenpair

$$e^{(-1-I)t} \cdot \left[\begin{array}{c} -I \\ -1 \\ I \\ 1 \end{array}\right]$$

- Use Euler identity to write solution in terms of sin and cos

$$e^{-t} \cdot (\cos(t) - I\sin(t)) \cdot \left[\begin{array}{c} -I \\ -1 \\ I \\ 1 \end{array}\right]$$

- Simplify expression

$$\mathrm{e}^{-t}\cdot\begin{bmatrix}-\mathrm{I}(\cos(t)-\mathrm{I}\sin(t))\\ -\cos(t)+\mathrm{I}\sin(t)\\ \mathrm{I}(\cos(t)-\mathrm{I}\sin(t))\\ \cos(t)-\mathrm{I}\sin(t)\end{bmatrix}$$

- Both real and imaginary parts are solutions to the homogeneous system

$$\left[\underline{\vec{x}}_3(t)=\mathrm{e}^{-t}\cdot\begin{bmatrix}-\sin(t)\\ -\cos(t)\\ \sin(t)\\ \cos(t)\end{bmatrix},\underline{\vec{x}}_4(t)=\mathrm{e}^{-t}\cdot\begin{bmatrix}-\cos(t)\\ \sin(t)\\ \cos(t)\\ -\sin(t)\end{bmatrix}\right]$$

- General solution of the system of ODEs can be written in terms of the particular solution $\underline{\vec{x}}$

$$\underline{\vec{x}}(t)=c_1\underline{\vec{x}}_1+c_2\underline{\vec{x}}_2+c_3\underline{\vec{x}}_3(t)+c_4\underline{\vec{x}}_4(t)+\underline{\vec{x}}_p(t)$$

☐ Fundamental matrix

- Let $\phi(t)$ be the matrix whose columns are the independent solutions of the homogeneous syst

$$\phi(t)=\begin{bmatrix}-1 & \mathrm{e}^{2t} & -\mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\cos(t)\\ 1 & \mathrm{e}^{2t} & -\mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t)\\ -1 & \mathrm{e}^{2t} & \mathrm{e}^{-t}\sin(t) & \mathrm{e}^{-t}\cos(t)\\ 1 & \mathrm{e}^{2t} & \mathrm{e}^{-t}\cos(t) & -\mathrm{e}^{-t}\sin(t)\end{bmatrix}$$

- The fundamental matrix, $\Phi(t)$ is a normalized version of $\phi(t)$ satisfying $\Phi(0)=I$ where I is th

$\Phi(t)=\phi(t)\cdot\frac{1}{\phi(0)}$

- Substitute the value of $\phi(t)$ and $\phi(0)$

$$\Phi(t)=\begin{bmatrix}-1 & \mathrm{e}^{2t} & -\mathrm{e}^{-t}\sin(t) & -\mathrm{e}^{-t}\cos(t)\\ 1 & \mathrm{e}^{2t} & -\mathrm{e}^{-t}\cos(t) & \mathrm{e}^{-t}\sin(t)\\ -1 & \mathrm{e}^{2t} & \mathrm{e}^{-t}\sin(t) & \mathrm{e}^{-t}\cos(t)\\ 1 & \mathrm{e}^{2t} & \mathrm{e}^{-t}\cos(t) & -\mathrm{e}^{-t}\sin(t)\end{bmatrix}\cdot\frac{1}{\begin{bmatrix}-1 & 1 & 0 & -1\\ 1 & 1 & -1 & 0\\ -1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}}$$

- Evaluate and simplify to get the fundamental matrix

$$\Phi(t)=\begin{bmatrix} \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} \\ -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} \\ -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} & \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} \\ \frac{e^{2t}}{4}-\frac{1}{4}+\frac{e^{-t}\sin(t)}{2} & -\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4}+\frac{1}{4} & -\frac{e^{-t}\sin(t)}{2}-\frac{1}{4}+\frac{e^{2t}}{4} & \frac{1}{4}+\frac{e^{-t}\cos(t)}{2}+\frac{e^{2t}}{4} \end{bmatrix}$$

- ☐ Find a particular solution of the system of ODEs using variation of parameters
  - Let the particular solution be the fundamental matrix multiplied by $\vec{v}(t)$ and solve for $\vec{v}(t)$

    $\vec{x}_p(t)=\Phi(t)\cdot\vec{v}(t)$
  - Take the derivative of the particular solution

    $\vec{x}_p{}'(t)=\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)$
  - Substitute particular solution and its derivative into the system of ODEs

    $\Phi'(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
  - The fundamental matrix has columns that are solutions to the homogeneous system so its der

    $A\cdot\Phi(t)\cdot\vec{v}(t)+\Phi(t)\cdot\vec{v}'(t)=A\cdot\Phi(t)\cdot\vec{v}(t)+\vec{f}(t)$
  - Cancel like terms

    $\Phi(t)\cdot\vec{v}'(t)=\vec{f}(t)$
  - Multiply by the inverse of the fundamental matrix

    $\vec{v}'(t)=\frac{1}{\Phi(t)}\cdot\vec{f}(t)$
  - Integrate to solve for $\vec{v}(t)$

    $\vec{v}(t)=\int_0^t \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds$
  - Plug $\vec{v}(t)$ into the equation for the particular solution

    $\vec{x}_p(t)=\Phi(t)\cdot\left(\int_0^t \frac{1}{\Phi(s)}\cdot\vec{f}(s)\,ds\right)$
  - Plug in the fundamental matrix and the forcing function and compute

$$\vec{x}_p(t)=\begin{bmatrix} -\frac{3}{32}+\frac{(3\sin(t)-\cos(t))e^{-t}}{4}-\frac{t^4}{16}-\frac{7t^3}{24}+\frac{t^2}{16}-\frac{11t}{16}+\frac{11e^{2t}}{32} \\ -\frac{35}{32}+\frac{(\sin(t)+3\cos(t))e^{-t}}{4}+\frac{t^4}{16}-\frac{11t^3}{24}+\frac{t^2}{16}-\frac{3t}{16}+\frac{11e^{2t}}{32} \\ -\frac{19}{32}+\frac{(-3\sin(t)+\cos(t))e^{-t}}{4}-\frac{t^4}{16}+\frac{5t^3}{24}-\frac{15t^2}{16}+\frac{5t}{16}+\frac{11e^{2t}}{32} \\ \frac{13}{32}+\frac{(-\sin(t)-3\cos(t))e^{-t}}{4}+\frac{t^4}{16}+\frac{t^3}{24}-\frac{7t^2}{16}-\frac{19t}{16}+\frac{11e^{2t}}{32} \end{bmatrix}$$

- Plug particular solution back into general solution

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3(t) + c_4\vec{x}_4(t) + \left[\begin{array}{c} -\frac{3}{32} + \frac{(3\sin(t)-\cos(t))e^{-t}}{4} - \frac{t^4}{16} - \frac{7t^3}{24} + \\ -\frac{35}{32} + \frac{(\sin(t)+3\cos(t))e^{-t}}{4} + \frac{t^4}{16} - \frac{11t^3}{24} + \\ -\frac{19}{32} + \frac{(-3\sin(t)+\cos(t))e^{-t}}{4} - \frac{t^4}{16} + \frac{5t^3}{24} - \\ \frac{13}{32} + \frac{(-\sin(t)-3\cos(t))e^{-t}}{4} + \frac{t^4}{16} + \frac{t^3}{24} - \end{array}\right.$$

- Substitute in vector of dependent variables

$$\left[\begin{array}{c} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{array}\right] = \left[\begin{array}{c} -\frac{3}{32} + \frac{((-1-4c_4)\cos(t)+(3-4c_3)\sin(t))e^{-t}}{4} + \frac{(11+32c_2)e^{2t}}{32} - \frac{t^4}{16} - \frac{7t^3}{24} + \frac{t^2}{16} - \frac{11t}{16} - c_1 \\ -\frac{35}{32} + \frac{((3-4c_3)\cos(t)+\sin(t)(1+4c_4))e^{-t}}{4} + \frac{(11+32c_2)e^{2t}}{32} + \frac{t^4}{16} - \frac{11t^3}{24} + \frac{t^2}{16} - \frac{3t}{16} + c_1 \\ -\frac{19}{32} + \frac{((1+4c_4)\cos(t)+\sin(t)(4c_3-3))e^{-t}}{4} + \frac{(11+32c_2)e^{2t}}{32} - \frac{t^4}{16} + \frac{5t^3}{24} - \frac{15t^2}{16} + \frac{5t}{16} - c_1 \\ \frac{13}{32} + \frac{((4c_3-3)\cos(t)+(-1-4c_4)\sin(t))e^{-t}}{4} + \frac{(11+32c_2)e^{2t}}{32} + \frac{t^4}{16} + \frac{t^3}{24} - \frac{7t^2}{16} - \frac{19t}{16} + c_1 \end{array}\right]$$

- Solution to the system of ODEs

$$\left\{x_1(t) = -\frac{3}{32} + \frac{((-1-4c_4)\cos(t)+(3-4c_3)\sin(t))e^{-t}}{4} + \frac{(11+32c_2)e^{2t}}{32} - \frac{t^4}{16} - \frac{7t^3}{24} + \frac{t^2}{16} - \frac{11t}{16} - c_1, x_2(t) = -\frac{35}{32}\right.$$

✓ Solution by Maple

Time used: 0.203 (sec). Leaf size: 273

```
dsolve([diff(x__1(t),t)=x__2(t)+x__3(t)+1,diff(x__2(t),t)=x__3(t)+x__4(t)+t,diff(x__3(t),t)=
```

$$x_1(t) = \frac{t^2}{16} - \frac{7t^3}{24} - \frac{t^4}{16} + \frac{c_1 e^{2t}}{2} - \frac{11t}{16} + c_4 + \frac{e^{-t}\sin(t)\, c_2}{2} - \frac{e^{-t}\sin(t)\, c_3}{2} - \frac{e^{-t}\cos(t)\, c_2}{2} - \frac{e^{-t}\cos(t)\, c_3}{2}$$

$$x_2(t) = \frac{t^4}{16} + \frac{e^{-t}\sin(t)\, c_2}{2} + \frac{e^{-t}\sin(t)\, c_3}{2} + \frac{e^{-t}\cos(t)\, c_2}{2} - \frac{e^{-t}\cos(t)\, c_3}{2} - \frac{11t^3}{24} + \frac{c_1 e^{2t}}{2} + \frac{t^2}{16} - c_4 - \frac{3t}{16} - \frac{19}{16}$$

$$x_3(t) = -\frac{t^4}{16} - \frac{e^{-t}\sin(t)\, c_2}{2} + \frac{e^{-t}\sin(t)\, c_3}{2} + \frac{e^{-t}\cos(t)\, c_2}{2} + \frac{e^{-t}\cos(t)\, c_3}{2} + \frac{5t^3}{24} + \frac{c_1 e^{2t}}{2} - \frac{15t^2}{16} + c_4 + \frac{5t}{16} - \frac{1}{2}$$

$$x_4(t) = \frac{t^4}{16} - \frac{e^{-t}\sin(t)\, c_2}{2} - \frac{e^{-t}\sin(t)\, c_3}{2} - \frac{e^{-t}\cos(t)\, c_2}{2} + \frac{e^{-t}\cos(t)\, c_3}{2} + \frac{t^3}{24} + \frac{c_1 e^{2t}}{2} - \frac{7t^2}{16} - c_4 - \frac{19t}{16} + \frac{5}{16}$$

✓ Solution by Mathematica

Time used: 1.491 (sec). Leaf size: 442

```
DSolve[{x1'[t]==x2[t]+x3[t]+1,x2'[t]==x3[t]+x4[t]+t,x3'[t]==x1[t]+x4[t]+t^2,x4'[t]==x1[t]+x2
```

$$
\begin{aligned}
\mathrm{x1}(t) \to \frac{1}{96}e^{-t}\Big(&e^t\big(-6t^4-28t^3+6t^2-66t \\
&+3\big(8c_1\left(e^{2t}+1\right)+8c_2\left(e^{2t}-1\right)+8c_3e^{2t}+8c_4e^{2t}-3+8c_3-8c_4\big)\big) \\
&+48(c_1-c_3)\cos(t)+48(c_2-c_4)\sin(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{x2}(t) \to \frac{1}{96}e^{-t}\Big(&e^t\big(6t^4-44t^3+6t^2-18t \\
&+3\big(8c_1\left(e^{2t}-1\right)+8c_2\left(e^{2t}+1\right)+8c_3e^{2t}+8c_4e^{2t}-35-8c_3+8c_4\big)\big) \\
&+48(c_2-c_4)\cos(t)-48(c_1-c_3)\sin(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{x3}(t) \to \frac{1}{96}e^{-t}\Big(&e^t\big(-6t^4+20t^3-90t^2+30t \\
&+3\big(8c_1\left(e^{2t}+1\right)+8c_2\left(e^{2t}-1\right)+8c_3e^{2t}+8c_4e^{2t}-19+8c_3-8c_4\big)\big) \\
&-48(c_1-c_3)\cos(t)-48(c_2-c_4)\sin(t)\Big)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{x4}(t) \to \frac{1}{96}e^{-t}\Big(&e^t\big(6t^4+4t^3-42t^2-114t \\
&+3\big(8c_1\left(e^{2t}-1\right)+8c_2\left(e^{2t}+1\right)+8c_3e^{2t}+8c_4e^{2t}+13-8c_3+8c_4\big)\big) \\
&-48(c_2-c_4)\cos(t)+48(c_1-c_3)\sin(t)\Big)
\end{aligned}
$$